Dynamic ray tracing in triangulated subsurface models

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ABSTRACT

The simplicity and efficiency of dynamic ray tracing is enhanced by triangulated models of the Earth's subsurface. When the inverse of velocity-squared varies linearly within each triangle of such models, rays are piecewise parabolic and the dynamic ray tracing equations may be solved analytically. This paper provides these solutions and demonstrates one application of dynamic ray tracing in triangulated subsurface models — the computation of synthetic seismograms via the method of Gaussian beams.

INTRODUCTION

The triangulated subsurface model illustrated in Figure 1 was constructed using the methods described by Hale and Cohen (1991). Such models have been proposed for seismic ray tracing (e.g., Chapman, 1985), in part, because rays are circular in triangles within which velocity varies linearly. Here, we let the inverse of velocity-squared vary linearly, as suggested by Červený (1987), so that the rays shown in Figure 2 are parabolic within each triangle. Note that velocity gradients in each triangle permit rays to refract and even turn beyond 90 degrees before and after reflection from the salt dome.

Ray tracing is particularly simple in triangulated subsurface models, like that shown in Figure 1, that are represented by data structures that contain the adjacency topology of the model (Hale and Cohen, 1991). The adjacency topology enables one to easily determine the triangles that are adjacent to any edge, and this information is used repeatedly in ray tracing as rays leave one triangle by crossing an edge and entering the adjacent triangle. Information associated with the edge data structure is also used to determine whether or not a particular edge is a reflector. The intersection of rays with reflectors, which can be an awkward computation for some subsurface representations, is simple and efficient for triangulated subsurface models.
Fig. 1. Triangulated two-dimensional model of the Earth's subsurface. (The model extends to the right of that portion displayed here.) Dark shading within the triangles corresponds to high velocity; e.g., the velocity in the sediments increases with depth.

One important application of seismic ray tracing in triangulated subsurface models is the computation of synthetic seismograms, as suggested, for example, by Müller (1984) and Weber (1988). They advocate the method of Gaussian beams (Červený et al., 1982), based on ray tracing in triangles with linear velocity. The Gaussian beam method is attractive for several reasons, not the least of which is that no two-point ray tracing is required. One simply shoots a fan of rays, like that illustrated in Figure 2, and then computes seismograms as a weighted sum of the beams represented by each ray.

Figure 3 illustrates a common-shot gather corresponding to the experiment depicted in Figure 2, in which slowness-squared (inverse of velocity-squared) is linear in each triangle. For these seismograms, 601 rays were traced, with take-off angles ranging from -90 to 90 degrees. Each of these 601 rays contributes to those receivers nearest the location where the ray emerges at the surface, according to a frequency-dependent beam width. Each seismogram corresponds to a receiver located at the distance indicated along the surface.

As one test of the accuracy of the Gaussian beam method used here, the common-midpoint (CMP) gather in Figure 4 was computed for a source-receiver midpoint at
Fig. 2. Rays and wavefronts for the subsurface model of Figure 1, with a source located at the surface at a horizontal distance of 5 km. Rays are plotted in white, and wavefronts (points with equal traveltimes) in black. Note the rays that turn past 90 degrees before and after reflection from the salt overhang. The waves that pass through the focus at a depth of about 2.3 km correspond to the strong reflection arriving after 3.7 s in Figure 3. For simplicity, the triangles in Figure 1 are not shown here.

5 km. Reciprocity requires that these seismograms be symmetric about zero-offset, and, in this case, reciprocity appears to be reasonably well-satisfied. Although not sufficient to guarantee accuracy, a reciprocity test such as this might have indicated that the method had failed, if significant asymmetry had been observed. (See, for example, Nowack and Aki, 1984.) Note that the reflection from the salt at 4.3 s exhibits virtually no moveout in this CMP gather, as expected for a more or less vertical reflector embedded in a subsurface with almost no lateral velocity variation.

Figures 1 through 4 serve as an introduction to ray tracing in triangulated subsurface models, but they leave several questions unanswered. What equations describe the ray paths and traveltimes? What equations describe the geometric spreading effects on wave amplitudes? How does one account for 90-degree phase shifts? How does one compute the beam corresponding to each ray? Do interfaces with discontinuous slopes, formed by linear edges of triangles, pose any problems in ray tracing?
FIG. 3. Synthetic seismograms computed via the method of Gaussian beams, for the experiment depicted in Figure 2. The strongest events correspond to refracted direct waves. The strongest reflection, between 3.7 and 4.6 s, exhibits a 90-degree phase shift, and corresponds to the waves that pass through the focus seen in Figure 2 at a depth of about 2.3 km. The weaker reflection between 2.5 and 4 s, with a symmetric wavelet, corresponds to waves that did not pass through a focus after reflection from the salt.

Some of these questions have been answered elsewhere, and will be addressed below with only a brief reference or summary. The primary purpose of this paper is to fill in some gaps in previous publications regarding the dynamic ray tracing equations and to discuss practical aspects of their solution in triangulated models. Solutions to these equations are important, because dynamic ray tracing facilitates the computation of, for example, wavefront curvature, reflection amplitude and phase, reflection moveout in CMP gathers, and Gaussian beams.

RAY TRACING

Following Červený (1987), we express the ray tracing equations in two dimensions as follows:

$$\frac{dx}{d\sigma} = p_x$$
FIG. 4. Synthetic common-midpoint gather, for a source-receiver midpoint located at 5 km. As a test of accuracy, seismograms for both negative and positive source-receiver offsets were computed. Here, reciprocity is well-satisfied, since asymmetries about zero-offset are relatively small.

\[ \frac{dz}{d\sigma} = p_z, \]
\[ \frac{dp_z}{d\sigma} = \frac{1}{2} \frac{\partial}{\partial x} v^{-2}, \]
\[ \frac{dp_z}{d\sigma} = \frac{1}{2} \frac{\partial}{\partial z} v^{-2}, \]
\[ \frac{dt}{d\sigma} = v^{-2}, \]

where \( v \) denotes velocity, \( x \) and \( z \) denote the Cartesian coordinates of the ray, \( p_x \) and \( p_z \) are components of the slowness vector in the \( x \) and \( z \) dimensions, respectively, and \( t \) is travelt ime. \( \sigma \) is a parameter that increases monotonically along the ray, as implied by the last of equations (1).

The reason for writing the ray tracing equations in this form, is that the partial derivatives of \( v^{-2} \) in equations (1) are constants when \( v^{-2} \) is a linear function of \( x \) and \( z \). Specifically, if we define "sloth", the inverse of velocity-squared, as follows

\[ v^{-2}(x, z) \equiv s(x, z) = s_{00} + s_{,x}x + s_{,z}z, \]
then the solutions to equations (1) are

\[ x(\sigma) = x(\sigma_0) + p_x(\sigma_0)(\sigma - \sigma_0) + \frac{1}{4}s_{xx}(\sigma - \sigma_0)^2 \]
\[ z(\sigma) = z(\sigma_0) + p_z(\sigma_0)(\sigma - \sigma_0) + \frac{1}{4}s_{zz}(\sigma - \sigma_0)^2 \]
\[ p_x(\sigma) = p_x(\sigma_0) + \frac{1}{2}s_{xx}(\sigma - \sigma_0) \]
\[ p_z(\sigma) = p_z(\sigma_0) + \frac{1}{2}s_{zz}(\sigma - \sigma_0) \]
\[ t(\sigma) = t(\sigma_0) + \left[s_{00} + s_{xx}x(\sigma_0) + s_{xz}z(\sigma_0)\right](\sigma - \sigma_0) + \frac{1}{2}\left[s_{xx}p_x(\sigma_0) + s_{xz}p_z(\sigma_0)\right](\sigma - \sigma_0)^2 + \frac{1}{12}(s_{xx}^2 + s_{xz}^2)(\sigma - \sigma_0)^3. \] (3)

Červený (1987) noted that these solutions, requiring only the evaluation of polynomials, are much simpler than the solutions one obtains when velocity (instead of slant) varies linearly.

The raypath described by equations (3) is a parabola, and the intersection of this raypath with the linear edge of triangle can be easily computed by solving an equation that is quadratic in \( \sigma \). The number of real roots of this quadratic equation is either zero, one, or two, corresponding to the number of possible intersections. To determine the edge at which a ray exits a triangle, one must solve three quadratic equations for \( \sigma \), one for each edge of the triangle. Then, letting \( \sigma_0 \) denote the value of \( \sigma \) at the point where a ray enters the triangle, the value of \( \sigma \) where the ray exits is the smallest real root of these three quadratic equations that is greater than \( \sigma_0 \). The value of \( \sigma \) at the exit point then yields the parameters in equations (3) at that point.

**Dynamic ray tracing equations**

In addition to the solution of equations (1), dynamic ray tracing in two dimensions requires the solution of the following system of coupled differential equations (Červený, 1987):

\[ \frac{dq}{d\sigma} = p \]
\[ \frac{dp}{d\sigma} = -\frac{v_{nn}}{v^3}q, \] (4)

where \( v_{nn} \) denotes the second partial derivative of velocity \( v \) with respect to distance \( n \) in a direction normal to the ray. Given a central ray described by equations (3), parameters \( p \) and \( q \) describe a nearby (paraxial) ray. Specifically, \( p \) is the partial derivative of traveltime \( t \) with respect to \( n \) for the paraxial ray, and \( q \) is the normal distance \( n \) from the central ray to the paraxial ray. The ratio \( p/q \) is the second partial
derivative of $t$ with respect to $n$, and is related to the the curvature of the wavefront normal to the ray.

Noting that $p_xv$ and $p_zv$ are the $x$ and $z$ components of the unit vector tangent to the ray, and using the definition of linear slope in equation (2) above, equations (4) may be expressed as

$$\frac{dq}{d\sigma} = p$$
$$\frac{dp}{d\sigma} = -\frac{3}{4}\left(\frac{s_xp_z - s_zp_x}{s^2}\right)^2q.$$  \hspace{1cm} (5)

The solutions to these equations are

$$q(\sigma) = \frac{1}{\sqrt{s_0s}} \left\{ \left[ s_0 + \frac{s_1(\sigma - \sigma_0)}{2} \right] q(\sigma_0) + p(\sigma_0)(\sigma - \sigma_0) \right\}$$
$$+ \left( \frac{s_1^2}{4s_0} - s_2 \right) q(\sigma_0)(\sigma - \sigma_0)^2 \right\}$$

$$p(\sigma) = \frac{1}{\sqrt{s_0s}} \left\{ \left[ s_0 + s_1(\sigma - \sigma_0) \right] p(\sigma_0) + \left[ \frac{s_1}{2} + \left( \frac{s_1^2}{2s_0} - 2s_2 \right)(\sigma - \sigma_0) \right] q(\sigma_0) \right\}$$
$$- \left[ \frac{s_1 + 2s_2(\sigma - \sigma_0)}{2s} \right] q(\sigma),$$  \hspace{1cm} (6)

where $s_0$, $s_1$, and $s_2$ are constants defined by

$$s_0 \equiv s[x(\sigma_0), z(\sigma_0)],$$
$$s_1 \equiv s_xp_x(\sigma_0) + s_zp_z(\sigma_0),$$
$$s_2 \equiv \frac{1}{4}\left( s_x^2 + s_z^2 \right),$$

so that

$$s = s[x(\sigma), z(\sigma)] = s_0 + s_1(\sigma - \sigma_0) + s_2(\sigma - \sigma_0)^2.$$  Given the values $q(\sigma_0)$ and $p(\sigma_0)$ at the point where a ray enters a triangle, equations (6) yield the values $q(\sigma)$ and $p(\sigma)$ at the point where the ray exits that triangle.

Although the derivation of solutions (6) to the dynamic ray tracing equations (5) is tedious and will therefore be omitted here, the validity of the solutions may be verified by direct substitution of the solutions into the differential equations. Even this verification is tedious if performed manually; I used a computer program to differentiate and verify the solutions symbolically.

**The KMAH index**

Most applications of dynamic ray tracing equations require the solutions $q(\sigma)$ and $p(\sigma)$ of equations (5) for the initial conditions $q(0) = 0$ and $p(0) = 1$. Following
Červený et al. (1982) the solution corresponding to these initial conditions are usually labeled $q_2(\sigma)$ and $p_2(\sigma)$. The computation of Gaussian beams requires, in addition, the solutions corresponding to $q(0) = 1$ and $p(0) = 0$, which are usually labeled $q_1(\sigma)$ and $p_1(\sigma)$. The solutions $q_1(\sigma)$ and $p_1(\sigma)$ correspond to a plane wave source, and the solutions $q_2(\sigma)$ and $p_2(\sigma)$ correspond to a point source.

Evaluation of the phase of synthetic seismograms requires that one count the number of zero crossings of the function $q_2(\sigma)$. This number is called the KMAH index (e.g., Chapman, 1985) and equals the number of times that a paraxial ray from a point source crosses its central ray. A $\pi/2$ phase shift is accumulated for each such crossing.

Because the raypaths for linearly varying slope are parabolic, zero, one, or two zero crossings are possible inside a triangle, as indicated in Figure 5. Therefore, the KMAH index increases by zero, one, or two as a ray travels within any triangle. This observation is consistent with the solution $q(\sigma)$ of equations (6), which has a numerator that is quadratic in $\sigma$. The number of zero crossings within any triangle equals the number of real roots $\sigma$ of the quadratic equation $q(\sigma) = 0$ that are greater than $\sigma_0$.

![Figure 5](image)

**Fig. 5.** Because raypaths are parabolic when slope varies linearly inside a triangle, they may cross once, twice, or not at all. The number of times a paraxial ray crosses its central ray (the KMAH index) equals the number of $\pi/2$ phase shifts that must be accumulated in the computation of synthetic seismograms.

Equivalently, letting $q_0$ and $p_0$ denote the values of $q_2(\sigma)$ and $p_2(\sigma)$ when a ray enters a triangle, and letting $q$ and $p$ denote the corresponding values when the ray exits that triangle, the following algorithm may be used to update the KMAH index:

\[
\begin{align*}
\text{if } (q_0 q \geq 0 \text{ and } p_0 p < 0 \text{ and } q_0 p_0 < 0) \\
\text{KMAH} &= \text{KMAH} + 2 \\
\text{else if } (q = 0 \text{ or } q_0 q < 0 \text{ or } (q_0 = 0 \text{ and } p_0 p < 0 \text{ and } q p > 0)) \\
\text{KMAH} &= \text{KMAH} + 1
\end{align*}
\]

This algorithm was derived by enumerating all of the possible ray crossings inside a triangle, including those illustrated in Figure 5, but also accounting for zero crossings...
that occur exactly at either the entry or exit points. Those wishing to verify the algorithm above should note that rays are diverging if \( pq > 0 \), converging if \( pq < 0 \), and parallel if \( p = 0 \).

**Tangent vectors and curvatures**

The computer program for dynamic ray tracing that was used to compute synthetic seismograms for the salt dome model above consists of three basic functions that (1) trace a ray inside a triangle, (2) trace a ray across an edge, and (3) reflect a ray from an edge. The discussion above pertains to the first function, that of dynamic ray tracing inside any triangle in which sloth varies linearly. A thorough derivation of transmission and reflection conditions for dynamic ray tracing at an interface was provided by Červený and Pšencík (1984) and will not be repeated here. Rather, the discussion below pertains to the manner in which one computes the parameters necessary to implement these conditions when performing dynamic ray tracing in triangulated subsurface models.

As shown by Červený and Pšencík (1984), transmission and reflection of dynamic rays at an interface require the tangent vector and curvature of the interface at the ray intersection point. In triangulated subsurface models, an interface consists of linear edges of triangles, so that the tangent vector is simply the edge itself and the curvature is zero. However, the discontinuity of the tangent vector along such an interface may be unacceptable for seismic ray tracing. Although Weber (1988) suggests that such discontinuities may be tolerable in the computation of synthetic seismograms via the method of Gaussian beams, most applications of ray tracing in triangulated subsurface models will require smooth approximations to interfaces.

Smooth approximation of an interface requires sufficient sampling of the interface so that changes in adjacent edge slopes are less than a specified tolerance. For example, the maximum discontinuity in edge slope for adjacent edges in the interfaces (black curves) of Figure 1 are less than the tangent of 5 degrees. These interfaces were specified by parametric cubic splines with continuous first and second derivatives, so that fine sampling of each interface is guaranteed to eventually reduce the maximum discontinuity in slope to a value below some finite tolerance. As indicated in Figure 1, the finest sampling (hence, the greatest number of vertices) is required where an interface exhibits high curvature.

After a satisfactory approximation to each interface has been obtained, the triangulation is constructed using the methods outlined by Hale and Cohen (1991). During this construction, the tangent vector and curvature (determined from the spline coefficients) at each vertex along the interface are attached as attributes of the edge-uses corresponding to that vertex. These attributes are attached to the edge-uses instead of to the vertices themselves, because each vertex is used by more than one edge, i.e., two or more interfaces may be joined at any given vertex, as illustrated in Figure 1.

Figure 6 illustrates the data structures used to determine the tangent vector and curvature at any point along an edge. Each edge is used by two triangles, and the tangent vector and curvature of an interface at the endpoints of an edge are stored
as attributes of the edge-uses of that edge. The tangent vector and curvature at any point between the edge endpoints are computed by linear interpolation of these attributes. These values are then used to accomplish transmission or reflection via dynamic ray tracing at the interface.

**Fig. 6.** The tangent vector and curvature at the endpoints of an edge are stored as attributes of the uses of that edge. Each edge-use knows the edge-use and, hence, the attributes of the edge-use on the other side of the edge. Linear interpolation is used to compute the attributes between the endpoints of the edge. Attributes are stored only for edges that are part of an interface, across which sloth may be discontinuous.

The edge-use attributes illustrated in Figure 6 are computed and stored only along interfaces, the black edges in Figure 1, because discontinuities in sloth are permitted to occur only across these edges. For most edges in triangulated subsurface models, such as the white edges in Figure 1, the tangent vector and curvature attributes are not required for dynamic ray tracing. For example, curvature is required in dynamic ray tracing across an interface only if sloth is discontinuous across that interface.

**OVERTHRUST MODEL**

The salt dome model discussed in the introduction (Figure 1) represents a fairly simple subsurface model, in contrast with the model of overthrust geology in Figure 7. The latter model contains numerous discontinuities in velocity that severely distort the raypaths and wavefronts plotted in Figure 8.

Synthetic seismograms for the overthrust model are plotted in Figure 9. As for the salt model, this common-shot gather was obtained by tracing 601 rays, with takeoff
Fig. 7. An overthrust model with significant velocity variations. Dark shading corresponds to high velocity. Within each layer, velocity increases with depth. The lowest velocity (at the surface) is 2.5 km/s, and the highest velocity (at a depth of 2.4 km) is 5.2 km/s.

angles between -90 and 90 degrees. Each ray contributed to the receivers nearest its emergence point, according to a frequency-dependent Gaussian beam width.

Although the synthetic seismograms in Figure 9 seem plausible (for example, amplitudes are strongest where emerging rays are focused), they are suspect because the synthetic common-midpoint gather plotted in Figure 10 is not symmetric about zero-offset, thereby violating the principle of reciprocity. There are at least two reasons for this failure. First, no attempt has yet been made to account for energy reflected at interfaces representing discontinuities in acoustic impedance. While the effects on geometrical spreading losses due to such interfaces have been accounted for, the loss of energy reflected by impedance discontinuities has not been included in the computation of these seismograms. Enhancements to the modeling software to account for reflected energy are not difficult, and will be made in the near future.

Even with such enhancements, however, there is a second reason to expect that subsurface models like the overthrust model in Figure 7 may pose difficulties in the computation of synthetic seismograms via Gaussian beams. The requirement that subsurface parameters vary smoothly relative to the width of a Gaussian beam (e.g.,
Fig. 8. Rays and wavefronts for the subsurface model of Figure 7, with a source located at the surface at horizontal distance of 1.6 km. Rays are plotted in white, and wavefronts (points with equal traveltime) in black. Note that both downgoing and upgoing rays are terminated when they are incident with an angle greater than the critical angle.

White et al. (1987) is not well satisfied for such models. A beam attached to a ray trajectory that passes near (but does not actually cross) a velocity discontinuity violates this assumption, and does not provide an accurate description of the wavefield in the neighborhood of the ray.

CONCLUSION

Analytical solutions to the dynamic ray tracing equations and the simple algorithm for updating the KMAH index provided here facilitate dynamic ray tracing for triangulated subsurface models. When such models are represented using the data structures described by Hale and Cohen (1991), dynamic ray tracing is particularly simple and efficient.

When velocity varies smoothly, as in the salt dome model of this paper, dynamic ray tracing enables the accurate computation of synthetic seismograms via the method of Gaussian beams. However, for models with significant discontinuities in velocity, such as the overthrust model shown here, synthetic seismograms are not accurate.
Fig. 9. Synthetic seismograms computed via the method of Gaussian beams, for the experiment depicted in Figure 8. The reciprocity test in Figure 10 suggests that these seismograms are inaccurate. This inaccuracy is caused, in part, by not accounting for the loss of energy reflected by discontinuities in impedance at some interfaces.

In particular, these seismograms violate the principle of reciprocity. This failure for discontinuous models is caused, in part, to the fact that reductions in wave amplitude due to transmission across discontinuities in impedance have not yet been included in synthetic seismogram computations.

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Fig. 10. Synthetic common-midpoint gather, for a source-receiver midpoint located at 1.6 km. Asymmetry here suggests that the seismograms of Figure 9 are not accurate.

REFERENCES


