Joint traveltime inversion of $P$- and $PS$-waves in orthorhombic media

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ABSTRACT

Reflection traveltimes recorded over azimuthally anisotropic fractured media can provide valuable information for reservoir characterization. As shown by Grechka and Tsvarkin, normal-moveout (NMO) velocity of any pure mode is described by an ellipse in the horizontal plane and, therefore, depends on only three combinations of the medium parameters. Hence, it is advantageous to use moveout velocities of several modes in resolving the coefficients of realistic orthorhombic or lower-symmetry fractured models.

Joint inversion of $P$ and $PS$ traveltimes is especially attractive because it does not require expensive shear-wave excitation. Here, we show that the azimuthal dependence of NMO velocity of converted waves in arbitrary anisotropic media also represents an ellipse that is not (in contrast with the pure-mode case) necessarily centered at the CMP location. This shift, however, vanishes for models composed of plane layers with a horizontal symmetry plane because the converted-wave traveltime becomes reciprocal with respect to the source and receiver positions. The converted-wave NMO ellipse for these models can be expressed through the NMO ellipses of the pure $P$- and $S$-waves, which provides a basis for obtaining shear-wave information from $P$ and $PS$ data. For orthorhombic models, combination of the reflection traveltimes of the $P$-wave and two split $PS$-waves makes it possible to reconstruct the azimuthally dependent NMO velocities of the shear waves and obtain the anisotropic parameters which cannot be found from $P$-wave data alone.

The method is applied to a physical-modeling data set acquired over a block of orthorhombic material Phenolite XX-324. The inversion of conventional-spread $P$ and $PS$ moveout data (in combination with the known reflector depth) allowed us to find the orientation of the vertical symmetry planes and eight (out of nine) elastic parameters of the medium. The remaining coefficient ($c_{12}$ or $\delta^{(3)}$ in Tsvarkin's notation) is obtained from the direct $P$-wave arrival in the horizontal plane. The inversion results accurately predict moveout curves of the pure $S$-waves and are in excellent agreement with direct measurements of the horizontal velocities.

Key words: converted waves, orthorhombic media, inversion of NMO ellipses, physical modeling of reflection data

Introduction

Analysis of stacking (moveout) velocities obtained from reflection traveltimes is routinely used in building starting velocity models for migration and inversion. Moveout velocities in 3-D surveys are often found to be azimuthally dependent, which is usually attributed to the influence of subsurface structure or lateral velocity variation. Another possible reason for azimuthal changes in reflection traveltimes is the presence of azimuthal anisotropic...
sotropy associated with fracture systems or transversely isotropic (TI) layers with a tilted symmetry axis. Substantial azimuthal variations in normal moveout of P-waves\textsuperscript{*} caused by fracture-induced azimuthal anisotropy were demonstrated on field data by Lynn et al. (1996). An analytic tool for quantitative interpretation of azimuthal moveout anomalies was provided by Grechka and Tsvankin (1996), who showed that the normal-moveout (NMO) velocity of any pure (non-converted) reflection mode typically represents an ellipse in the horizontal plane. The orientation and semi-axes of the NMO ellipse are influenced by both the reflector geometry and properties of the overburden and, therefore, can be inverted for the medium parameters. Closed-form expressions for the NMO ellipses from horizontal and dipping interfaces in arbitrary anisotropic media were given by Grechka et al. (1997).

Here, we focus on normal moveout of pure and converted waves in Media with orthorhombic (or orthotropic) symmetry. Orthorhombic anisotropy is believed to be typical for fractured formations and can be caused, for instance, by two orthogonal vertical crack systems or parallel vertical cracks in a TI matrix with a vertical symmetry axis (Wild and Crampin, 1991; Schoenberg and Helbig, 1997). Orthorhombic media have three mutually orthogonal symmetry planes, in which the Christoffel equation has the same form as in TI models with a vertical symmetry axis (VTI media). This analogy was used by Tsvankin (1996) to introduce a notation for orthorhombic media based on the same principle as the well-known Thomsen parameters for vertical transverse isotropy. Tsvankin’s notation is especially advantageous for moveout inversion because it reduces the number of parameters responsible for P-wave traveltime from nine to six and simplifies description of normal-moveout velocity both within and outside symmetry planes. As shown by Grechka and Tsvankin (1996), the semi-axes of the P-wave NMO ellipse in a horizontal orthorhombic layer are aligned with the symmetry planes and depend on just three parameters introduced by Tsvankin (1996): the vertical velocity $V_{T0}$ and the anisotropic coefficients $\delta^{(1,2)}$. (If the vertical velocity is unknown, P-wave NMO velocity from horizontal reflectors can be inverted for the orientation of the symmetry planes and the NMO velocities $V_{nmo}^{(1,2)}$ within them.) Additional information for traveltime inversion can be provided by P-wave NMO velocities of dipping events (discussed in detail by Grechka and Tsvankin, 1997), which depend on $V_{nmo}^{(1,2)}$ and three “anelipticity” coefficients $\eta^{(1,2,3)}$ defined in the symmetry planes. Grechka and Tsvankin (1997) developed an inversion scheme designed to obtain these five parameters from P-wave NMO ellipses from horizontal and dipping reflectors in vertically inhomogeneous orthorhombic media.

Still, P-wave traveltime data cannot be used to recover the three parameters ($V_{T0}$, $\gamma^{(1,2)}$) responsible for shear-wave velocities in the coordinate directions (Tsvankin, 1996) and, moreover, conventional-spread P-wave moveout from horizontal reflectors depends only on $V_{T0}$ and $\delta^{(1,2)}$. Therefore, it is important to investigate the possibility of using moveout of shear or converted waves to constrain the remaining anisotropic parameters. Combining $P$ and $PS$ data from a conventional source represents a viable alternative to the direct generation of $S$-waves because of the high cost of multicomponent excitation and often poor quality of shear data. If the medium is isotropic, shear-wave velocity can be obtained in a straightforward way from normal moveout of $P$ and $PS$-waves (Tesmer and Behle, 1988). Serriff and Sriram (1991) presented a relationship between the NMO velocities of $P$, $SV$ and $PSV$ waves in VTI media that can be used to extend this methodology to vertical transverse isotropy. The transition from the NMO to the vertical velocity in VTI models, however, is impossible without knowledge of the reflector depth. Tsvankin and Thomsen (1994) obtained the quartic moveout coefficient for $P - SV$ waves and studied the distortions in moveout-velocity estimation due to nonhyperbolic moveout. They found that SV-wave moveout becomes strongly nonhyperbolic for models with negative $\epsilon - \delta$, thus hampering direct estimation of the SY-wave NMO velocity. PSV moveout curves on conventional moderate spreads, however, are typically close to a hyperbola even in this case, and SV-wave NMO velocity can still be recovered from $P$ and $PSV$ moveouts in a relatively stable fashion. These results for vertical transverse isotropy remain entirely valid in the vertical symmetry planes of orthorhombic media.

Here, extending the approach developed by Grechka and Tsvankin (1996) for pure modes, we show that azimuthally dependent normal-moveout velocity of converted waves is also described by an elliptical curve in the horizontal plane centered at the minimum of the traveltime field. Then we focus on the practically important model composed of horizontal layers with a horizontal symmetry plane and prove that converted-wave traveltime in such a medium becomes reciprocal and the center of the NMO ellipse coincides with the CMP location. Application of the generalized Dix equation of Grechka et al. (1997) leads to a simple representation of the converted-wave NMO ellipse through the ellipses of the pure modes.

\* For brevity, the qualifiers in “quasi-P-wave” and “quasi-S-wave” will be omitted.
We use this general result to develop an inversion scheme for the model of a horizontal orthorhombic layer. Explicit relationships between the NMO ellipses of the \( P \), \( S \), and converted waves make it possible to obtain normal-moveout velocities of shear waves from azimuthally dependent \( P \) and \( PS \)-data. Then the NMO ellipses of \( P \) and \( S \)-waves are combined with the vertical velocities (or reflector depth) to obtain the anisotropic coefficients of the orthorhombic model. This algorithm is applied to the inversion of physical-modeling data acquired over a composite orthorhombic material Phenolite XX-324 (Gibson and Theophanis, 1996). Normal-moveout velocities of the \( P \)-wave and two split \( PS \)-waves, obtained from semblance analysis, allowed us to find the orientation of the symmetry planes and (using the known layer thickness) eight elastic coefficients, while the remaining parameter \([\delta]^{(3)}\) in Tvankin’s (1996) notation was recovered from the direct \( P \)-wave arrival in the horizontal plane.

3-D NMO equation for converted waves

**Arbitrary anisotropic inhomogeneous media**

We consider a converted wave recorded on a suite of differently oriented common-midpoint (CMP) lines with the same CMP location. The derivation of the pure-mode NMO equation in Grechka and Tvankin (1996) is based on replacing the CMP reflection traveltime with the one-way traveltimes between the zero-offset reflection point and the surface. Then expansion of the traveltime field in a Taylor series leads to an expression of the NMO ellipse in terms of the spatial derivatives of the slowness vector.

Generalization of this formalism for converted waves is discussed in Appendix A. Due to the nature of mode conversions and the influence of reflection-point dispersal on the NMO velocity, their moveout can no longer be described by one-way traveltimes. The main complication in treating two-way reflection traveltimes for converted waves is that the extremum of the traveltime surface \( t(x_1, x_2) \) is generally shifted from the CMP location \( x_1 = x_2 = 0 \) to a different position \( \xi \equiv \{ \xi_1, \xi_2 \} \). This implies that the representation of \( t(x_1, x_2) \) and \( t'(x_1, x_2) \) by a Taylor series expansion in the vicinity of the common midpoint contains odd (linear, cubic, etc.) terms in offset. Linear terms, however, can be eliminated by considering the traveltime as a function of the new variables \( x_1 - \xi_1 \) and \( x_2 - \xi_2 \) computed with respect to the extremum point \( \xi \) (see Appendix A). It should be emphasized that in practice this procedure requires prior identification of the extremum from azimuthally-dependent normal moveout. Then the converted-wave NMO velocity for the “shifted” traveltime field can be obtained as an ellipse similar to the one for pure modes [equation (A12)]

\[
V_{nmo}^{-2}(\alpha) = W_{11} \cos^2 \alpha + 2W_{12} \sin \alpha \cos \alpha + W_{22} \sin^2 \alpha,
\]

where \( \alpha \) is the azimuth of the CMP gather, \( W \) is a symmetric matrix determined by

\[
W_{ij} = t_0 \frac{\partial^2 t}{\partial x_i \partial x_j} \bigg|_{x_1 = \xi_1, x_2 = \xi_2},
\]

\( t \) is the (two-way) reflection traveltime, and \( t_0 = t(\xi_1, \xi_2) \).

Therefore, the converted-wave NMO equation (1) for the new CMP location has the same form as that for pure modes (Grechka and Tvankin, 1996). Repeating the analysis in Grechka and Tvankin (1996) and Grechka et al. (1997), we can represent \( V_{nmo} \) in the equivalent form

\[
\frac{1}{V_{nmo}^2(\alpha)} = \lambda_1 \cos^2 (\alpha - \beta) + \lambda_2 \sin^2 (\alpha - \beta),
\]

where \( \lambda_1 \) and \( \lambda_2 \) are the eigenvalues of the matrix \( W \) and

\[
\beta = \tan^{-1} \left[ \frac{W_{22} - W_{11} + \sqrt{(W_{22} - W_{11})^2 + 4W_{12}^2}}{2W_{12}} \right].
\]

is the azimuth of one of its eigenvectors with respect to the axis \( x_1 \). Typically, the traveltime surface \( t(x_1, x_2) \) has a minimum at point \( \xi \) and, therefore, the squared NMO velocity is positive in all azimuthal directions [see equation (A11)]. This implies that both \( \lambda_1 \) and \( \lambda_2 \) should be positive as well, and equation (3) or (1) describes an ellipse in the horizontal plane.

As is the case with the pure-mode NMO equation in Grechka and Tvankin (1996), this conclusion does not involve any assumptions about the model and, therefore, is valid for general anisotropic inhomogeneous velocity fields. Our representation of the conventional-spread moveout in terms of NMO velocity, however, may break down in structurally complicated media or areas of shear-wave cusps where reflection traveltime cannot be approximated by the Taylor series expansion as a whole (see also the discussion in Grechka and Tvankin, 1996).

Although the NMO velocities of both pure and converted waves have the same elliptical shape in the horizontal plane [equations (1) and (3)], the matrix \( W \) for pure modes has a simpler form because it depends on one-way traveltimes from the zero-offset reflection point. As shown by Grechka and Tvankin (1996),

\[
W_{ij} [\text{pure modes}] = t_0 \frac{\partial^2 \tau}{\partial x_i \partial x_j} \bigg|_{x_{\text{CMP}}} = t_0 \frac{\partial p_i}{\partial x_j} \bigg|_{x_{\text{CMP}}},
\]

\( i, j = 1, 2 \).

Here, \( \tau(x_1, x_2) \) is the one-way traveltime from the zero-offset reflection point to the location \( x \{ x_1, x_2 \} \) at the surface, \( t_0 \) is the one-way zero-offset traveltime, \( p_i \) are
the components of the slowness vector corresponding to the ray emerging at point \( \mathbf{x} \), and \( x_{\text{CMP}} \) is the CMP location. The spatial derivatives of the slowness vector in equation (5) can be expressed through the slowness components of the zero-offset ray, which leads to a compact representation of the NMO ellipse for any pure mode (Grechka et al., 1997).

In contrast, the two-way traveltimes in equation (2) for converted waves are much more difficult to relate to the slowness vector and, eventually, to the model parameters. Therefore, the most convenient way to include shear information into moveout inversion is to obtain the \( S \)-wave NMO ellipses (generally, there are two split shear waves in anisotropic media) from the NMO ellipses of \( P \) and \( PS \)-waves. Then the orientation and semi-axes of the pure-mode ellipses can be inverted for the elastic coefficients. Such an algorithm represents a generalization of the well-known method of shear-wave velocity estimation in isotropic media based on the NMO velocities of \( P \) and \( PS \)-waves (Tessmer and Behle, 1988). In arbitrary anisotropic inhomogeneous media, this methodology can be implemented only numerically, for example, the approach developed by Grechka et al. (1997).

**Stratified media with a horizontal symmetry plane**

Description of normal moveout of converted waves can be significantly simplified if the medium is composed of homogeneous layers with a horizontal symmetry plane. The layers can be, for instance, orthorhombic (then one of the symmetry planes should be horizontal) or transversely isotropic with a vertical (VTI) or horizontal (HTI) symmetry axis. Here, we will restrict ourselves to a single-layer model, but equations (6) and (8), given below, remain valid for multilayered media as well.

If the slowness and group-velocity surfaces of all modes are symmetric with respect to the horizontal plane, the traveltimes of converted waves remains the same when we interchange the source and receiver (Appendix B). The reciprocity implies that the traveltimes series for converted waves [equation (A2)] does not contain odd terms and the traveltine minimum corresponds to the CMP location. The hyperbolic approximation for converted-wave moveout in this model can be written as [see equation (A7)]

\[
\left[ t_0^{(PS)}(x) \right]^2 = \left[ t_0^{(P)} \right]^2 + \sum_{i,j=1}^{2} W_{ij}^{(PS)} x_i x_j ,
\]

where \( t_0^{(PS)} \) is the zero-offset traveltine of the \( PS \)-wave that can be expressed through the one-way \( P \) and \( S \) traveltimes as

\[
t_0^{(PS)} = t_0^{(P)} + t_0^{(S)} .
\]

The matrix \( W^{(PS)} \) defines the converted-wave NMO ellipse [equation (1)] around the original CMP location. Using the generalized Dix equation (Grechka et al., 1997), \( W^{(PS)} \) can be related to the NMO ellipses of the pure \( P \)- and \( S \)-waves (Appendix B):

\[
t_0^{(PS)} \left[ W^{(PS)} \right]^{-1} = t_0^{(P)} \left[ W^{(P)} \right]^{-1} + t_0^{(S)} \left[ W^{(S)} \right]^{-1} .
\]

Equation (8) makes it possible to obtain the shear-wave NMO ellipse (i.e., the matrix \( W^{(S)} \)) from azimuthally-dependent \( P \) and \( PS \) moveout data.

**Orthorhombic layer**

In the following, we concentrate on the model of a homogeneous orthorhombic layer used in the physical-modeling experiment discussed below. An orthorhombic medium with a horizontal symmetry plane also has two vertical symmetry planes that determine the directions of the axes of the pure-mode NMO ellipses (Grechka and Tsvankin, 1996). Indeed, by choosing the vertical symmetry planes of the medium as the coordinate planes \([x_1, x_2] \) and \([x_2, x_3] \), we eliminate the off-diagonal terms of the pure-mode matrices \( W^{(P)} \) and \( W^{(S)} \) [equation (5)]. It is clear from equation (1) that a diagonal matrix \( W \) describes an NMO ellipse with the axes in the coordinate directions. The matrix \( W^{(PS)} \) [equation (8)] in this case becomes diagonal as well, which implies that the axes of the \( PS \)-wave NMO ellipse are also aligned with the symmetry planes. Therefore, the matrices \( W \) are fully defined by the normal-moveout velocities in the symmetry planes \( V_{nmo}^{(1,2)} \):

\[
W_{11}^{(Q)} = \left[ \frac{1}{V_{Q,nmo}^{(2)}} \right]^2 , \quad W_{12}^{(Q)} = 0 ,
\]

\[
W_{22}^{(Q)} = \left[ \frac{1}{V_{Q,nmo}^{(1)}} \right]^2 , \quad (Q = P, S, \text{ or } PS) .
\]

Here and below, the superscript “1” corresponds to the \([x_2, x_3] \)-plane and “2” — to the \([x_1, x_3] \)-plane (the superscripts denote the axis normal to each plane).

Since all \( W^{(P)} \) became diagonal, equation (8) splits into two separate equations for the symmetry-plane NMO velocities:

\[
t_0^{(PS)} \left[ V_{PS,nmo}^{(1)} \right]^2 = t_0^{(P)} \left[ V_{P,nmo}^{(i)} \right]^2 + t_0^{(S)} \left[ V_{S,nmo}^{(i)} \right]^2 ,
\]

\((i = 1, 2)\).

As could be expected from the kinematic equivalence between the symmetry planes of orthorhombic and VTI media (Tsvankin, 1996), equation (10) has the same form as the relationship between the NMO velocities of
$P$-, $SV$-, and $PSV$-waves for vertical transverse isotropy (Seriff and Sriram, 1991). If we have obtained the NMO velocities of the $P$-wave and two split converted waves in both symmetry planes, the symmetry-plane NMO velocities of the shear modes can be found from equation (10):

$$
\frac{V_{S, \text{nmo}}^{(2)}}{V_{P, \text{nmo}}} = \sqrt{\frac{t_0^{(PS)} t_0^{(P)}}{t_0^{(PS)} - t_0^{(P)}}} \left[ \frac{V_{V, \text{nmo}}^{(1)}}{V_{P, \text{nmo}}} \right]^2.
$$

Although this approach looks straightforward, application of equation (11) is complicated by the fact that only one ($PSV$) converted wave can be generated in each symmetry plane; this will be discussed in more detail below.

### NMO velocities of $P$- and $S$-waves in an orthorhombic layer

After obtaining the relationships between the NMO ellipses of pure and converted modes in orthorhombic media, we need to discuss the dependence of NMO velocities of $P$- and $S$-waves on the anisotropic parameters. As discussed above, normal-moveout velocity of each pure mode represents an ellipse with the semi-axes equal to the NMO velocities in the symmetry planes:

$$
\frac{1}{V_{Q, \text{nmo}}^2(\alpha)} = \frac{\cos^2 \alpha}{V_{Q, \text{nmo}}^{(2)}} + \frac{\sin^2 \alpha}{V_{Q, \text{nmo}}^{(1)}},
$$

where $Q = P$, $S_1$, or $S_2$.

Due to the identical form of the Christoffel equation in the symmetry planes of orthorhombic and VTI media, all symmetry-plane kinematic signatures, including NMO velocity, can be obtained by analogy with vertical transverse isotropy (Tsvankin, 1996). (The only exception is cuspidal shear wavefronts near point singularities that can introduce additional group-velocity branches in orthorhombic media.) Tsvankin’s (1996) notation is especially convenient for adapting VTI equations because it is based on the same principle as Thomsen’s parameters for vertical transverse isotropy. For $P$-waves, the symmetry-plane NMO velocities are given by (Tsvankin, 1996; Grechka and Tsvankin, 1996)

$$
V_{P, \text{nmo}}^{(2)} = V_0 \sqrt{1 + 2 \delta^{(2)}},
$$

$$
V_{P, \text{nmo}}^{(1)} = V_0 \sqrt{1 + 2 \delta^{(1)}},
$$

where $V_0$ is the $P$-wave vertical velocity and $\delta^{(1,2)}$ are the anisotropy coefficients defined in Appendix C.

Before giving the corresponding expressions for $S$-waves, it is necessary to review some relevant polarization properties of shear waves in orthorhombic media. A cartoon of typical phase-velocity sheets in orthorhombic media is shown in Figure 1. The outer ($P$-wave) phase-velocity surface is usually separated from the other two sheets corresponding to split shear waves. The shear-wave phase velocities in orthorhombic media always always coincide in certain directions corresponding to the so-called “point singularities” (Crampin and Yedlin, 1981), such as point $A$ in Figure 1. We assume that the singularities are far enough from vertical so that the fast shear wave $S_1$ can be distinguished from the slow wave $S_2$ in some vicinity of the vertical direction sufficient to obtain normal-moveout velocities.

While applying the equivalent with VTI media to $S$-waves, we have to remember that the polarization of shear waves with respect to the vertical incidence plane varies with azimuth. Suppose that the fast vertically traveling shear wave $S_1$ is polarized in the $x_2$ direction and, therefore, represents a pure transverse ($SH$) wave for any phase direction in the $[x_1, x_3]$ plane. As we move along the phase-velocity surface of the $S_1$-wave around the $x_3$ axis to the $[x_2, x_3]$-plane (Figure 1), its polarization changes from transverse (cross-plane) to in-plane (in other words, from $SH$ in $[x_1, x_3]$-plane to $SV$ in $[x_2, x_3]$-plane). Thus, according to the kinematic analogy with vertical transverse isotropy, the $S_1$-wave propagating in the $[x_1, x_3]$-plane is equivalent to the $SH$-wave in VTI media, while in the $[x_2, x_3]$-plane it is equivalent to the $SV$-wave. Likewise, the polarization of the $S_2$-wave changes from $SV$ in the $[x_1, x_3]$-plane to $SH$ in the $[x_2, x_3]$-plane. This property of $S$-waves has a direct bearing on the form of their NMO velocities listed below (again, the superscript “1” corresponds to the $[x_2, x_3]$-plane and “2” to the $[x_1, x_3]$-plane):
\[ V^{(2)}_{S_1, nmw} = V_{S_1} \sqrt{1 + 2 \gamma^{(2)}}, \]
\[ V^{(1)}_{S_1, nmw} = V_{S_1} \sqrt{1 + 2 \sigma^{(1)}}, \]
\[ V^{(2)}_{S_2, nmw} = V_{S_2} \sqrt{1 + 2 \sigma^{(2)}}, \]
\[ V^{(1)}_{S_2, nmw} = V_{S_2} \sqrt{1 + 2 \gamma^{(1)}} = \sqrt{c_{0w}/\mu}, \]

where
\[ \sigma^{(2)} = \left( \frac{V_{P_0}}{V_{S_2}} \right)^2 (\epsilon^{(2)} - \delta^{(2)}), \]
\[ \sigma^{(1)} = \left( \frac{V_{P_0}}{V_{S_1}} \right)^2 (\epsilon^{(1)} - \delta^{(1)}). \]

The vertical shear velocities \( V_{S_1} \) and \( V_{S_2} \) in equations (14)-(16) are related to Tsvankin’s (1996) vertical velocity \( V_{S_0} \) (Appendix C) as
\[ V_{S_2} = V_{S_0}, \quad V_{S_1} = V_{S_0} \sqrt{1 + 2 \gamma^{(1)}}. \]

Equations (11), (13)-(16) provide an analytic basis for obtaining the parameters of an orthorhombic layer from moveout measurements. Note that the NMO velocities of three pure modes depend on eight (out of nine) parameters of the orthorhombic layer. The only coefficient that makes no contribution to the NMO velocities is \( \delta^{(3)} \) (or \( c_{12} \)); this parameter is defined in the horizontal symmetry plane and cannot be found from near-vertical measurements of normal moveout. Another important observation relates to the fact that the \( SH \)-wave velocities in the symmetry planes are identical [see equations (14) and (15)], i.e.,
\[ V^{(2)}_{S_1, nmw} = V^{(1)}_{S_2, nmw}. \]

This indicates a possible redundancy in the moveout measurements that can be used to increase the accuracy of the inversion procedure.

Suppose we have found the NMO velocities and zero-offset traveltimes of all three pure modes (\( P, S_1, S_2 \)) in the symmetry planes (in our case, using \( P \) and \( PS \)-waves). Since the ratios of the vertical velocities can be obtained from the zero-offset traveltimes, only one vertical velocity has to be determined. Therefore, for a total of seven unknowns (\( V_{P_0}, \epsilon^{(1,2)}, \delta^{(1,2)}, \gamma^{(1,2)} \)), there are only six moveout equations (13)-(15). Separating out the \( SH \)-wave equations that depend on \( \gamma^{(1,2)} \) (or a single stiffness coefficient \( c_{0w} \)), we have four moveout equations for \( P \)- and \( SV \)-waves that include five unknown parameters (\( V_{P_0}, \epsilon^{(1,2)}, \delta^{(1,2)} \)). Clearly, the number of unknowns exceeds the number of equations by one, and the inversion is impossible without including some additional information. The same underdetermined inverse problem, as discussed in detail by Tsvankin and Thomsen (1995), arises for vertical transverse isotropy (or, equivalently, in each vertical symmetry plane of orthorhombic media). Here, we assume that the layer thickness is known, and all vertical velocities can be found from the zero-offset traveltimes. Another possible alternative is use of dipping events, but this information was not available in our physical-modeling experiment.

In the discussion above, we assumed that the combination of \( P \) and \( PS \)-waves is sufficient to determine the symmetry-plane NMO velocities for all three pure modes. Polarization properties of shear waves, however, have serious implications for the generation of the converted modes used in our algorithm. Since the phase-velocity sheets (and the corresponding wavefronts) of the waves \( P, S_1, \) and \( S_2 \) are continuous, from the kinematic viewpoint the converted waves \( PS_1 \) and \( PS_2 \) can be recorded in all azimuthal directions. However, the \( P \)-wave propagates in either vertical symmetry plane of a horizontal orthorhombic layer; it cannot generate the converted \( PSH \)-wave because the particle motion should be confined to the incidence plane. Thus, the \( PS_1 \)-wave does not exist in the \([x_1, x_3]\)-plane, while the \( PS_2 \)-wave cannot be excited in the \([x_1, x_3]\)-plane. The \( PSH \)-type reflections will also be weak near the symmetry planes because shear-wave polarizations change in a continuous fashion. However, the NMO velocities of these (physically nonexistent) converted waves in the symmetry planes can be reconstructed from moveout measurements in other azimuthal directions by taking advantage of the elliptical shape of the moveout function in the horizontal plane.

**Inversion of physical modeling data**

Scaled physical modeling over anisotropic materials has proved to be useful in testing theoretical predictions of various wave-propagation phenomena. A number of publications is devoted to simulations of fracture-induced anisotropy and measurements of shear-wave splitting (e.g., Tatham et al., 1987; Ebrum et al., 1990). In their experiments with orthorhombic phenolic laminate, Chedale et al. (1991) and Brown et al. (1991) recorded transmitted waves propagating in different directions and determined the stiffness coefficients of the material. Here, we simulate a reflection seismic survey over an azimuthally anisotropic layer by recording pure and converted reflected waves generated in a rectangular block of orthorhombic Phenolite XX-324. Previous experiments with the same material were described by Gibson and Theophanis (1996).

**Laboratory set-up**

The ultrasound model was constructed of six 30.0×5.0×14.81 cm blocks of Phenolite XX-324, a com-
**Figure 2.** Seismograms of the vertical displacement component recorded at azimuths 0° (a), 45° (b), and 90° (c) and the corresponding zero-offset traveltime (d, e, f). The composite material had orthorhombic symmetry. “Slow” (in terms of P-wave velocity) axes of the blocks were aligned along a direction that will be called the z1-axis of the model. The blocks were bonded with epoxy to make a sample with dimensions 30.0×30.0×14.81 cm (z = 14.81 cm is the thickness). The epoxy bonding of the joints was performed under a uniform pressure of 15 psi. The reflection coefficient for the epoxy joints between layers was tested with both P- and S-waves of the appropriate frequencies and found to be unmeasurably small.
Taking into account the seismic velocities and size of the Phenolite model, ultrasonic transducers were chosen to produce an acoustic wavelength close to 1/6 of the model thickness. This wavelength corresponds to frequencies of approximately 100 kHz for S-waves and 200 kHz for P-waves. Vertical and horizontal contact transducers (supplied by Panametrics, Waltham, MA), employed as source and receiver, had a diameter of 1.0 inch. The central frequency and bandwidth of the transducers can be adjusted by varying the width of the excitation pulse. The pulse was excited by a Hewlett Packard 214B high voltage pulse generator which has independent control of voltage, pulse width and repetition rate. Data were collected directly from the receiving transducer with a LeCroy 9304A oscilloscope which has real-time signal averaging capability for noise reduction, as well as a disk drive for data storage.

Common-midpoint surveys were simulated by recording arrivals reflected from the bottom of the block (free surface). Gather were acquired along differently oriented CMP lines with offsets for each line ranging from 5 to 25 cm with 2 cm increment. Data were collected sequentially with P-P transducer pairs at azimuths 0°, 45°, 90°, and 135° with respect to the axis x1 and with P-S transducer pairs along azimuths 0°, 30°, 60°, and 90°.

P-wave processing

We begin with processing of reflection data recorded by vertical transducers along azimuths 0°, 45°, 90°, and 135° (Figure 2). The seismogram for azimuth 135° is not shown in Figure 2 because it looks very similar to that for azimuth 45°. The most prominent feature of the seismograms is a significant azimuthal variation in the P-wave traveltimes. Clearly, the moveout velocity increases as the azimuth changes from 0° to 90°, which leads to a corresponding decrease in the P-wave traveltime for any fixed offset. To reconstruct the azimuthal dependence of moveout velocity, we performed conventional hyperbolic semblance analysis and displayed the results in the semblance panels (Figure 2d-f). (We used a non-linear scale to enhance semblance maxima and muted out all semblance values below a certain level.) The position of the semblance maximum on the velocity axis varies with azimuth by more than 1.0 km/s (42%), indicating a high degree of P-wave azimuthal anisotropy. Having picked the moveout velocities from the semblance panels, we calculated the corresponding hyperbolic moveout curves for each azimuth and plotted them on the seismograms (dashed lines). Since the zero-offset traveltimes of the semblance maxima is higher than the time of the first break, we introduced an appropriate correction in t0 that allowed us to match the actual arrival times on the seismograms.

Clearly, the hyperbolic moveout approximation is close to the actual arrival times in all azimuthal directions, except for relatively small deviations at the very far offsets. Therefore, nonhyperbolic moveout on the spreads used in the experiment (the maximum offset-to-depth ratio is 1.7) is relatively weak for the Phenolite model (see the discussion section).

Using the P-wave moveout velocities measured in all four azimuthal directions, we reconstructed the P-wave NMO ellipse (Figure 3). The data points, which correspond to finite-spread moveout velocities, lie very close to the best-fit elliptical curve (the maximum deviation is 1.6% at azimuth 90°). This serves as another corroboration of the small magnitude of nonhyperbolic moveout and high accuracy of the hyperbolic moveout equation parameterized by the analytic NMO velocity. The azimuths of the axes of the best-fit NMO ellipse are equal to 0.6° and 90.6°. As discussed above, in orthorhombic media these axes should be aligned with the vertical symmetry planes; indeed, our results are in good agreement with the expected symmetry-plane orientation. The values of the NMO velocities along the axes (i.e., in the symmetry planes; see Table 1) were combined with the vertical velocity Vp0 = 2s/t0(0) = 3.569 km/s to compute the anisotropic parameters δ(1,2) = 0.073, δ(2) = −0.218 from equations (13). No other information can be obtained from P-wave moveout measurements on conventional-length spreads.
Processing and inversion of converted waves

To obtain moveout velocities of converted waves, we processed seismograms recorded by horizontal transducers oriented along four CMP lines at azimuths 0°, 30°, 60°, and 90° (Figures 4 and 5). Although P-waves can be seen on all four sections of horizontal displacement, the largest semblance maximum corresponds to the second arrival that we identified as a converted wave. Our interpretation is based on the zero-offset traveltime of this wave, its relatively low moveout velocity and predominantly horizontal polarization. Since the wavefield was excited by a vertical source, we did not expect to record strong pure shear waves at moderate offsets used in the experiment. Although $S_1S_1$ and $S_2S_2$ reflections can be seen on the horizontal displacement component,
Figure 5. Seismograms of the horizontal in-line displacement component recorded at azimuths 60° (a) and 90° (b) and the corresponding semblance panels (c,d). Dashed lines mark PS1-wave moveouts picked from the semblance panels. Moveouts of the pure P- and S1-wave reflections, computed using the inversion results, are marked by dots.

their semblance maxima are not focused enough for accurate velocity picking. Therefore, for each section we determined the moveout velocity of a single (PS) mode corresponding to the most pronounced semblance maximum and used it in moveout inversion. (After obtaining the model parameters, we computed the reflection moveout of pure S-waves and plotted it against the actual arrivals to verify the accuracy of our algorithm.)

It is clear from both the seismograms and semblance panels that we observe two different converted waves in different azimuthal directions. While the second arrival at azimuths 0° and 30° (Figure 4) has the zero-offset traveltime of about 0.15 ms, the value of $t_0$ for the second arrival at 60° and 90° (Figure 5) is noticeably smaller. The difference in the zero-offset traveltimes allows us to identify the PS reflection in Figure 4 as the slow conver-
tended wave $PS_2$ and the corresponding event in Figure 5 as the fast wave $PS_1$. The magnitude of shear-wave splitting in the vertical direction is sufficient for the two converted waves to be well separated in time. The vertical shear-wave velocities $V_{S_2}$ and $V_{S_1}$ can be calculated from the zero-offset traveltimes and the known layer thickness using the equation $t_0^{(PS_i)} = z/V_{S_i} + z/V_{P_0}$, where we obtained the values $V_{S_2} = 1.391$ km/s and $V_{S_1} = 1.911$ km/s.

Figures 4 and 5 confirm our prediction that only one converted wave ($PSV$) can be observed in a vicinity of each symmetry plane. The fast wave $PS_1$ has the SV polarization in the $[x_2, x_3]$-plane (azimuth 90°) and, therefore, is recorded at azimuths 60° and 90°. In contrast, the sections at 0° and 30° contain an intensive $PS_2$ arrival with the SV polarization in the $[x_1, x_3]$-plane. It is likely that the arrival with the zero-offset traveltine close to 0.125 ms that can be seen at small offsets in Figure 4b is the $PS_1$ reflection, but its amplitude is not sufficient to generate a focused semblance maximum.

Note that even if the $SH$-type waves were excited at the reflecting boundary, they would not be recorded by our in-line horizontal receiver. In general, $PS$-wave processing in azimuthally anisotropic media requires two orthogonal horizontal geophones (in-line and cross-line) that would record the horizontal displacement of both split converted modes. Then the in-line and cross-line seismograms should be simultaneously rotated to separate the converted arrivals; this algorithm in application to pure $S$-waves excited by a single source was described in detail by Thomsen (1988). Due to the high degree of $S$-wave splitting in our model, the converted waves were separated in time at all offsets, and rotation was not necessary.

Using the semblance maximum of the most intensive arrival on the semblance panels (Figures 4c,d and 5c,d), we obtained the NMO velocities of each converted wave in two azimuthal directions and plotted the corresponding moveouts with dashed lines in Figures 4a,b and 5a,b. As was the case with $P$-waves, the best-fit hyperbolic moveout curve is close to the $PS$ arrivals, with deviations becoming noticeable only at far offsets. Most importantly, the velocities corresponding to the semblance maxima for each converted wave exhibit a pronounced azimuthal dependence and can be used to build the NMO ellipses. For instance, there is a significant change in the best-fit moveout velocity of the $PS_1$ reflection from an azimuth of 60° to that of 90°.

In general, a minimum of three different azimuthal moveout measurements is necessary to recover the NMO ellipse for each mode. However, since the orientation of the ellipses (i.e., the azimuths of the symmetry planes) has already been determined from $P$-wave data (Figure 3), two sufficiently separated azimuths provide enough information to find the values of the elliptical semi-axes.†

The NMO ellipses of both converted waves, reconstructed from the data, are shown in Figure 6. Then, substituting the symmetry-plane NMO velocities of the converted waves into equation (11), we computed the corresponding velocities of the pure shear modes (Table 1) and plotted their NMO ellipses with dashed lines in Figure 6. Note that although the time delay between the vertically travelling shear and converted waves is very significant, the NMO ellipses of the $S$-waves (as well as the ellipses of the converted waves) intersect each other due to the influence of the anisotropic coefficients in equations (14) and (15).

<table>
<thead>
<tr>
<th>Reflection</th>
<th>Azimuth</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>90°</td>
</tr>
<tr>
<td>$PP$</td>
<td>2.679</td>
</tr>
<tr>
<td>$PS_1$</td>
<td>1.924</td>
</tr>
<tr>
<td>$PS_2$</td>
<td>2.100</td>
</tr>
<tr>
<td>$SiS_1$</td>
<td>1.357</td>
</tr>
<tr>
<td>$S_0S_2$</td>
<td>1.825</td>
</tr>
</tbody>
</table>

Table 1. Normal-moveout velocities (in km/s) in the symmetry planes of the model. The velocities of the $P$ and $PS$-waves were obtained from the reconstructed NMO ellipses; the velocities of the pure shear reflections were computed using equation (11).

The most non-trivial part of this parameter-estimation procedure is determination of the moveout velocities of the physically non-existing reflected $P$/$SH$-waves in the symmetry planes ($PS_1$ in the $[x_1, x_3]$-plane and $PS_2$ in the $[x_2, x_3]$-plane). These velocities have been obtained essentially by extrapolating the moveout measurements of the waves $PS_1$ and $PS_2$ in other azimuthal directions using the known (elliptical) shape of the NMO velocity in the horizontal plane.

Since the vertical velocities and coefficients $\delta^{(1,2)}$ are already known, equations (14) and (15) make it possible to find the anisotropic parameters $\epsilon^{(1,2)}$ and $\gamma^{(1,2)}$ from the symmetry-plane NMO velocities of the pure shear

† If the medium does not have vertical symmetry planes (e.g., the symmetry is lower than orthorhombic), the NMO ellipses of pure and converted modes may have a different orientation.
waves. The set of Tsvankin’s (1996) parameters (Appendix C) of the Phenolite model obtained from the moveout inversion is as follows:

\[ V_{p0} = 3.569 \text{ km/s}, \quad V_{s0} = 1.911 \text{ km/s}, \]
\[ \delta^{(2)} = -0.218, \quad \delta^{(1)} = 0.073, \]
\[ \epsilon^{(2)} = -0.163, \quad \epsilon^{(1)} = 0.109, \]
\[ \gamma^{(2)} = -0.248, \quad \gamma^{(1)} = -0.028. \]

The difference between the coefficients \( \delta^{(1)} \) and \( \delta^{(2)} \) is responsible for the azimuthal variation in the P-wave NMO velocity, while \( \epsilon^{(1)} - \epsilon^{(2)} \) determines the variation in the P-wave horizontal velocity between the \( x_2 \) and \( x_1 \)-axis. \( \gamma^{(1)} \) and \( \gamma^{(2)} \) define the velocity anisotropy of the \( SH \)-waves in the symmetry planes (\( S_1 \) in the [\( x_1, x_3 \)]-plane and \( S_2 \) in the [\( x_2, x_3 \)]-plane).

Note that the only anisotropic coefficient that has not been determined is \( \delta^{(3)} \). As mentioned above, this parameter has no influence on either the vertical velocities or the NMO velocities of any mode and, therefore, cannot be found from conventional-spread reflection data. \( \delta^{(3)} \), however, does make a contribution to the traveltime of the direct P-wave arrival in the horizontal plane that can be identified at relatively large offsets in Figures 4a,b and 5a,b. Like any other \( \delta \) coefficient, \( \delta^{(3)} \) influences the phase velocities only away from the coordinate directions of the orthorhombic model. Therefore, we estimated \( \delta^{(3)} \) from the group velocity of the direct P-wave arrival at azimuths 30° and 60° (Figure 7). We found the value of \( \delta^{(3)} = -0.211 \) gives the best fit to the group-velocity measurements in both azimuthal directions.

Thus, we have determined the full set of Tsvankin’s parameters of the model. Using the expressions of these parameters in terms of the stiffness coefficients \( c_{ij} \) (Appendix C), we calculated the following stiffness matrix for the Phenolite sample:

\[
\frac{c_{ij}}{\rho} = \begin{pmatrix}
8.575 & 2.741 & 2.007 & 0 & 0 & 0 \\
2.741 & 15.511 & 9.755 & 0 & 0 & 0 \\
2.007 & 9.755 & 12.735 & 0 & 0 & 0 \\
0 & 0 & 0 & 3.652 & 0 & 0 \\
0 & 0 & 0 & 0 & 1.934 & 0 \\
0 & 0 & 0 & 0 & 0 & 1.841
\end{pmatrix}
\]

(19)

**Verification of the inversion results**

There are several ways to check the accuracy of the inversion algorithm. First, as discussed above, we have some redundancy in the moveout measurements since the NMO velocity of the fast mode \( S_1S_1 \) at azimuth 0° (\( x_1, x_3 \)-plane) should be equal to the \( S_2S_2 \)-wave NMO velocity at azimuth 90° [see equation (18)]. Note that these velocities

![Figure 6](image1.png)

**Figure 6.** The converted-wave NMO ellipses (solid lines) reconstructed from the picked moveout velocities (dots) and the NMO ellipses of pure shear reflections (dashed) computed from the inversion results.

![Figure 7](image2.png)

**Figure 7.** The seismograms of the horizontal displacement component at azimuths 30° (a) and 60° (b). The straight dashed lines mark the direct P-wave arrival. The group velocities determined from the slopes of the dashed lines are 2.860 km/s (30°) and 3.200 km/s (60°).
correspond to the \( SH \)-waves in the symmetry planes and, therefore, could not be measured from the data directly due to the absence of \( PSH \) reflections. Nevertheless, the values \textit{computed} using the extrapolated NMO ellipses of both converted waves are remarkably close to each other (1.357 and 1.352 km/s, Table 1).

<table>
<thead>
<tr>
<th>( \sqrt{c_{11}/\rho} ) (km/s)</th>
<th>Inverted velocity</th>
<th>Measured velocity</th>
<th>Difference (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.928</td>
<td>2.923</td>
<td>0.2</td>
<td></td>
</tr>
<tr>
<td>3.938</td>
<td>4.016</td>
<td>-1.9</td>
<td></td>
</tr>
<tr>
<td>( \sqrt{c_{22}/\rho} )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \sqrt{c_{66}/\rho} )</td>
<td>1.357</td>
<td>1.393</td>
<td>-2.6</td>
</tr>
</tbody>
</table>

\textbf{Table 2.} Comparison of the horizontal velocities obtained from moveout inversion with the directly measured values.

Another way of verifying the inversion results is to compute the traveltimes of reflection arrivals not used for parameter estimation and check whether the predicted moveouts match the data. \( PP \)-wave moveout curves \textit{calculated} for the model parameters given above are shown in Figures 4b and 5a (azimuths 30\(^\circ\) and 60\(^\circ\)). Although the \( P \)-wave NMO ellipse was built from moveout data at azimuths 0\(^\circ\), 45\(^\circ\), 90\(^\circ\), and 135\(^\circ\), it accurately predicts \( P \)-wave moveout in these intermediate directions. In addition, we also computed the traveltimes of both pure shear reflections which are excited by the source together with \( P \)-waves and essentially represent a by-product of the experiment. Figures 4a,b and 5a,b show that the relatively weak shear-wave arrivals, not used in the moveout inversion, lie sufficiently close to the predicted moveout curves.

The most direct check of the inversion procedure is a comparison between the predicted and measured group velocities in certain directions. The reflection data used in our experiment illuminate a limited range of group angles with vertical, with the maximum incidence angle for pure reflections reaching about 40\(^\circ\). Still, the inversion of shear data allowed us to obtain the coefficients \( \varepsilon^{(1)} \) and \( \varepsilon^{(2)} \) responsible for the \( P \)-wave horizontal velocity in the \( x_1 \) (\( \sqrt{c_{11}/\rho} \)) and \( x_2 \) (\( \sqrt{c_{22}/\rho} \)) directions. These results are in excellent agreement with the \textit{direct} measurements of the \( P \)-wave horizontal velocities given in Table 2. Also, the NMO velocities of the \( SH \)-waves in the symmetry planes should be equal to the corresponding horizontal velocity \( \sqrt{c_{66}/\rho} \) [see equations (14) and (15)]. Although the moveout of the \( SH \)-waves was estimated by extrapolating the NMO ellipses of the converted waves rather than actual \( SH \)-reflections, the inverted \( SH \)-wave horizontal velocity is close to the directly measured value. The horizontal velocities of the \( SV \)-waves in each vertical symmetry plane are equal to the corresponding vertical velocities and, therefore, are not constrained by moveout inversion.

We conclude that the underlying assumption about the orthonhombic symmetry of the model is valid, and the accuracy of the moveout inversion is sufficient for obtaining the orientation of the symmetry planes and the elastic parameters of the medium.

\textbf{Discussion}

Despite the excellent results of the inversion procedure, the model used in our experiment was relatively simple (a single layer) and strongly anisotropic, which helped to separate the split converted waves in a straightforward fashion. Below, we discuss the main potential problems in the application of this algorithm to processing of multicomponent data acquired over typical vertically inhomogeneous fractured formations.

\textit{Nonhyperbolic moveout}

Our inversion scheme operates with normal-moveout velocities obtained from the data using the hyperbolic approximation for reflection traveltimes. One may believe that for converted waves this approximation is inherently contradictory because \( PS \)-wave moveout is known to be nonhyperbolic even in a homogeneous isotropic medium. However, as shown by Tsvankin and Thomsen (1994) for vertical transverse isotropy, \( PS \) moveout curves do not deviate much from hyperbolas on conventional-length spreads close to the reflector depth, especially for positive values of the anisotropic parameter \( \sigma \). This conclusion remains entirely valid in the symmetry planes of orthorhombic media, if we use the appropriate \( \sigma \) coefficient [equation (16)]. Both \( \sigma^{(1)} \) and \( \sigma^{(2)} \) were positive (and small) in our physical-modeling experiment, and nonhyperbolic moveout for \( PS \) arrivals was not significant, despite a relatively large offset-to-depth ratio of 1.7.

The magnitude of nonhyperbolic moveout for \( P \)-waves mostly depends on the degree of “aneilipticity” of the model measured by the parameters (Tsvankin, 1996)

\[ \eta^{(i)} = \frac{\varepsilon^{(i)} - \delta^{(i)}}{1 + 2\delta^{(i)}} \quad (i = 1, 2). \]

Both \( \eta^{(1)} \) and \( \eta^{(2)} \) did not exceed 10% in our model (\( \eta^{(1)} = 0.042, \eta^{(2)} = 0.098 \)), and \( P \)-wave moveout on
the spreadlengths used in the experiment was close to
hyperbolic as well. Another factor that helped to mit-
giate the influence of nonhyperbolic moveout for both P
and PS-waves is a rapid decrease of reflection ampli-
ditudes with offset (Figures 4a,b and 5a,b).

In realistic layered media, the magnitude of non-
hyperbolic moveout may be enhanced by vertical in-
homogeneity. In this case, stable recovery of NMO ve-
locities requires either muting out offsets exceeding the
reflector depth (as conventionally done in seismic pro-
cessing) or applying nonhyperbolic semblance analysis
based, for instance, on the equation developed by Ts-
vankin and Thomsen (1994). Although this equation may
not be accurate enough for converted waves in its ana-
lytic form, it still provides a good moveout approxima-
tion with the analytic NMO velocity and fitted coeffi-
cients of the nonhyperbolic term. Therefore, application
of this equation to converted-wave moveout may substan-
tially increase the accuracy in the estimation of normal-
moveout velocity.

Strength of azimuthal anisotropy and shear-wave
splitting

In the inversion for the medium parameters, we used the
zero-offset traveltimes and azimuthally-dependent NMO ve-
locities of the converted waves PS1 and PS2. These
measurements require prior separation of the split con-
verted waves on the seismograms. The relative differ-
ce in the zero-offset traveltimes of shear and converted
waves can be described by the shear-wave splitting para-
ter \( \gamma(S) \) which is conventionally defined as the frac-
tional difference between the squared vertical shear-wave
velocities \( \gamma(S) = (V_{S1}^2/V_{S2}^2 - 1)/2 \). For the Phenolite
sample, the splitting coefficient turned out to be uncom-
monly large \( \gamma(S) = 0.444 \), and the arrivals PS1 and PS2
were well separated at all offsets used in the experiment.
If \( \gamma(S) \) were smaller, the converted waves would interfere
near vertical, making the moveout velocities much more
difficult to obtain. In this case, it is necessary to deploy
two horizontal geophones and use a rotation algorithm
(Thomsen, 1988) to separate orthogonally polarized split
converted waves. This operation may reduce the accu-
cy of moveout inversion even in a single layer.

The situation becomes much more complicated in stratified media, especially if the target layer is overlaid by an azimuthally anisotropic overburden. In principle, the Dix-type equations (B3) and (B4) can still be used to obtain the effective and interval NMO ellipses of pure shear modes from those of \( P \) and \( PS \)-waves. However, the data may be too corrupted by multiple splitting in the overburden for a reliable identification of the split converted waves.

Shear-wave point singularities

The problem of shear-wave singularities is directly re-
lated to the issue of the magnitude of shear-wave splitting
discussed above. We assumed that the phase-velocity (or slowness) sheets of the waves \( S1 \) and \( S2 \) do not inter-
sect in some vicinity of the vertical direction. In other
words, shear-wave point singularities (point \( A \) in Fig-
ure 1), which always exist in orthorhombic media, are
supposed to be sufficiently far from vertical. The posi-
tion of singularities with respect to the vertical axis is
determined by the value of \( \gamma(S) \); with increasing \( \gamma(S) \),
the distance between the shear-wave sheets near vertical be-
comes bigger, and shear-wave singularities move closer
to the horizontal plane. Due to the large value of \( \gamma(S) \)
in the Phenolite model, the singularity closest to the vertical
axis corresponds to a phase angle (with vertical) of 68.5°.
Clearly, the assumption that the phase-velocity sheets of
split \( S \)-waves intersect far from the vertical is satisfied.
However, if \( \gamma(S) \) were smaller and the reflected rays for a
certain range of offsets and azimuthal angles crossed a
singularity area, we would record complicated converted
wavefields (including multiple arrivals) that would be dif-
ficult to separate into distinct \( PS1 \) and \( PS2 \) reflections
(Crampin and Yedlin, 1981; Grechka and Obolentseva,
1993).

Polarization of shear waves

The physical modeling confirmed our theoretical predic-
tion that each of the converted waves does not exist in a
close vicinity of one the symmetry planes, where its po-
larization becomes close to that of an \( SH \)-wave (Figure 1).
This fact imposes additional requirements on the number of
CMP lines needed to reconstruct the NMO ellipses.
While three well separated CMP lines are generally suf-
ficient to obtain \( P \)-wave NMO ellipses (Grechka and Ts-
vankin, 1996), there is no guarantee that both \( PS \) re-
flexions will be recorded along at least two of those lines.
(We need at least two measurements of the NMO ve-
cocity of each \( PS \) wave to build the NMO ellipses, assuming
that their orientation has been determined from \( P \)-wave
data.) For instance, if any one of azimuths 0°, 30°, 60°,
or 90° (see Figure 4 and 5) were missing, we would not
have enough data to recover one of the converted-wave
NMO ellipses. Generally, acquisition of converted-wave
data along four well separated CMP lines should prob-
ably be accepted as a minimum requirement.

The complications outlined above show that the
problem of joint inversion of \( P \) and \( PS \) traveltimes in
azimuthally anisotropic media may become much more
involved under less favorable circumstances, especially in
the presence of vertical inhomogeneity.
Conclusions

Here, we derived a general moveout equation for mode-
converted waves in inhomogeneous anisotropic media by
generalizing the approach developed for pure modes by
Grechka and Tsvankin (1996). Due to the presence of
odd terms in offset in the moveout series for converted
waves, the minimum of reflection traveltime is shifted
from the common-midpoint (CMP) location. To intro-
duce the normal-moveout velocity for converted waves,
we consider reflection moveout with respect to the travel-
time minimum and, following the approach of Grechka
and Tsvankin (1996), apply a Taylor series expansion in
the horizontal coordinates. The NMO velocity, defined
in this fashion, has the same azimuthal dependence as
that for pure modes and also represents an ellipse in
the horizontal plane. In contrast to pure-mode reflec-
tions, however, normal moveout of converted waves cannot be
expressed through one-way traveltimes from the zero-
offset reflection point.

Converted-wave moveout can be significantly sim-
plicated if the model consists of homogeneous layers with
a horizontal symmetry plane. Since reflection traveltime
in such a model becomes reciprocal with respect to the
source and receiver locations, the traveltime series for
converted waves contains only even terms, and the NMO
ellipse is centered at the CMP location. Furthermore,
the NMO ellipse for converted waves can be expressed
through the pure-mode NMO ellipses using the general-
ized Dix equation of Grechka et al. (1997). This rela-
tionship between the NMO velocities of pure and con-
verted modes provides an analytic basis for obtaining
shear-wave information from P- and PS-moveout data.

We tested this general methodology on physical-
modeling data acquired over a block of azimuthally an-
isotropic phenolic material which is known to have or-
thorhombic symmetry. By combining normal-moveout
velocities and zero-offset traveltimes of P- and two split
PS-waves, we determined the orientation of the vertical
symmetry planes and eight (out of nine) elastic param-
ters of the material. In the inversion procedure we used
the known layer thickness to obtain the vertical veloc-
ities that otherwise would not be constrained by the re-
fection data. It should be emphasized that the theory
even allowed us to reconstruct the moveout velocities of
the converted PSH-waves in the symmetry planes, al-
though these arrivals cannot be physically excited in our
model. The only elastic coefficient that could not be re-
covered from moveout data \( [\delta^{(3)} \text{ in Tsvankin's (1996)}
notation], was evaluated using the group velocity of the
direct P-wave arrival. The high accuracy of the inversion
procedure was verified in several ways, including a com-
parison of the inverted and directly measured horizontal
velocities of pure modes.

Although the physical-modeling experiment was per-
fomed for a single orthorhombic layer, the theory can be
applied to more complicated vertically inhomogeneous
azimuthally anisotropic models. The main problem in the
practical implementation of this algorithm is the recovery
of reflection moveout of converted waves in the presence
of multiple splitting in azimuthally anisotropic layers.

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APPENDIX A: 3-D NMO equation for converted waves in anisotropic inhomogeneous media

Here, we derive a general equation for NMO velocity of converted waves valid in any sufficiently smooth inhomogeneous anisotropic media. Let us consider a converted mode of arbitrary type recorded on common-midpoint (CMP) lines with different azimuthal orientation but the same CMP location (Figure A1). Following the approach of Grechka and Tsvankin (1996), we expand the two-way reflection traveltimes in the vicinity of the common midpoint:

\[
t(x) = \tau_0 + \sum_{i=1}^{2} t_{x_i} x_i + \frac{1}{2} \sum_{i,j=1}^{2} t_{x_i x_j} x_i x_j + \ldots ,
\]

(A1)

where

\[
t_{x_i} = \frac{\partial \tau}{\partial x_i} \bigg|_{x_1=x_2=0}, \quad t_{x_i x_j} = \frac{\partial^2 \tau}{\partial x_i \partial x_j} \bigg|_{x_1=x_2=0},
\]

and \(\tau_0 = \tau(0)\) is the two-way zero-offset travelt ime.

In the pure-mode case, normal-moveout velocity in CMP geometry is not influenced by reflection-point dispersal, and reflected rays can be assumed to travel from the zero-offset reflection point (Hubral and Krey, 1980). Therefore, the moveout expansion similar to (A1) can be rewritten for pure modes through the one-way traveltimes, and the spatial derivatives of \(\tau\) can be expressed through the horizontal components of the slowness vector (Grechka and Tsvankin, 1996). Unfortunately, this convenient simplification is no longer valid for converted waves since the reflection travelt ime is calculated along two segments corresponding to different modes. Also, the offset dependence of the coordinates of the reflection point \(y\) [that lies on the reflector \(f(y) = 0\), Figure A1] cannot be ignored in the derivation of the NMO velo-
Joint traveltime inversion of $P$- and PS-waves

Keeping only the quadratic and lower-order terms in the Taylor series expansion (A1), we can represent the squared two-way CMP traveltime as

$$t^2(\mathbf{x}) = t_0^2 + 2 \sum_{i=1}^{2} \hat{h}_i x_i + \sum_{i,j=1}^{2} W_{ij} x_i x_j,$$

(A2)

where

$$\hat{h}_i = 2\hat{t}_0 t_{e_i}$$

and $\mathbf{W}$ is a symmetric matrix given by

$$W_{ij} = t_{e_i} t_{e_j} + \hat{t}_0 t_{e_i} t_{e_j}.$$  

(A3)

Equation (A2) represents a hyperbolic approximation of the converted-mode reflection traveltime. In general, the moveout of converted waves is not reciprocal with respect to the source and receiver positions, and expansion (A2) contains linear terms in offset. As a result, the extremum of the squared CMP traveltime $t^2(\mathbf{x})$, defined by

$$\frac{\partial^2 t}{\partial x_1^2} = \frac{\partial^2 t}{\partial x_2^2} = 0,$$

(A5)

is generally shifted from the zero offset $x_1 = x_2 = 0$. The asymmetry of the traveltime curve, which does not exist for pure modes, causes obvious complications in moveout analysis. To eliminate the linear terms in equation (A2), let us change variables

$$\hat{\mathbf{x}} = \mathbf{x} - \xi.$$  

(A6)

Equation (A2) now can be rewritten as

$$t^2(\hat{\mathbf{x}}) = t_0^2 + 2 \sum_{i,j=1}^{2} W_{ij} \hat{x}_i \hat{x}_j,$$

(A7)

where $t_0$ is the two-way traveltime at the extremum point and

$$W_{ij} = t_{e_i} \frac{\partial^2 t}{\partial x_i \partial x_j} \bigg|_{x_1 = x_2 = 0}.$$  

(A8)

Equation (A7) has exactly the same form as the one for pure modes [see equation (2) in Grechka and Tsvankin, 1996], but we describe the squared CMP traveltime for converted waves in the vicinity of its extremum $\hat{\mathbf{x}} = 0$ rather than near the zero-offset point $\mathbf{x} = 0$. In the polar coordinates centered at the extremum point, $\hat{\mathbf{x}}$ is expressed through the distance $\tilde{h}$ from the extremum and the polar azimuth $\alpha$ as

$$\hat{x}_1 = \tilde{h} \cos \alpha, \quad \hat{x}_2 = \tilde{h} \sin \alpha.$$  

(A9)

Equation (A7) then becomes

$$t^2(\tilde{h}, \alpha) = t_0^2 + \left( W_{11} \cos^2 \alpha + 2 W_{12} \sin \alpha \cos \alpha \right) \tilde{h}^2 + W_{22} \sin^2 \alpha \tilde{h}^2.$$  

(A10)

According to the definition of the normal-moveout velocity $V_{nmo}$,

$$t^2(\tilde{h}, \alpha) = t_0^2 + \frac{\tilde{h}^2}{V_{nmo}^2(\alpha)} + \ldots.$$  

(A11)

Combining equations (A10) and (A11) yields

$$V_{nmo}^2(\alpha) = W_{11} \cos^2 \alpha + 2 W_{12} \sin \alpha \cos \alpha + W_{22} \sin^2 \alpha,$$

which is identical to the expression for azimuthally-dependent normal-moveout velocity of pure modes given in Grechka and Tsvankin (1996).

**APPENDIX B: Moveout of converted waves in a plane homogeneous layer**

Here, we show that the reflection traveltime of converted waves in a plane layer with a horizontal symmetry plane is reciprocal with respect to the source and receiver positions. The reciprocity implies the absence of uneven terms in the traveltime series and helps to extend the generalized Dix equation of Grechka et al. (1997) to converted waves.

To prove that the reflection moveout $t^{(PS)}$ is an even function of offset, let us consider the converted PS-wave traveling along ray $SRG$ and build a reciprocal ray $G_{1} R_{1} G$, with $G$ being in the middle of $SG_{1}$ (Figure B1). Due to the presence of the horizontal symmetry plane, the upgoing and downgoing rays with the same values of the horizontal slowness components $p_1$ and $p_2$ will be symmetric with respect to the horizontal plane. The same symmetry holds for two downgoing rays with the horizontal slownesses of the same magnitude but opposite signs; also, the group velocities along these rays are equal to each other. Hence, to find the reflected PS-wave from $G_{1}$ to $G$, we generate a downgoing $P$-ray $G_{1} R_{1}$ (Figure B1) with the horizontal slownesses ($-p_1$) and ($-p_2$), where $p_1$ and $p_2$ correspond to ray $SR$. Then $G_{1} R_{1}$ will represent a mirror image of $SR$ with respect to the horizontal plane, and the traveltimes along these two rays will be identical. Also, as illustrated by the plan view in Figure B1, the projections of $R_{1}$ and $R$ on the surface are symmetric with respect to $G$. Therefore, ray $R_{1} G$ and the shear-wave reflected ray $R G$ are symmetric with respect to the horizontal plane as well. In accordance with the above discussion of $P$-waves, this implies that $R_{1} G$ corresponds to the ray parameters ($-p_1$) and ($-p_2$) and, therefore, represents the $S$-wave reflected at $R_{1}$. Thus, the $P$ and $S$ segments of the $PS$ reflection from $G_{1}$ to $G$ represent mirror images with respect to horizontal of the corresponding segments of ray $SRG$. As a result, the traveltimes along $SRG$ and $G_{1} R_{1} G$ coincide with each
other, and the PS moveout is reciprocal with respect to the source and receiver positions.

Next, we show that NMO velocity of the converted wave can be obtained from the generalized Dix equation of Grechka et al. (1997), originally developed for pure-mode reflections in stratified media. Because of the source-receiver reciprocity, the quadratic approximation for the PS-wave reflection traveltimes takes the form (A7) without any shift of the zero offset:

$$[t_0^{(PS)} (x)]^2 = [t_0^{(PS)}]_0^2 + \sum_{i,j=1}^2 W^{(PS)}_{i,j} |x_i x_j|$$

(B1)

where $t_0^{(PS)}$ is the PS-wave zero-offset traveltimes and the matrix $W^{(PS)}$ determines the PS-wave NMO ellipse.

To relate $W^{(PS)}$ to the pure-mode NMO ellipses described by the matrices $W^{(P)}$ and $W^{(S)}$, let us add a second identical layer beneath the first one (Figure B1) and construct a PSSSP reflected wave from the bottom of this artificial model. Since the intermediate interface represents a symmetry plane, the refracted ray $R\tilde{G}$ should coincide with $RG$ and, therefore, hit point $R_1$. Finally, due to the preservation of the horizontal skewness along the raypath, rays $\tilde{G}R_1$ and $R_1G$ correspond to the same skewness components $p_1$ and $p_2$, and $R_1G$ will represent the refracted P-wave excited by $\tilde{G}R_1$ at $R_1$.

Hence, the PS-wave traveltimes along ray $SRG$ can be replaced with the traveltimes along $SR\tilde{G}$, equal to one half of the reflection traveltimes along $SR\tilde{G}R_1G$. Since the offset $SG = SG_1/2$, the NMO velocities (and, therefore, the matrices $W$) for the PS-converted wave $SRG$ and the reflected wave $SR\tilde{G}R_1G$ are equal to each other. Note that the PSSSP reflection is analogous to a pure mode and its NMO ellipse can be obtained from the generalized Dix equation (Grechka et al., 1997),

$$t_0^{(PSSP)} [W^{(PSSP)}]^{-1} = 2t_0^{(P)} [W^{(P)}]^{-1} + 2t_0^{(S)} [W^{(S)}]^{-1},$$

(B2)

where the matrices $W^{(P)}$ and $W^{(S)}$ describe the NMO ellipses that correspond to the pure $P$- and $S$-reflections, and $t_0^{(P)}$ and $t_0^{(S)}$ are the one-way zero-offset traveltimes of the $P$- and $S$-waves. Taking into account that $t_0^{(PSSP)} = t_0^{(PS)}$ and $W^{(PSSP)} = W^{(PS)}$, we find

$$t_0^{(PS)} [W^{(PS)}]^{-1} = t_0^{(P)} [W^{(P)}]^{-1} + t_0^{(S)} [W^{(S)}]^{-1},$$

(B3)

Repeating the above derivation for a stack of plane layers with a horizontal symmetry plane shows that the reciprocity relationship for PS-waves and equation (B3) remain valid in layered media as well. In this case, however, the matrices $W^{(P)}$ and $W^{(S)}$ become effective quantities that can be expressed through a Dix-type average of the interval values $W^{(P)}_{i,j}$ and $W^{(S)}_{i,j}$ (Grechka et al., 1997). Then equation (B3) takes the following form:

$$t_0^{(PS)} [W^{(PS)}]^{-1} = \sum_{i=1}^N t_0^{(P)} [W^{(P)}_{i,j}]^{-1}$$

where $t_0^{(P)}$ and $t_0^{(S)}$ are the interval one-way zero-offset traveltimes.

APPENDIX C: Tsvakin's notation for orthorhombic media

Due to the identical form of the Christoffel equation, the velocities and polarizations in the symmetry planes of orthorhombic media are given by the same equations as for vertical transverse isotropy. This equivalence was used
by Tsvankin (1996) to introduce anisotropic parameters similar to the well-known Thomsen’s (1986) coefficients $\epsilon$, $\delta$, and $\gamma$ for vertical transverse isotropy. Expressions for these parameters in terms of the stiffness components $c_{ij}$ and density $\rho$ are given below.

- $V_{p0}$ – $P$-wave vertical velocity:

$$V_{p0} = \sqrt{\frac{c_{33}}{\rho}}. \quad (C1)$$

- $V_{s0}$ – the vertical velocity of the $S$-wave polarized in the $x_1$-direction:

$$V_{s0} = \sqrt{\frac{c_{55}}{\rho}}. \quad (C2)$$

- $\varepsilon^{(2)}$ – the VTI parameter $\epsilon$ in the $[z_1, z_3]$ symmetry plane normal to $x_2$-axis (this explains the superscript $^{(2)}$):

$$\varepsilon^{(2)} = \frac{c_{11} - c_{33}}{2c_{33}}. \quad (C3)$$

$\varepsilon^{(2)}$ is close to the fractional difference between the $P$-wave velocities in the $x_1$ and $x_3$ directions.

- $\delta^{(2)}$ – the VTI parameter $\delta$ in the $[x_1, x_3]$ plane:

$$\delta^{(2)} = \frac{(c_{33} + c_{55})^2 - (c_{33} - c_{55})^2}{2c_{33}(c_{33} - c_{55})}. \quad (C4)$$

$\delta^{(2)}$ is responsible for near-vertical $P$-wave velocity variations, also influences $SV$-wave velocity anisotropy.

- $\gamma^{(2)}$ – the VTI parameter $\gamma$ in the $[x_1, x_3]$ plane:

$$\gamma^{(2)} = \frac{c_{66} - c_{44}}{2c_{44}}. \quad (C5)$$

$\gamma^{(2)}$ is close to the fractional difference between the $SH$-wave velocities in the $x_1$ and $x_3$ directions.

- $\epsilon^{(1)}$ – the VTI parameter $\epsilon$ in the $[x_2, x_3]$ symmetry plane:

$$\epsilon^{(1)} = \frac{c_{22} - c_{33}}{2c_{33}}. \quad (C6)$$

- $\delta^{(1)}$ – the VTI parameter $\delta$ in the $[x_2, x_3]$ plane:

$$\delta^{(1)} = \frac{(c_{33} + c_{44})^2 - (c_{33} - c_{44})^2}{2c_{33}(c_{33} - c_{44})}. \quad (C7)$$

- $\gamma^{(1)}$ – the VTI parameter $\gamma$ in the $[x_2, x_3]$ plane:

$$\gamma^{(1)} = \frac{c_{66} - c_{55}}{2c_{55}}. \quad (C8)$$

- $\delta^{(3)}$ – the VTI parameter $\delta$ in the $[x_1, x_2]$ plane ($x_1$ plays the role of the symmetry axis):

$$\delta^{(3)} = \frac{(c_{12} + c_{66})^2 - (c_{11} - c_{66})^2}{2c_{11}(c_{11} - c_{66})}. \quad (C9)$$