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Introduction

This report on the Consortium Project at the Center for Wave Phenomena summarizes much of the research conducted within CWP, which is now entering its twentieth year. As in past years, the papers in this report and those that will be presented orally during the Annual Project Review Meeting, May 12-15, 2003, only partially overlap. Also, in addition to these papers, a number of last-minute manuscripts will be distributed during the Meeting and mailed to representatives of sponsor companies.

Papers in This Report

The 22 papers contained herein span a range of research areas grouped into the following seven categories: modeling, migration and imaging, velocity analysis, data processing, VSP data, scattering, and experimental studies. Similarities between these categories and those of the past few years indicate continuity with research of the recent past, while differences reflect the flow and expanding breadth of our research. Readily, many of the papers could have been placed in any of several of these categories, and the categorization could have been different from that selected.

Two of the papers on modeling are studies of how reflections from faults can be used to understand properties of those faults, e.g., are they barriers to or conduits for flow? Reflections from fault zones can differ substantially from those from the boundaries between layers in welded contact. The reflection coefficients can be large and frequency-varying, dependent on the compliance of the fault zone. One of the papers shows that finite-element modeling is required for numerical computation of waves reflected at and transmitted across faults modeled as slip discontinuities. In another paper, finite-difference simulations of the wave equation support a theoretical treatment that demonstrates the equivalence between radiative transfer and the O’Doherty-Anstey formula for waves transmitted through a strongly scattering 1D random-layer medium. In the final paper in the modeling section, an analytic expression derived for geometric spreading in orthorhombic media can be used to correct wide-azimuth AVO data for the influence of the anisotropic overburden.

The use of one-way wave equations for imaging has always suffered from its inability to preserve amplitude. One of the papers in the migration-and-imaging section obtains amplitude that is correct to leading asymptotic order for the one-way wave equation, in both wave-equation migration and Kirchhoff migration, for common-shot data. The second paper in this section demonstrates with numerical computations the limitations in the expected accuracy of conventional DMO and AMO where the subsurface exhibits lateral velocity variation. The next two papers develop closed-form expressions for map time-migration and time-demigration in VTI media, and also demonstrate the applicability of map depth-migration even where caustics are present. Map time-migration, which continues to have its place in seismic data processing, is made considerably more efficient with use of these closed-form expressions.

Two of the papers in the velocity-analysis section pertain to the information that can be derived from moveout in the presence of anisotropy. Where the medium is anisotropic, flattening of events in common-image gathers is not a sufficient condition for estimating the subsurface parameters needed for accurate imaging in depth. One paper studies the quality of parameter
estimation where the subsurface consists of piecewise factorized VTI regions. The second of these papers examines conditions under which the sign and magnitude of the quartic moveout coefficient as a function of azimuth can help constrain the estimated anisotropic velocity field. The third paper in this section develops a method of wave-equation reflection tomography, based on single scattering of body waves, that explicitly takes into account the bandlimited character of data.

The first of the three papers in the data-processing section addresses the ‘\( PP + PS = SS \)’ method presented in past CWP meetings by extending the theory to 3D and showing an implementation applied to four-component OBC data acquired in the Gulf of Mexico. The next paper shows application of a 3D wave-theoretical method, as an alternative to short-wavelength static corrections, to experimental data acquired in John Scales’s Physical Acoustics Laboratory. The method, based on an integral-equation formulation of scattering in the near-surface, was developed by Xander Campman at Delft University. The third paper is a field-data case study aimed at diagnosing pressure differences in reservoirs from fault-plane reflections. This is work directly supported by and done in collaboration with Shell Research.

Two papers in the vertical seismic programming section address issues of estimation of absorption from spectral ratios in VSP data. One deals with distortions in such estimates caused by thin layering, and the other attempts to quantify uncertainties in estimated absorption.

Multiple scattering of waves, in both theory and experiment, has received considerable attention within CWP over the past three years. One paper focuses on analysis of surface-wave scattering in laboratory-acquired data acquired on an aluminum block that contains grooves that could represent random variations, for example, in the near-surface of the earth. Movies that will be shown in the Project Review Meeting show clearly the surprisingly strong scattering of an incident Rayleigh wave into body waves, which could be one source of dispersion of the surface waves. Two papers that analyze seismic coda are aimed at using such data to obtain localized information. One paper describes a method that likely has most application to constraining the location of earthquake sources, but may also apply to micro-seismic sources caused by fracturing near a borehole. The second of these papers derives and tests the theory for the sensitivity kernel appropriate for coda wave interferometry aimed at obtaining localized changes in a medium, such as from time-lapse data.

Three papers treat experimental data obtained in the Physical Acoustics Laboratory. The first analyzes the coherent and incoherent scattering in randomly heterogeneous rock samples. Movies of data acquired with a laser vibrometer allow one to visualize the complex dynamics of diffractions, multiple scattering, mode conversion, and whispering gallery modes. Using a radiative transfer model one can quantify scattering properties of rocks with similar constitution but differing grain size. A second paper shows and analyzes the transition from ballistic-mode to diffusive propagation caused by strong scattering, again with laboratory physical-model data. The final paper treats in depth the theory and application to experimental data of ultrasound spectroscopy for estimation of elastic parameters and the quality-factor \( Q \) for different man-made materials and rocks.

**OVERVIEW OF DEVELOPMENTS IN CWP**

Following is an overview of other developments within CWP during the past year.
Center Support

Despite corporate mergers and unfavorable economic conditions for many companies within the industry, sponsorship of CWP has held fairly steady, with losses of some companies and gains of others. During the past year, three companies joined as new sponsors of CWP — Anadarko, Encana, and GX Technology — leaving CWP membership at 25 companies going into year 20. Two more companies may join prior to the Annual Project Review Meeting. A full list of sponsors over the term of the past year appears on the acknowledgment page at the beginning of this volume.

We benefit from approximately the same level of government support as last year. A list of those sponsor agencies also appears in the acknowledgment page.

Our industrial and government support for both research and education complement one another; each gains from, and strengthens, the other. As a net result, for the present annual fee of $43,000, a company participates in a research project whose total funding level is about $1.44M — continuing at a leverage factor of about 33.

The SEG Foundation has continued to provide support for Seismic Unix (SU), under John Stockwell’s leadership.

John Scales’s Changed Status

Over the past five years or so, John has become a part-time experimentalist while also working on projects directly related to the consortium. After returning from his sabbatical, he decided that he wanted to switch to doing experimental research full-time. He in fact did just that starting last summer, and the results of that work — both his and that of students — have been truly exciting.

John has decided that his opportunities to pursue and concentrate on his experimental work full-time will be enhanced if he goes it alone, no longer as part of CWP. He and we look forward to continued, fruitful collaborations with CWP faculty and students on research of mutual interest, but neither John nor the Physical Acoustics Laboratory will have any formal connection with CWP or its industry-sponsored consortium.

Joint Project with Shell Research

Roel Snieder and CWP student, Matt Haney, are collaborating with Jon Sheiman at Shell Research in Houston on a project to relate physical properties of fault zones to the seismic response from these structures. Though this research is not strictly part of the Consortium Project, Shell is willing for CWP to share results with Consortium sponsors, within constraints of Shell’s research agreement. We encourage similar types of directly sponsored research with other companies that could lead to prospective sharing of results with the Consortium.

Papers at SEG and EAGE

CWP students and faculty presented an unusually large number of papers at the 2002 SEG Annual Meeting in Salt Lake City — a total of 23 oral presentations, poster papers, and workshop presentations. One factor in this large number of presentations is the relatively recent CSM Department of Geophysics requirement that Ph.D. students must complete research
papers in two different areas with two separate faculty members. The two goals of this policy are to broaden students’ educational background in geophysics and to get students into doing research early in their Ph.D. studies.

CWP personnel presented several papers at the EAGE Meeting in Florence, with the paper on angle tomography by Sverre Brandsberg-Dahl, Bjorn Ursin, and Martijn de Hoop recognized as one of the ‘Best of the EAGE’ and invited for presentation at the SEG Annual Meeting.

CWP on the Web

Samizdat Press, the Internet archive that distributes free books and sets of lecture notes, has grown to a listing of 21. The newest addition is “How to be a Programmer” by Rob Read.

Every few months, you may wish to check the latest offerings of Samizdat Press at samizdat@dix.mines.edu or www.cwp.mines.edu/bookshelf.html.

Computing Environment

In February 2003, CWP purchased a 32-processor Linux cluster system. Each of the 16 nodes is a dual processor pentium 2.4 GHz Pentium Xeon system with 2GB of RAM available per processor, and 160 GB of diskspace available for each node. The system has been tested with PVM and MPI based modeling codes and performs as expected.

An ongoing plan for expansion of CWP’s computer infrastructure is to upgrade the student facilities so that each CWP member has a 2-Ghz or better system.

Students have access to the following commercial packages: Mathematica, Matlab, NAG95 (fortran 90/95 compiler).

Software Releases

CWP releases both free software and software that is offered as a proprietary deliverable to the Consortium. Most proprietary codes depend heavily on the free software environment, so both are relevant to the Consortium. The proprietary period is for three years.

Proprietary Software –

The two most recent proprietary software releases were:

• U-52: ScreenMod3D, Jerome Le Rousseau, 3D Generalized Screen seismic modeling code. This is a parallelized implementation using PVM.

• U-53: Psdm\topo, Baoniu Han, Phase shift finite difference depth migration with topography for OBS data in VTI media. This is a parallelized implementation using PVM.

These were released on 30 May 2002 and are proprietary until 30 May 2005.

New software will likely include a number of Matlab applications.

The most current release of SU is SU 36, released on 16 Dec 2002. SU has been installed at more than 3000 sites in 62 countries (where a country is defined by a unique Internet country code).
For his development, maintenance, and support of Seismic Unix, John Stockwell was awarded the SEG Special Commendation Award, along with Jack Cohen, Shuki Ronen, and Einar Kjartansson.

Visitors to CWP

CWP has benefited again this year from visits by a number of scientists and friends from other universities and industry. We strongly encourage visits from our sponsor representatives, whether it be for a single day, or for an extended period. Below is a list of those who spent time at CWP.

- Bjorn Ursin, one-year sabbatical from NTNU, Trondheim, Norway - through August, 2002
- Art Gautesen, one-year sabbatical from the University of Iowa - August, 2002, to August, 2003
- Stig-Kyrre Foss, PhD student from NTNU - August, 2001, to July, 2002
- Michael Oberguggenberger, University of Innsbruck - one week of collaborative research with Martijn
- Guenther Hoermann, University of Innsbruck - August-September, 2002, collaborative research with Martijn
- Xander Campman, Delft University of Technology - September-October 2002, collaborative research in the Physical Acoustics Lab with John Scales and Kasper van Wijk
- Sergey Goldin, Professor, member of Russian Academy of Sciences, Novosibirsk, Russia - October-November, 2002, collaborative research with Martijn de Hoop, under CRDF grant
- Adrianus T. de Hoop, Professor, Delft University of Technology - one week of collaborative Research with Martijn de Hoop
- Joe Dellinger, BP, Houston - one week: seminars, BP/CWP interaction, brainstorming session for research directions
- Nizar Chemingui, ExxonMobil - seminar and two-day visit with CWP people
- David Eckert, TotalFinaElf, Pau, France - Visiting Scientist for 16 months
- Bart von Tiggelen, Université Joseph Fourier, Grenoble, France - seminar and several-day visit with John Scales and students
- Victor Isakov, Professor, Wichita State University - CWP seminar and math colloquium in October, 2002
- Rob van der Hilst (Utrecht University), Hans Peter Bunge (Princeton Univ.), Michael Bostock (Univ. British Columbia), Art Weglein (UH), and Jerry Harris (Stanford), among others, presented Geophysics Department Heiland Lectures this past year
Travels by CWP People

Interactions and collaborations with others have taken place away from Golden and across the Net as well as in Golden. Collaborations and activities elsewhere include the following.

Norm Bleistein –

- presentation at Veritas DGC in Calgary
- EAGE meeting in Florence
- two-day short course at BHP Billiton, and two-day visit to Veritas DGC, both in Houston.
- China:
  - Tongji University - two days of presentation and interactions
  - Chinese University of Geosciences, Wuhan - three weeks of teaching in the Geophysics Department
  - Academica Sinica, Beijing - collaboration with Professor GuanQuan Zhang and Yu Zhang, Veritas DGC
  - University of Petroleum, Beijing - lecture
- Stanford University, two-day workshop in honor of Joseph B. Keller’s 80th birthday; also two-day visit to SEP

Martijn de Hoop –

- EAGE and SEG Annual Meetings, Florence and Salt Lake City
- Institut de Physique du Globe, Paris - lectures
- Univ. of British Columbia - lectures at Institute of Applied Mathematics
- Wichita State University - lectures at Department of Mathematics
- PanAmerican Advanced Studies Institute, Chile - lectured for a month
- MIT, EAPS - visiting professor (part time second year, full time this half year)
- Institute for Pure and Applied Mathematics, UCLA - extended stay this coming Fall
- invited speaker at five other conferences and workshops

Ken Larner –

- Utrecht University, The Netherlands - invited lectures of the Vening Meinesz Geodynamics School
- TotalFinaElf, Pau, France - lectures and interactions
- Abu Dhabi National Oil Company and WesternGeco
• SEG Annual Meeting, Salt Lake City - presentation

*Roel Snieder –*

• First Pan-American/Iberian Meeting on Acoustics, Cancun, Mexico (invited speaker)
• Workshop on Inverse Problems and Quantification of Uncertainty, Institute for Mathematics and its Application, Minneapolis (invited speaker)
• New Mexico Institute of Mining and Technology, Socorro NM (two invited lectures)
• SEG Annual Meeting, Salt Lake City
• USGS Golden (seminar)
• Amaranda Hess, Houston (seminar)
• Waterways Experimental Station, Vicksburg (seminar)
• Shell International Exploration and Production Inc., Houston (two work visits)
• Stanford University (two seminars)
• Univ. of California, Santa Cruz (seminar)
• National Science Foundation, Washington (member of review panel)
• Utrecht University (two trips, PhD defense and opening of the new building of the Geological Survey)

*Ilya Tsvankin –*

• Houston and Calgary - taught the SEG course on anisotropy with Vladimir Grechka
• Tutzing, Germany - 10th Intl. Workshop on Seismic Anisotropy; chaired a session and gave several papers.
• Houston - two invited talks at ExxonMobil
• Mexico City - visited IMP and gave invited lecture

It should also be noted that our students also traveled quite a bit. Matt Haney, Alex Grêt, and Kasper van Wijk received travel grants to attend and to make presentations at the two-week NATO Summer School in Cargese, Corsica on “Wave Scattering in Complex Media.” Also, Alison Malcolm was awarded a grant to attend the March 2003 SIAM Conference on Mathematical and Computational Issues in Geophysics.
Students

Three CWP students completed their studies this past year: Valmore Celis, graduated December 2002 (MSc), is now working with PDVSA Intevep in Caracas, Venezuela; Reynaldo Cardona, graduated December 2002 (PhD), is now working with ChevronTexaco in Bellaire, TX; and Andrés Pech, finished his degree in March 2003 (PhD), has returned to Mexico as an employee of Pemex.

Four new students joined CWP in the past year: Carlos Pacheco, MSc, Venezuela; Mirjana Vuletic, PhD, Yugoslavia; Greg Wimpey, MSc, USA; Xiaoxia (Ellen) Xu, PhD, The People’s Republic of China.

Confirmed new CWP students who will start Fall, 2003: Matt Reynolds, graduating with a B.S. in geosciences from Colorado University; and Kurang Mehta, receiving a M.S. in electrical engineering from North Carolina State University.

During the 2002-2003 academic year, CWP provided full or partial funding for 15 students.

Publications

We recently distributed three PhD theses to our sponsors written by our recent graduates: Valmore Celis, Reynaldo Cardona and Andrés Pech.

During the 2002-2003 term, 22 papers authored or co-authored by CWP faculty and students have been published and over 30 papers are currently under review or in press for publication in a variety of journals. In your meeting folder, you can find a complete listing of distributed reports in the CWP list of “Available Papers.” The list is also on our web site at http://www.cwp.mines.edu/bookshelf.html.

Welcome

As always, we greatly welcome representatives of our sponsor companies to the Annual Project Review Meeting. We look forward to the opportunity to exchange with you ideas and thoughts about this past year’s projects.

Ken Larner, Director
Center for Wave Phenomena
May 2003
Numerical modeling of waves incident on slip discontinuities

Matthew Haney and Roel Snieder
Center for Wave Phenomena, Colorado School of Mines, Golden, CO 80401

ABSTRACT
Interfaces in elastic media need not be in welded contact. For instance, fractures allow a small amount of slip to occur along their surfaces during the passage of a seismic wave. By slip, we mean that the displacement across the interface can be discontinuous. Reflection and transmission of plane waves in this case are frequency dependent.

Previous numerical studies have chosen the finite-difference method to simulate slip discontinuities. The approach suffers from difficulties in incorporating boundary conditions into the strong form of the equations of motion. We show in detail that only a finite-element formulation of the $\sigma$-$v$ (stress and velocity) equations can overcome these problems. Numerical examples illustrate the method for an $SH$-wave normally incident on a slip discontinuity and a $P$-wave incident at an angle.

1 INTRODUCTION

Full-waveform forward modeling in seismology has traditionally been dominated by finite-difference methods. Perhaps the intuitive appeal of such methods has led to their popularity. Kelly and Marfurt (1990), in reviewing the numerical literature from the exploration geophysics community, cited at least four possible reasons for the relative neglect of other methods, specifically finite-elements. We complement their list by stating that finite-elements require additional numerical detail to implement properly. For instance, explicit finite-difference algorithms do not require the inversion of a matrix. In contrast, all finite-element methods eventually lead to a matrix equation that must be solved.

Finite-elements offer more flexibility than finite-differences with respect to the shape of the numerical mesh; however, it is less widely accepted that finite-elements are superior to finite-differences in the way they account for boundary conditions. Such an advantage can easily be seen in the formula for integration by parts: an integral is replaced by another integral and a term representing the values of the function at the boundaries. This basic fact shows up in the formulation of finite-elements since they approximate the integrated, or weak, equations of motion.

Finite-differences, as approximations to the strong form of the equations of motion, require ad hoc techniques to take into account, for instance, a welded boundary between two different elastic media. David Boore, one of the pioneers of the finite-difference method in seismology, summarized the problem by stating that in finite-difference schemes “the interface displacements must satisfy the continuity of displacement and stress, but are not explicitly required to satisfy the equation of motion” (Boore, 1970).

The motivation of this paper is to explore the advantages of the finite-element method for the modeling of slip discontinuities. The slip discontinuity has been proposed by Schoenberg (1980) as an interface condition applicable to cracks. The boundary condition can be thought of as a generalized interface condition since it supplies a parameter, the compliance $\eta$, that, over its range of physical values, takes an elastic interface from a welded contact ($\eta = 0$) to a free surface ($\eta = \infty$). Mathematically, a slip discontinuity for an $SH$-wave is expressed as

$$u_y^+ - u_y^- = \eta \sigma_{yz},$$

(1)

$$\sigma_{yz}^+ - \sigma_{yz}^- = 0,$$

(2)

where $(-)$ refers to the side of the interface on which the wave is incident, $(+)$ the other side of the interface, $u_y$ refers to the displacement normal to the plane of propagation, and $\sigma_{yz}$ is the shear stress. In this paper, a slip discontinuity extends infinitely; we do not consider crack-tips. Physically, boundary conditions (1) and (2) model the transmission and reflection of a thin, low shear zone (Schoenberg, 1980). Faults generally fall within this category.
In this paper, we will argue three main points:

1. Finite-element modeling is needed to incorporate interlace boundary conditions into the equations of motion.
2. The equations of motion should be the first-order $\sigma$-$v$ (stress and velocity) system of partial differential equations instead of the second-order wave equation in $u$ (displacement) to avoid any finite-difference-like approximations to spatial derivatives.
3. Only an implicit, unconditionally stable time integration scheme can handle the zero-length elements needed to accurately model a slip discontinuity.

We conclude this discussion of the finite-element method with numerical examples of the reflection and transmission of plane $SH$- and $P$-waves at a slip discontinuity.

\section{A FInite-ElemEnt SCheme For the 1D SaCULAR WaVE EquatIon}

The beginning of any numerical study must be an analysis of the two competing limitations imposed on finite systems: numerical dispersion and stability. An understanding of these concepts should steer the subsequent implementation of a particular numerical scheme. Ideally, any numerical simulation should satisfy a stability criterion with the least amount of dispersion possible.

In this section, we present the most simple case: a finite-element implementation of the scalar wave equation in 1D. We choose this problem since, in 1D, the finite-element implementation of the scalar wave equation yields

\begin{equation}
\frac{c^2}{s^2} \frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 u}{\partial t^2} - s = 0,
\end{equation}

and assume Dirichlet (essential) boundary conditions at $z = 0$ and $z = L$. The displacement, $u$, is approximated by a finite series of spatial basis functions with coefficients that depend on time

\begin{equation}
u(z, t) = \sum_{K=1}^{N} \alpha_K(t) \phi_K(z).
\end{equation}

In this paper, the nodal basis will always be used. For a spatial discretization step $h$ and a uniform partition of the interval $[0, L]$ into $N+1$ subintervals, these basis functions are mathematically defined as

\begin{equation}
\phi_K(z) = \begin{cases}
(z - z_{K-1})/h & \text{if } z_{K-1} \leq z \leq z_K, \\
(z_{K+1} - z)/h & \text{if } z_{K} \leq z \leq z_{K+1}, \\
0 & \text{otherwise.}
\end{cases}
\end{equation}

Fig. 1 shows these piecewise linear functions graphically.

The Galerkin formulation of the finite-element method seeks to minimize the weighted average of the error induced by the incompleteness of the finite set of basis functions in equation (5) The essence of the Galerkin method is that it weights the errors with the same basis functions used in the approximation of the displacement, $u$ (Marfurt, 1984)

\begin{equation}
\int_0^L c^2 \sum_{K=1}^{N} \alpha_K(t) \phi_K(z) - \int_0^L \phi_J(z) dz = 0
\end{equation}

for $J=1, 2, ..., N$.

Rearranging the order of the sums and integrals in this equation yields

\begin{equation}
0 = -c^2 \sum_{K=1}^{N} \alpha_K \left[ \int_0^L \frac{\partial^2 \phi_K}{\partial z^2} \phi_J dz \right] + \sum_{K=1}^{N} \frac{\partial^2 \alpha_K}{\partial t^2} \left[ \int_0^L \phi_K \phi_J dz \right] + \int_0^L s \phi_J dz.
\end{equation}

Since the basis functions are piecewise linear over their range, the first integral on the r.h.s. of equation (7), containing a second derivative of the basis function, must be altered (Haltiner and Williams, 1980). It turns out that in altering this term by integrating by parts, boundary terms appear explicitly in the equation of motion

\begin{equation}
\int_0^L \frac{\partial^2 \phi_K}{\partial z^2} \phi_J dz = - \int_0^L \frac{\partial \phi_K}{\partial z} \phi_J dz.
\end{equation}

To implement Dirichlet boundary conditions at $z = 0$ and $z = L$, the weighting function, $\phi_J$, is set to zero at the boundaries (Wang, 2000). Hence, the boundary term in equation (8) disappears and the elements $\phi$ are fixed to be zero on the boundaries. More complicated boundary conditions, such as absorbing boundary conditions, are not discussed here.

Without the boundary term, inserting equation (8)
Numerical modeling of waves incident on slip discontinuities

into equation (7) yields the weak form of the scalar wave equation

$$0 = c^2 \sum_{K=1}^{N} \alpha_K \left[ \int_0^L \frac{\partial \phi_K}{\partial z} \frac{\partial \phi_J}{\partial z} \, dz \right] + \sum_{K=1}^{N} \left[ \int_0^L \phi_K \phi_J \, dz \right] + \int_0^L s \phi_J \, dz. \quad (9)$$

The bracketed terms in equation (9) can be represented as matrices multiplying the vectors $\vec{a}$ and $\vec{a}$. The matrix multiplying the second time derivative, $K$, is usually called the mass matrix, $M$, and the matrix acting on $\vec{a}$ is referred to as the stiffness matrix, $S$. From equation (5), the entries of these matrices can be calculated exactly. For the mass matrix, $M_{K,J}$, the non-zero entries are

$$\int_0^L \phi_K \phi_J \, dz = 2h/3 \quad \text{for } K = J \quad (10)$$

$$\int_0^L \phi_K \phi_J \, dz = h/6 \quad \text{for } K = J + 1, J - 1. \quad (11)$$

This means that $M$ is symmetric and tridiagonal. Similarly, the stiffness matrix, $S_{K,J}$, is also symmetric tridiagonal and the non-zero entries are

$$\int_0^L \frac{\partial \phi_K}{\partial z} \frac{\partial \phi_J}{\partial z} \, dz = 2/h \quad \text{for } K = J \quad (12)$$

$$\int_0^L \frac{\partial \phi_K}{\partial z} \frac{\partial \phi_J}{\partial z} \, dz = -1/h \quad \text{for } K = J + 1, J - 1. \quad (13)$$

The finite-element scheme can now be written in matrix form

$$0 = c^2 S \vec{a} + M \vec{a}. \quad (14)$$

Note that the source term has been omitted in equation (14). In our numerical simulations, we use an initial condition as a source instead of a forcing term in the equations of motion. To include a forcing term, $s$ has to be approximated in the nodal basis

$$s(z,t) = \sum_{K=1}^{N} \phi_K(t) \phi_K(z). \quad (15)$$

Substituting this into equation (9), it can be seen that the mass matrix multiplies the source vector.

3 STABILITY ANALYSIS OF AN EXPLICIT FINITE-ELEMENT SCHEME

The finite-element approach has served to integrate the above matrix equation in space, but a suitable time integration scheme has yet to be determined. As a first example, we use an explicit time integration known as central differences (Zienkiewicz and Taylor, 2000) to approximate the time behavior of $\vec{a}$

$$0 = c^2 S \vec{a} + M \left( \vec{a}_{n+1} - 2 \vec{a}_n + \vec{a}_{n-1} \right) \Delta t^2 \quad (16)$$

where the subscripts refer to the time step $n + 1$, to be calculated, and the previous time steps $n$ and $n - 1$. The time discretization interval is shown as $\Delta t$. Notice that equation (16) is symmetric in the $n + 1$ and $n - 1$ terms. This ensures that the resulting dispersion relation on the numerical grid is real-valued. Such a property is desirable since a complex dispersion relation would yield solutions to the wave equation that grow or decay exponentially with distance. In contrast, real-valued solutions do not grow exponentially, so long as they satisfy a stability criterion.

To calculate the stability criterion of this explicit time integration scheme, first use equations (10), (11), (12), and (13) to carry out the matrix multiplications in equation (16). What results is the so-called finite-element stencil

$$0 = \frac{h}{6 \Delta t^2} \alpha_{m-1,n+1} + \frac{2h}{3 \Delta t^2} \alpha_{m,n+1} + \frac{h}{6 \Delta t^2} \alpha_{m+1,n+1} - \frac{h}{3 \Delta t^2} \left( \frac{c^2}{h} \right) \alpha_{m-1,n} - \left( \frac{4h}{3 \Delta t^2} - \frac{2c^2}{h} \right) \alpha_{m,n} - \left( \frac{h}{3 \Delta t^2} + \frac{c^2}{h} \right) \alpha_{m+1,n} + \frac{h}{6 \Delta t^2} \alpha_{m-1,n-1} + \frac{2h}{3 \Delta t^2} \alpha_{m,n-1} + \frac{h}{6 \Delta t^2} \alpha_{m+1,n-1}, \quad (17)$$

where the first subscripts refer to gridpoints in space and the second subscripts are still the time steps. To investigate the stability of the central difference scheme, insert a harmonic function for $\alpha$ (Alterman and Loewenthal, 1970)

$$\alpha_{m,n} = e^{ikhz} \zeta^n \quad \zeta = e^{ik \Delta t}, \ z = mh, \ t = n \Delta t, \quad (18)$$

where the integers $m$ and $n$ multiply the spatial and temporal interval lengths $h$ and $\Delta t$, and $k$ is the wavenumber.

Inserting equation (18) into equation (17), and organizing terms yields

$$0 = \left[ \frac{h}{6 \Delta t^2} (e^{-ikh} + 4 + e^{ikh}) \zeta^2 - \frac{c^2}{h} (e^{-ikh} - 2 + e^{ikh}) \zeta - \frac{h}{3 \Delta t^2} (e^{-ikh} + 4 + e^{ikh}) \zeta + \frac{h}{6 \Delta t^2} (e^{-ikh} + 4 + e^{ikh}) \right] e^{ikh} \zeta^{n-1}. \quad (19)$$

In order to have non-trivial solutions, the terms inside the brackets in equation (19) must equal zero. The resulting equation contains all the information about
the dispersion and stability properties of the numerical scheme. Note that in higher dimensions, equation (19) is a matrix equation and the condition for non-trivial solutions is that the determinant of the matrix equals zero.

Dividing equation (19) by the coefficient of the $\zeta^2$ term and setting the terms inside the brackets to zero gives

$$0 = \zeta^2 - \frac{6c^2\Delta t^2}{h^2} \left( \frac{e^{-i\Delta t h} - 2 + e^{i\Delta t h}}{e^{-i\Delta t h} + 4 + e^{i\Delta t h}} \right) \zeta - 2\zeta + 1. \tag{20}$$

Using the Euler formula and the relation $\cos(kh) - 1 = -2\sin^2(kh/2)$, we can simplify equation (20)

$$0 = \zeta^2 - \left[ \frac{2\Delta \zeta}{\Delta t^2} + \frac{4c^2}{\Delta t^2} \sin^2(kh/2) \right] - \frac{2\Delta \zeta}{\Delta t^2} \sin^2(kh/2) \zeta + 1. \tag{21}$$

For the numerical scheme to be stable, the roots of equation (21) should have magnitudes less than or equal to 1. This means that the solutions are not exponentially growing in time. For a quadratic of the form

$$0 = x^2 - xz + 1, \tag{22}$$

the magnitude of the two roots, $x_1$ and $x_2$, are both less than or equal to 1 if

$$-1 \leq \frac{z}{2} \leq 1. \tag{23}$$

Inserting equation (21) into this expression yields

$$-1 \leq \frac{2h}{3\Delta \zeta} - \frac{4c^2}{\Delta t^2} \sin^2(kh/2) \leq 1. \tag{24}$$

Scanning over all possible values of $\sin^2(kh/2)$, the strongest condition on the spatial and temporal discretizations $h$ and $\Delta t$ occurs when $\sin^2(kh/2) = 1$ (Alterman and Loewenthal, 1970). In this case, equation (24) becomes:

$$-1 \leq \frac{2h}{3\Delta \zeta} - \frac{4c^2}{\Delta t^2} \leq 1. \tag{25}$$

From equation (25), the right inequality:

$$\frac{2h}{3\Delta \zeta} \geq \frac{4c^2}{\Delta t^2} \tag{26}$$

is trivially true. The left inequality provides a more meaningful relation:

$$\frac{2h}{3\Delta \zeta^2} \leq \frac{2h}{3\Delta \zeta^2} - \frac{4c^2}{h},$$

$$\frac{4c^2}{h} \leq 4h \Delta t^2,$$

$$c\Delta t \sqrt{3} \leq h. \tag{27}$$

This is the Courant-Friedrichs-Levy (CFL) stability condition for this implementation of explicit finite elements (Marfurt, 1984).

Above, we stated that equation (19) contains all the information about the dispersion and stability properties of the numerical scheme. We have the stability condition - what about the dispersion relation? To obtain this, set $b$ in equation (18) to $i\omega$ so that $\zeta = e^{i\omega \Delta t}$.

Substituting this into equation (19), setting the terms in the brackets to zero, and simplifying a bit yields the dispersion relation

$$\cos(\omega \Delta t) = 1 + \frac{i3\Delta \zeta \Delta t^2}{h^2} \left( \frac{\cos(kh) - 1}{\cos(kh) + 2} \right). \tag{28}$$

For wavelengths much larger than the grid-spacing, $\omega \Delta t$ and $kh$ are small parameters. Keeping the lowest order terms in a series expansion of equation (28) gives the well-known relation

$$\omega^2 = c^2 k^2. \tag{29}$$

In other words, the finite-element scheme should do an excellent job propagating waves well-sampled by the numerical grid.

## 4 ZERO-LENGTH ELEMENTS FOR FRACTURES

The finite-element example explored above shows the essence of the numerical implementation of the weak form of the equations of motion. It is relatively easy to code since all that is needed is the inversion of a tridiagonal matrix. Modeling interfaces in elastic media, though, introduces several complications.

To model true boundaries, some of the elements should have zero length. In this way, two points on opposite sides of a crack, but infinitesimally close to one another, could be modeled precisely. Explicit finite-elements of the second-order wave equation fail this requirement for two reasons. First, entries in the stiffness matrix, equations (12) and (13), go to infinity as the length of any element went to zero. This statement assumes that the CFL-condition acts at each point in the computational domain for an irregular grid (which has been verified numerically).

In addition, the boundary conditions in elastic media concern the continuity or relationship of displacement and stress at an interface. Discrete approximations of the stress would have to do for an implementation of the wave equation in displacement. By formulating a finite-element analysis for the $\sigma$-$v$ (stress and velocity) system of equations, the stress would be available in the numerical scheme without approximation. Furthermore, it turns out that the mass and stiffness matrices in the $\sigma$-$v$ formulation can accommodate elements of zero length. The next section will explore this in more detail.

The solution to the problem of the CFL condition
for an element of zero length is to use an implicit time integration scheme that is unconditionally stable. To illustrate unconditional stability, consider this alternative time scheme for equation (16)

\[ 0 = c^2 S \left( \frac{\bar{v}_{n+1} + \bar{v}_{n-1}}{2} \right) + M \left( \frac{\bar{v}_{n+1} - 2\bar{v}_n + \bar{v}_{n-1}}{\Delta t^2} \right), \]

(30)

where now the value of \( \bar{v} \) at the time \( n \) is replaced by a simple average of its values at times \( n+1 \) and \( n-1 \). Note again that this expression is symmetric in the terms at \( n+1 \) and \( n-1 \) so that the dispersion relation is real-valued. By going through the same steps outlined previously to obtain the stability of this time integration scheme (which we call forward-backward time integration), the CFL-like condition that results is

\[-2h^2 \leq 3c^2 \Delta t^2,\]

(31)

which is always true. Hence, the forward-backward time scheme is unconditionally stable. This sort of time scheme makes the modeling of zero-length elements possible.

5 A FINITE-ELEMENT SCHEME FOR STRESS VELOCITY IN 1D

In 1D, all three elastic wave types decouple since they are normally incident on interfaces. Here, we focus the analysis to the SH-wave. The system of partial differential equations for velocity and stress in this case is

\[ \frac{\partial \bar{v}_k}{\partial t} = \frac{1}{\rho} \frac{\partial \sigma_{yz}}{\partial z} = 0, \]

(32)

\[ \frac{\partial \sigma_{yz}}{\partial t} - \mu \frac{\partial \bar{v}_y}{\partial z} = 0. \]

(33)

These equations are recognized as Newton’s and Hooke’s Laws, respectively. The same nodal basis as above is used to approximate the velocity and stress

\[ \bar{v}_y(z,t) = \sum_{K=1}^{N} \alpha_K(t) \phi_K(z), \]

(34)

\[ \sigma_{yz}(z,t) = \sum_{K=1}^{N} \beta_K(t) \phi_K(z), \]

(35)

and Dirichlet boundary conditions are assumed to apply on the ends of some domain \( z = 0 \) to \( z = L \). Applying the Galerkin method to equation (32), as was done to the scalar wave equation, and integrating by parts gives the resulting weak form

\[ \sum_{K=1}^{N} \frac{\partial \alpha_K}{\partial t} \left[ \int_0^L \phi_K \phi_J dz \right] = -\frac{1}{\rho} \sum_{K=1}^{N} \beta_K \left[ \int_0^L \phi_K \frac{\partial \phi_J}{\partial z} dz \right]. \]

(36)

In this equation, the mass matrix (multiplying \( \dot{\alpha} \)) is identical to the one for the scalar wave equation; however, the “stiffness” matrix (multiplying \( \beta \)) has a different form. Using equation (5), the entries of this new stiffness matrix, \( S_{K,J} \), can be calculated

\[ \int_0^L \phi_K \frac{\partial \phi_J}{\partial z} dz = \frac{1}{2} \quad \text{for} \quad K = J + 1 \]

(37)

\[ \int_0^L \phi_K \frac{\partial \phi_J}{\partial z} dz = \frac{1}{2} \quad \text{for} \quad K = J - 1. \]

(38)

The matrix form of equation (36) is thus

\[ M \ddot{\alpha} = -\frac{1}{\rho} S \ddot{\beta}. \]

(39)

Note that the entries of this stiffness matrix are constants, independent of the spatial discretization \( h \). This is in contrast to the expression for the second-order wave system. The independence of this stiffness matrix from \( h \) allows zero length elements to be modeled in the finite-element implementation of the \( \sigma-v \) system.

For the second equation in the \( \sigma-v \) system, equation (33), we apply the Galerkin method but stop short of integrating by parts

\[ \sum_{K=1}^{N} \frac{\partial \beta_K}{\partial t} \left[ \int_0^L \phi_K \phi_J dz \right] = \mu \sum_{K=1}^{N} \alpha_K \left[ \int_0^L \frac{\partial \phi_K}{\partial z} \phi_J dz \right]. \]

(40)

The term on the l.h.s. of equation (40) is similar to what has been encountered up to this point. The new challenge is to properly treat the first term on the r.h.s., since it contains information about slip discontinuities. That is the subject of the next section.

6 THE INCLUSION OF A SLIP DISCONTINUITY

The first step in evaluating the term on the r.h.s. of equation (40) involves breaking the integral into two - one running up to the fault (or fracture), and the other continuing on the other side of the fault. In the limit of...
 whose positions eventually coincide with each other.

The fault having no thickness (slip discontinuity)

\[ \mu \sum_{K=1}^{N} \alpha_K \left[ \int_0^L \frac{\partial \phi_K}{\partial z} \phi_J dz - \int_{f^-}^f f^+ \frac{\partial \phi_K}{\partial z} dz \right] = \mu \sum_{K=1}^{N} \alpha_K \left[ \int_0^L \frac{\partial \phi_K}{\partial z} \phi_J dz + \int_{f^-}^f f^+ \frac{\partial \phi_K}{\partial z} dz \right]. \] (41)

After integrating by parts and combining the resulting two integrals back into a single integral from \( z = 0 \) to \( z = L \), we obtain for the r.h.s. of equation (41)

\[ \mu \sum_{K=1}^{N} \alpha_K \left[ \phi_K \phi_J |_{f^-}^f + \phi_K \phi_J |_{f^+}^f - \int_0^L \phi_K \frac{\partial \phi_J}{\partial z} dz \right]. \] (42)

For simplicity at this point, assume Dirichlet boundaries at \( z = 0 \) and \( z = L \), and distribute the sums

\[ \mu \sum_{K=1}^{N} \alpha_K \left[ \phi_K (f^-) \phi_J (f^-) - \phi_K (f^+) \phi_J (f^+) \right] - \mu \sum_{K=1}^{N} \alpha_K \int_0^L \phi_K \frac{\partial \phi_J}{\partial z} dz. \] (43)

The term with the integral is ready to go into the equations of motion. Notice that it has the same stiffness matrix as in equation (36). The other term represents the slip discontinuity explicitly. To evaluate this term, we first place the nodes of our basis functions at \( f^- \) and \( f^+ \), (Fig. 2). The weighting functions \( \phi_J \) in equation (43) are continuous at the fault, and we set them to their values at the nodes \( f^- \) and \( f^+ \) (unity). We then take the limit as \( f^- \rightarrow f^+ \), creating a zero-length element at the fault. The leftmost term of equation (43) becomes

\[ -\mu \sum_{K=1}^{N} \alpha_K \left( \phi_K (f^+) - \phi_K (f^-) \right). \] (44)

The series in this equation is the slip in the nodal basis. Taking the time-derivative of equation (1), we replace the terms inside the series with the nodal basis representation of the time-derivative of the stress at the fault multiplied by the compliance, \( \eta \)

\[ -\mu \sum_{K=1}^{N} \beta_K \phi_K (f), \] (45)

where \( f \) is located halfway between \( f^- \) and \( f^+ \) (see Fig. 2). Since we have put two nodes centered at \( f^- \) and \( f^+ \), the \( \phi_K \) in equation (45) are all zero except for the two that straddle the fault. When evaluated at \( f \), both of these nodal basis functions are equal to 1/2, making the previous equation a simple average

\[ -\frac{\mu \eta}{2} \frac{\partial}{\partial t} \left( \beta^+ - \beta^- \right), \] (46)

where the superscripts \( f^- \) and \( f^+ \) denote that these terms are non-zero for only the two nodes nearest the fault. In taking the limit as the fault thickness goes to zero, these two nodes represent displacement and stress infinitely close to, but on opposite sides of, a slip discontinuity.

Substituting equation (46) into equations (43) and (40), we obtain the weak form of the second equation in the \( \sigma-v \) system

\[ \sum_{K=1}^{N} \frac{\partial \beta_K}{\partial t} \left[ \int_0^L \phi_K \phi_J dz \right] = -\frac{\mu \eta}{2} \frac{\partial}{\partial t} \left( \beta^+ - \beta^- \right), \] (47)

Putting this equation into matrix notation using the mass and stiffness matrices, the finite-element system in the \( \sigma-v \) formulation is

\[ M \ddot{\alpha} = -\frac{1}{\rho} S \beta, \] (48)

\[ M \ddot{\beta} = -\mu S \ddot{\alpha} - \frac{\mu \eta}{2} \frac{\partial}{\partial t} \left( \beta^+ - \beta^- \right). \] (49)

7 A NUMERICAL EXAMPLE IN 1D

In implementing equations (48) and (49), we chose an implicit time integration scheme so that zero-length elements at the fault could be modeled without stability issues. Denoting the values of displacement
Numerical modeling of waves incident on slip discontinuities

Figure 3. Snapshots of the velocity and stress fields at five different times. There is a slip discontinuity at $z = 50$ m.

and stress at a future timestep by $n + 1$ and at previous timesteps as $n$ and $n - 1$, the system of equations is

\begin{align*}
M \frac{\alpha_{n+1} - \alpha_{n-1}}{2\Delta t} &= -\frac{1}{\rho} S \frac{\beta_{n+1} + \beta_{n-1}}{2}, \\
M \frac{\beta_{n+1} - \beta_{n-1}}{2\Delta t} &= -\mu S \frac{\alpha_{n+1} + \alpha_{n-1}}{2} - \frac{\mu \eta}{2} \frac{\beta_{f+1} - \beta_{f-1}}{2\Delta t} - \frac{\mu \eta}{2} \frac{\beta_{f+1} - \beta_{f-1}}{2\Delta t}. \\
\end{align*}

(50)

(51)

This system of equations is solved in Matlab using LU decomposition to obtain the values of stress and displacement at time $n + 1$. Note that no additional constraints to account for boundary conditions are needed - the boundary conditions are integrated in equations (50) and (51).

Figure 3 shows the velocity and stress, calculated from equations (50) and (51), at several moments in time. The simulation had an initial condition representing a plane wave incident from the left. At the center, $z = 50$ m, is a slip discontinuity with compliance $\eta$ equal to $.5 \, \text{s}^2 \text{m}^2 / \text{kg}$. The media are identical on either side of the slip discontinuity. For a compliance of zero, the two media would be in welded contact and there would be no reflection. The wave reflected from the slip discontinuity is the derivative of the incident wave, as pointed out by Widess (1973), and hence has a higher frequency content. The slip discontinuity is actually an infinitesimally thin bed; in the parlance of scattering theory, it is a 1D Rayleigh point scatterer.

Assuming a high velocity thin bed, Widess concluded that the reflection of the thin bed would become negligible for a bed thickness less than one-eighth of a wavelength (Widess, 1973). So why is it that the slip discontinuity simulated here gave such a strongly reflected wave? From Schoenberg (1980), we know that a slip discontinuity models the reflection from a thin, low shear layer. For a sufficiently low shear velocity, there is no lower limit to the thickness of a bed that would produce a significant S-wave reflection. Extremely low shear wave velocities have been observed in laboratory measurements of fluid pressurized sands (Zimmer et al., 2002) and can be expected to occur in “blown” faults that have high pore pressures in the fault itself. Hence, the slip discontinuity should be an excellent model for a “blown” fault.

Note that the stress is continuous and the velocity discontinuous across the imperfect interface. This has been accomplished with the zero-length elements at the slip discontinuity. From Schoenberg (1980) and Pyrak-Nolte et. al, 1990, we obtain excellent agreement over the frequency band used in the simulation.

8 EXTENSION TO THE P-SV CASE IN 2D

Of more interest to seismic surveys is the reflection and transmission of $P - SV$-waves at a slip discontinuity. Two compliances, the normal and tangential compliances ($\eta_N$ and $\eta_T$), characterize a crack in this case. Analogous to equations (1) and (2), the interface boundary conditions are

\begin{align*}
\sigma_{xz}^{+} - \sigma_{xz}^{-} &= \eta_T \sigma_{xz}, \\
\sigma_{zz}^{+} - \sigma_{zz}^{-} &= 0,
\end{align*}

(52)

(53)
The horizontal slowness, $p_x$, does not change when the background velocity changes. With this information

$$\frac{\partial \sigma_{xx}}{\partial t} = -\frac{1}{p_x} \frac{\partial \sigma_{xx}}{\partial x},$$

$$\frac{\partial v_x}{\partial x} = -p_x \frac{\partial v_x}{\partial t},$$

and substituting this into equation (60) yields

$$\frac{\partial \sigma_{xx}}{\partial x} - p_x^2 (\lambda + 2\mu) \frac{\partial v_x}{\partial t} - p_x \lambda \frac{\partial v_x}{\partial z} = 0.$$  

Equation (64) can be used to eliminate $\sigma_{xx}$ in equation (56). Notice that we have also replaced a derivative w.r.t. $x$ with a $t$-derivative in equation (63). Using this same trick on equations (57)-(59), $P-SV$ propagation in a $c(z)$ medium can be written as a system of 4 PDEs with derivatives in $t$ and $z$ only

$$\frac{\partial v_x}{\partial t} \left(1 - p_x^2 V_p^2\right) - \frac{1}{\rho} \frac{\partial \sigma_{xx}}{\partial z} + p_x \frac{\lambda}{\rho} \frac{\partial v_x}{\partial z} = 0,$$

$$\frac{\partial \sigma_{xx}}{\partial t} - \mu \frac{\partial v_x}{\partial z} + p_x \mu \frac{\partial v_x}{\partial t} = 0,$$

$$\frac{\partial v_z}{\partial t} - \frac{1}{\rho} \frac{\partial \sigma_{zz}}{\partial t} = 0,$$

$$\frac{\partial \sigma_{zz}}{\partial t} - \lambda \frac{\partial v_z}{\partial z} - \frac{\partial \sigma_{xx}}{\partial z} = 0,$$

where $V_p = \sqrt{\frac{\lambda + 2\mu}{\rho}}$ is the velocity of the $P$-wave. Note that for normal incidence ($p_x = 0$), the system of 4 PDEs decouples into 2 systems of 2 PDEs in the form of equations (32) and (33).

Applying the same Galerkin method to equations (65)-(68), we use the expansions

$$v_x(z,t) = \sum_{K=1}^{N} \alpha_K(t) \phi_K(z),$$

$$\sigma_{zz}(z,t) = \sum_{K=1}^{N} \beta_K(t) \phi_K(z),$$

$$v_z(z,t) = \sum_{K=1}^{N} \gamma_K(t) \phi_K(z),$$

$$\sigma_{zz}(z,t) = \sum_{K=1}^{N} \delta_K(t) \phi_K(z).$$

Including a slip discontinuity from equations (52) and (53), the matrix equations are

$$(1 - p_x^2 V_p^2) M \ddot{\alpha}_{n+1} - \ddot{\alpha}_{n-1} =$$

$$\frac{1}{\rho} S \ddot{\beta}_{n+1} + \ddot{\beta}_{n-1} + \frac{\lambda}{p_x^2} \ddot{\gamma}_{n+1} + \ddot{\gamma}_{n-1},$$

for the incident angle $\theta$ from the vertical and medium velocity $c$, is constant. Symbolically, this means that plane wave solutions for any of the displacements or stresses in a $c(z)$ medium would have the form $S(t - p_x x - p_z z)$. 

Figure 4. Detail of the velocity and stress fields near the crack at $t = 15$ s. The discontinuity is easily seen in the velocity at the crack, whereas the stress is continuous.

$$u^+ - u^- = \eta_N \sigma_{zz},$$

$$\sigma_{zz}^+ - \sigma_{zz}^- = 0.$$  

In what follows, we assume that $\eta_N = 0$, i.e. the crack is a thin low shear zone, but not a zone of low bulk modulus.

The equations of motion for $P-SV$-waves couple two components of displacement with three stresses

$$\frac{\partial v_x}{\partial t} - \frac{1}{\rho} \left[ \frac{\partial \sigma_{xx}}{\partial z} + \frac{\partial \sigma_{zz}}{\partial z} \right] = 0,$$

$$\frac{\partial \sigma_{xx}}{\partial t} - \mu \left[ \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right] = 0,$$

$$\frac{\partial v_z}{\partial t} - \frac{1}{\rho} \left[ \frac{\partial \sigma_{xx}}{\partial z} + \frac{\partial \sigma_{zz}}{\partial z} \right] = 0,$$

$$\frac{\partial \sigma_{zz}}{\partial t} - \lambda \frac{\partial v_z}{\partial z} - (\lambda + 2\mu) \frac{\partial v_x}{\partial z} = 0,$$

$$\frac{\partial \sigma_{xx}}{\partial t} - (\lambda + 2\mu) \frac{\partial v_x}{\partial x} - \lambda \frac{\partial v_x}{\partial z} = 0.$$
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9 CONCLUSIONS

The exercise of realizing slip discontinuities in a simple 1D finite-element algorithm should assist in the implementation of slip interfaces in 2D numerical codes, which are further complicated by meshing and assembly issues. We find that proper modeling of fractures demands an implicit, unconditionally stable time integration scheme. Future work should explore if an explicit time scheme is at all possible, since this would increase the computational efficiency needed for an extension to a 2D or 3D code. In 2D or 3D, the repeated use we made of integration by parts in this paper would be replaced by Green’s Theorem. Simulations in a higher dimension would allow the modeling of crack tips.

In the numerical examples, we kept the background medium homogeneous except for the slip discontinuities. The finite-element scheme can also be worked out in the...
case of smoothly-varying changes in the material properties. An example of this would be a zone of linearly increasing velocity bounded by two slip discontinuities. A zero-length element is also necessary in simulating a welded interface.

We limited our study to the reflection and transmission of a pure slip (displacement) discontinuity. Schoenberg (1980) has also suggested the possibility of a viscous slip condition. Other, more exotic interface conditions exist, such as the Maxwell and Kelvin versions of the combined displacement and velocity discontinuity (Pyrak-Nolte, 1996). This interface condition models the reflection and transmission from a thin, low shear zone with attenuation and has been validated in laboratory experiments (Pyrak-Nolte et al., 1990). The same finite-element methods we have described here are needed to numerically model these other types of interfaces.

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Reflected and transmitted waves from fault zones

Yaping Zhu and Roel Snieder
Center for Wave Phenomena, Dept. of Geophysics, Colorado School of Mines, Golden, CO 80401

ABSTRACT
Waves reflected from and transmitted across fault zones can be used to extract information on those zones. Fault zones, characterized as either a low-velocity zone or fracture systems, modify the phase of the waveform and make the seismic response frequency dependent. In this paper we develop an extrapolation scheme for simulating fractures described as linear slip boundaries. Two \(9 \times 9\) linear systems need to be solved if the fracture stiffnesses and the background medium parameters are constant along the fractures. Numerical examples support the validity and efficiency of this approach. Based on the numerical simulation and theoretical arguments, we show that various internal properties of fault zones have differing influence on reflected waves. Fracture stiffness and fracture spacing change significantly the magnitude of the reflected waves, while small-scale random variation in velocity or fracture stiffness has little influence on the wavefields; hence fault zones with small-scale random variation can be modeled with equivalent media.

1 INTRODUCTION
Fault zones play a major role in accommodating the deformation in the Earth (structural geology and basin development), in determining the strength of Earth materials (mining applications), and in transporting or acting as barriers to fluid and gas (hydrocarbon exploitation). For a variety of applications, it is useful to determine the properties of fault zones remotely using wavefields.

The seismic description of a fault zone can be classified in terms of velocity change and fractures. A fault zone is usually thought to contain highly damaged material that has a lower velocity than do the surrounding rocks. Based on this, a number of studies have shown that waves can be guided along fault zones (Li et al., 1990; Li et al., 1994; Li and Vidale, 1996; Igel et al., 1997). Li and Vidale (1996) observed fault-zone guided waves and discussed the effects of overlaying sediments, fault zone step-overs, and hypocentral offset from the fault on the guided waves. Igel et al. (1997) and (2001) considered fault zones as low-velocity layers, and discussed various factors that influence trapped waves based on numerical simulation. They found that velocity variations along the fault zone alter significantly the trapping efficiency, while moderate geometric variations (e.g., changes in fault zone width) or small-scale scatterers do not.

Fractures have also been recognized as an important potential property of fault zones. In tight reservoir rocks, fractures serve as major conduits for oil and gas (Yi et al., 1998; Schoenberg, 1998). Extensive laboratory work has been carried out for the seismic response of synthetic fractures (Pyrak-Nolte et al., 1990a) and natural fractures in rocks (Pyrak-Nolte et al., 1990b). A challenge is to extract the information on the geometrical and physical properties of fractures such as fracture strength, orientation, and spacing.

In addition to trapping waves, fault zones also generate reflected and transmitted waves. In this paper, we focus the discussion on reflected and transmitted waves from fault zones.

We analyze four basic models with different degree of complexity for characterizing fault zones. A fault zone can be modeled as 1) a low-velocity zone, 2) an idealized fracture, 3) a system of cracks, or 4) a complex shear zone that includes cracks, voids, and velocity perturbation. As stated in Schoenberg (1980), an idealized fracture is equivalent to a thin low-velocity layer with appropriate choice of layer thickness and impedance. In this paper, most emphasis is placed on fault zones characterized as fractures or cracks.

We develop an extrapolation scheme for the staggered-grid finite-difference simulation of fractures described as linear slip boundaries. In the update of the stresses at each time step in the region of fractures, particle velocities are extrapolated from neighboring points to the point where the stress needs to be evaluated. The difference between the extrapolated particle velocities
across the fracture gives time derivative of the stresses and for updating the stresses.

In the extrapolation scheme, two $9 \times 9$ linear systems are designed for evaluating the particle velocities and stresses at the fracture interface. If the fracture stiffnesses and the background medium parameters are constant along the fractures, only two $9 \times 9$ linear systems are required for all the fractures; otherwise different linear systems need to be solved for each grid point on the fractures where either the fracture stiffness or the background medium parameter changes. Since coefficient matrices in all those linear systems are time-invariant and need to be solved only once, extra computation cost for simulating the fractures is negligible. Numerical examples support the validity and efficiency of this approach.

Based on numerical simulation and theoretical arguments, we find that the internal properties of fault zones have various degrees of influence on reflected waves. For example, fracture stiffness and fracture spacing change significantly the magnitude of the reflected waves. As the fracture stiffness increases to infinity, the fracture behaves as a welded interface, but as the fracture stiffness decreases to zero, the fracture degenerates to a pair of free surfaces. For fault zones modeled as a system of cracks oriented perpendicular to the wavefront, waves propagate as if there is no crack and hence no reflected wave is excited. Other properties such as small-scale random variation in the fault zone have little influence on the wavefields; hence fault zones can be modeled with equivalent media.

The paper is organized as follows. In the next section, we introduce various types of models for characterizing fault zones and discuss the analytic solution for reflection and transmission coefficients from an idealized fracture. We then propose an extrapolation scheme for the simulation of fractures and test the approach by comparing the numerical results with the analytic solution for an idealized fracture. In the final section, we investigate various properties of the wavefields, including fracture stiffness, fracture spacing, crack distribution pattern, and random variation.

2 THEORETICAL BACKGROUND

2.1 Models for characterizing fault zones

Fault zones can be characterized by models in order of increasing complexity (Figure 1): 1) a low-velocity zone, 2) an idealized fracture, 3) a system of cracks, and 4) a complex shear zone.

A fracture is a non-welded interface across which the tractions are continuous while the displacements are discontinuous. A simple way to describe the displacement discontinuity is the elastic boundary condition,

$$\tau_{xz} = K_x(u_x^{II} - u_x^{I}), \quad \tau_{yz} = \tau_{yz}^{I},$$

where the fracture interface is in the horizontal $x - y$ plane, the $z$ axis is perpendicular to the fracture interface, $u$ and $\tau$ are the particle displacement and stress, respectively, and $K$ is the fracture stiffness (in units of Pa/m), which is an indicator of the strength of the fracture. The elastic boundary condition is also denoted as a linear-slip interface condition (Schoenberg, 1980; Fan et al., 1996, Schoenberg, 1998; Myer, 1998), which is valid when the thickness of the fracture is much smaller than the dominant wavelength of the seismic wave. Hence we consider an idealized fracture as having zero thickness in the context of linear slip theory.

A more complicated model of the fault zone is a system of cracks possibly with varying length scales and strengths. The cracks typically have a certain dominant orientation. Here we limit the discussion to parallel cracks. More complicated fault zones result from complex shearing. Such zones consist of many fractures, with velocity and strength variation, and the presence of fault gouge and fluids.

2.2 Reflection and transmission coefficients of an idealized fracture

The reflection and transmission coefficients of SH-waves that interact with a planar infinite fracture are given by

$$\hat{S}_S = \frac{1 - i\Delta - \rho_2 \beta_2 \cos j_2}{\rho_1 \beta_1 \cos j_1},$$

$$\hat{S}_T = \frac{2}{1 - i\Delta - \rho_2 \beta_2 \cos j_2},$$

where

$$\Delta = \omega \rho_2 \beta_2 \cos j_2 / K_y,$$

subscripts 1 and 2 denote the media on both sides of the fracture. $j$ is the angle that the raypath of the SH-waves
make with the normal to the fracture, \( \rho \) and \( \beta \) are the density and SH-wave velocity, \( \omega \) is the radial frequency, and \( i \) represents the imaginary unit.

The presence of the fracture is manifested in the factor \( \Delta \), which depends on properties of the medium, the frequency of the wave, and more significantly, the fracture stiffness \( K \). Since \( \Delta \) appears in the combination \( i\Delta \), both the reflection and transmission coefficients are complex numbers for all angles of incidence. Hence, the presence of a fracture changes both the amplitude and the phase of the waveform. If there is no fracture, \( \Delta = 0 \) and the expressions (2) and (3) degenerate to the familiar coefficients resulting from an impedance contrast. In that case, both the reflection and transmission coefficient are real valued and independent of frequency for incidence angle less than critical.

For the special case where the media on both sides of the fracture have the same density and velocity, expressions (2) and (3) simplify to

\[
\tilde{S} \tilde{S} = \frac{-1}{2 - i\Delta}, \quad \tilde{S} \tilde{S} = \frac{2}{2 - i\Delta}.
\] (5)

Figure 2 shows the reflection coefficient for SH-waves at a velocity contrast across an interface and at an idealized fracture. In both cases, the density is constant: \( \rho_1 = \rho_2 = 1500 \text{ kg/m}^3 \). For the velocity contrast (Figure 2a), SH-wave velocities are \( \beta_1 = 1732 \text{ m/s} \) for the upper medium and \( \beta_2 = 1270 \text{ m/s} \) for the lower medium. For the fracture interface (Figure 2b), SH-wave velocity is constant across the interface, with \( \beta_1 = \beta_2 = 1732 \text{ m/s} \), and the fracture stiffness \( K_y = 2.096 \times 10^6 \text{ Pa/m} \). In this test, the fracture stiffness has the same order of the magnitude as in Fan et al. (1996). Reasonable magnitudes for the fracture stiffness have also been discussed in Myer (1998) and Pyrak-Nolte and Morris (2000). For the influence of fracture stiffness on reflected waves, refer to section “Internal properties of fault zones” below.

For a pure velocity contrast, the reflection coefficient is independent of frequency. In contrast, for the idealized fracture the reflection coefficient varies substantially with frequency. For our choice of the fracture stiffness, the magnitude of the reflection coefficients is large (0.1 to 0.2), which means that with favorable source-receiver geometry these reflected waves can be observed in reflection data, perhaps better than reflection from impedance contrasts. Note that for the idealized fracture, the waveform changes because of the phase change which is also dependent on frequency.

The analytic formulas of the reflection and transmission coefficients are more complicated for P- and SV-waves than for SH-waves, they also exhibit frequency dependence and phase change.

### 3 NUMERICAL SCHEME AND VERIFICATION

#### 3.1 Extrapolation scheme for simulating fractures

Although the analytic solution for scattered waves is known for a planar fracture, a stack of parallel fractures (Appendix B), and a finite crack (Sánchez-Sesma and Iturrarán-Viveros, 2001), the analytic solution for systems of fractures is hard to get, especially when the background medium is heterogeneous. Hence a numerical approach is required to assess wave propagation through fractures.

We developed an extrapolation scheme for simulating fractures described as linear slip boundaries (Appendix A). It is based on velocity-stress finite-differencing on a staggered-grid (Virieux, 1984; Virieux, 1986), which updates the particle velocities and stresses alternatively at each time step.

In our approach, two \( 9 \times 9 \) linear systems, expressions (A9) and (A10), are designed for evaluating the
The medium parameters are constant background velocity, P-wave velocity \( v_p = 3000 \text{ m/s} \), SV-wave velocity \( v_s = 1732 \text{ m/s} \), and density \( \rho = 1500 \text{ kg/m}^3 \). The fracture stiffnesses are \( K_x = 4.192 \times 10^9 \text{ Pa/m} \) and \( K_z = 2.096 \times 10^9 \text{ Pa/m} \).

Figure 3. Reflection and transmission coefficients of waves excited from an idealized fracture: a) \( PP \), b) \( PS \), c) \( PP \), and d) \( PS \). The medium parameters are constant background velocity, P-wave velocity \( v_p = 3000 \text{ m/s} \), SV-wave velocity \( v_s = 1732 \text{ m/s} \), and density \( \rho = 1500 \text{ kg/m}^3 \). The fracture stiffnesses are \( K_x = 4.192 \times 10^9 \text{ Pa/m} \) and \( K_z = 2.096 \times 10^9 \text{ Pa/m} \).

3.2 Numerical example and verification

To test the validity of the finite-difference code, we consider P-SV waves in a two-layer model with an idealized planar fracture embedded in constant background media (Figure 4). An idealized planar fracture is placed at a depth of 2000 m. The fracture stiffnesses are set as \( K_x = 1.2985 \times 10^9 \text{ Pa/m} \) and \( K_z = 5.2564 \times 10^9 \text{ Pa/m} \). The grid spacing is 5 m, the time step is 0.25 ms, and the number of grid points in the test example is 1201×1401. A P-wave was excited from a point source at the position marked by an asterisk. The source wave is a Ricker wavelet with the dominant frequency of 12 Hz. An array of receivers is located at a depth of 1800m, with horizontal offsets extending from -500 m to 500 m. Another array of receivers is located below the source, with the depth extending from 1800 m to 2200m to record snapshots of the waves propagating through the fracture. In Igel et al. (2001), the number of points per dominant wavelength is set as 20 in the numerical simulation of fault zones characterized as low-velocity zones. In our numerical test, we recommend that within the fracture region the number of points in the dominant wavelength be set at \( \geq 40 \).

In Figure 5, we show snapshots of the modeled vertical particle velocity \( v_z \) and normal stress \( \tau_{xz} \) across the fracture. The P-wave propagates through the fracture at the depth of 2000 m at normal incidence. Discontinuity is visible for \( v_z \) across the fracture, while \( \tau_{xz} \) is continuous across the fracture. The figure clearly shows how a fracture is modeled as a linear-slip boundary condition.

Figure 6a) shows the modeled PP reflected field at normal incidence. Compared to the analytical solution for the PP reflection at normal incidence (Chaisri and...
Reflected and transmitted waves

Figure 4. Geometry of the model with an idealized planar fracture.

Figure 5. Different space-time snapshots of a) the normalized vertical component of the particle velocity $v_z$ and b) the normal stress $\tau_{zz}$, during the wave propagation through a fracture at normal incidence. The difference in time between subsequent plots is 0.02 s. The fracture is located at the depth of 2000 m and the vertical grid spacing in the simulation is 5 m. Discontinuity is observed for $v_z$ across the fracture, while $\tau_{zz}$ keeps continuous across the fracture. Waveforms near the fracture are highlighted with the plus sign (+).

Figure 6. Waveforms of PP and P-SV reflections: a) offset=0 m and b) offset=200 m. For offset=200 m, the lower trace is the $x$-component of the particle velocity, $v_x$, and the upper trace is the $z$-component of the particle velocity, $v_z$. Receivers are located at the depth of 1800 m.

Krebes, 2000), the error of the peak amplitude in the numerical simulation is less than 5%. For larger offset, as shown in Figure 6 b), the particle polarization for PP and P-SV reflections needs to be considered in order to evaluate the amplitude of PP and P-SV reflections from $x$- and $z$-component data. Comparison between the numerical and analytic solution also indicates the validity of the finite difference scheme in simulating waves excited by fractures.

3.3 Further discussion

Currently, a popular way to model wave propagation in fractures is to use an equivalent medium to represent a fracture(Igel et al., 1997; Coates and Schoenberg, 1995), e.g., a low-velocity layer. Schoenberg (1980) discussed the equivalence between a thin low-velocity layer and an idealized fracture. The underlying assumptions for the equivalence between the low-velocity layer and the fracture are: 1) the thickness of the low-velocity layer...
should be much smaller than the wavelength; 2) the impedance in the layer should be much smaller than that in the background media.

The advantage of the low-velocity-layer model in finite-difference simulation is that it requires no special treatment of the boundary conditions. However, since it requires the thickness to be much smaller than the wavelength and the impedance in the layer to be much smaller than that in the background media, both grid size and the time step must be very small. In our approach, in the regions of fractures 40 grid points per dominant wavelength are good enough to model waveforms of the reflection and transmission accurately.

To make the extrapolation stable for a fracture, other fractures should not be present at neighboring points of this fracture, which requires that the number of grid points between two adjacent fractures be at least 4 for second-order extrapolation. That explains why in the region of fractures the number of points per dominant wavelength is relatively large, e.g., no less than 40 in our numerical tests.

4 INTERNAL PROPERTIES OF FAULT ZONES

Fault-zone reflected and transmitted waves can be influenced by the internal structures as well as by the fault-zone geometry. For a fault zone consisting of a system of fractures (cracks) or a complex shear zone, important parameters include fracture strength and fracture spacing.

In the following discussion, these parameters are listed in order of decreasing importance to reflected waves. For example, we discuss the fracture stiffness first since its magnitude varies substantially with different rock samples and it largely governs the magnitude of the waves reflected from fractures.

Here we discuss fault zones consisting of fractures. For discussions on fault zones characterized as low-velocity zones, refer to Igel et al. (1997).

4.1 Fracture stiffness

The fracture strength, or the magnitude of fracture stiffness, can vary over many orders of magnitudes, strongly influencing the magnitude of the reflected waves. Laboratory experiments suggest a range of fracture stiffness from $10^7$ to $10^{13}$ Pa/m (Myer, 1998; Schoenberg, 1998; Pyrak-Nolte and Morris, 2000).

Consider an SH-wave in a constant background. According to expression (5), as $K_y \to 0$, a fracture degenerates to a free surface and the reflection coefficient for displacement approaches unity. As $K_y \to \infty$, a fracture becomes a welded interface, and the reflection coefficient for displacement approaches 0, as shown in Figure 7. Note that reflection and transmission coefficients shown in Figures 7~10 are for the particle displacement (or velocity), not for the potential of the waves.

Now consider P- and SV-waves in constant background. Here we change the normal component of the fracture stiffness $K_z$ from $10^7$ to $10^{12}$ Pa/m and set the shear component as $K_x = K_z/2$. When both $K_x$ and $K_z$ decrease to a small value, e.g., $10^7$ Pa/m, the fracture degenerates to a free surface. The reflection coefficient for a P-wave approaches unity for normal incidence. However, for oblique angles, it approaches a value less than 1 because of energy conversion between P- and SV-waves, as shown in Figure 8. For a reflected SV-wave, the reflection coefficient vanishes for normal incidence no matter how large the fracture stiffness, but for oblique angles, it increases because of the energy conversion, as shown in Figure 9.

When both $K_x$ and $K_z$ increase to a large value, e.g., $10^{12}$ Pa/m, the fracture behaves as a welded interface and the reflection coefficients for both P and SV reflections vanish.

For $K_z = 0$, the fracture becomes a hydraulic fracture, which cannot carry any shear stress. Consider the SS transmission coefficients. As shown in Figure 10, for a dry fracture ($K_z \neq 0$), the SS-transmission coefficients vanish as both $K_x$ and $K_z$ decrease to a small value, e.g., $10^7$ Pa/m and approach unity as both $K_x$ and $K_z$ increase to a large value, e.g., $10^{12}$ Pa/m. For a hydraulic fracture ($K_z = 0$), however, the SS-transmission coefficients vanish for normal incidence no matter how large the normal component of the fracture stiffness $K_x$, and vary in a small range (e.g., $0 \sim 0.2$) for oblique angles, which is indicated as shear-wave shadowing (Groenenboom and Fokkema, 1998; Wills et al., 1992).

4.2 Fracture spacing

Fracture spacing has a significant influence on the reflected waves. Here we consider a stack of fractures. All fractures are identical and evenly spaced with the frac-
Figure 8. Reflection amplitude versus fracture strength for PP reflected waves. Left: normal incidence; right: incidence angle=30°. The medium parameters are constant background velocity, P-wave velocity $\alpha=3000$ m/s, SV-wave velocity $\beta=1732$ m/s, constant density $\rho=1500$ kg/m$^3$, and $K_x = K_z / 2$.

Figure 9. Reflection amplitude versus fracture strength for P-SV reflected waves. Left: normal incidence; right: incidence angle=30°. The medium parameters are the same as in Figure 8.

To analyze the influence of fracture spacing on reflected waves, we fix the fault-zone width while changing the number of fractures in the zone. Figure 11 shows the configuration of fracture spacing. Waves propagate downward to the fault zone, which consists of horizontal fractures in a constant background medium. As fracture spacing decreases, the number of fractures increases; hence the reflection amplitude increases, as shown in Figure 12.

Figure 13 shows the reflected waveforms for various fracture spacings based on reflection coefficients given by the analytic formulas (B15) in Appendix B. In this analysis, the fault zone width is set as 1 m and the fracture stiffness $K_q = 2.096 \times 10^9$ Pa/m. As the fracture spacing decreases from 1 m to 2 cm, more fractures are included in the fault zone, reducing the effective strength of the fault zone and increasing the reflection coefficient. Note that the reflection coefficients also depend on the fault-zone width.

Figure 10. Transmission amplitude versus fracture strength for SS transmitted waves: a) dry fracture, incidence angle=0°, b) dry fracture, incidence angle=30°, c) hydraulic fracture, incidence angle=0°, d) hydraulic fracture, incidence angle=30°. For the dry fracture, the medium parameters are the same as in Figure 8; for the hydraulic fracture, the shear component of the fracture stiffness is set at 0.

Figure 11. Configuration of fracture spacing. All the fractures are identical, horizontal, and densely but evenly distributed. Waves propagate downward to the horizontal fault zone. The medium parameters are constant SH-wave velocity $\beta=1732$ m/s, constant density $\rho_1 = \rho_2=1500$ kg/m$^3$, and fracture stiffness $K_y = 2.096 \times 10^9$ Pa/m. The fault zone width is set as 6 m, and the dominant wavelength of the source is set at 40 Hz.
Figure 12. Dependence of the waveform and amplitude of SH-waves reflected at normal incidence on fracture spacing. Each trace represents the response from a fault zone with different vertical fracture density.

Figure 13. SH reflection coefficients at normal incidence for different fracture spacing, based on the analytic solution (B15). Velocity, density, and fracture stiffness are the same as in Figure 11. The fault-zone width is set as 1 m.

4.3 Crack distribution pattern

Compared to other properties, the lateral crack distribution pattern has less influence on the reflected waves. Consider a system of parallel cracks with the different distribution patterns shown in Figure 14. All the cracks are identical for all distribution patterns. The horizontal spacing between cracks is set as four times the crack length. In pattern 1, all layers of cracks are identical. In pattern 2, the cracks in each layer shift by one half of the horizontal spacing relative to the layer above. In pattern 3, the cracks in each layer shift by a distance of one crack length relative to the layer above.

Figure 15 shows how the reflected waveform changes with different pattern. The data are noise free. The difference among maximum amplitudes corresponding to different patterns is only 20% in this numerical test; in practice, such a small amplitude variation will often be obscured by noise.

4.4 Random variation

The least important properties for waves reflected from fault zones is the small-scale random variation in density and velocity, or in fracture stiffness. Figure 16 shows SH reflected waveforms for different internal structures of a fault zone. In plots a) and b), waveforms are compared for a constant low-velocity layer and a low-velocity layer with 25% of Gaussian distribution added to the constant background velocity in the layer. No significant change is observed for the two zones.

Similar results hold for the fault zone consisting of cracks, as shown in plots c) and d). Only a negligible change on the wavefield is detected. The waveshape of the reflection keeps almost unaltered. The results suggest that small-scale structure (compared to the dominant wavelength) in the complex shear zone can be
Reflected and transmitted waves largely smoothed out and modeled with equivalent media in the observed range of wavelengths.

5 DISCUSSION AND CONCLUSIONS

Reflected and transmitted waves provide a potential tool for diagnosing fault zones when trapped waves cannot be measured. Studies based on fault-zone trapped waves, carried out for large-scale tectonic fault zones, give estimates of the shear velocities in fault zones (Li et al., 1994). In the interpretation of fault zones close to a reservoir, fracture strength and density, among other factors, hold the key for the ability of a fault zone to excite reflected and transmitted waves. A fault zone, whether characterized as a low-velocity layer or comprised of fractures, can excite significant reflected and transmitted waves. Apart from changing the amplitudes of reflected and transmitted waves, fault zones characterized as either a low-velocity zone or fracture (crack) systems modify the phase of the waveform and make the seismic response frequency-dependent.

Using the staggered-grid finite-difference, we developed an extrapolation scheme for simulating wave propagation influenced by fractures described as linear slip boundaries. In the update of the stresses at each time step in the region of fractures, particle velocities are extrapolated from neighboring points to the point where stresses need to be evaluated and the difference between the extrapolated particle velocities across the fracture is calculated to get the time derivative of the stress. To make the extrapolation stable, relationship between the stress and particle velocity are taken as constraints on the extrapolation. Two $9 \times 9$ linear systems are designed for evaluating the particle velocities and stresses at the fracture interface, one for $v_x$ and $\tau_{xz}$ and another for $v_y$ and $\tau_{zz}$. Wherever the fracture stiffnesses and the background medium parameters stay the same along the fractures, only two $9 \times 9$ linear systems are required; otherwise different $9 \times 9$ linear systems need to be solved for each grid point on the fracture where either the fracture stiffness or the background medium parameter changes. Since all the linear systems are time-invariant and need to be solved only once, extra computation cost for simulating the fractures are negligible. Numerical examples support the validity of this approach. A standard staggered-grid scheme is applied to regions where there is no fracture.

We discussed relative importance of internal properties of fault zones on reflected waves. Fracture stiffness and fracture spacing change significantly the magnitude of the waves, while crack distribution pattern and small-scale random variation on velocity or fracture stiffness have little influence on the wavefields; hence, fault zones can be modeled with equivalent media in a typical range of seismic wavelengths. Another important factor, crack orientation, is not covered in this paper.

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APPENDIX A: EXTRAPOLATION SCHEME FOR THE FINITE DIFFERENCE SIMULATION OF FRACTURES ON A STAGGERED-GRID

Consider the velocity-stress representation of a 2-D wave equation. In the presence of a horizontal fracture (in x-direction), we have

\[
\frac{\partial v_x}{\partial t} = \frac{1}{\rho} \left( \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} \right),
\]

\[
\frac{\partial v_z}{\partial t} = \frac{1}{\rho} \left( \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{zz}}{\partial z} \right),
\]

\[
\frac{\partial \tau_{xx}}{\partial t} = (\lambda + 2\mu) \frac{\partial v_x}{\partial x} + \lambda \frac{\partial v_x}{\partial z},
\]

\[
\frac{\partial \tau_{zz}}{\partial t} = K_z[v_z],
\]

\[
\frac{\partial \tau_{xz}}{\partial t} = K_z[v_x],
\]

where \( v \) denotes the particle velocity, \( \tau \) denotes the stress, and \( \lambda \) and \( \mu \) are the Lamé parameters. The square brackets here denote the difference between the particle velocities across the fracture interface.

In the calculation of the stresses given by expression (A1), we extrapolate the particle velocity from neighboring points to the point where the stress needs to be evaluated as illustrated in Figure A1, to calculate...
the difference between the extrapolated particle velocities across the fracture, which accounts for the time derivative of the stress, and finally to get the updated stress at that point.

A direct extrapolation would be a natural way to evaluate the stresses at the fracture interface, but it is not constrained since different extrapolation schemes lead to different value of the particle velocities, which greatly change the value of stresses. Hence further constraints need to be considered. Our approach is to extrapolate the particle velocity to the fracture interface while taking the relationship between the stress and particle velocity as a constraint on the extrapolation.

For medium I, two second-order polynomials are used to approximate the particle velocities near the fracture. Hence we have

\[
\begin{align*}
  a'_{j+\frac{1}{2}}z_{j+\frac{1}{2}} + b'_{j+\frac{1}{2}}z_{j+\frac{1}{2}} + c' &= v_{xj+\frac{1}{2}}, \\
  a'_{j+1}z_{j+1} + b'_{j+1}z_{j+1} + c' &= v_{xj+1}, \\
  a'_{j}z_{j} + b'_{j}z_{j} + c' &= v'_{xj+\frac{1}{2}}, \\
  a'_{j-\frac{1}{2}}z_{j-\frac{1}{2}} + b'_{j-\frac{1}{2}}z_{j-\frac{1}{2}} + c' &= v'_{xj-\frac{1}{2}},
\end{align*}
\]

for the particle velocity \(v_x\) and

\[
\begin{align*}
  a_{j+\frac{1}{2}}z_{j+\frac{1}{2}} + b_{j+\frac{1}{2}}z_{j+\frac{1}{2}} + c &= v_{yj+\frac{1}{2}}, \\
  a_{j+1}z_{j+1} + b_{j+1}z_{j+1} + c &= v_{yj+1}, \\
  a_{j}z_{j} + b_{j}z_{j} + c &= v'_{yj+\frac{1}{2}}, \\
  a_{j-\frac{1}{2}}z_{j-\frac{1}{2}} + b_{j-\frac{1}{2}}z_{j-\frac{1}{2}} + c &= v'_{yj-\frac{1}{2}},
\end{align*}
\]

for the particle velocity \(v_y\). The stress components \(\tau_{xz}\) and \(\tau_{zx}\) are governed by the following constitutive relationship

\[
\begin{align*}
  \frac{\partial \tau_{xz}}{\partial t} &= \mu_{j+\frac{1}{2}}(2a_{j+\frac{1}{2}}z_{j+\frac{1}{2}} + b) + \mu_{j+\frac{1}{2}}\left(\frac{\partial v_x}{\partial x}\right)_{j+\frac{1}{2}}, \\
  \frac{\partial \tau_{zx}}{\partial t} &= (\lambda_j + 2\mu_j)(2a'_{j+1}z_{j+1} + b') + \lambda_j\left(\frac{\partial v_y}{\partial x}\right)_{j+\frac{1}{2}}.
\end{align*}
\]

where \(a\) through \(c\) and \(a'\) through \(c'\) are coefficients of the second-order polynomials.

Similarly, for medium II we have

\[
\begin{align*}
  dz_{j+1} + e_{j+1}z_{j+1} + f &= v_{xj+1}, \\
  dz_{j+2} + e_{j+2}z_{j+2} + f &= v_{xj+2}, \\
  dz_{j+\frac{1}{2}} + e_{j+\frac{1}{2}}z_{j+\frac{1}{2}} + f' &= v_{xj+\frac{1}{2}}, \\
  dz_{j+\frac{1}{2}} + e_{j+\frac{1}{2}}z_{j+\frac{1}{2}} + f'' &= v_{xj+\frac{1}{2}},
\end{align*}
\]

for the particle velocity \(v_x\), and

\[
\begin{align*}
  dz'_{j+1} + e'_{j+1}z'_{j+1} + f' &= v'_{xj+1}, \\
  dz'_{j+2} + e'_{j+2}z'_{j+2} + f'' &= v'_{xj+2}, \\
  dz'_{j+\frac{1}{2}} + e'_{j+\frac{1}{2}}z'_{j+\frac{1}{2}} + f' &= v'_{xj+\frac{1}{2}},
\end{align*}
\]

for the particle velocity \(v_x\), and

\[
\begin{align*}
  \frac{\partial \tau_{xz}}{\partial t} &= \mu_{j+\frac{1}{2}}(2dz_{j+\frac{1}{2}} + e) + \mu_{j+\frac{1}{2}}\left(\frac{\partial v_x}{\partial x}\right)_{j+\frac{1}{2}}, \\
  \frac{\partial \tau_{zx}}{\partial t} &= (\lambda_j + 2\mu_j)(2d'z_{j+1} + e') + \lambda_j\left(\frac{\partial v_y}{\partial x}\right)_{j+\frac{1}{2}},
\end{align*}
\]

where \(d\) through \(f\) and \(d'\) through \(f'\) are coefficients of the second-order polynomials.

Furthermore, the stress components \(\tau_{xz}\) and \(\tau_{zx}\) are governed by the difference between particle displacements (and hence particle velocities) across the fracture interface

\[
\begin{align*}
  \frac{\partial \tau_{xz}}{\partial t} &= \mathcal{K}_{xj+\frac{1}{2}}[v_x]_{j+\frac{1}{2}} = \mathcal{K}_{xj}[v'_{xj+\frac{1}{2}}], \\
  \frac{\partial \tau_{zx}}{\partial t} &= \mathcal{K}_{yj}[v_y]_{j} = \mathcal{K}_{yj}(v'_{xj} - v'_{j+\frac{1}{2}}),
\end{align*}
\]

Re-organizing the above expressions (A2~A8) leads to two \(9 \times 9\) linear systems:

\[
\begin{align*}
  a_{xj-1}z_{xj-1} + b_{xj-1}z_{xj-1} + c &= v_{xj-1}, \\
  a_{xj}z_{xj} + b_{xj}z_{xj} + c &= v_{xj}, \\
  a'_{xj}z'_{xj} + b'_{xj}z'_{xj} + c &= v'_{xj+\frac{1}{2}}, \\
  a'_{xj+1}z'_{xj+1} + b'_{xj+1}z'_{xj+1} + c &= v'_{xj+\frac{1}{2}}, \\
  a'_{xj+2}z'_{xj+2} + b'_{xj+2}z'_{xj+2} + c &= v'_{xj+1}, \\
  a'_{xj+\frac{1}{2}}z'_{xj+\frac{1}{2}} + b'_{xj+\frac{1}{2}}z'_{xj+\frac{1}{2}} + c &= v'_{xj+\frac{1}{2}}, \\
  a'_{xj-\frac{1}{2}}z'_{xj-\frac{1}{2}} + b'_{xj-\frac{1}{2}}z'_{xj-\frac{1}{2}} + c &= v'_{xj-\frac{1}{2}}, \\
  a'_{xj-1}z'_{xj-1} + b'_{xj-1}z'_{xj-1} + c &= v'_{xj-1}, \quad (A9)
\end{align*}
\]

\[
\begin{align*}
  \frac{\partial \tau_{xz}}{\partial t} &= (2a_{xj+\frac{1}{2}}z_{xj+\frac{1}{2}} + b) + \left(\frac{\partial v_x}{\partial x}\right)_{j+\frac{1}{2}},
\end{align*}
\]
where the nine unknowns are \( a \) through \( f \), \( v_{xj} \), and \( v_{xj}^I \), and

\[
\begin{align*}
(a')^*_{zj-\frac{d}{2} + b'z_{j-\frac{d}{2}} + c'} &= v_{xj-\frac{d}{2}} \\
(a')^*_{zj+\frac{d}{2} + b'z_{j+\frac{d}{2}} + c'} &= v_{xj+\frac{d}{2}} \\
(a')^*_{zj} + b'z_{j} + c' &= v_{xj} \\
d'z_{j+\frac{d}{2}} + e'z_{j+\frac{d}{2}} + f' &= v_{xj+\frac{d}{2}} \\
d'z_{j} + e'z_{j} + f' &= v_{xj} \quad (A10)
\end{align*}
\]

where the nine unknowns are \( a' \) through \( f' \), \( v_{xj}' \), and \( v_{xj}^I \), and \( \left( \frac{\partial \tau_{xz}}{\partial t} \right)_j \). The stress components are normalized by the Lamé parameters: \( \tilde{\tau}_{xz} = \frac{\tau_{xz}}{\mu_j} \) and \( \tilde{\tau}_{xz} = \frac{\tau_{xz}}{\lambda + 2\mu_j} \).

**APPENDIX B: SH REFLECTION AND TRANSMISSION COEFFICIENTS FOR A STACK OF FRACTURES**

**B1 SH reflection and transmission coefficients for two parallel fractures**

Here we consider two identical, parallel fractures with infinite length. The spacing between these two fractures is denoted as \( d \). The fracture stiffness is denoted as \( K_y \). The parameters of the media above and below the layer are denoted by index 1 and 2, respectively, while parameters of the layer bounded by the two fractures are denoted by index 3.

Summing over all possible intra-bed multiples in the layer yields

\[
\begin{align*}
R &= R_{13} + T_{13}D^2R_{32}T_{31} + T_{13}D^4R_{32}R_{31}T_{31} + \ldots \\
T &= T_{13}DT_{32} + T_{13}D^3R_{32}T_{31} + T_{13}D^5R_{32}R_{31}T_{31} + \ldots \\
\end{align*}
\]

where \( D = e^{i\omega \cos j \frac{d}{\beta_3}} \) denotes the phase shift of waves propagating in the bounded layer. Reflection and transmission coefficients from different boundaries are given by

\[
\begin{align*}
R_{13} &= \frac{(Z_1 - Z_3) - iZ_1Z_3 \frac{\omega}{K_y}}{(Z_1 + Z_3) - iZ_1Z_3 \frac{\omega}{K_y}} \quad (B3) \\
R_{31} &= \frac{(Z_3 - Z_1) - iZ_3Z_1 \frac{\omega}{K_y}}{(Z_3 + Z_1) - iZ_3Z_1 \frac{\omega}{K_y}} \quad (B4) \\
T_{13} &= \frac{2Z_1}{(Z_1 + Z_3) - iZ_1Z_3 \frac{\omega}{K_y}} \quad (B5) \\
T_{31} &= \frac{2Z_3}{(Z_3 + Z_1) - iZ_3Z_1 \frac{\omega}{K_y}} \quad (B6) \\
R_{32} &= \frac{(Z_3 - Z_2) - iZ_3Z_2 \frac{\omega}{K_y}}{(Z_3 + Z_2) - iZ_3Z_2 \frac{\omega}{K_y}} \quad (B7) \\
T_{32} &= \frac{2Z_3}{(Z_3 + Z_2) - iZ_3Z_2 \frac{\omega}{K_y}} \quad (B8)
\end{align*}
\]

where \( \omega \) denotes the angular frequency, \( Z_m = \rho_m \beta_m \cos j \) denotes the impedance in medium \( m \), and \( j \) denotes the angle that SH waves make with the normal to the fracture interface. For reflection and transmission coefficients, the first index denotes the incident medium, and the second index denotes the medium where the reflected and transmitted waves propagate.

In the special case of constant background (\( \rho_1 = \rho_2 = \rho_3 = \rho \), \( \beta_1 = \beta_2 = \beta_3 = \beta \), and hence \( j_1 = j_2 = j_3 = j \), we have

\[
\begin{align*}
R_{13} &= R_{31} = R_{32} = \frac{-iZ_1 \frac{\omega}{K_y}}{2 - iZ_1 \frac{\omega}{K_y}} = \frac{-iB}{2 - iB} \quad (B9) \\
T_{13} &= T_{31} = T_{32} = \frac{2}{2 - iZ_1 \frac{\omega}{K_y}} = \frac{2}{2 - iB} \quad (B10)
\end{align*}
\]

where \( Z = \rho \beta \cos j \) is much smaller than the wavelength, the phase shift term \( D \approx 1 + i \frac{\omega \cos j \frac{d}{\beta}}{\beta} \).
Reflected and transmitted waves

\[ R = \frac{-iB}{2 - 2iB + B^2 \frac{\omega \cos jd}{\beta}} + 4 \frac{\omega \cos jd}{\beta}, \quad (B13) \]

\[ T = \frac{2 \omega \cos jd}{2 - 2iB + B^2 \frac{\omega \cos jd}{\beta}}. \quad (B14) \]

**B2  SH reflection and transmission coefficients for a stack of fractures**

For a stack of fractures, the reflection and transmission coefficients can be derived recursively. Suppose the reflection and transmission coefficients for fractures 1 through \( m - 1 \) are \( R_{(m-1),m} \), \( R_{m,(m-1)} \), \( T_{(m-1),m} \), and \( T_{m,(m-1)} \), respectively, where the first index denotes the incident medium and the second index denotes the reflecting or transmitting medium, and brackets denote fractures 1 through \( m - 1 \), as shown in Figure A2.

Hence the reflection and transmission coefficients for the system consisting of fractures 1 through \( m \) are (Snieder, 2001)

\[ R_{(m),m+1} = R_{(m-1),m} + \frac{T_{(m-1),m} D^2 R_{m,m+1} T_{m,(m-1)}}{1 - D^2 R_{m,m+1} R_{m,(m-1)}}, \quad (B15) \]

\[ T_{(m),m+1} = \frac{T_{(m-1),m} D T_{m,m+1} + 1 - D^2 R_{m,m+1} R_{m,(m-1)}}, \quad (B16) \]

where \( D = e^{i \omega \cos j d / \beta_m} \) denotes the phase shift of waves propagating in the medium between fracture \( m - 1 \) and \( m \).
Radiative transfer in 1D and connection to the O’Doherty-Anstey formula

Matthew Haney, Kasper van Wijk, and Roel Snieder
Center for Wave Phenomena, Colorado School of Mines, Golden, CO 80401

ABSTRACT
There is a growing interest in incorporating multiply scattered waves into modeling the Earth’s interior using radiative transfer. We examine radiative transfer in a layered medium with general scattering and directional properties of the source. This allows us to demonstrate in detail the nature of energy propagation in the presence of strong scattering. At its most basic level, radiative transfer predicts that, after a distance known as the mean free path, the wavefield breaks into a coherent, or wave-like part and an incoherent, or diffusive flow. The dynamic properties of both aspects are linked.

For 1D point scatterers, or thin beds, we derive the equivalence of the exponential decay of the transmitted wave predicted by the O’Doherty-Anstey formula with the coherent, or direct, wave obtained from the radiative transfer equation. The equivalence shows an underlying relationship between mean field theory and radiative transfer.

Turning to the incoherent wave intensity, we make the well-known diffusion approximation to the late-time radiative transfer behavior. A finite-difference simulation of the wave equation with random scatterers corroborates the theoretical results for the incoherent energy.

Key words: multiple scattering, attenuation, diffusion

1 INTRODUCTION
Radiative transfer has its origins in the kinetic theory of gases and is sometimes referred to as the Boltzmann transport equation in honor of its earliest proponent. In the earth sciences, it first appeared within the context of light propagation through the atmosphere (Schuster, 1905). Recently, geophysicists have begun to address the applicability of radiative transfer to multiply-scattered seismic waves (Hennino et al., 2001; Campillo & Paul, 2003; van Wijk et al., 2003).

By squaring a wavefield and averaging over many realizations of random disorder, the phase information of the underlying wavefield is, for the most part, lost. What remains is the average intensity, or squared amplitude. Radiative transfer is a phenomenological theory for the spatial and temporal evolution of a wavefield’s average intensity. The theory’s strengths lie in the ability to provide statistical information about the structure of a medium at scales less than a wavelength and the description of the decoupling of scattering and absorption effects for incoherent wave energy.

Here, we give the complete solution of the radiative transfer equation in one dimension (1D) for general directional sources and general scattering. Such generality is relevant for plane wave propagation in layered media, and recently became important for describing physical experiments of surface wave propagation through 1D disordered grooves with a directional source (van Wijk et al., 2003). We derive results from radiative transfer that agree with results from mean field theory, namely the O’Doherty-Anstey formula. Such an equivalence suggests that radiative transfer is a proper extension of mean field theory (a “variance field” theory) for the fluctuating, multiply-scattered waves.

At late times, we demonstrate that radiative transfer can be simplified even further by approximating its behavior as the solution to a diffusion equation. Results of finite-difference simulations of the 1D wave equation with random scatterers are presented to support the ac-
accuracy of this approximation. Using correct values for
the parameters needed to describe the scattering, the
average intensity of the numerical simulations is seen to
approach the diffusive limit with time.

2 THE RADIATIVE TRANSFER EQUATION

The radiative transfer equation can be derived from en-
ergy balance considerations (Morse & Feshbach, 1953;
Ishimaru, 1978; Turner, 1994). Heuristically, the equa-
tion takes the form:

\[ \partial I / \partial t + v \cdot \nabla I = \text{source} - \text{loss} + \text{gain}. \] (1)

The left-hand side of equation (1) is the total time
derivative of the intensity. On the right-hand side, loss
gain mechanisms in addition to sources determine
the dynamic behavior. In the absence of loss or gain, this
equation becomes the advection, or one-way wave, equa-
tion. Scattering and absorption show up as loss me-
chanisms since both remove energy from the forward di-
rection. Only scattering can put energy back into the
original direction of propagation. Hence, scattering and
absorption enter equation (1) in fundamentally differ-
ent ways. This fact leads to the ability to separate their
effects within radiative transfer theory.

Using the same form as equation (1), here is a gen-
eral radiative transfer equation valid for any dimension:

\[ \frac{\partial I(\vec{r}, \Omega, t)}{\partial t} + v_\Omega \cdot \nabla I(\vec{r}, \Omega, t) = S(\vec{r}, \Omega, t) - \frac{1}{\tau_s} I(\vec{r}, \Omega, t) - \frac{1}{\tau_a} I(\vec{r}, \Omega, t) + \int \frac{1}{\sigma_s} \frac{\partial \sigma_s}{\partial \Omega} (\vec{r}, \Omega', t) d\Omega', \] (2)

where \( I(\vec{r}, \Omega, t) \) is the intensity, or average squared wave-
field, at position \( \vec{r} \) propagating in direction \( \Omega \), \( v \)
is the group velocity of the average (coherent) wavefield, \( \hat{n} \)
is the unit vector in the direction of propagation, and
\( S(\vec{r}, \Omega, t) \) is the angle-resolved source function.
The differential scattering cross section, \( \sigma_s / \partial \Omega' \), describes the
exchange of energy traveling from direction \( \Omega \) into di-
rection \( \Omega' \). The characteristic time between these ex-
changes is \( \tau_s \), the scattering mean free time. The total
scattering cross section, \( \sigma_s \), is the energy exchanged into
all directions:

\[ \sigma_s = \int \frac{\partial \sigma_s}{\partial \Omega'} d\Omega'. \] (3)

We have allowed for attenuation by including the char-
acteristic absorption time \( \tau_a \).

Using terminology originally coined by Clausius in
1858, it is common to define mean free paths for scatter-
ing and absorption, \( \ell_s \) and \( \ell_a \), according to the relations
\( \ell_s = v \tau_s \) and \( \ell_a = v \tau_a \). The scattering mean free path,
\( \ell_s \), can be thought of as the typical distance a wave
travels between scatterings. Under most circumstances,
\( \ell_s \) is inversely proportional to the number density of
scatterers, \( \rho \), and their scattering cross section:

\[ \ell_s = \frac{1}{\rho \sigma_s}. \] (4)

This equation is called the independent scattering ap-
proximation (ISA) and it holds when the scatterers are
separated by more than a wavelength. It can be obtained
from a stationary phase argument applied to the aver-
age wavefield in random media (Ishimaru, 1978). From
equation (4), \( \ell_s \) contains information about the product
of \( \rho \) and \( \sigma_s \) in a way analogous to a wave reflected from
an interface containing information about the acoustic
impedance.

3 RADIATIVE TRANSFER IN 1D

Since in 1D only two directions of propagation exist,
a general expression for the differential scattering cross
section, appearing under the integral in equation (2), is:

\[ \frac{\partial \sigma(\Omega, \Omega')}{\partial \Omega'} = E_f \delta(\Omega' - \Omega) + E_b \delta(\Omega' - \Omega - 180^\circ). \] (5)

where \( E_b \) and \( E_f \) represent amounts of energy back-
scattered and forward-scattered divided by the energy of
the incident wave. Their sum is equal to the total
scattering cross section:

\[ \sigma_s = E_b + E_f. \] (6)

Hence, in equation (2), the differential scattering cross
section divided by the total scattering cross section be-
comes:

\[ \frac{1}{\sigma_s} \frac{\partial \sigma_s(\Omega, \Omega')}{\partial \Omega'} = \frac{E_f}{E_b + E_f} \delta(\Omega' - \Omega) + \frac{E_b}{E_b + E_f} \delta(\Omega' - \Omega - 180^\circ). \] (7)

For the rest of this paper, we denote the ratios \( E_f / (E_b +
E_f) \) and \( E_b / (E_b + E_f) \) by \( B \) and \( F \) respectively. These
ratios satisfy \( B + F = 1 \). In the case of isotropic scat-
ttering, \( B = F = \frac{1}{2} \) (Paaschens, 1997).

For a general 1D scatterer, \( B \) and \( F \) can be related to
the total transmission and reflection coefficients of a
thin bed, \( T_i \) and \( R_i \) (Sheng, 1995):

\[ B = \frac{|R_i|^2}{|R_i|^2 + |T_i - 1|^2} \]
\[ F = \frac{|T_i - 1|^2}{|R_i|^2 + |T_i - 1|^2}. \] (8)

Note that a thin bed consists of two interfaces, and
hence \( R_i \) and \( T_i \) are not simple reflection and transmis-
sion coefficients. The quantities \( R_i \) and \( T_i \) can be related to
a geometric summation of the interface reflection and
transmission coefficients via generalized rays (Aki &
Richards, 1980).
Integrating equation (7) into equation (2), we obtain:
\[
\frac{\partial I(x, \Omega, t)}{\partial t} + v \vec{n}(I) \frac{\partial I(x, \Omega, t)}{\partial x} = S(x, \Omega, t) - \frac{1}{\tau_s} I(x, \Omega, t) - \frac{1}{\tau_a} I(x, \Omega, t) + \frac{1}{\tau_a} \int [F \delta(\Omega' - \Omega) + B \delta(\Omega' - \Omega - 180^\circ)] I(\vec{r}', \Omega', t) \, d\Omega'
\]
\[
= \frac{B}{\tau_s} I(x, \Omega + 180^\circ, t) - \frac{B}{\tau_a} I(x, \Omega, t) - \frac{1}{\tau_a} I(x, \Omega, t) + S(x, \Omega, t),
\]
where we have used the fact that \( B + F = 1 \). Equation (9) can be evaluated for the two possible directions in 1D, \( \Omega = 0^\circ \) or \( 180^\circ \). In this paper, we will refer to these directions as right and left, respectively. For simplicity, the total intensity propagating in direction \( \Omega = 0^\circ \), \( I(\vec{r}, 0^\circ, t) \), will be represented by \( I_r \), \( I(\vec{r}, 180^\circ, t) \) will be \( I_l \), and the source function will be split into \( S_r \) and \( S_l \). The coordinate system is defined such that \( \tilde{n}(0^\circ) = 1 \) and \( \tilde{n}(180^\circ) = -1 \). The two equations that describe the propagation of right-going and left-going intensities are:
\[
\frac{\partial I_r}{\partial t} + v \frac{\partial I_r}{\partial x} = \frac{B}{\tau_s} (I_l - I_r) - \frac{1}{\tau_a} + S_r,
\]
\[
\frac{\partial I_l}{\partial t} - v \frac{\partial I_l}{\partial x} = \frac{B}{\tau_s} (I_l - I_r) - \frac{1}{\tau_a} + S_l.
\]
This system of partial differential equations comprises radiative transfer in 1D and has been derived by other methods (Goecke, 1977). In Appendix A, the system of partial differential equations is solved for both \( I_r \) and \( I_l \). For now, we solve for the total intensity, \( I_t = I_r + I_l \), since this is commonly measured in practice.

Two new quantities emerge from adding and subtracting equations (10) and (11). In addition to the total intensity, \( I_t \), the net right-going intensity, \( I_n = I_r - I_l \), appears. Similarly, the source function can be expressed as its total and net right-going components: \( S_t = S_r + S_l \) and \( S_n = S_r - S_l \). The result of adding equations (10) and (11) is:
\[
\frac{\partial I_t}{\partial t} + v \frac{\partial I_t}{\partial x} = -\frac{I_t}{\tau_a} + S_t.
\]
Subtracting equations (10) and (11) yields:
\[
\frac{\partial I_n}{\partial t} + v \frac{\partial I_n}{\partial x} = -2B \frac{I_n}{\tau_s} - \frac{I_n}{\tau_a} + S_n.
\]
From these two equations, we derive a single partial differential equation in terms of what we measure, \( I_t \), by taking the spatial derivative of equation (13):
\[
\frac{\partial}{\partial t} \frac{\partial I_t}{\partial x} + v \frac{\partial^2 I_t}{\partial x^2} = \left[ -\frac{2B}{\tau_s} - \frac{1}{\tau_a} \right] \frac{\partial I_t}{\partial x} + \frac{\partial S_t}{\partial x}.
\]
But we know from equation (12) that:
\[
\frac{\partial I_t}{\partial x} = \frac{1}{v} \left[ -\frac{I_t}{\tau_a} + \frac{1}{\tau_a} \right].
\]
Substituting equation (15) into equation (14) yields a single partial differential equation in \( I_t \). Omitting some algebraic manipulation, we obtain:
\[
\frac{\partial^2 I_t}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 I_t}{\partial t^2} + \frac{2B}{v \ell_a} + \frac{1}{v^2} \frac{\partial I_t}{\partial t} + \frac{1}{v} \frac{2B}{\ell_a} + \frac{1}{v} \frac{1}{\ell_a} I_t - \frac{2B}{\ell_a} \frac{1}{v} \frac{\partial S_t}{\partial t} + \frac{1}{v} \frac{\partial S_n}{\partial x}.
\]

Equation (16) encapsulates a wealth of information. First of all, in the absence of a source, the scattering and attenuation show up in both the first and zeroth order time derivatives of the total intensity. For a medium with no scattering or attenuation, \( \ell_a^{-1} = \ell_m^{-1} = 0 \), we are left with the 1D wave equation. Also, in order to solve for the Green’s function of the total intensity, we cannot simply insert a \( \delta \)-source into the homogeneous form of equation (16). Instead, a more complicated combination of the source and its time and spatial derivatives must be inserted.

4 THE GREEN’S FUNCTION OF THE TOTAL INTENSITY

To solve for the Green’s function of the total intensity, we find the Green’s function of the homogeneous form of equation (16) and construct the total intensity Green’s function from it. First, take an impulsive total source function:
\[
S_t = \delta(x) \delta(t),
\]
and a general form for its net right-going component:
\[
S_n = c S_t,
\]
where \( c \in [-1, 1] \). The parameter \( c \) allows the radiation pattern of the impulsive source function to directionally vary from left-going (\( c = -1 \)), to isotropic (\( c = 0 \)), to right-going (\( c = 1 \)), and to any combination in between. After inserting this source into equation (16), we find that the “effective source”, denoted \( S_e \), is a combination of a \( \delta \)-function in time, its time derivative, and its \( x \)-derivative:
\[
S_e = \frac{1}{v} \frac{\partial S_t}{\partial t} + \frac{\delta(x) \delta(t)}{v^2} + \frac{1}{v} \delta(x) \delta'(t) - \frac{c \delta(x) \delta(t)}{v}.
\]
This effective source can be constructed from the knowledge of the Green’s function, \( P \), of the homogeneous form of equation (16):
\[
\frac{\partial^2 P}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 P}{\partial t^2} + \frac{2B}{v \ell_a} + \frac{1}{v^2} \frac{\partial P}{\partial t} + \frac{1}{\ell_a} \left[ \frac{2B}{\ell_a} + \frac{1}{\ell_a} \right] P - \delta(x) \delta(t).
\]
Note that $P$ is not the Green’s function for the total intensity. This equation is a variation of the telegraph equation, there being a zeroth order derivative appearing due to the presence of attenuation. In Morse and Feshbach (1953), the Green’s function of the telegraph equation is solved via a spatial Fourier transform and a Laplace transform over time. Applying the same technique, the Green’s function of equation (20) can be readily obtained by generalizing the solution stated in Morse and Feshbach (1953):

$$
P(x, t) = \frac{v}{2} \exp(-Bvt/\ell_s - vt/\ell_a) \times \left[ J_0 \left( \frac{B}{\ell_s} \sqrt{x^2 - v^2 t^2} \right) u(vt - |x|), \right. \tag{21}$$

where $u(vt - |x|)$ is the unit step-function, guaranteeing causality. This Green’s function only differs from the one for the telegraph equation by the exponential damping factor due to attenuation. The Green’s function for the total intensity, denoted $I_t$, can be expressed in terms of the above Green’s function through equation (19):

$$I_t = \left[ \frac{2B}{v\ell_s} + \frac{1}{v\ell_a} \right] P + \frac{1}{v^2} \frac{\partial P}{\partial t} - \frac{c}{v} \frac{\partial P}{\partial x} \tag{22}$$

Taking the necessary derivatives of $P$, we obtain for $B \in [0, 1]$ and $c \in [-1, 1]$:

$$I_s(x, t) = \frac{1}{2} \exp(-Bvt/\ell_s - vt/\ell_a) \times \left[ (1 - c)\delta(vt + x) + (1 + c)\delta(vt - x) + \frac{B}{\ell_s} u(vt - |x|) \left[ J_0 \left( \frac{B}{\ell_s} \sqrt{v^2 t^2 - x^2} \right) + \frac{vt + cx}{\sqrt{v^2 t^2 - x^2}} \right] \right]. \tag{23}$$

where $I_0$ and $I_1$ are the modified Bessel functions of the zeroth and first orders. These should not be confused with the symbols used for the various intensities ($I_i, I_r, I_t, I_s, J_0,$ and $J_1$). A previous result by Hemmer (1961) is obtained from equation (23) for the case of an isotropic source ($c = 0$) and isotropic scattering, $B = \frac{1}{2}$.

The Green’s function for the total intensity can be broken up into two parts. The term with the $\delta$-function propagates like a wave and is called the coherent intensity. The rest of the total intensity is referred to as the incoherent intensity. It does not propagate ballistically, and, in later sections of this paper, we show that at late times it propagates diffusively. Also, in Appendix A we show that each Bessel function represents a different direction of propagation for the incoherent energy.

An interesting result in equation (23) is that the decay of coherent intensity due to scattering, described by the first exponential term, goes with distance by the factor $\ell_s/B$ and not $\ell_s$. This new length scale, determining the decay of the coherent energy, is called the extinction mean free path, $\ell_{ext}$. The fact that $\ell_{ext} \neq \ell_s$ is unique to 1D (Paaschens, 1997).

5 THE COHERENT INTENSITY AND THE O’DOHERTY-ANSTEY FORMULA

In the field of exploration geophysics, a well known result for waves multiply scattered by a 1D layering is that obtained by O’Doherty and Anstey (O’Doherty & Anstey, 1970). The O’Doherty-Anstey formula has subsequently been derived from mean field theory (Banik et al., 1985). One outcome of O’Doherty-Anstey is that the amplitude of a wave transmitted through a stack of layers decays exponentially with distance as (Shapiro & Zien, 1993):

$$|T| \sim \exp(-\tilde{R}(k)x), \tag{24}$$

where $\tilde{R}(k)$ represents the power spectrum of the average reflection coefficient series normalized by two-way travel distance (Banik et al., 1985). From the solution for the total intensity obtained in the last section, equation (23), radiative transfer also predicts an exponential decay for the transmitted, or coherent, wave with distance:

$$|T| \sim \exp(-Bx/2\ell_s), \tag{25}$$

where the distance $x$ has replaced $vt$ in equation (23) since the $\delta$-function is only non-zero at $x = vt$. The factor of $1/2$ in the exponent of this equation shows up since radiative transfer predicts decay of the transmitted intensity - the square of the true transmission coefficient. We investigate the equivalence of these two theories for the transmission of normally incident waves through assemblages of weak 1D point scatterers (thin beds). The two theories are equivalent if:

$$\tilde{R}(k) = B/2\ell_s. \tag{26}$$

Depicted in Fig. 1 is the random medium we will consider: a series of thin layers of varying strength are
embedded in a constant velocity background medium. In the parlance of O’Doherty-Anstey, this would be called a “cyclic” sequence. It happens to be the type of medium that radiative transfer, and scattering theory, are geared for. The reflection coefficient series, $RC(x)$, for such a medium would be a series of delta functions of oscillating plus and minus sign:

$$RC(x) = \sum_{j=1}^{N} R_j \{\delta(x - d_j) - \delta(x - h - d_j)\}, \quad (27)$$

where $h$ is the thickness of the beds, $R_j$ and $d_j$ represent the reflection coefficient and location of the $j$-th bed, respectively, and $N$ is the number of beds.

To calculate $\tilde{R}(k)$, we take the Fourier transform of equation (26), square its magnitude to get the power spectrum, and divide by the two-way travel distance:

$$|\tilde{R}(k)|^2 = \frac{1}{2L} \int_{-\infty}^{\infty} \sqrt{\sum_{j=1}^{N} R_j e^{-|i2k|d_j}(1 - e^{-i2kh})^2} dx|^2. \quad (28)$$

Note that the Fourier transform is with respect to $2k$ and not $k$, similar to a Born inversion formula in 1D (Bleistein et al., 2001). This is evident from standard references in the literature (Banik et al., 1985; Shapiro & Zien, 1993).

Inserting equation (27) into equation (28) results in:

$$\tilde{R}(k) = \frac{1}{2L} \sum_{j=1}^{N} R_j e^{2ikd_j} (1 - e^{2ikh})^2. \quad (29)$$

For thin layers, $kh << 1$ and a first order Taylor series expansion in $h$ leads to $1 - e^{2ikh} \approx -2ikh$. Pulling it out of the summation yields:

$$\tilde{R}(k) = \frac{1}{2L} 4k^2 h^2 \sum_{j=1}^{N} R_j e^{2ikd_j} |^2. \quad (30)$$

We now use a standard argument from the theory of multiple scattering: if $d_j$, the spacing of the thin beds, is a random variable, the cross terms in the square of the summation in equation (30) cancel in the average and the squaring can be brought inside the summation:

$$\tilde{R}(k) = \frac{1}{2L} 4k^2 h^2 \sum_{j=1}^{N} |R_j e^{2ikd_j}|^2. \quad (31)$$

Now, inside the summation, the exponential does not contribute to the magnitude and we are left with:

$$\tilde{R}(k) = \frac{1}{2L} 4k^2 h^2 \sum_{j=1}^{N} |R_j|^2 = \frac{1}{L} 2k^2 h^2 N \langle |R_j|^2 \rangle, \quad (32)$$

where $\langle |R_j|^2 \rangle$ is the mean-square of the interface reflection coefficients.

Returning to equation (26), to prove that radiative transfer and the O’Doherty-Anstey formula predict the same exponential decay for the transmitted wave, we set equation (32) to:

$$\frac{B}{2\ell_s} = \frac{1}{L} 2k^2 h^2 N \langle |R_j|^2 \rangle. \quad (33)$$

For (Rayleigh) point scatterers in 1D, the radiation is isotropic. Hence, $B = 1/2$. Rearranging equation (33):

$$\ell_s = \frac{1}{8k^2 h^2 \langle |R_j|^2 \rangle} \frac{1}{2}. \quad (34)$$

The quantity $N/L$ is simply the number density of the thin beds, $\rho$. In the limit of weak scatterers (such that $R_j << 1$) $8k^2 h^2 \langle |R_j|^2 \rangle = \sigma_s$, the scattering cross section (Sheng, 1995). The presence of weak reflection coefficients is an underlying assumption in the O’Doherty-Anstey result (Banik et al., 1985), so that equation (34) can now be rewritten in a familiar form:

$$\ell_s = \frac{1}{\rho \sigma_s}. \quad (35)$$

This is recognized as equation (4), the independent scattering approximation. Previously, we stated that for this relation to hold, the scatterers (thin beds) had to be separated by at least a wavelength. Hence, in this model, no reflections from below the recording depth interfere with the transmitted wave. All the interference resulting in the exponential decay of the direct wave originates from peg-leg multiples within the thin beds, not between them (Fig. 1). Equation (35) demonstrates that, for this model, the exponential decay of the transmitted wave from O’Doherty-Anstey, or mean-field theory, is equivalent to that predicted by radiative transfer.

A conceptual diagram of this equivalence is shown in Fig. 2. From mean field theory, both the phase and the amplitude of the transmitted wave can be obtained; however, the incoherent energy, for which the mean is zero, falls out. Similarly, 1D radiative transfer can address the amplitude of the transmitted wave and the behavior of the incoherent intensity, but phase information is lost. Both theories agree in their region of over-
lap, as demonstrated by the case of random layering we considered here.

6 THE DIFFUSION APPROXIMATION IN INFINITE 1D MEDIA

In addition to the coherent intensity, physical insight can be gained on the incoherent part of the total intensity. The general expression for the Green’s function for radiative transfer in 1D, equation (23), shows that for late times the coherent term is zero and the incoherent field, defined by a combination of Bessel functions, approximates the solution to the diffusion equation (Ishimaru, 1978). Especially in optics, where it is hard to obtain phase information, inferences on the statistical properties of the medium are often based on this late-time diffusive behavior (Boas et al., 1995). In elastic wave-scattering, the incoherent field is used to decipher the different mechanisms of attenuation (Margerin et al., 1999).

To derive the diffusion approximation from equation (23), all we need is that vt >> x. Noting that the zeroth and first modified Bessel functions have the asymptotic forms:

\[ I_0(z) \approx I_1(z) \approx (2\pi z)^{-1/2} \exp(z) \quad \text{for } z >> 1, \]

we can write equation (23) in the late-time limit as:

\[ I_t(x,t) = \exp(-Bvt/\ell_s - vt/\ell_a) \times \frac{B}{\ell_s} \left[ \exp\left( \frac{B}{\ell_s} \sqrt{v^2 t^2 - x^2} \right) \right]. \]  

(37)

In this expression, the delta functions from equation (23) have fallen out.

Organizing terms in equation (37), expanding the square root in the exponential as a Taylor series in the square root, we get:

\[ I_t(x,t) = \exp(-Bvt/\ell_s - vt/\ell_a) \exp\left( \frac{B}{\ell_s} \sqrt{v^2 t^2 - x^2} \right) \frac{\exp\left( \frac{Bvt}{\ell_s} (1 - \frac{1}{2} (x/vt)^2) \right)}{\sqrt{2\pi \frac{Bvt}{\ell_s}}} \]  

(38)

Two of the exponentials cancel in equation (38) and, after isolating the term \( \ell_s/2B \), the late time limit of the radiative transfer equation can finally be written:

\[ I_t(x,t) = \frac{\exp\left( -\frac{x^2}{4(\frac{B}{\ell_s})vt} - \frac{vt}{\ell_a} \right)}{\sqrt{4\pi \left( \frac{B}{\ell_s} \right) vt}}. \]  

(39)

In the case of no attenuation (\( \ell_a \to \infty \)), equation (39) can be identified as the Green’s function for the 1D diffusion equation with the diffusion constant \( D = (\ell_s/2B)v \). This implies that the movement of energy at late times has an effective mean free path different from \( \ell_a \) or \( \ell_{\text{ext}} \). This effective mean free path is called the transport mean free path, \( \ell_{\text{tr}} = \ell_s/2B \). In 1D, \( \ell_{\text{tr}} = \frac{1}{4} \ell_{\text{ext}} \), since \( \ell_{\text{ext}} = \ell_s/B \). Note that the transport mean free path can be determined from the extinction mean free path without knowledge of the underlying details of the scattering.

It is common to relate \( \ell_{\text{tr}} \) to \( \ell_s \) via:

\[ \ell_{\text{tr}} = \frac{\ell_s}{1 - \langle \cos \theta \rangle}. \]  

(40)

where \( \langle \cos \theta \rangle \) represents the average scattered energy in all directions weighted by the cosine of that direction. For isotropic scattering, \( \langle \cos \theta \rangle = 0 \) and the two mean free paths are identical. However, using the general relation \( \langle \cos \theta \rangle = F - B \) (Hendrich et al., 1994) and the fact that \( F + B = 1 \), equation (40) can be rewritten:

\[ \ell_{\text{tr}} = \frac{\ell_s}{1 - F + B} = \frac{\ell_s}{2B}, \]  

(41)

which is exactly the relationship we have derived from the diffusion approximation.

7 THE DIFFUSION APPROXIMATION IN FINITE 1D MEDIA

The above derivation of the diffusion approximation showed how the solution of the radiative transfer equation approaches that of the diffusion equation at late times. In this section, we prove that the underlying governing equation for the total intensity at late times also becomes the diffusion equation. While the radiative transfer equation cannot be analytically solved for in a finite geometry, its late time equivalent - the diffusion equation - can be solved with boundary conditions.

Neglecting absorption (\( \ell_a \to \infty \)), we can rearrange equation (13) as:

\[ \frac{\partial I_n}{\partial t} + \frac{2B}{\tau_a} I_n = -v \frac{\partial I_n}{\partial x}. \]  

(42)

In the diffusive regime, we assume that (Morse & Feshbach, 1953):

\[ \frac{2B}{\tau_a} I_n >> \frac{\partial I_n}{\partial t}, \]  

(43)

meaning that the time rate of change of the right and left-going intensities is relatively small. Under this condition, equation (42) becomes:

\[ \frac{2B}{\tau_a} I_n = -v \frac{\partial I_n}{\partial x}. \]  

(44)

Substituting equation (44) into equation (12) for \( I_n \) yields:

\[ \frac{\partial I_t}{\partial t} + v \frac{\partial}{\partial x} \left[ -\tau_a v \frac{\partial I_t}{\partial x} \right] = 0. \]  

(45)
Under the assumption that \( v \) and \( \tau_s \) do not depend on position, equation (45) takes the form:
\[
\frac{\partial I_t}{\partial t} = v \left( \frac{\ell_s}{2B} \right) \frac{\partial^2 I_t}{\partial x^2}.
\] (46)
which we recognize as the 1D diffusion equation with the same diffusion constant \( D = v(\ell_s/2B) = \nu \ell_{tr} \) we obtained in the previous section.

Now assume there is a boundary at \( x = 0 \) where scattering occurs to the right (positive values of \( x \)), but not to the left (negative values of \( x \)). Then, at \( x = 0 \), there is no intensity coming into the scattering region, i.e. the right-going intensity is zero. We can express the right-going intensity as the sum of the total intensity and the net right-going intensity (flux) and set it to zero at \( x = 0 \):
\[
I_r = \frac{1}{2} I_t + \frac{1}{2} I_n = 0 \quad \text{at} \quad x = 0. \quad (47)
\]
Using the approximation we derived in equation (44), the \( I_n \)-term can be replaced by a spatial derivative of \( I_t \):
\[
\frac{1}{2} I_t + \frac{1}{2} \left( -\frac{\ell_s}{2B} \frac{\partial I_t}{\partial x} \right) = 0. \quad (48)
\]
From this equation, we learn that:
\[
I_t = \frac{\ell_s}{2B} \frac{\partial I_t}{\partial x} = \ell_{tr} \frac{\partial I_t}{\partial x}. \quad (49)
\]
The solution to equation (49) states that, near \( x = 0 \), \( I_t \) has the form:
\[
I_t \sim x + \ell_{tr}. \quad (50)
\]
Extrapolating away from the boundary according to equation (50), \( I_t = 0 \) at \( x = -\ell_{tr} \). Hence, the presence of a boundary that radiates energy out of a finite scattering region can be approximated by a Dirichlet boundary condition a distance \( \ell_{tr} \) outside the scattering region.

Suppose there is a region of length \( L \) extending from \( x = 0 \) to \( x = L \). Then, at late times, the Green’s function for the total intensity should obey the boundary value problem:
\[
\frac{\partial I_t}{\partial t} = D \frac{\partial^2 I_t}{\partial x^2} + \delta(x-x')\delta(t)
\]
\[I_t = 0 \quad \text{at} \quad x = -\ell_{tr} \quad \text{and} \quad x = \ell_{tr} + L. \quad (51)
\]
where \( D = v \ell_{tr} \). In 1D, this PDE can be solved by expanding over the modes of the Laplacian:
\[
I_t(x,x',t) = \sum_{m=1}^{\infty} \exp \left( -\frac{m^2 \pi^2 D t}{(L + 2\ell_{tr})^2} \right) \times 
\sin \left( \frac{m \pi (x + \ell_{tr})}{L + 2\ell_{tr}} \right) \sin \left( \frac{m \pi (x' + \ell_{tr})}{L + 2\ell_{tr}} \right). \quad (52)
\]
The PDE could have equivalently been solved by the method of images.

8 NUMERICAL SIMULATIONS

These late-time solutions of the total intensity have been tested with finite-difference simulations of the wave equation in the presence of random discrete scatterers. The setup of the numerical experiment is shown in Fig. 3. A source \( S \) is excited in the center of a finite 1D random medium, of size \( L \), containing identical low velocity (1 km/s) thin beds. By “thin” in this experiment, we mean that their thickness is approximately one-tenth of the dominant wavelength. The background medium in which they are embedded has a velocity of 2 km/s. A receiver \( R \) is placed just outside the scattering region. The experiment is repeated for \( L = 80 \text{m} \),

Figure 3. The geometry of the 1D numerical scattering experiments. The source was at the center of a region with thin random layers and a receiver was positioned above the layers for each experiment. The size \( L \) of the scattering region varied between experiments with the values 80m, 120m, 160m, 200m, and 240m.

Figure 4. The average intensity measured as a function of time for the 5 experiments. Note how the direct wave decays due to the scattering with offset.
Figure 5. The maximum of the direct wave (coherent intensity) as a function of offset for the 5 experiments. Note that this is a log-linear plot. A linear fit to the data gives the characteristic exponential decay due to scattering - the extinction mean free path, $\ell_{ext}$. We estimate $\ell_{ext} = 45.9$ m.

120m, 160m, 200m, and 240m. The number of scatterers per unit length is constant for each size of the scattering region and the number of scatterers per dominant wavelength is 2. All interfaces are numerically put into welded contact (Boore, 1970) and there is no intrinsic absorption ($\ell_a \to \infty$). For each of the five sizes of the scattering region, the average intensity was obtained by performing the experiment for 20 realizations of the randomness, squaring each of the 20 wavefields, and adding them. Of course, in practice we only have one realization of the model, the Earth. By assuming the Earth to be ergodic at some scale, we can obtain ensemble averages from earthquake data, seismic exploration, or rock physics (Scales & Malcolm, 2003).

The results are plotted in Fig. 4. The average intensities contain a large direct wave traveling at nearly the background velocity (2 km/s). This is the coherent intensity. Following the coherent intensity is the incoherent multiply scattered energy. If, in the averaging process, the wavefields were added (stacked) before they were squared, the incoherent energy would cancel out and leave only the coherent intensity. From the move-out of the direct wave, we know that the group velocity entering the radiative transfer equation, $v$, is the background velocity (2 km/s).

To fully characterize the scattering in the radiative transfer model, the extinction mean free path, $\ell_{ext}$, must be measured. This parameter describes the exponential decay of the maximum of the coherent intensity with distance. The decay is depicted in Fig. 5. By doing a linear regression on a logarithmic plot, the characteristic distance over which the direct wave decays exponentially can be estimated. For this model, $\ell_{ext} = 45.9 \pm 2.1$ m. In the previous sections, we showed that $\frac{1}{\ell_a} \ell_{ext} = \ell_{tr}$. Therefore $\ell_{tr} = 22.9 \pm 1.0$ m. Getting an estimate of $\ell_a$ would require knowledge of the degree of back-scattering the individual scatterers radiated relative to their forward-scattering (van Wijk et al., 2003).

With $v$ and $\ell_{tr}$ estimated from the numerical results, the theoretical prediction of equation (52) can be compared with the simulated total intensities. In Fig. 6, the solution of the diffusion equation asymptotically approaches the numerical intensities with time, as it should. Note that the approximation fails severely for early times since it is acausal. The late time exponential decay is correctly predicted by the diffusion approximation. Such behavior verifies our radiative transfer model for late times and hints at the fact that, in 1D systems, there is an intermediate range of distances where the incoherent intensity can be described by diffusion instead of localization (Sheng, 1995).
9 DISCUSSION

In higher dimensions, the radiative transfer equation becomes considerably more difficult since there are an infinite number of directions to scatter into, as compared to 2 in 1D (Paaschens, 1997). However, even in 1D, the rich character of radiative transfer is evident. Exponential decay is experienced by the direct wave due to scattering and absorption. Aspects of both wave and diffusive behavior emerge in the average total intensity, and, in the presence of both, a "mesoscopic" picture of the scattering medium can be formed.

The theory of radiative transfer has its limitations. The most severe is that it does not include wave interference. As a result of this, there exists a distance between source and receiver, known as the localization length, past which radiative transfer is incorrect. Sheng (1995) estimates that in 1D the localization length is approximately 4 mean free paths. This offers the possibility of an intermediate range (1 to 4 mean free paths) when radiative transfer holds. Future work should attempt to find good bounds on this range in practice.

10 CONCLUSIONS

Radiative transfer is a relative newcomer to the field of exploration seismology. By formulating the theory in 1D, we have attempted to make the connection with familiar concepts such as reflection/transmission coefficients, thin beds, and the O’Doherty-Anstey formula. In the process, new features have emerged, such as the diffusion approximation and incoherent intensity.

The link between radiative transfer and the O’Doherty-Anstey formula can be extended beyond the 1D point scatterer approximation we made in this paper. To do so implies moving into the more complicated Mie scattering regime, where the wavelength is on the order of the size of the scatterer. Additionally, we considered a “cyclic” sequence, which radiative transfer and scattering theory are designed for. It remains to be seen what radiative transfer can do for “transitional” sequences when the interfaces cannot be grouped into pairs that define scatterers.

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REFERENCES


APPENDIX A: THE GREEN’S FUNCTION FOR THE DIRECTIONAL INTENSITY

Expressions (10) and (11) show that the 1D radiative transfer equation can be split into a system of PDEs...
in terms of the left and right-going intensities. So far, only the reduced PDE governing the total intensity has been studied. This is due to the fact that measuring either the left or right-going intensity entails splitting the wavefield into left and right-going waves. Such a decomposition requires dense spatial sampling to perform the type of filtering routinely done in Vertical Seismic Profiling: separating up from down-going waves. Here, we show that knowledge of the individual left and right-going energies can give us more detailed insight into the incoherent energy.

Assuming that the wavefield has been decomposed into left and right-going waves, we now solve the system of 2 partial differential equations that comprise the full radiative transfer equation. To begin, we write equations (10) and (11) in matrix form:

\[
\begin{align*}
\frac{\partial \vec{I}}{\partial t} + M \frac{\partial \vec{I}}{\partial x} &= N \vec{I} + \vec{S}, \\
N &= \begin{bmatrix}
    \frac{v}{\tau_a} & -\frac{1}{\tau_a} \\
    -\frac{\beta}{\tau_a} & \frac{1}{\tau_a}
\end{bmatrix}, \\
\vec{S} &= \begin{bmatrix}
    S_r \\
    S_l
\end{bmatrix}.
\end{align*}
\]

There exists no general theory for solving systems of PDEs as there is for systems of ODEs. Hence, we proceed by Fourier transforming equation (A1) over space, solving the system of ODEs, and inverse Fourier transforming back to spatial coordinates. With the Fourier conventions:

\[
\vec{I}(x) = \int_{-\infty}^{\infty} \vec{I}(k)e^{-ikx} dk \quad \text{(A3)}
\]

\[
\frac{\partial \vec{I}}{\partial t} = (N + i k M) \vec{I} + \vec{S}. 
\]

For the source function, we again take a general directional point source with right and left-going components \( S_r \) and \( S_l \). Allowing the parameter \( c \) to govern the directivity of the source as we did previously, the source vector is:

\[
\vec{S} = \begin{bmatrix}
    1 + c \\
    1 - c
\end{bmatrix} \frac{\delta(x)\delta(t)}{2}.
\]

The solution of the system of ODEs follows that given in standard texts on differential equations (Boyce & DiPrima, 1997). Here we give the solution in the \( k \)-domain:

\[
I_r(k, t) = \frac{1}{4\pi} e^{-\frac{B vt}{\ell_x}} e^{-\frac{v t}{\tau_a}} 
\]

\[
\left( 1 - c \right) \frac{B}{\tau_a} + i(1 + c) k v \right) \sinh \left( t \sqrt{\frac{B^2}{\tau_a^2} - k^2 v^2} \right) 
\]

\[
\left( 1 + c \right) \cosh \left( t \sqrt{\frac{B^2}{\tau_a^2} - k^2 v^2} \right) 
\]

\[
I_l(k, t) = \frac{1}{4\pi} e^{-\frac{B vt}{\ell_x}} e^{-\frac{v t}{\tau_a}} 
\]

\[
\left( 1 + c \right) \frac{B}{\tau_a} - i(1 - c) k v \right) \sinh \left( t \sqrt{\frac{B^2}{\tau_a^2} - k^2 v^2} \right) 
\]

\[
\left( 1 - c \right) \cosh \left( t \sqrt{\frac{B^2}{\tau_a^2} - k^2 v^2} \right) 
\]

To get the directional intensities in the spatial domain, we must inverse Fourier transform equations (A6) and (A7). Two identities are needed for this inversion:

\[
ix \int_{-\infty}^{\infty} \vec{I}(k)e^{-ikx} dk = \int_{-\infty}^{\infty} \frac{\partial \vec{I}(k)}{\partial k} e^{-ikx} dk,
\]

and from the theory of Bessel functions (Hemmer, 1961):

\[
\int_{-\infty}^{\infty} \cos(kx) \sin \left( t \sqrt{\frac{k^2 v^2 - \frac{B^2}{\tau_a^2}}{k^2 v^2}} \right) dk = 
\]

\[
\pi \int_{0}^{\pi/2} \frac{B}{\ell_x} \sqrt{\frac{B^2}{\ell_x^2} - \frac{v^2 t^2}{x^2}} \, u(vt - |x|)
\]

After inverting the Fourier transform, we obtain for the right-going intensity:

\[
I_r(x, t) = \frac{1}{4} e^{-\frac{B vt}{\ell_x}} e^{-\frac{v t}{\tau_a}} [2(1 + c) \delta(vt - x) + 
\]

\[
\frac{B}{\ell_x} \left( \frac{B}{\ell_x} \sqrt{\frac{v^2 t^2}{x^2} - \frac{v^2 t^2}{x^2}} \right) + 
\]

\[
\left( 1 + c \right) \sqrt{\frac{v^2 t^2 + x^2}{v^2 t^2 - x^2}} I_1 \left( \frac{B}{\ell_x} \sqrt{\frac{v^2 t^2 - x^2}{x^2}} \right)
\]

and for the left-going intensity:

\[
I_l(x, t) = \frac{1}{4} e^{-\frac{B vt}{\ell_x}} e^{-\frac{v t}{\tau_a}} [2(1 - c) \delta(vt + x) + 
\]

\[
\frac{B}{\ell_x} \left( \frac{B}{\ell_x} \sqrt{\frac{v^2 t^2}{x^2} - \frac{v^2 t^2}{x^2}} \right) + 
\]

\[
\left( 1 - c \right) \sqrt{\frac{v^2 t^2 - x^2}{v^2 t^2 + x^2}} I_1 \left( \frac{B}{\ell_x} \sqrt{\frac{v^2 t^2 + x^2}{x^2}} \right)
\]

These equations for the two intensities show that the two Bessel functions that make up the incoherent intensity are sensitive to different aspects of the source radiation pattern. For instance, if the source were unidirectional, \( c = -1 \) or \( c = 1 \) and the zero order Bessel
function would come from one direction and the first order Bessel from the other. It can also be verified that adding equations (A10) and (A11) gives the total intensity, equation (23). In the absence of phase information, perhaps the directional intensities can yield important information about spatial variations in the material properties.
Analytic description of geometrical spreading in azimuthally anisotropic media

Xiaoxia Xu, Ilya Tsvankin and Andrés Pech

Center for Wave Phenomena, Dept. of Geophysics, Colorado School of Mines, Golden, CO 80401

ABSTRACT

Geometrical spreading is highly sensitive to elastic anisotropy and may strongly influence the amplitude signature of reflected waves recorded over anisotropic formations. For purposes of processing and inversion of reflection data, it is convenient to express geometrical spreading through the reflection traveltime measured at the earth surface. Such expressions are particularly important for azimuthally anisotropic models in which variations of geometrical spreading with both offset and azimuth may significantly distort the results of AVO analysis.

Here, we obtain the inverse geometrical-spreading factor $L^{-1}$ for horizontally layered anisotropic media with a horizontal symmetry plane as a simple function of the spatial derivatives of reflection traveltime. By combining this result with the Tsvankin-Thomsen nonhyperbolic moveout equation, the factor $L^{-1}$ is represented through the moveout coefficients which can be estimated from surface seismic data. This formulation is then applied to P-wave reflections in an orthorhombic layer to evaluate the distortions of the geometrical spreading in a typical azimuthally anisotropic model.

The spreading factor $L^{-1}$ in orthorhombic media is controlled by five parameters which are also responsible for time processing of P-wave data: the NMO velocities $V_{nmo}^{(1)}$ and $V_{nmo}^{(2)}$ in the vertical symmetry planes and the anellipticity coefficients $\eta^{(1)}$, $\eta^{(2)}$, and $\eta^{(3)}$. The weak-anisotropy approximation, verified by numerical tests, shows that azimuthal velocity variations make a significant contribution to the geometrical spreading, and the existing VTI equations for the factor $L^{-1}$ cannot be applied even in the vertical symmetry planes. For the $[x_1, x_3]$ symmetry plane, the influence of the azimuthal anisotropy is governed (for weak anisotropy) by the combination $(\eta^{(2)} - \eta^{(1)} + \eta^{(3)})$, and for the $[x_2, x_3]$–plane by $(\eta^{(1)} - \eta^{(2)} + \eta^{(3)})$.

The shape of the azimuthally dependent function $L^{-1}$ is close to an ellipse for offsets smaller than the reflector depth but becomes more complicated for larger offset-to-depth ratios. The overall magnitude of the azimuthal variation of the inverse geometrical spreading for the moderately anisotropic model used in the tests rapidly increases with offset and exceeds 25% for a wide range of offsets. While the formulation developed here is helpful in modeling and analyzing the anisotropic geometrical-spreading factor, its main practical application is in correcting the wide-azimuth AVO signature for the influence of the anisotropic overburden.

Key words: Geometrical spreading, azimuthal anisotropy, wide-azimuth AVO
1 INRODUCTION

Inversion of prestack amplitude variation with offset and azimuth (AVO analysis) represents one of the most effective tools for characterization of naturally fractured reservoirs. The presence of preferentially oriented fractures and/or horizontal stresses makes the reservoir formation azimuthally anisotropic, and wide-azimuth reflection amplitudes can be used to estimate the fracture orientation and, in some cases, map the lateral variations of the fracture density (Mallick et al., 1998; Lynn et al., 1999; Bakulin et al., 2000; Rüger, 2001). The main advantage of amplitude methods compared to traveltime inversion is their high vertical resolution that makes AVO analysis applicable to relatively thin reservoir layers.

The amplitude signature of reflected waves is controlled by the radiation pattern of the source, geometrical spreading, attenuation, and the reflection/transmission coefficients along the raypath (Martinez, 1993; Maultzsch et al., 2003). Since AVO analysis operates with the reflection coefficient at the target horizon, the critically important element of AVO processing is the removal of the influence of all other factors from the measured amplitude. If the medium does not possess strong attenuation, geometrical spreading makes the most significant contribution to amplitude distortions above the reflector (Martinez, 1993; Ursin & Hokstad, 2002). In particular, if the overburden is anisotropic (as in a sand-shale sequence), it acts like a 3-D focusing lens that changes the amplitude distribution along the wavefront of the reflected wave (Tsvankin, 1995). The results of Tsvankin (1995, 2001) show that for transversely isotropic media the influence of the geometrical spreading on the AVO signature may be comparable to that of the anisotropic term in the reflection coefficient. Therefore, if the overburden contains anisotropic layers, estimation of the reflection coefficient is impossible without an accurate geometrical-spreading-correction method.

The most straightforward way to compute anisotropic geometrical spreading is by performing dynamic ray tracing (Gajewski & Pšenčík, 1987). For simple homogeneous models it is possible to use analytic approximations of the Green’s function, such as that presented by Tsvankin (1995). Using the stationary-phase method and the weak-anisotropy approximation, Tsvankin (1995) obtained explicit expressions for the geometrical spreading and the source directivity factor of P- and SV-waves in a transversely isotropic layer. Modeling methods, however, require accurate information about the anisotropic velocity field for the whole overburden, which is seldom available in practice.

An alternative approach, more suitable for purposes of AVO analysis, is based on relating geometrical spreading to the traveltimes of reflection events recorded at the surface. Ursin & Hokstad (2002) expressed the geometrical spreading in VTI (transversely isotropic media with a vertical symmetry axis) media in terms of the reflection traveltime and the group angle in the subsurface layer. For horizontally layered VTI models, P-wave traveltime can be accurately described by the Tsvankin-Thomsen nonhyperbolic moveout equation (Tsvankin & Thomsen, 1994) parameterized by just two moveout coefficients — the effective NMO velocity $V_{nmo}$ and the effective anellipticity parameter $\eta$ (Alkhalifah & Tsvankin, 1995). The best-fit parameters $V_{nmo}$ and $\eta$ can be estimated, for example, by a 2-D semblance scan (Grechka & Tsvankin, 1998), which makes it possible to compute geometrical spreading using solely surface reflection data (Ursin & Hokstad, 2002). This approach can be also used to find analytic expressions for geometrical spreading in terms of the parameters $V_{nmo}$ and $\eta$.

The distortions caused by geometrical spreading in reflection amplitudes are even more pronounced for azimuthally anisotropic media (Rüger & Tsvankin, 1997; Maultzsch et al., 2003). Here, we apply ray theory to obtain a general expression for geometrical spreading in terms of reflection traveltime for azimuthally anisotropic models composed of horizontal layers with a horizontal symmetry plane. Substitution of the Tsvankin-Thomsen traveltime equation then yields the geometrical-spreading factor as a function of the azimuthally varying moveout coefficients.

While this result is well-suited for correcting reflection amplitudes for geometrical spreading, the goal of this paper is to evaluate the magnitude of the anisotropic geometrical-spreading factor and its variation with the source-receiver azimuth and the anisotropic parameters. For the model of a single horizontal orthorhombic layer, geometrical spreading is expressed through the medium parameters using the explicit representation of the moveout coefficients given by Al-Dajani et al. (1998). Application of the weak-anisotropy approximation helps to explain the dependence of the geometrical-spreading factor on the anisotropic parameters both within and outside the symmetry planes of the model. Numerical tests verify the accuracy of the analytic results and illustrate the character of the amplitude distortions caused by the azimuthally-varying geometrical spreading.

2 GEOMETRICAL SPREADING AS A FUNCTION OF REFLECTION TRAVELTIME

Geometrical spreading is defined as the amplitude decay of an elastic wave caused by the expansion of its wavefront away from the source. The geometrical-spreading factor $L$ can be computed as the ratio of the wavefront curvatures at the source and receiver locations. In the high-frequency limit of the generalized ray approximation, the amplitude along a ray can be written as (Cerveny, 2001; Schleicher et al., 2001):
where $Y$ denotes the geometrical spreading between the source location $x^s$ and the receiver location $x^r$, $\rho(x^r)$ and $V(x^r)$ are the density and group velocity at the source, $\omega$ is the circular frequency, $k(x^r, x^s)$ is an index that accounts for possible caustics, and $\prod_k R(k)$ is the product of the plane-wave reflection or transmission coefficients along the raypath. $L(x^r, x^s)$ can be expressed in terms of the second-order traveltime derivatives with respect to the local coordinates associated with the wavefront normal (the normal to the wavefront is parallel to the slowness or phase-velocity vector). Schleicher et al. (2001) show that by applying a coordinate rotation, an expression for geometrical spreading can be derived in the so-called “local ray coordinates” associated with the group-velocity vector:

$$L(x^r, x^s) = \det Y(x^r, x^s)^{-1/2}. \quad (2)$$

The matrix $Y(x^r, x^s)$ is defined as

$$Y(x^r, x^s) = \left[ \begin{array}{c}
\frac{\partial^2 T(x^r, x^s)}{\partial x^r_1 \partial x^s_1} \\
\frac{\partial^2 T(x^r, x^s)}{\partial x^r_2 \partial x^s_1} \\
\frac{\partial^2 T(x^r, x^s)}{\partial x^r_1 \partial x^s_2} \\
\frac{\partial^2 T(x^r, x^s)}{\partial x^r_2 \partial x^s_2}
\end{array} \right], \quad (3)$$

where $T$ is the traveltime along the ray, $g^s_1$ and $g^s_2$ are the local coordinates in the plane normal to the ray at the source, and $g^r_1$ and $g^r_2$ are the local coordinates defined in the same way at the receiver.

Here, we consider P-wave propagation in a horizontally layered anisotropic medium with a horizontal symmetry plane in each layer (for example, the layers can be orthorhombic). For a single homogeneous layer with a horizontal symmetry plane (Figure 1), the reflected rays are confined to the vertical incidence plane that contains the source and the receiver. Although the rays can deviate from the incidence plane in a multilayered medium, these deviations can usually be ignored for models with moderate azimuthal anisotropy (Al-Dajani and Tsankin, 1998).

If the incident and reflected rays lie in the incidence plane, it is convenient to rotate the matrix $Y(x^r, x^s)$ from the local ray coordinates into the global Cartesian coordinate system, which results in a matrix denoted by $B$:

$$B = \left[ \begin{array}{c}
\frac{\partial^2 T(x^r, x^s)}{\partial x^r_1 \partial x^r_1} \\
\frac{\partial^2 T(x^r, x^s)}{\partial x^r_2 \partial x^r_1} \\
\frac{\partial^2 T(x^r, x^s)}{\partial x^r_1 \partial x^r_2} \\
\frac{\partial^2 T(x^r, x^s)}{\partial x^r_2 \partial x^r_2}
\end{array} \right] = S A Y(x^r, x^s) R. \quad (4)$$

Here, $S$, $A$, and $R$ are rotation matrices defined as

$$S = \begin{bmatrix}
\cos \phi^s & 0 \\
0 & 1
\end{bmatrix}, \quad A = \begin{bmatrix}
\cos \alpha & \sin \alpha \\
-\sin \alpha & \cos \alpha
\end{bmatrix},$$

and

$$R = \begin{bmatrix}
\cos \phi^r & 0 \\
0 & 1
\end{bmatrix}. \quad (5)$$

$\phi^s$ and $\phi^r$ are the angles between the ray and the vertical axis $x_3$ at the source and receiver locations, respectively, and $\alpha$ is the azimuth of the source-receiver line with respect to the $x_1$-axis. The matrix $A$ rotates $Y(x^r, x^s)$ around the $x_3$-axis from the incidence plane to the $[x_1, x_2]$-plane, while $S$ and $R$ rotate $Y(x^r, x^s)$ around the $x_2$-axis from the local ray coordinates into the Cartesian coordinates $x_1$ and $x_3$ at the source and receiver, respectively. From equations (2)–(5) it follows that

$$L(x^r, x^s) = [\cos \phi^s \cos \phi^r]^{-1/2} \det B^{-1/2}. \quad (6)$$

If the sources and receivers are located on the same horizontal surface, the reflection traveltime depends only on the offset $x$ and the azimuth $\alpha$ of the source-receiver line (Figure 1). Then, as shown in Appendix A, equation (6) becomes the following function of the traveltime $T$:

$$L(x^r, x^s) = [\cos \phi^s \cos \phi^r]^{1/2} \left[ \frac{\partial^2 T}{\partial x^r_1 \partial x^r_1} \frac{\partial T}{\partial x^r_1} \frac{1}{x^r_1} \right]^{-1/2}. \quad (7)$$

For the special case of a single horizontal layer with a horizontal symmetry plane, the incidence and reflection angles are equal to each other (i.e., $\phi^s = \phi^r = \phi$), and

$$L(x^r, x^s) = \cos \phi \left[ \frac{\partial^2 T}{\partial x^r_1 \partial x^r_1} \frac{\partial T}{\partial x^r_1} \frac{1}{x^r_1} \right]^{-1/2}. \quad (8)$$

Equation (8) not only gives a concise representation of the factor $L(x^r, x^s)$ in terms of the reflection traveltime $T(x, \alpha)$, it also allows us to separate the contributions of azimuthal and polar anisotropy to the geometrical spreading. Indeed, the first term in the brackets coincides with the geometrical spreading factor for VTI media (Ursin & Hostad, 2002), where the traveltime $T$ is independent of the azimuth $\alpha$. The remaining two terms
3 NONHYPERBOLIC MOVEOUT EQUATION IN ORTHORHOMBIC MEDIA

Reflection moveout, as well as other signatures of reflected waves in orthorhombic media, are convenient to describe using the notation suggested by Tsvankin (1997, 2001). Tsvankin’s parameter definitions are based on the analogous form of the Christoffel equation in the symmetry planes of orthorhombic (Figure 2) and VTI media. The anisotropic parameters \( \epsilon(1) \), \( \delta(1) \), and \( \gamma(1) \) play the roles of Thomsen’s (1986) VTI coefficients \( \epsilon, \delta, \) and \( \gamma \) in the vertical symmetry plane \([x_2, x_3]\) (the superscript denotes the orthogonal axis \( x_1 \)). The similar set of the anisotropic coefficients in the \([x_1, x_3]\)-plane includes \( \epsilon(2), \delta(2), \) and \( \gamma(2) \), and one more anisotropic coefficient, \( \delta(3) \), is defined in the horizontal plane \([x_1, x_2]\). The parameter \( V_{\text{PS}} \) denotes the vertical P-wave velocity, and \( V_{\text{PS}} \) is the velocity of the vertically propagating S-wave polarized in the \( x_1 \)-direction.

Although orthorhombic symmetry is described by a total of nine independent parameters (for a fixed orientation of the symmetry planes), time-domain signatures of P-waves depend only on five combinations of these parameters. As shown by Grechka and Tsvankin (1999), P-wave reflection traveltimes and the operators for dip-moveout (DMO) correction and time migration in homogeneous orthorhombic media are controlled by the NAM velocities of horizontal events in the vertical
The general form of equation (13) makes it sufficiently accurate for P-wave moveout in models with substantial azimuthal anisotropy, provided the azimuthal variation of $A_2$, $A_4$, and $A$ is taken into account (Al-Dajani & Tsvankin, 1998; Al-Dajani et al., 1998).

For P-waves in VTI media, the parameters $A_2$, $A_4$, and $A$ can be expressed through the NMO velocity of horizontal events,

$$V_{\text{nmo}} = V_{P0} \sqrt{1 + 2\alpha},$$

and the anellipticity parameter $\eta$,

$$\eta \equiv \frac{\epsilon - \delta}{1 + 2\alpha}.$$  

Equation (13) then reduces to the form given by Alkhalifah and Tsvankin (1995):

$$T^2(x) = T^2_0 + \frac{x^2}{V_{\text{nmo}}^2} - \frac{2\eta x^4}{V_{\text{nmo}}^2 [T^2_0 V_{\text{nmo}}^2 + (1 + 2\eta) x^2]}.$$  (18)

As shown below, equation (18) can be extended in a straightforward way to a weakly anisotropic orthorhombic layer.

For orthorhombic media, the hyperbolic coefficient $A_2$ was obtained by Grechka and Tsvankin (1998a), who proved that the azimuthal variation of NMO velocity has a simple elliptical form even in arbitrarily anisotropic, heterogeneous media. Since an orthorhombic layer has two orthogonal vertical symmetry planes, the axes of the NMO ellipse should be aligned with the symmetry-plane directions, which yields (for P-waves):

$$A_2(\alpha) = A_2^{(1)} \sin^2 \alpha + A_2^{(2)} \cos^2 \alpha,$$  (19)

$$A_2^{(1)} = \frac{1}{[V_{\text{nmo}}^{(1)}]^2} = \frac{1}{V_{P0}^2 (1 + 2\delta^{(1)}]},$$  (20)

$$A_2^{(2)} = \frac{1}{[V_{\text{nmo}}^{(2)}]^2} = \frac{1}{V_{P0}^2 (1 + 2\delta^{(2)})];}$$  (21)

$\alpha$ is the azimuth with respect to the $x_1$-axis.

The azimuthally dependent P-wave quartic moveout coefficient $A_4$ was derived by Al-Dajani et al. (1998) as

$$A_4(\alpha) = A_4^{(1)} \sin^4 \alpha + A_4^{(2)} \cos^4 \alpha + A_4^{(x)} \sin^2 \alpha \cos^2 \alpha,$$  (22)

$$A_4^{(1)} = -\frac{2\eta^{(1)}}{T^2_0 [V_{\text{nmo}}^{(1)}]^2],$$  (23)

$$A_4^{(2)} = -\frac{2\eta^{(2)}}{T^2_0 [V_{\text{nmo}}^{(2)}]^2},$$  (24)

$$A_4^{(x)} = \frac{2}{T^2_0 [V_{\text{nmo}}^{(1)}]^2 [V_{\text{nmo}}^{(2)}]^2}
\left[1 - \frac{(1 + 2\eta^{(1)}) (1 + 2\eta^{(2)})}{1 + 2\eta^{(1)}},$$  (25)

where $A_4^{(1)}$ and $A_4^{(2)}$ are the symmetry-plane coefficients and $A_4^{(x)}$ is a cross-term that contributes to nonhyperbolic moveout in off-symmetry directions.

While equation (13) with the moveout coefficients listed above provides an accurate description of longspread moveout, it is too complicated to elucidate the dependence of geometrical spreading on the anisotropic parameters. A simplified form of equation (13) can be obtained by assuming that the magnitude of velocity anisotropy is weak and employing the approximate kinematic equivalence between vertical planes in orthorhombic and VTI media. As discussed by Tsvankin (2001, p. 164), in the limit of weak anisotropy all out-of-plane phenomena in a horizontal orthorhombic layer can be ignored. Also, the P-wave phase velocity in any vertical plane of orthorhombic media can be described by Thomson’s (1986) VTI equation with azimuthally-dependent coefficients $\epsilon$ and $\delta$. Therefore, P-wave reflection moveout in a horizontal orthorhombic layer can be approximated by the VTI equation (18) with the appropriate parameters $V_{\text{nmo}}$ and $\eta$ for each azimuth:

$$T^2(x, \alpha) = T^2_0 + \frac{x^2}{V_{\text{nmo}}^2(\alpha)} - \frac{2\eta(\alpha) x^4}{V_{\text{nmo}}^2(\alpha) [T^2_0 V_{\text{nmo}}^2(\alpha) + (1 + 2\eta(\alpha)) x^2]}.$$  (26)

In equation (26), $V_{\text{nmo}}(\alpha)$ is determined from equations (19)–(21),

$$V_{\text{nmo}}^2(\alpha) = \frac{[V_{\text{nmo}}^{(1)}]^2 [V_{\text{nmo}}^{(2)}]^2}{[V_{\text{nmo}}^{(1)}]^2 \cos^2 \alpha + [V_{\text{nmo}}^{(2)}]^2 \sin^2 \alpha},$$  (27)

and the azimuthally dependent $\eta$ is given by (Pech and Tsvankin, 2003)

$$\eta(\alpha) = \eta^{(1)} \sin^2 \alpha - \eta^{(2)} \sin^2 \alpha \cos^2 \alpha + \eta^{(3)} \cos^2 \alpha.$$  (28)

Although the linearization in the anisotropic parameters implied by the weak-anisotropy approximation requires dropping the coefficient $\eta(\alpha)$ from the denominator of equation (26), the complete denominator of the original VTI equation (18) can be retained to increase the accuracy at large source-receiver offsets.

4 GEOMETRICAL-SPREADING FACTOR IN AN ORTHORHOMBIC LAYER

The derivatives of the traveltime with respect to offset and azimuth needed to obtain the geometrical spreading $L(x', x^*)$ from equation (8) can be found using the nonhyperbolic moveout equation (13). Explicit expressions for the traveltime derivatives in terms of the azimuthally dependent moveout coefficients $A_2$, $A_4$, and $A$ are given in Appendix B. Substitution of equations (19), (22), and (15) for $A_2(\alpha)$, $A_4(\alpha)$, and $A(\alpha)$ yields the geometrical spreading factor as a function of the parameters of orthorhombic media. Equation (8) also contains the term $\cos \phi$ ($\phi$ is the group angle) that can be written as $(T^2_0 V_{P0})/\sqrt{x^2 + T^2_0 V_{P0}^2}$. The final equation for the geometrical-spreading factor, which is too lengthy...
to be included in the paper, is used in the numerical tests below.

While the derived expression is well-suited for numerical modeling of the factor \( L(x', x^a) \), it does not provide insight into the dependence of the geometrical spreading on the anisotropic parameters. Therefore, next we apply the weak-anisotropy approximation based on equation (26) to obtain the inverse geometrical-spreading factor \( L^{-1}(x', x^a) \):

\[
L^{-1}(x', x^a) = \cos^{-1} \phi \left[ \frac{\partial T}{\partial x} \frac{\partial T}{\partial x} \frac{1}{x} + \frac{\partial^2 T}{\partial x^2} \frac{\partial^2 T}{\partial x^2} \frac{1}{x^2} - \left( \frac{\partial T}{\partial x} \right)^2 \frac{1}{x^4} \right]^{1/2}
\]

(29)

The traveltime derivatives in equation (29) are found from equation (26) and then linearized in the anellipticity parameters \( \eta^{1,2,3} \). Further linearization of equation (29) gives the weak-anisotropy approximation for \( L^{-1}(x', x^a) \) analyzed below.

5 ANALYSIS OF THE WEAK-ANISOTROPY APPROXIMATION

5.1 Geometrical spreading in the symmetry planes

While the full linearized expression for geometrical spreading is still rather long, it takes a much more concise form in the vertical symmetry planes. For the symmetry plane \([x_1, x_3]\), we find

\[
L^{-1} = \cos^{-1} \phi \left( \frac{A + Bx^2 + Cx^4}{V_{nmo}^{(1)} V_{nmo}^{(2)} (T_0 V_{nmo}^{(2)})^2 + x^4} \right),
\]

(30)

where

\[
\cos \phi = \frac{T_0 V_{p0}}{\sqrt{x^2 + T_0^2 V_{p0}^2}},
\]

(31)

\[
A = T_0^3 V_{nmo}^{(2)} \eta^{(2)},
\]

(32)

\[
B = T_0^3 \left( V_{nmo}^{(2)} \right)^2 \left\{ 2 \left[ 1 - 4 \eta^{(2)} \right] V_{nmo}^{(2)} \right\},
\]

(33)

\[
C = T_0 \left\{ 1 + \eta^{(2)} \right\} \left( V_{nmo}^{(2)} \right)^2
\]

(34)

For the single-layer model, the term \( \cos^{-1} \phi \) in equation (30) is perfectly isotropic and, for fixed values of the zero-offset time \( T_0 \) and offset \( x \), depends just on the vertical velocity \( V_{p0} \).

At zero offset, the factor \( L^{-1} \) becomes \( 1/(T_0 V_{nmo}^{(1)} V_{nmo}^{(2)}) \), which is an exact expression that can be obtained directly from the wavefront curvatures for any strength of the anisotropy. From equations (20) and (21) for the NMO velocities it is clear that the geometrical spreading at vertical incidence is governed by two anisotropic coefficients, \( \delta^{(1)} \) and \( \delta^{(2)} \), which also determine the NMO ellipse. For VTI media, \( V_{nmo}^{(1)} = V_{nmo}^{(2)} \), and \( L^{-1} \) at zero offset reduces to \( 1/(T_0 V_{nmo}^2) \), this result was previously obtained by Tsvankin (1995) and Ursin & Hostad (2002). If the medium is isotropic, \( L^{-1} \) further simplifies to the well-known expression \( 1/(T_0 V_{p0}^2) \).

The factors \( B \) and \( C \) in equation (30) can be called the “near-offset” and “far-offset” spreading coefficients, respectively. It should be emphasized that \( B \) and \( C \) contain terms dependent on both in-plane and out-of-plane traveltime (and, therefore, velocity) variations. P-wave reflection traveltime in the incidence plane is controlled just by the NMO velocity \( V_{nmo}^{(2)} \), and the anisotropic parameter \( \eta^{(2)} \) (Grechka and Tsvankin, 1999, Tsvankin, 2001). Hence, the term \( 1 - 4 \eta^{(2)} \) \( V_{nmo}^{(2)} \) in the coefficient \( B \) represents the in-plane contribution that coincides with the corresponding (near-offset) spreading factor for VTI media. The other term in the expression for \( B \), \( (\cos \alpha - \eta^{(1)} + \eta^{(3)}) V_{nmo}^{(1)} \), is entirely due to azimuthal anisotropy and a nonzero value of the second traveltime derivative with respect to \( \alpha \). Similarly, the far-off coefficient \( C \) contains the in-plane term \( 1 + \eta^{(3)} \) \( V_{nmo}^{(2)} \) and exactly the same out-of-plane term as the one in the expression for \( B \).

The inverse spreading factor \( L^{-1} \) in the symmetry plane \([x_2, x_3]\) can be obtained from equation (30) by simply switching the superscripts (1) and (2) in the NMO velocities and the coefficients \( \eta \). A more detailed comparison of the geometrical spreading in the symmetry planes of orthorhombic media with that in VTI media is given in the numerical examples below.

5.2 Azimuthal variation of geometrical spreading

Since azimuthal AVO analysis often operates with prestack amplitudes measured at a fixed offset, it is of interest to analyze the azimuthally-varying spreading factor \( L^{-1}(\alpha) \) for \( x = \text{const} \). To facilitate the analysis of the function \( L^{-1}(\alpha) \), it is convenient to replace the NMO velocities in the hyperbolic moveout term of equation (26) with their expressions in terms of the coefficients \( \delta^{(1)} \) and \( \delta^{(2)} \). Linearizing both the \( x^2 \)- and \( x^4 \)-terms in equation (26) in the anisotropic parameters yields

\[
T^2(\alpha, x) = T_0^2 + x^2 \left[ 1 - \delta^{(1)} - \delta^{(2)} + (\delta^{(2)} - \delta^{(1)}) \cos 2\alpha \right]
\]

\[
- 2x^4 \frac{(\cos^2 \alpha + \eta^{(1)} \sin^2 \alpha - \eta^{(3)} \cos^2 \alpha \sin^2 \alpha)}{T_0^2 \eta^{(2)} (1 + \frac{\alpha^2}{\eta^{(2)} V_{p0}^2})}
\]

(35)
Substituting the moveout equation (35) into equation (29) and carrying out further linearization in the anisotropic parameters, we obtain the inverse geometrical spreading as

\[
L^{-1}(x, \phi) = D(x) + E(\alpha) \left[ \frac{x}{T_0 V_{P0}} \right]^2 + F(\alpha) \left[ \frac{x}{T_0 V_{P0}} \right]^4 + \ldots \tag{36}
\]

Here, \(D(x)\) is an azimuthally-independent term that would coincide with \(L^{-1}\) in VTI media if the anisotropic coefficients in the vertical symmetry planes were identical, and \(\eta^{(3)} = 0\). The azimuthally-varying terms were expanded in \(x^2\), and powers of \(x\) higher than four were neglected. The coefficients \(E(\alpha)\) and \(F(\alpha)\) are given by

\[
E(\alpha) = T_0^2 V_{P0} \left\{ 3 \left[ \eta^{(1)} - \eta^{(2)} \right] - \left[ \delta^{(1)} - \delta^{(2)} \right] \right\} \cos 2\alpha; \tag{37}
\]

\[
F(\alpha) = T_0^2 V_{P0} \left\{ \left[ \frac{3}{2} (\delta^{(1)} - \delta^{(2)}) + 9(\eta^{(1)} - \eta^{(2)}) \right] \cos 2\alpha + \frac{9}{8} \eta^{(3)} \cos 4\alpha \right\}. \tag{38}
\]

The coefficient \(E(\alpha)\) in equation (37) determines the azimuthal dependence of the geometrical spreading at near offsets. Since \(E(\alpha)\) is proportional to \(\cos 2\alpha\), for small \(x\) the function \(L^{-1}(\alpha)\) traces out a curve close to an ellipse. In contrast, the far-offset coefficient \(F\) contains both \(\cos 2\alpha\) and \(\cos 4\alpha\), and the form of \(L^{-1}(\alpha)\) may substantially deviate from elliptical; this is illustrated by numerical examples in the next section. The magnitude of the azimuthal variation of geometrical spreading is controlled by the differences \((\delta^{(1)} - \delta^{(2)})\), \((\eta^{(1)} - \eta^{(2)})\) and, at far offsets, by the coefficient \(\eta^{(3)}\). If \(\delta^{(1)} = \delta^{(2)}\), \(\eta^{(1)} = \eta^{(2)}\), and \(\eta^{(3)} = 0\), P-wave velocity becomes azimuthally independent, and for purposes of computing P-wave geometrical spreading the orthorhombic medium becomes equivalent to VTI.

6 NUMERICAL EXAMPLES

The numerical tests presented here are designed to illustrate the following properties of the inverse geometrical-spreading factor \(L^{-1}\) in an orthorhombic layer:

- The influence of azimuthal anisotropy on \(L^{-1}\) in the vertical symmetry planes.
- The azimuthal variation of \(L^{-1}\) at a fixed source-receiver offset.
- The spatial variation of \(L^{-1}\) plotted as a function of offset and azimuth.
- The accuracy of the weak-anisotropy approximation for \(L^{-1}\).

In the examples below we use an orthorhombic model formed by parallel vertical penny-shaped cracks embedded in a VTI background. The stiffness coefficients for this model are given in Schoenberg and Helbig (1997), and the corresponding anisotropic parameters, listed in the caption of Figure 3, are taken from Tsvankin (1997). Although this model has substantial azimuthal velocity variations, it is dominated by the VTI component, with both \(\epsilon\) coefficients close to 30%.

As before, we assume that the coordinate planes coincide with the symmetry planes of the orthorhombic layer. The inverse geometrical spreading factor \(L^{-1}\) is computed using both the “exact” formalism and the weak-anisotropy approximation. The “exact” approach is based on combining equation (29) for \(L^{-1}\) with the Tsvankin-Thomsen moveout equation [using equations (19), (22), and (15) for the moveout coefficients] without making any approximations in computing the traveltime derivatives and the spreading factor itself. The word “exact” is put in quotes because the Tsvankin-Thomsen equation, despite its high accuracy for P-waves, is not an exact representation of the reflection traveltine. The weak-anisotropy approximation, as described in the previous section, is obtained by linearizing all equations in the anisotropic coefficients.

The inverse geometrical-spreading factor \(L^{-1}\) in the vertical symmetry planes of the model is displayed in Figure 3. In addition to the “exact” (solid) line and weak-anisotropy (dashed line) solutions, we plot the “exact” values of \(L^{-1}\) for the reference VTI models (dotted line) in each symmetry plane. The reference VTI model has the same vertical velocity \(V_{P0}\) as the actual orthorhombic medium and the anisotropic parameters \(\epsilon\) and \(\delta\) equal to the corresponding parameters in a given symmetry plane. For example, the reference VTI parameters for the \([x_1, x_3]\) symmetry plane are \(\epsilon = \delta^{(2)}\) and \(\delta = \delta^{(3)}\).

All three curves are normalized by the factor \(L^{-1}\) in the corresponding isotropic model with the velocity \(V_{P0}\). Clearly, the influence of anisotropy leads to significant distortions of geometrical spreading in a wide range of offsets for both symmetry planes. As shown by Tsvankin (1995, 2001) for VTI media, the influence of anisotropy causes the amplitude (e.g., the inverse geometrical-spreading factor) to decrease away from the vertical if the difference \(\epsilon - \delta\) is positive (i.e., \(\eta > 0\)). Figure 3 confirms that this conclusion remains valid for the symmetry planes of orthorhombic media with moderate azimuthal anisotropy. Indeed, the \(\eta\) coefficients in both vertical symmetry planes (\(\eta^{(1)}\) and \(\eta^{(2)}\)) are positive, and the normalized factor \(L^{-1}\) rapidly decreases with offset at near-vertical incidence.

Comparison with the geometrical spreading in the reference VTI medium helps to quantify the influence of azimuthal anisotropy in both symmetry planes. It is interesting that azimuthal anisotropy changes the geometrical-spreading factor even at vertical incidence, where for orthorhombic media \(L^{-1} = 1/(T_0 V_{nmo}^{(1)} V_{nmo}^{(2)})\), while for VTI media \(L^{-1} = 1/(T_0 V_{nmo}^{2})\). If we substi-
trolled by the combination \((\eta^{(2)} - \eta^{(1)} + \eta^{(3)})\) of the anellipticity coefficients. Since for our model this combination is positive and relatively large \((0.38)\), \(L^{-1}\) in the \([x_1, x_3]\) plane initially decreases with offset slower than in the corresponding VTI media (Figure 3a). For offset-to-depth ratios exceeding two, however, the factor \(L^{-1}\) almost coincides with the VTI value, which contradicts the weak-anisotropy result.

Similarly, the factor \(L^{-1}\) in the \([x_2, x_3]\) symmetry plane contains the “out-of-plane” term proportional to \((\eta^{(1)} - \eta^{(2)} + \eta^{(3)})\). For the model at hand, however, this term is close to zero \((0.006)\), and the offset dependence of the geometrical spreading in the \([x_2, x_3]\)-plane is close to that in the reference VTI medium (Figure 3b).

Figure 3 also helps to evaluate the accuracy of the weak-anisotropy approximation for a model that can be characterized as moderately-to-strongly anisotropic in terms of the magnitude of P-wave velocity variations. While the weak-anisotropy solution is exact at zero offset (because we did not linearize the NMO velocities in the denominator of the hyperbolic term of \(L^{-1}\)), it rapidly deviates from the exact factor \(L^{-1}\) with increasing offset. Still, the weak-anisotropy approximation correctly predicts the general character of the function \(L^{-1}(x)\) and remains accurate for offset-to-depth ratios of up to about one.

The azimuthal variation of the normalized factor \(L^{-1}\) at two different offsets is plotted in Figure 4. Since the geometrical spreading in an orthorhombic layer is symmetric with respect to both vertical coordinate planes, the signature of \(L^{-1}\) is repeated in each quadrant. For the offset equal to the reflector depth, the azimuthal variation of \(L^{-1}\) is close to elliptical, as predicted by the weak-anisotropy approximation (Figure 4a). The fractional difference of \(L^{-1}\) between the symmetry planes, which determines the overall magnitude of the azimuthal variation of the inverse geometrical spreading, is about 30%. Hence, for this model the eccentricity of the “geometrical-spreading ellipse” exceeds that of the NMO ellipse \((18\%)\). For larger offset-to-depth ratios, the shape of the curve \(L^{-1}(\alpha)\) becomes more complicated and, in agreement with the weak-anisotropy approximation \((38)\) for the \(x^4\)-term, deviates from an ellipse (Figure 4b). The magnitude of the azimuthal variation of \(L^{-1}(\alpha)\) does not noticeably increase with offset for offset-to-depth ratios between one and two.

A complete picture of the spatial variations of the geometrical spreading in our model is given in Figure 5a, where the factor \(L^{-1}\) is computed as a function of both offset and azimuth. The combined influence of polar and azimuthal anisotropy creates a rather complicated pattern of the normalized factor \(L^{-1}\), with substantial azimuthal variations and pronounced deviations from the corresponding isotropic values. The largest anisotropy-induced distortions of the geometrical spreading reach-
Figure 4. Azimuthal variation of the normalized factor $L^{-1}$ for offset-to-depth ratios of one (a) and two (b). The azimuth $\alpha$ (numbers on the perimeter) is measured with respect to the $x_1$-axis. The solid line is the “exact” solution, the dashed line is the weak-anisotropy approximation.

Figure 5. Map of the normalized factor $L^{-1}$ as a function of offset and azimuth. The offset-to-depth ratio varies from zero to four. The model in plot (a) is the same as that in Figures 3 and 4; in plot (b), the sign of the parameter $\delta^{(2)}$ was changed from negative to positive (i.e., $\delta^{(2)} = 0.078$).

7 DISCUSSION AND CONCLUSIONS

Although geometrical spreading of reflected waves is determined by the medium properties along the whole raypath, it can be obtained from the reflection traveltime and the group angles at the surface. We showed that for a stack of horizontal azimuthally anisotropic layers with a horizontal symmetry plane, the inverse geometrical-spreading factor $L^{-1}$ can be expressed as a simple function of the traveltime derivatives with
respect to offset and azimuth. Describing the travel-time by the Tsvankin-Thomsen nonhyperbolic moveout equation (Tsvankin and Thomsen, 1994) helps to represent the factor \( L^{-1} \) through the azimuthally varying moveout coefficients.

For a horizontal orthorhombic layer, the factor \( L^{-1} \) was related to the medium parameters by using the explicit expressions for the moveout coefficients given by Al-Dajani et al. (1998). P-wave traveltime and, therefore, the geometrical spreading for this model is governed by the NMO velocities \( V_{\text{nmo}}^{(1)} \) and \( V_{\text{nmo}}^{(2)} \) in the vertical symmetry planes and the anellipticity coefficients \( \eta^{(1)}, \eta^{(2)}, \) and \( \eta^{(3)} \). To explain the dependence of the factor \( L^{-1} \) on these parameters, we employed the weak-anisotropy approximation based on linearization in the anisotropic coefficients. The analytic results were verified by numerical tests for an orthorhombic model formed by vertical penny-shaped cracks embedded in a VTI matrix.

Although the geometrical-spreading signature in a single orthorhombic layer is repeated in each quadrant, the variation of the factor \( L^{-1} \) with offset and azimuth has a rather complicated character. For the model used here, the error of the isotropic equation for the geometrical spreading reaches a maximum of 40% in the intermediate offset range (i.e., for the offset-to-depth ratio between one and two). The azimuthal variation \( L^{-1}(\alpha) \) for a fixed offset is close to elliptical at relatively small offsets-to-depth ratios close to one. For larger offsets, \( L^{-1}(\alpha) \) deviates from an ellipse and may have intermediate minima or maxima between the symmetry planes.

Both analytic and numerical results show that the spreading factor \( L^{-1} \) is substantially influenced by azimuthal velocity variations even in the vertical symmetry planes. At zero offset (vertical incidence), the exact inverse geometrical spreading is given by a simple equation that involves only the NMO velocities in both symmetry planes: \( L^{-1} = 1/(T_0 V_{\text{nmo}}^{(1)} V_{\text{nmo}}^{(2)}) \). The offset-dependent part of \( L^{-1} \) in the symmetry planes can be separated (in the weak-anisotropy approximation) into the in-plane term, identical to the factor \( L^{-1} \) in the corresponding VTI medium, and the out-of-plane term associated with azimuthal anisotropy. In the \([x_1, x_3]\)-plane, the contribution of azimuthal velocity variation is proportional to the combination \((\eta^{(2)} - \eta^{(1)} + \eta^{(3)})\), and in the \([x_1, x_2]\)-plane to \((\eta^{(1)} - \eta^{(2)} + \eta^{(3)})\).

The large magnitude of the anisotropy-induced distortions of the factor \( L^{-1} \) means that reliable interpretation of the AVO response for media with azimuthally anisotropic overburden is impossible without properly correcting for the geometrical spreading. The estimation and removal of geometrical spreading can be accomplished by applying the Tsvankin-Thomsen moveout equation with fitted (i.e., estimated from the data) coefficients. The methodology of the geometrical-spreading correction for azimuthally anisotropic media will be discussed in a sequel paper.

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REFERENCES


APPENDIX A: GEOMETRICAL SPREADING OVER AN AZIMUTHALLY ANISOTROPIC MEDIUM WITH A HORIZONTAL SYMMETRY PLANE

As shown in the main text [equation (6)], the geometrical-spreading factor for a stack of horizontal anisotropic layers with a horizontal symmetry plane can be written in the following form:

\[ L(x', x^*) = \left[ \cos \phi \cos \phi' \right]^{1/2} |\det B|^{-1/2}, \]  

where

\[ B = \begin{bmatrix} \frac{\partial^2 T(x', x^*)}{\partial x \partial x} & \frac{\partial^2 T(x', x^*)}{\partial x \partial x^*} \\ \frac{\partial^2 T(x', x^*)}{\partial x^* \partial x} & \frac{\partial^2 T(x', x^*)}{\partial x^* \partial x^*} \end{bmatrix}. \]  

When the source and receiver lie on the same horizontal surface, the traveltime \( T \) depends only on the distance \( x \) between the source and the receiver and the azimuth \( \alpha \) of the source-receiver line with respect to the \( x_1 \)-axis:

\[ x = \sqrt{(x_1^* - x_1^2)^2 + (x_2^* - x_2^2)^2}, \]  

\[ \alpha = \tan^{-1} \left( \frac{x_2^* - x_2^2}{x_1^* - x_1^2} \right). \]  

If the traveltime \( T \) is expressed as a function of \( x \) and \( \alpha \), the elements of the matrix \( B \) become

\[ \frac{\partial^2 T}{\partial x_1^2 \partial x_1} = \frac{\partial^2 T}{\partial x \partial x} + \frac{\partial^2 T}{\partial x \partial x^*} + \frac{\partial^2 T}{\partial x^* \partial x}, \]  

\[ \frac{\partial^2 T}{\partial x_1^2 \partial x_2^*} = \frac{\partial^2 T}{\partial x_1 \partial x_2^*} + \frac{\partial^2 T}{\partial x_2 \partial x^*} + \frac{\partial^2 T}{\partial x_2 \partial x}, \]  

\[ \frac{\partial^2 T}{\partial x_2^2 \partial x_1} = \frac{\partial^2 T}{\partial x_2 \partial x_1}, \]  

\[ \frac{\partial^2 T}{\partial x_2^2 \partial x_2^*} = \frac{\partial^2 T}{\partial x_2 \partial x^*} + \frac{\partial^2 T}{\partial x_2 \partial x} + \frac{\partial^2 T}{\partial x \partial x^*} + \frac{\partial^2 T}{\partial x \partial x}. \]

The derivatives of \( x \) and \( \alpha \) with respect to the source and receiver coordinates can be obtained from equations (A3) and (A4):

\[ \frac{\partial x}{\partial x_1} = \frac{x_1^* - x_1}{x}, \quad \frac{\partial x}{\partial x_2} = \frac{x_2^* - x_2}{x}, \quad (i = 1, 2) \]  

\[ \frac{\partial \alpha}{\partial x_1} = \frac{x_2^* - x_1^2}{x^2}, \quad \frac{\partial \alpha}{\partial x_2^*} = \frac{x_1^* - x_1^2}{x^2}, \]  

\[ \frac{\partial^2 x}{\partial x_1^2 \partial x_1} = \frac{-(x_2^* - x_2^2)^2}{x^3}, \]  

\[ \frac{\partial^2 x}{\partial x_1^2 \partial x_2^*} = \frac{-(x_1^* - x_1^2)(x_2^* - x_2^2)}{x^3}, \]  

\[ \frac{\partial^2 x}{\partial x_2^2 \partial x_1} = \frac{-(x_1^* - x_1^2)^2}{x^3}, \]  

\[ \frac{\partial^2 x}{\partial x_2^2 \partial x_2^*} = \frac{-(x_1^* - x_1^2)(x_2^* - x_2^2)}{x^3}, \]  

\[ \frac{\partial^2 \alpha}{\partial x_1^2 \partial x_1} = \frac{2(x_1^* - x_1^2)(x_2^* - x_2^2)}{x^4}, \]  

Substituting equations (A9)–(A17) into equations (A5)–(A8) yields

\[ \frac{\partial^2 T}{\partial x_1^2 \partial x_1} = -\frac{\partial^2 T}{\partial x^2}(x_1^* - x_1^2)^2 \frac{\partial^2 T}{\partial x \partial x_1} + \frac{\partial^2 T}{\partial x^* \partial x_1} + \frac{\partial^2 T}{\partial x \partial x_1} + \frac{\partial^2 T}{\partial x^* \partial x_1} + \frac{\partial T}{\partial x_1} \frac{\partial^2 T}{\partial x \partial x}, \]
\[ \frac{\partial^2 T}{\partial x_1 \partial x_2} = \frac{\partial^2 T - (x_1 - x_1')(x_2' - x_2)}{x^2} + \frac{\partial T \, (x_1 - x_1')(x_2' - x_2)}{x^3} \]
\[ + \frac{\partial^2 T \, (x_1 - x_1')(x_2' - x_2)^2}{x^4} + \frac{\partial T \, (x_1 - x_1')^2 - (x_2' - x_2)^2}{x^4} . \tag{A19} \]

\[ \frac{\partial^2 T}{\partial x_2^2} = \frac{\partial^2 T \, (x_2' - x_2)^2}{x^2} - \frac{\partial T \, (x_1 - x_1')^2}{x^3} - \frac{\partial^2 T \, (x_1 - x_1')^2}{x^4} + \frac{\partial T \, 2(x_1 - x_1')(x_2' - x_2)}{x^4} \tag{A20} . \]

The determinant of the matrix \( B \) is then found as
\[ \det(B) = \frac{\partial^2 T \, \partial T}{\partial x^2} \frac{1}{x} + \frac{\partial^2 T \, \partial^2 T}{\partial x^2 \, \partial \alpha^2} \frac{(\partial T)^2}{x^4} \tag{A22} . \]

Finally, using equation (A22), the geometrical-spreading factor (A1) can be expressed through the traveltime derivatives with respect to the offset \( x \) and azimuth \( \alpha \):
\[ L(x', x^*) = (\cos \phi^* \cos \phi')^{1/2} \left[ \frac{\partial^2 T \, \partial T}{\partial x^2} \frac{1}{x} + \frac{\partial^2 T \, \partial^2 T}{\partial x^2 \, \partial \alpha^2} \frac{(\partial T)^2}{x^4} \right]^{-1/2} . \tag{A23} \]

**APPENDIX B: TRAVELTIME DERIVATIVES FROM THE NONHYPERBOLIC MOVEOUT EQUATION**

P-wave nonhyperbolic (long-spread) reflection traveltime can be accurately described by the Tsvankin-Thomsen (1994) moveout equation:
\[ T^2(x, \alpha) = T_0^2 + A_2(\alpha) \, x^2 + \frac{A_4(\alpha) \, x^4}{1 + A(\alpha) \, x^2} . \tag{B1} \]

where the moveout coefficients \( A_2, A_4, \) and \( A \) generally vary with the azimuth \( \alpha \).

The derivatives of the traveltime with respect to the offset \( x \) are given by
\[ \frac{\partial T}{\partial x} = \frac{1}{T} \left[ A_2 x + \frac{2 A_4 x^3}{1 + A x^2} - \frac{A A_4 x^5}{(1 + A x^2)^2} \right] \tag{B2} \]
and
\[ \frac{\partial^2 T}{\partial x^2} = \frac{1}{T^2} \left[ f(x) - \left( \frac{\partial T}{\partial x} \right)^2 \right] ; \tag{B3} \]

\[ f(x) \equiv A_2 + 6 A_4 x^2 + \frac{9 A A_4 x^4}{1 + A x^2} + \frac{4 A_4 A^2 x^6}{(1 + A x^2)^3} . \tag{B4} \]

Differentiating equation (B1) with respect to azimuth yields
\[ \frac{\partial T}{\partial \alpha} = \frac{1}{2 T} \left[ A_2' x^2 + \frac{A_4' x^4}{1 + A x^2} - \frac{A A_4' x^6}{(1 + A x^2)^2} \right] \tag{B5} \]
and
\[ \frac{\partial^2 T}{\partial \alpha^2} = \frac{\partial T}{\partial \alpha} \frac{1}{2 T^2} \left[ A_2' x^2 + \frac{A_4' x^4}{1 + A x^2} - \frac{A A_4' x^6}{(1 + A x^2)^2} \right] + \frac{1}{2 T^2} \left[ A_2'' x^2 + \frac{A_4'' x^4}{1 + A x^2} - \frac{A A_4'' x^6}{(1 + A x^2)^2} - \frac{2 A A_4' x^6}{(1 + A x^2)^2} + \frac{2 A_4 A^2 x^8}{(1 + A x^2)^3} \right] . \tag{B6} \]

Here, \( A_2', A_4', A_2'', A_4'' \) and \( A'' \) are the first and second derivatives of the moveout coefficients with respect to \( \alpha \). For the model of a single orthorhombic layer, these derivatives can be found from the explicit expressions for \( A_2, A_4 \), and \( A \) given in the main text.
Wave equation migration arising from true amplitude one-way wave equations

Yu Zhang,* Guanquan Zhang,† & Norm Bleistein‡

*Veritas DGC Inc., 10300 Town Park Drive, Houston, TX 77072
†Institute of Computational Mathematics & Sci/Eng. Computing, Academy of Mathematics & System Sciences, Chinese Academy of Sciences, Beijing, 100080 P. R. China
‡Center for Wave Phenomena, Department of Geophysics, Colorado School of Mines, Golden, CO 80401-1887 USA

ABSTRACT

One-way wave operators are powerful tools for forward modeling and inversion. Their implementation, however, involves introduction of the square-root of an operator as a pseudo-differential operator. Furthermore, a simple factoring of the wave operator produces one-way wave equations that yield the same travel-times of the full wave equation, but do not yield accurate amplitudes except for homogeneous media and for almost all points in heterogeneous media. Here, we present augmented one-way wave equations. We show that these equations yield solutions for which the leading order asymptotic amplitude as well as the traveltime agree with satisfy the same differential equations as do the corresponding functions for the full wave equation. Exact representations of the square-root operator appearing in these differential equations are elusive, except in cases in which the heterogeneity of the medium is independent of the transverse-spatial variables. Here, singling out depth as the preferred direction of propagation, we introduce a representation of the square-root operator as an integral in which a rational function of the transverse Laplacian appears in the integrand. This allows us to carry out explicit asymptotic analysis of the resulting one-way wave equations. To do this, we introduce an auxiliary function that satisfies a lower dimensional wave equation in transverse-spatial variables only. We prove that ray theory for these one-way wave equations leads to one-way eikonal equations and the correct leading order transport equation for the full wave equation. We then introduce appropriate boundary conditions at \( z = 0 \) to generate waves at depth whose quotient leads to a reflector map and estimate of the ray-theoretical reflection coefficient on the reflector. Thus, these true amplitude one-way wave equations lead to a “true amplitude wave equation migration (WEM)” method. In fact, we prove that applying the WEM imaging condition to these newly defined wavefields in heterogeneous media leads to the Kirchhoff inversion formula for common-shot data. This extension enhances the original WEM. The objective of that technique was a reflector map, only. The underlying theory did not address amplitude issues. Computer output using numerically generated data confirms the accuracy of this inversion method. However, there are practical limitations. The observed data must be a solution of the wave equation. Therefore, the data over the entire survey area must be collected from a single common-shot experiment. Multi-experiment data, such as common-offset data, cannot be used with this method as presently formulated. Research on extending the method is ongoing at this time.

1 INTRODUCTION

One-way wave equations provide fast tools for modeling and migration. These one-way equations allow us to separate solutions of the wave equation into downgoing and upgoing waves except in the limit of near-horizontal propagation. The original one-way wave equations used for wave equation migration (WEM)
[Claerbout, 1971, 1985] were designed to produce accurate traveltimes, but were never intended to produce accurate amplitudes, even at the level of leading order asymptotic WKBJ or ray-theoretic amplitudes. As such, that WEM provides a reflector map consistent with the background propagation model, but with unreliable amplitude information.

The objective of this paper is to describe a modification of those one-way wave equations to produce equations that provide accurate leading order WKBJ or ray-theoretic amplitude as well as accurate traveltimes. The necessary modification of the basic one-way wave equations can be motivated by considering depth-dependent \( v(z) \) medium. In this case, through the use of Fourier transform in time and transverse spatial coordinates \((x,y)\), we reduce the problem of modifying the one-way equations to the study of ordinary differential equations. There, it is relatively simple to see how to modify the equations used by Claerbout in order to obtain equations that provide leading order WKBJ amplitudes, as well. This leading order amplitude is what we mean by “true amplitude” for forward modeling.

For heterogeneous media, \( v = v(x,y,z) \), the same one-way wave equations still provide true amplitudes. However, now the transverse wave vector \((k_x, k_y)\) must be interpreted as differentiations in the corresponding dual spatial variables. Further, our modified one-way wave equations involve square-roots and divisions by functions of this transverse wave vector. We provide an interpretation of these operators through some basic ideas from the theory of pseudo-differential operators.

We provide a relatively simple representation of the one-way differential operators. This, in turn allows us to prove that the ray-theoretic solutions of these equations satisfy the separate eikonal equations for downgoing (increasing \( z \)) and upgoing waves, but the leading order amplitudes also satisfy the same equation—the transport equation—as does the leading order amplitude for the full wave equation. It is in this sense that describe the solutions of these one-way wave equations as “true amplitude” solutions.

Having these true amplitude one-way equations allows us to develop a “true amplitude” WEM for heterogeneous media. To date, we only have numerical checks on this method for \( v(z) \) media, where the pseudo-differential operators revert to simple multiplications in the temporal/transverse-spatial Fourier domain. However, we are able to prove that the reflection amplitude agrees with the amplitudes generated by Kirchhoff inversion (true amplitude Kirchhoff migration) as developed by one of the authors [Bleistein, 1987, Bleistein et al. 2001] and colleagues. This proof is valid in heterogeneous media. Thus, at this time, the proof of validity is ahead of the computer implementation in terms of generality and it anticipates a reliable computer implementation in general heterogeneous media. It confirms that the output of this method is a reflector map with the peak amplitude on the reflector being in known proportion to an angularly dependent reflection coefficient at a specular reflection angle.

This type of inversion requires common-shot data with the receiver array covering the entire domain of the survey. This is a serious obstacle for practical implementation; such data gathers are still relatively rare. To date, we do not have an extension of this true amplitude WEM to other source/receiver configurations.

In the next section, we provide motivation for the modification of the simple one-way wave equations with the objective of developing true amplitude one-way equations for forward modeling. We start from homogeneous media where the basic one-way equations do provide true amplitude. We then proceed to analysis of modeling for \( v(z) \) media. That leads to a modification of the basic one-way equations in order to assure that the appropriate transport equation is satisfied, as well.

Following that, we introduce the idea of using the same equations for heterogeneous media—\( v = v(x,y,z) \)—with functions of transverse wave vectors now reinterpreted as pseudo-differential operators. It is for this new interpretation that we provide a confirmation that these one-way wave equations provide true amplitude forward modeling. The proof of the claim that this extension leads to the appropriate transport equation is provided in Appendix A. That extension and the proof were originally developed by the second author in Zhang [1993]. Here, we present an update of that proof with attention to the application to WEM.

Following the discussion of forward modeling, we develop true-amplitude WEM. This requires a modification of the basic one-way wave equations of Claerbout’s WEM and also a modification of the boundary conditions of that WEM, which corrects the phase as well as the amplitude of the downgoing wave used in WEM. We also show how to modify Claerbout’s original WEM equations in order to turn those into true amplitude equations. There is a subtlety of scaling by a pseudo-differential operator in the comparison. This leads to a slightly different one-way wave equation for the downgoing waves in Claerbout’s approach when compared to the one-way equation that we use in the new theory.

We then provide a proof that this new approach leads to the same common-shot Kirchhoff inversion formula as is found in Bleistein [1987] and Bleistein et al. [2001], as expressed by Hanitzsch [1997].

Following that, we present our numerical check of true amplitude WEM. As noted above, the example we present is for the case of a \( v(z) \) medium where implementation of the pseudo-differential operators appearing in our method reduces to a multiplication in the temporal/transverse-spatial domain.
2 MOTIVATION

In this section we provide motivation for modifying the standard one-way wave equations that are used in WEM. We do this by starting with the standard operator factoring scheme for the wave equation in homogeneous media, separating off the $z$-dependence so that we can identify upgoing and downgoing waves. We identify the separate waves and confirm that they are solutions of first order wave equations obtained by factoring the operator. We then show that the solutions of the derived one-way wave equations are no longer solutions of the full (two-way) wave equation when the medium is allowed to depend on $z$; that is, $v = v(z)$. In a WKBJ solution in a $v(z)$ medium, the amplitudes of the first order equations do not agree with the amplitudes of the two one-way solutions of the full wave equation. By modifying the one-way wave equations, we obtain new equations whose eikonal and transport equations agree with the eikonal and transport equations for the full wave equation; each one-way equation governing one-way propagating waves of the two-way or full wave equation.

We begin by introducing the wave equation,

$$\frac{1}{v^2} \frac{\partial^2 W}{\partial t^2} - \nabla^2 W = 0. \quad (1)$$

We will use the following definition of the Fourier transform:

$$F(x,y,z,t) = \frac{1}{(2\pi)^3} \int dk_x dk_y d\omega F(k_x, k_y, z, \omega) \cdot \exp\left\{i [\omega t - k_x x - k_y y] \right\}. \quad (2)$$

In this equation, the wave number integration ranges over all space. The frequency domain integral has the range $-\infty < \text{Re} \omega < \infty$, with $\text{Im} \omega$ large enough that the contour of integration passes below all singularities of the integrand in the complex $\omega$-plane. Typically, this is just below the real axis in $\omega$, with the integrand defined on the axis only through analytic continuation from below. Consequently, when viewed as an integral on the real $\omega$ axis from $-\infty$ to $\infty$, we must interpret multi-valued functions, such as square-roots, as if we had passed under their singular points—branch points and poles—to go from one side of one of these points to the other.

Consistent with the Fourier transform in (2), when we write down WKBJ solutions, they will take the form,

$$W = A \exp\left\{i [\omega t - \Phi(x,y,z,\omega)] \right\} \quad \text{or} \quad W = A \exp\left\{i \omega [t - \varphi(x,y,z)] \right\} \quad (3)$$

or

$$W = A \exp\left\{-i \omega \varphi(x,y,z) \right\},$$

depending on the context. In these forms, $\nabla \Phi$ or $\nabla \varphi$ points in the direction of propagation of the wavefronts.

In particular, $\text{sign}(\omega \partial \Phi / \partial z) = 1$ or $\text{sign}(\partial \varphi / \partial z) = 1$ indicates waves in the direction of increasing $z$; that is, *downgoing*. Of course, then, for upgoing waves $\text{sign}(\omega \partial \Phi / \partial z) = -1$ or $\text{sign}(\partial \varphi / \partial z) = -1$. We could alternatively represent upgoing waves by

$$W = A \exp\left\{i \omega t + \Phi(x,y,z,\omega) \right\} \quad \text{or} \quad W = A \exp\left\{i \omega [t + \varphi(x,y,z)] \right\} \quad (4)$$

$$W = A \exp\left\{i \omega \varphi(x,y,z) \right\},$$

with $\text{sign}(\omega \partial \Phi / \partial z) = 1$ or $\text{sign}(\partial \varphi / \partial z) = 1$. Both alternatives for representing upgoing waves are used in this paper and in the literature.

For constant wavespeed, we can rewrite this equation in the frequency-wave-vector domain in the form

$$\mathcal{L} W = \frac{\partial^2 W}{\partial z^2} + k^2 W \quad (5)$$

$$= \left[ \frac{\partial}{\partial z} \pm i k_z \right] \left[ \frac{\partial}{\partial z} \pm i k_z \right] W = 0. \quad (6)$$

Here,

$$k_z = \text{sign}(\omega) \sqrt{\frac{\omega^2}{v^2} - \bar{k}^2} = \frac{\omega}{v} \sqrt{1 - \left(\frac{v \bar{k}}{\omega} \right)^2}, \quad (7)$$

and $\bar{k}$ is the transverse wave vector,

$$\bar{k} = (k_x, k_y), \quad \bar{k}^2 = k_x^2 + k_y^2. \quad (8)$$

Further, the solutions of the full second order wave equation are actually solutions of the two one-way wave equations

$$\begin{cases} \frac{\partial}{\partial z} \pm i k_z \end{cases} A \exp\{\mp i k_z z\} = 0, \quad (9)$$

with the upper signs yield a downgoing solution and the lower signs yield an upgoing solution.

We would like one-way wave equations for the heterogeneous case, as well, similarly separating upward and downward propagating waves. For this generalization, we will content ourselves with ray-theoretic solutions that yield the same leading order amplitude as does the two-way wave equation, using the leftmost expression in (4).

First consider the case where $v = v(z)$ only and recast the leftmost differential operator in (4) in a form that lends itself to ray theory analysis. To this end, we introduce the slowness vector $\bar{p}$ by setting

$$\bar{p} = \frac{\bar{k}}{\omega}, \quad p_z = \frac{k_z}{\omega} = \frac{1}{v(z)} \sqrt{1 - (v(z)\bar{p})^2} \quad (10)$$

and rewrite the wave equation in (4) as

$$\frac{\partial^2 W}{\partial z^2} + \omega^2 p_z^2 W = 0. \quad (11)$$

When we substitute the standard form
\[ W = A(z, \rho) e^{-i \omega \varphi(z, \rho)} \]  
\hspace{1cm} (10)

into (9), we find
\begin{align*}
\left\{ -\omega^2 \left[ \frac{d^2 \varphi}{dz^2} \right] - p_z^2 \right\} A \\
- i \omega \left[ \frac{d^2 A}{dz^2} + \frac{d^2 \varphi}{dz^2} A \right] + O(1) \right\} e^{-i \omega \varphi} = 0.
\end{align*}
\hspace{1cm} (11)

Here, \( O(1) \) is in order of powers of \( \omega \).

This last equation leads to the familiar eikonal and transport equations for the rays, except that we are in a Fourier transform domain in transverse slowness vector, \((p_x, p_y)\). Those equations are
\begin{align*}
\left[ \frac{d^2 \varphi}{dz^2} \right] = p_z^2 \implies \frac{d \varphi}{dz} = \pm p_z,
\end{align*}
and
\[ \frac{d^2 A}{dz^2} + \frac{d^2 \varphi}{dz^2} A = 0 \]
or
\[ \pm \left[ \frac{2p_z}{A} - \frac{1}{v^3(z) p_z} \frac{dv(z)}{dz} A \right] = 0. \tag{12} \]
or
\[ \frac{d A}{dz} - \frac{1}{2 v^3(z) p_z^2} \frac{dv(z)}{dz} A = 0. \]

Here, we think of the upper sign solution as the one for which \( \text{sign}(p_z) = 1 \). In this case, the upper sign corresponds to the downgoing wave and the lower sign corresponds to the upgoing wave. Note that the transport equation is the same for the two waves because only \( p_z^2 \) appears in that equation.

Now consider the two one-way wave equations in (7) and corresponding WKBJ or ray-theoretic solutions. That is, consider
\begin{align*}
\left[ \frac{\partial}{\partial z} \pm i \omega p_z \right] A_{\pm} e^{-i \omega \varphi_{\pm}} \\
= \pm i \omega \left[ - \frac{d \varphi_{\pm}}{dz} \pm p_z \right] A_{\pm} e^{-i \omega \varphi_{\pm}} \\
+ \frac{d A_{\pm}}{dz} e^{-i \omega \varphi_{\pm}} = 0,
\end{align*}
\hspace{1cm} (13)

with consequent eikonal and transport equations,
\[ \frac{d \varphi_{\pm}}{dz} = \pm p_z \quad \text{and} \quad \frac{d A_{\pm}}{dz} = 0. \tag{14} \]

While the eikonal equations in (12) and (14) agree, the transport equations for amplitudes do not. Thus, we must consider modifying the one-way wave equations in (13) if we are to make the latter transport equation agree with the former, while keeping the same eikonal equation in both. We can achieve this goal for the one-way wave equations if we modify them by adding a term to the one-way operators appearing in the leftmost expression in (13). The key to doing this comes from examining the last form of the transport equation in (12). That is, we consider the new one-way wave equations
\begin{align*}
\left[ \frac{\partial}{\partial z} \mp i \omega p_z - \frac{1}{2 v^3(z) p_z^2} \frac{dv(z)}{dz} \right] W = 0.
\end{align*}
\hspace{1cm} (15)

For these equations
\begin{align*}
\left[ \frac{\partial}{\partial z} \pm i \omega p_z - \frac{1}{2 v^3(z) p_z^2} \frac{dv(z)}{dz} \right] A_{\pm} e^{-i \omega \varphi_{\pm}} =
\end{align*}
\begin{align*}
i \omega \left[ - \frac{d \varphi_{\pm}}{dz} \pm p_z \right] A_{\pm} e^{-i \omega \varphi_{\pm}} \\
+ \left[ \frac{d A_{\pm}}{dz} - \frac{1}{2 v^3(z) p_z^2} \frac{dv(z)}{dz} \right] e^{-i \omega \varphi_{\pm}} = 0,
\end{align*}
for which the transport equation in (14) is replaced by
\[ \frac{d A_{\pm}}{dz} - \frac{1}{2 v^3(z) p_z^2} \frac{dv(z)}{dz} A_{\pm} = 0 \tag{16} \]
while the eikonal equation remains unchanged. We have already seen that this is equivalent to the transport equation for the full wave equation. Hence, the one-way wave operators in (15) will produce the upgoing and downgoing traveltimes and leading order amplitudes of the full wave equation. In fact, these solutions are exact for the case of a medium that depends only on \( z \); that is, when \( v = v(z) \).

The question now arises as to how this insight can be extended to factoring the original wave operator in (1) of a fully heterogeneous medium, where \( v = v(x, y, z) \). In this case, we could not have derived differential equations in \( z \) alone through Fourier transform. If we disregard this obstacle, we could still examine the one-way wave equations in (15) with the hope of achieving a sensible interpretation. To that end, let us first recast the one-way wave equations (15) in the original Fourier variables. Thus, we first rewrite that equation as
\begin{align*}
\left[ \frac{\partial}{\partial z} \mp i k_z - \frac{\omega^2}{2 v^3(x, y, z) k_z^2} \frac{\partial v(x, y, z)}{\partial z} \right] \right\} W = 0. \tag{17}
\end{align*}

For reasons that will become clear in the next section, we prefer writing the multiplier on this additional term as follows.
\[ \frac{\omega^2}{2 v^3(x, y, z) k_z^2} \frac{\partial v(x, y, z)}{\partial z} - \frac{1}{2 v(x, y, z) k_z} \frac{\partial v(x, y, z)}{\partial z} \]
and then rewrite (17) as
\begin{align*}
\left[ \frac{\partial}{\partial z} \mp i k_z - \frac{1}{2 v(x, y, z) k_z} \frac{\partial v(x, y, z)}{\partial z} \right] \right\} \left[ 1 + \frac{(v(x, y, z) k_z)^2}{\omega^2 - (v(x, y, z) k_z)^2} \right] W = 0. \tag{18}
\end{align*}

Let us now think of \( \omega \) as a place-holder for the temporal derivative and \( -k = -(k_x, k_y) \) as a place-holder for a transverse gradient operator; that is,
\[ i\omega \Leftrightarrow \partial/\partial t \text{ and } i(k_x,k_y) \Leftrightarrow -(\partial/\partial x, \partial/\partial y). \]

Then we could easily give meaning to the expression \((v(x,y,z)\vec{k})^2\) as follows.

\[ (v(x,y,z)\vec{k})^2 \Leftrightarrow v(\partial/\partial x, \partial/\partial y) \cdot (v(\partial/\partial x, \partial/\partial y)). \]

However, symbolically, \(k_z\) involves taking the square-root of a differential operator, while the division in the last term requires that we give meaning to the reciprocal of a differential operator. Interpretation of such expressions is what the theory of pseudo-differential operators is all about. Thus, in the next section we address the interpretation of these terms in fully heterogeneous media where \(v = v(x,y,z)\). With an appropriate interpretation, it turns out that these modified one-way wave equations provide an asymptotic solution for fully heterogeneous media—\(v = v(x,y,z)\)—for which the resulting transport equations agree with the transport equation for the full two-way wave equation.

### 3 True amplitude wave equation migration

Motivated by the discussion of the previous section, we introduce the same one-way equations (18) for the heterogeneous medium in which \(v = v(x,y,z)\). Here, we will extend the definition of differentiation through the use of pseudo-differential operators so that we can give meaning to the pseudo-differential operator \(k_z\) in (17) and thereby give meaning to those one-way equations themselves.

Couched in the language of pseudo-differential operator theory, those one-way wave equations are

\[ \mathcal{L}_\pm W = \left[ \frac{\partial}{\partial z} \pm \Lambda \right] W - \Gamma W = 0 \quad (19) \]

Here, \(\Lambda\) and \(\Gamma\) are pseudo-differential operators with symbols \(\lambda\) and \(\gamma\), respectively:

\[ \lambda = i k_z = \frac{i\omega}{v} \sqrt{1 - \frac{(v\vec{k})^2}{\omega^2}}, \]

\[ \gamma = -\frac{1}{2k_z} \frac{\partial k_z}{\partial z} = \frac{1}{2v} \frac{\partial v}{\partial z} \left( 1 + \frac{(v\vec{k})^2}{\omega^2 - (v\vec{k})^2} \right) \]

\[ = \frac{v_z}{2v} \left( 1 + \frac{(v\vec{k})^2}{\omega^2 - (v\vec{k})^2} \right), \quad \vec{k} \equiv (k_x,k_y). \]

In this last expression, the subscripted variable \(v_z\) connotes derivative with respect to \(z\), in contrast to its use in \(k_x,k_y,k_z\) where it distinguishes the three components of the wave vector.

The symbol notation is somewhat inadequate here because it is really designed for leading order accuracy—propagation of discontinuities—only. The notation \((v\vec{k})^2\) is the symbol for the operator

\[ (v\nabla_{Tz})^2 = (v(\vec{\rho},z)\nabla_{Tz})^2 \]

\[ = \left( v(\vec{\rho},z) \frac{\partial}{\partial x} \right)^2 + \left( v(\vec{\rho},z) \frac{\partial}{\partial y} \right)^2 \]

\[ = v^2(\vec{\rho},z) \left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right] + v(\vec{\rho},z) \left[ \frac{\partial v(\vec{\rho},z)}{\partial x} \frac{\partial}{\partial x} + \frac{\partial v(\vec{\rho},z)}{\partial y} \frac{\partial}{\partial y} \right] \]

\[ + v(\vec{\rho},z) \left[ \frac{\partial^2 v(\vec{\rho},z)}{\partial x^2} + \frac{\partial^2 v(\vec{\rho},z)}{\partial y^2} \right]. \]

Thus, the subtleties of lower order corrections to this term will only affect lower order amplitude corrections that are not of interest to us here; we are only concerned with getting the leading order amplitude right.

Returning to the discussion of \(\Lambda\), we must distinguish between \((v\vec{k})^2\) and \((kv)^2\) for the operator \((\nabla_{Tz}v)^2\) defined by

\[ (\nabla_{Tz}v)^2 = \{ (v\nabla_{Tz})^2 \}^*, \quad (22) \]

given in more detail by

\[ (\nabla_{Tz}v)^2 = (v\nabla_{Tz}(v(\vec{\rho},z))^2 \]

\[ = \left( \frac{\partial}{\partial x} v(\vec{\rho},z) \right)^2 + \left( \frac{\partial}{\partial y} v(\vec{\rho},z) \right)^2 \]

\[ = v^2(\vec{\rho},z) \left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right] + 3v(\vec{\rho},z) \left[ \frac{\partial v(\vec{\rho},z)}{\partial x} \frac{\partial}{\partial x} + \frac{\partial v(\vec{\rho},z)}{\partial y} \frac{\partial}{\partial y} \right] + v(\vec{\rho},z) \left[ \frac{\partial^2 v(\vec{\rho},z)}{\partial x^2} + \frac{\partial^2 v(\vec{\rho},z)}{\partial y^2} \right] \]

\[ + v(\vec{\rho},z) \left[ \frac{\partial^2 v(\vec{\rho},z)}{\partial x^2} + \frac{\partial^2 v(\vec{\rho},z)}{\partial y^2} \right]. \]

However, in the current application, in the definition of \(\Lambda\), we need this operator to be valid to two orders in \(\omega\), rather than the usual leading order. The reason is that this operator is \(O(\omega)\). Thus, as in the \(v(z)\) case, it affects the eikonal equation to leading order, however, its correction term, \(O(1)\) in \(\omega\) is of the same order as the other contributions to the transport equation. On the other hand, \(\Gamma\) is already \(O(1)\) in \(\omega\) at leading order.

Thus, in the current application, we must distinguish between \((v\vec{k})^2\) and \((kv)^2\) for the operator \((\nabla_{Tz}v)^2\) defined by

\[ (\nabla_{Tz}v)^2 = \{ (v\nabla_{Tz})^2 \}^*, \quad (22) \]

given in more detail by

\[ (\nabla_{Tz}v)^2 = (v\nabla_{Tz}(v(\vec{\rho},z))^2 \]

\[ = \left( \frac{\partial}{\partial x} v(\vec{\rho},z) \right)^2 + \left( \frac{\partial}{\partial y} v(\vec{\rho},z) \right)^2 \]

\[ = v^2(\vec{\rho},z) \left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right] + 3v(\vec{\rho},z) \left[ \frac{\partial v(\vec{\rho},z)}{\partial x} \frac{\partial}{\partial x} + \frac{\partial v(\vec{\rho},z)}{\partial y} \frac{\partial}{\partial y} \right] \]

\[ + v(\vec{\rho},z) \left[ \frac{\partial^2 v(\vec{\rho},z)}{\partial x^2} + \frac{\partial^2 v(\vec{\rho},z)}{\partial y^2} \right] \]

\[ + v(\vec{\rho},z) \left[ \frac{\partial^2 v(\vec{\rho},z)}{\partial x^2} + \frac{\partial^2 v(\vec{\rho},z)}{\partial y^2} \right]. \]

Thus, we need to define the symbols associated with these operators more carefully by

\[ (v\vec{k})^2 \equiv v^2 \cdot \vec{k}^2 + v\{kv\} \cdot \vec{k}, \]

\[ (kv)^2 \equiv v^2 \cdot \vec{k}^2 + 3v\{kv\} \cdot \vec{k} + v\{k^2v\} + \{kv\} \cdot \{kv\}. \]

In these equations, we use the curly brackets \(\{\}\) to indicate that the symbol only operates on the function con-
tained within them. We think of the operator $\tilde{k}$ applied to $W$ as contain terms that are $O(\omega)$, or $O(1)k$ while $k^2$ applied to $W$ as containing terms that are $O(\omega^2)$, $O(\omega)$, $O(1)$. We need to keep terms of the two leading orders in these operators when dealing with $\Lambda$. This issue will arise in the proof contained in Appendix A.

The symbol $\lambda$ has an exact representation [Zhang, 1993] in terms of a rational function of the argument inside the square-root, that is,

$$\lambda = \frac{i\omega}{v} \left\{ 1 - \frac{1}{\pi} \int_{-1}^{1} \sqrt{1 - s^2} \frac{(v\tilde{k})^2}{\omega^2 - s^2 (v\tilde{k})^2} ds \right\}. \quad (24)$$

A proof of this identity using contour integration in the complex plane is provided in Appendix B.

It is fairly easy to think of the symbol, $\omega^2 - (v\tilde{k})^2$, as representing the two-way wave operator. That is,

$$\omega^2 - s^2 (v\tilde{k})^2 \iff \Lambda_T(s; \tilde{\rho}, z, t) = \frac{\partial^2}{\partial t^2} - s^2 (v\nabla T_x)^2, \quad (25)$$

in which case, this operator appearing in the denominator of the symbols would represent an inverse differential operator or convolution with a Green's function for the adjoint of this operator.

Further, if we neglect the transverse dependence in $\Lambda$, neglect the amplitude corrections in $\Gamma$ and revert to constant velocity for a moment, then the identity $(24)$ used in the one-way wave equations in $(19)$ leads to the equations

$$\left[ \frac{\partial}{\partial z} \pm \frac{1}{v} \frac{\partial}{\partial t} + \cdots \right] W = 0,$$

with solutions

$$W = F(z \mp vt + \cdots).$$

That is, the choices of signs that we have made in the symbolic operators assure the separation into downgoing and upgoing waves that we intended.

Symbolically then, we think of the operators $\Lambda$ and $\Gamma$ as follows.

$$\Lambda = \frac{1}{v} \frac{\partial}{\partial t} \left\{ I - \frac{1}{\pi} \int_{-1}^{1} \sqrt{1 - s^2} \right\}
\cdot \Lambda_T^{-1}(s; \tilde{\rho}, z, t)(v\nabla T_x)^2 W(\tilde{\rho}, z, t) ds \right\}, \quad (26)$$

$$\Gamma = \frac{v_z}{2v} \left( I + \Lambda_T^{-1}(1; \tilde{\rho}, z, t)(v\nabla T_x)^2 \right).$$

Now we can see that the representation $(18)$ leads to the same inverse of the operator $\Lambda_T$, evaluated at general $s$ or $s = 1$ in the two pseudo-differential operator expressions used in our one-way wave equations.

Let us introduce a function $q(s; \tilde{\rho}, z, t)$ that satisfies the equation

$$L_T(s; \tilde{\rho}, z, t)q = \left\{ \frac{\partial^2}{\partial t^2} - s^2 (v\nabla T_x)^2 \right\} q(s; \tilde{\rho}, z, t) \quad (27)$$

$$= (v\nabla T_x)^2 W(\tilde{\rho}, z) \quad z > 0, \ t > 0.$$  

Using this function plus the identity $(24)$ for $k_z$, $\Lambda W$ and $\Gamma W$ can be expressed as

$$\Lambda W = \frac{1}{v} \frac{\partial W}{\partial t} - \frac{1}{\pi v} \frac{\partial}{\partial t} \int_{-1}^{1} \sqrt{1 - s^2} q(s; \tilde{\rho}, z, t) ds, \quad (28)$$

$$\Gamma W = \frac{v_z}{2v} [W + q(1; \tilde{\rho}, z, t)].$$

Here, the arguments of $W$ are $(\tilde{\rho}, z, t)$. Furthermore, $(19)$ can be rewritten in the expanded form

$$L_{\pm} W = \frac{\partial W}{\partial z} \mp \frac{1}{v} \frac{\partial}{\partial t} \int_{-1}^{1} \sqrt{1 - s^2} q(s; \tilde{\rho}, z, t) ds \quad (29)$$

$$+ \frac{v_z}{2v} [W + q(1; \tilde{\rho}, z, t)] = 0.$$  

We assume that boundary data

$$W(\tilde{\rho}, 0, t) = W_0(\tilde{\rho}, t), \quad (30)$$

are given. Further, for the downgoing wave, we provide the initial condition $W(\tilde{\rho}, z, 0) = 0$, while for the upgoing wave, we provide a final condition of zero. That is, $W(\tilde{\rho}, z, t) = 0$ for $t$ greater than some finite time, $t > T$. For the inverse problem, we need both one-way operators. The source propagates downward, subject to the wave equation $(19)$ with upper signs and a boundary value at $z = 0$ that is equivalent to an impulsive source. On the other hand, the observed data is governed by the upward propagating one-way wave equation, lower signs in $(19)$ with the given data being the appropriate observed data at $z = 0$. That is the subject of the next section. Here, we have only introduced the governing equations for propagation.

It is for this expanded form of the one-way wave equations that second author has shown in Zhang [1993] that the transport equations for the one-way wave equations agree with the transport equation for the full wave equation. of course, this is in addition to the agreement of the eikonal equations, except for the separation into downgoing and upgoing provided explicitly by the one-way wave equations. A modified version of that proof is provided in Appendix A.

4 TRUE AMPLITUDE WAVE EQUATION MIGRATION

In this section, we describe the application of these true amplitude one-way wave equations to WEM. The objective is to derive a true-amplitude WEM. We begin by introducing Claerbout's [1971,1985] classic WEM and
explain how we modify the governing equations and boundary data to obtain our proposed true amplitude WEM.

The standard method uses the one-way propagators of (4), even for heterogeneous media. More specifically, suppose that the reflected wave field from a single source experiment is observed at \( z = 0 \) for all time. Then the source and observed wavefields are assumed to be solutions of the equations

\[
\begin{aligned}
\left( \frac{\partial}{\partial z} + \Lambda \right) D &= 0, \\
D(x, y, z = 0; \omega) &= -\delta(\vec{x} - \vec{x}_s), \tag{31}
\end{aligned}
\]

where \( D \) is the downgoing (source) wavefield and \( U \) is the upgoing (observed) wavefield. The image is then produced as an impedance or reflectivity function at every image point defined by

\[
R(\vec{x}, y, z) = \int \frac{U(\vec{x}; \omega)}{D(\vec{x}; \omega)} d\omega. \tag{33}
\]

The key to this imaging method is that the constructive/destructive interference between the phases of the two waves produces a large amplitude where the reflectors reside and a small amplitude where they do not. While this result produces a reflector map, it does not provide accurate amplitude information. To achieve that, we use the solutions of our modified true amplitude one-way wave equations (19). That is, we introduce \( p_D \) and \( p_U \) as solutions of the following problems.

\[
\begin{aligned}
\left( \frac{\partial}{\partial z} + \Lambda - \Gamma \right) p_D(\vec{x}; \omega) &= 0, \\
p_D(\vec{x}_s; \omega) &= -\frac{1}{2} \Lambda^{-1} \delta(\vec{x} - \vec{x}_s), \tag{34}
\end{aligned}
\]

and

\[
\begin{aligned}
\left( \frac{\partial}{\partial z} - \Lambda + \Gamma \right) p_U(\vec{x}; \omega) &= 0, \\
p_U(\vec{x}_s; \omega) &= Q(x, y; \omega). \tag{35}
\end{aligned}
\]

We have not only modified the governing one-way wave equations in accordance with the discussion of the previous sections, but we have also modified the boundary term for the downgoing wave from the source. The reason for this is that the boundary value only accounts for the impulsive nature of the source in the transverse direction. In the \( z \)-direction, we must account for an impulsive source by balancing the terms,

\[
\frac{\partial^2 u}{\partial z^2} \quad \text{and} \quad -\delta(z)
\]

\[
\text{or} \quad \frac{\partial u}{\partial z} \quad \text{and} \quad -H(z)
\]

\[
\text{or} \quad Au = \text{1}.
\]

(In the second balance, \( H(z) \) is the Heaviside function.)

We think of the impulsive source as sending half its energy in each direction in \( z \). Hence, in the positive \( z \)-direction we use half of the balance in this last expression as boundary value for the downgoing wave (35). Note that this modification introduces a phase shift in this wave since \( \Lambda \) is imaginary and carries the same sign as \( \omega \).

Also, we modify the imaging condition (33) to be the quotient of the wave fields \( p_D \) and \( p_U \):

\[
R(\vec{x}) = \int \frac{p_U(\vec{x}; \omega)}{p_D(\vec{x}; \omega)} d\omega. \tag{36}
\]

See Zhang et al. [2001, 2002].

It is important to recognize that the boundary data for \( p_D \) in (34) involves a pseudo-differential operator. Thus, it might be easier to solve a modified version of (31) for \( D \), after adjusting that equation to be equivalent to the true amplitude equation (34) for \( p_D \).

Let us, therefore, introduce a new \( D \) in (34),

\[
D = 2\Lambda p_D, \quad \Leftrightarrow \quad p_D = \frac{1}{2} \Lambda^{-1} D, \tag{37}
\]

for which the boundary data agrees with the data for \( D \) in (31). Now, let us substitute this choice into the differential equation (34) for \( p_D \):

\[
\left( \frac{\partial}{\partial z} + \Lambda - \Gamma \right) \Lambda^{-1} D = \Lambda^{-1} \frac{\partial D}{\partial z} - \Lambda^{-2} \frac{\partial \Lambda}{\partial z} D + D - \Gamma \Lambda^{-1} D = 0, \tag{38}
\]

We have omitted the constant factor of 2 throughout these manipulations. For the last two terms, the leading order contributions contribute to the leading order amplitude and the corrections affect only lower order amplitudes. Therefore we can set

\[
\Lambda \Gamma \Lambda^{-1} \approx \Gamma
\]

\[
\Lambda^{-1} \frac{\partial \Lambda}{\partial z} + \Gamma \Leftrightarrow \frac{1}{k_s} \frac{\partial k_s}{\partial z} - \frac{1}{2k_s} \frac{\partial k_s}{\partial z} = -\gamma.
\]

Consequently, the true amplitude equation for \( D \) is

\[
\left( \frac{\partial}{\partial z} + \Lambda + \Gamma \right) D = 0. \tag{39}
\]

Within the factor of 2, this use of \( D \) agrees with the earlier papers cited in the references. Thus, we can avoid apply a pseudo-differential operator to the two-dimensional Dirac delta function in the boundary condition for \( p_D \) in (34) by solving for \( D \), instead, using
this equation and the boundary condition in (31). Of course, we could tie $U$ and $p_U$ together in exactly the same way. That is,

$$U = 2\Lambda p_U,$$

leading to the differential equation

$$\left( \frac{\partial}{\partial z} - \Lambda + \Gamma \right) U = 0.$$  

However, now the pseudo-differential operator would appear in the boundary data for $U$; that is,

$$U(x', \omega) = 2\Lambda Q(x, y; \omega).$$

Furthermore, the correct imaging condition now becomes

$$R(x) = \int \frac{p_U}{p_D} d\omega = \int \frac{\Lambda^{-1} U}{\Lambda^{-1} D} d\omega.$$  

5 COMPARISON OF TRUE AMPLITUDE WEM OUTPUT AND KIRCHHOFF INVERSION OUTPUT

The previous section proposed the use of new one-way propagators for the surface data, a modification of the data for the downgoing wave and a new imaging condition in equation (36) to achieve true amplitude WEM. Here we show that the integration in this imaging condition agrees with the integration for Kirchhoff inversion in Bleistein [1987] and Bleistein et al. [2001]. Actually, we derive the representation of the Kirchhoff inversion formula as stated in Hanitzsch [1997]. In carrying out this comparison, we rely on the proof of Appendix A of the dynamic as well as the kinematic equivalence of the solutions of the one-way wave equations and the solutions of the two-way wave equation.

We start by considering the problem for $p_D$ as defined by (34). Relying on the proof of Appendix A, we know that this function is dynamically equivalent to the downgoing (outward radiating) Green’s function of the full wave equation. Therefore,

$$p_D(x, x; \omega) = A(x, x)e^{-i\omega \varphi(x, x')}.$$  

(40)

Here, $\varphi$ is the solution of the full eikonal equation for the full wave equation (1) with $\partial \varphi / \partial z > 0$ and $A$ is the solution of the transport equation for the full wave equation. Equivalently, $\varphi$ is a solution of the eikonal equation

$$\frac{d\varphi_\pm}{dz} = \sqrt{\frac{1}{v^2(x)} - p^2 - p^2_\pm},$$

deduced from (18) with upper sign, and $A$ is a solution of the transport equation

$$2\nabla \varphi_\pm \cdot \nabla A_\pm + A_\pm \Delta \varphi_\pm = 0,$$

deduced from (18) with upper sign. This is the essential conclusion of the proof of Appendix A.

To derive a representation of the function $p_U$, we have to work a little harder. Again, however, we rely on the equality between the leading order asymptotic solutions of the one-way wave equation, (35), and the full wave equation. In Appendix C, we show that the Green’s function representation for $p_U$ is given by

$$p_U(x; \omega) = 2i\omega \int \frac{\cos \alpha_r}{v(x_r)} A(x_r, x)e^{i\omega \varphi(x_r, x)} dx_r dy_r,$$  

(41)

$x_r = (x_r, y_r, 0)$.

In this equation, again, $\varphi$ and $A$ are the phase and amplitude of the free space Green’s function for the full wave equation. However, because $p_U$ is an upward (incoming) wave, we need the inward propagating Greens function. Hence the sign in the phase is opposite the sign of the phase of $p_D$ defined by (40). Further, $v_r = v(x_r)$ and $\alpha_r$ is the emergence angle of the ray with respect to the normal.

We use this last result and (40) in equation (36) for $R$ and obtain

$$R(x) = 2 \int i\omega \frac{\cos \alpha_r}{v_r} A(x_r, x) A(x, x_a) e^{i\omega \varphi(x_r, x) + \varphi(x, x_a)} dx_r dy_r d\omega.$$  

(42)

This is the result in Bleistein[1987] and Bleistein et al. [2001] as expressed by Hanitzch [1997].

This representation of the imaging condition for true amplitude WEM, being the same as the inversion formula for Kirchhoff inversion, confirms the claim of this paper, namely, that this formulation provides a true amplitude WEM. We can now be assured that the output of this new WEM will provide a reflectivity map with peak amplitude on the reflector being a known multiple of the geometrical optics reflection coefficient. The incidence angle in that reflection coefficient
is the angle defined by the specular pair of rays from a source/receiver pair. Note that this result is confirmed for heterogeneous media — \( v = v(\mathbf{z}) \). Previous verifications, as presented for example in Zhang et al. [2001, 2002], were only for the case of depth dependent media — \( v = v(z) \).

6 NUMERICAL TEST

To show how true amplitude common-shot migration works, we apply it to a 2-D horizontal reflector model in a medium with velocity \( v = 2000 + 0.3z \). Recall from the theory that in this case, the modeling and migration can be carried out in the transverse spatial and temporal Fourier domains, with \( \mathbf{k} \) being the simple transverse part of the wave vector.

The input data (Figure 1) is a single shot record over four horizontal reflectors from density contrast.

Figure 2 left shows the migrated shot record using the conventional common-shot migration algorithm (33). The peak amplitudes along the four migrated reflectors are shown in Figure 2 right, normalized to the geometrical optics reflection coefficient along the reflector. This method has a phase error: note the multiplication by \( i \) in \( \Lambda \) on the right side in (34) as opposed to the lack of such a phase shifting factor on the right side of (31). The consequent phase error has been corrected during the migration. However, the migrated amplitudes are poor, especially on the reflector at depth \( z = 1000m \) along which the reflection angles vary over a wide range. (This method has incorrect angular dependence when compared to true amplitude reflectivity or the geometrical optics reflection coefficient at each point.) The wide angle peak amplitudes decrease monotonically with increasing depth. The greatest error occurs at wide angle, with the result along the shallowest reflector being the worst. However, the error is zero at zero offset; in this limit, \( \hat{\mathbf{k}} = (0, 0) \) and \( \cos \alpha_r = 1 \).

Figure 3 left shows results of true amplitude common-shot migration (36). The peak amplitudes along the reflectors are shown in Figure 3 right.

From this plot, we clearly see that the true amplitude algorithm recovers the reflectivity accurately, aside from the edge effects and small jitters caused by interference with wraparound artifacts.

7 CONCLUSIONS

Common-shot migrations offer good potential of imaging complex structures, but the conventional formulations of such migrations produce incorrect migrated amplitudes. Here, we have described true-amplitude one-way wave equations that allow us to extend the standard method both for forward modeling and for wave equation migration. These modified one-way wave operators are developed with the aid of pseudo-differential operator theory. We have provided proofs that these new one-way wave equations provide solutions that agree dynamically, as well as kinematically, with the solutions of the full wave equation. Further, we have proposed a new approach to WEM, transforming it into a true amplitude process, meaning that it produces an inversion output that agrees asymptotically to Kirchhoff inversion: it produces a reflector map with peak amplitudes on the reflector in known proportion to the geometrical optics reflection coefficient. We have provided a proof of this claim. With the aid of a simple numerical example, we demonstrated that the migration method we proposed does calibrate common-shot migrations by correcting both their amplitude and phase behavior for an example in which the wave speed is depth-dependent — \( v = v(z) \). The new method actually builds a bridge between true amplitude common-shot Kirchhoff migration and the migrations based on one-way wavefield extrapolation.

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References


Figure 2. Left: finite difference migration using (33) for imaging. Right: peak amplitudes along the four reflectors. The wide angle error decreases with depth of the reflector.

Figure 3. Left: finite difference migration using (36) for imaging. Right: peak amplitudes along the four reflectors. The wide angle error decreases with depth of the reflector.

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Appendix A

In this appendix, we show that the one-way wave equations that we developed in this paper provided the same traveltime and amplitudes as does the full wave equation (1). More specifically, we will consider the one-way wave equation for the downward propagating wave defined by (19) in two spatial dimensions (to make some of the calculations simpler to follow). That equation is repeated here as

$$\left[ \frac{\partial}{\partial z} + \Lambda \right] D(x, y, z; t) + \Gamma D = 0,$$  \hspace{1cm} (A-1)

Motivated by the source adjustment introduced in (34), we introduce the downgoing wave of the inversion process through the scaling

$$p_D = \Lambda^{-1} D \quad \Leftrightarrow \quad \Lambda p_D = D.$$  \hspace{1cm} (A-2)
Our objective is to prove that the eikonal equation of (A-1) is the downgoing branch of the eikonal equation for the full wave equation (1) in heterogeneous media. This equations will be specified below after we introduce our notation for this asymptotic analysis. Clearly, we could define a corresponding function $p_D$ and carry out the same analysis for the upgoing wave. Below, we will see that the eikonal equation for the downgoing wave chooses the sign of the $z$ derivative of the traveltime to agree with the sign of $\omega$, guaranteeing downward propagation. For $p_D$, the sign of the $z$-derivative is opposite to the sign of $\omega$ and that is the only different in the analysis provided here.

We will then show that the leading order amplitude of $p_D$ satisfies the same transport equation as does the amplitude of the full wave equation. Further, the sign of the $z$-derivative will be a common multiplier of all terms of the derived transport equation, assuring that the same will be true for the solution $p_U$.

We start from equation (29), specialized to downgoing waves in two spatial dimensions:

$$
\frac{\partial D}{\partial z} + \frac{1}{v} \frac{\partial D}{\partial t} - \frac{1}{\pi v} \int_{-1}^{1} \sqrt{1 - s^2 q_D(s; x, z, t)} ds = 0.
$$

(A-3)

Here, $D = D(x, z, t)$ and $q_D(s; \cdot)$ satisfies

$$
\left\{ \frac{\partial^2}{\partial t^2} - s^2 \left( \frac{\partial}{\partial x} \right)^2 \right\} q_D(s; x, y, z; t) = \left( \frac{\partial}{\partial x} \right)^2 D(x, y, z; t).
$$

(A-4)

We use the definition of $p_D$ in (A-2) and the definition (28) for $A$ to write

$$
\frac{1}{v} \frac{\partial}{\partial t} p_D - \frac{1}{v} \frac{\partial}{\partial t} \left( \frac{1}{\pi} \int_{-1}^{1} \sqrt{1 - s^2 q_D(s; x, z, t)} ds \right) = D(x, z; t),
$$

(A-5)

where $q_R(s; \cdot)$ satisfies

$$
\left\{ \frac{\partial^2}{\partial t^2} - s^2 \left( \frac{\partial}{\partial x} \right)^2 \right\} q_R(s; x, z; t) = \left( \frac{\partial}{\partial x} \right)^2 p_R(x, y, z; t).
$$

(A-6)

**A-1 High frequency asymptotic expansion**

We consider only the downgoing wave equation (A-1). For the upgoing wave equation, all results can be obtained by the same approach.

We seek the solution of (A-1) in the form of following asymptotic expansion

$$
F(x, y, z; t) \sim e^{i \omega (t - \varphi(x, z))} \sum_{j=0} A^j(x, z) \omega^{-j}
$$

(A-7)

$A^0$ is simply written as $A(F)$ ($F$ can be either $D$ or $q_D$). We simply denote $p_D$ by $p$, to be determined in the form

$$
p(x, z; t) \sim e^{i \omega (t - \varphi(x, z))} \sum_{j=0} A^{j+1} \omega^{-(j+1)}.
$$

(A-8)

By substituting these asymptotic expansions into (A-4), we find that

$$
A(q(D))(s; \cdot) = \frac{v^2 \varphi_s^2}{1 - s^2 v^2 \varphi_s^2} A(D) = e(s) A(D),
$$

(A-9)

with

$$
e(s) = \frac{v^2 \varphi_s^2}{1 - s^2 v^2 \varphi_s^2}.
$$

(A-10)

Now we substitute the asymptotic expansions into (A-5) and (A-6) to obtain

$$
A(D) = E(D) A(p),
$$

(A-11)

with

$$
E(D) = \frac{1}{v} \left[ 1 - v^2 \varphi_s^2 \right]^{1/2}.
$$

(A-12)

For $e(s)$ in (A-10), we will also have need of $\partial e(s)/\partial x$. That result is

$$
e(s)_x = \frac{\partial}{\partial x} \left( \frac{v^2 \varphi_s^2}{1 - s^2 v^2 \varphi_s^2} \right)
$$

$$
= -\frac{1}{s^2} \frac{\partial}{\partial x} \left( 1 - \frac{1}{1 - s^2 v^2 \varphi_s^2} \right)
$$

$$
= \frac{1}{s^2} \left[ \frac{s^2 (v^2 \varphi_s^2)_x}{(1 - s^2 v^2 \varphi_s^2)^2} \right]
$$

$$
= \frac{(v^2 \varphi_s^2)_x}{(1 - s^2 v^2 \varphi_s^2)^2}.
$$

(A-13)

**A-2 Asymptotic solution of downgoing one-way wave equation**

By substituting the equation (A-8) into equations (A-3) and (A-4), we have

$$
\frac{i \omega}{v \pi} \int_{-1}^{1} \sqrt{1 - s^2 A(q_D(s; \cdot))} ds
$$

$$
- \frac{1}{v} \left[ \frac{\varphi_s - 1}{v} A(D) \right]
$$

$$
- \frac{1}{v} \left[ \frac{\varphi_s}{2v} (A(D) + A(q(1; \cdot))) \right]
$$

$$
+ \frac{1}{i \omega} \left[ \cdot \cdot \cdot \right] = 0,
$$

and
The following integrals are needed

\[ B(A) = 2v^2 \varphi_x A_x + A \left( \frac{\partial}{\partial x} \right)^2 \varphi. \]  

We will replace the integral operator on \( A(q_d(s; \cdot)) \) in (A-14) by the same type of operator on \( A(D) \), itself, since the latter is independent of \( s \). To do so, we integrate (A-15) on the interval \( s \in (-1, 1) \) with the weight \( \sqrt{1-s^2/(v \pi (1-s^2 v^2 \varphi_x^2))} \). Then, we add the result of that integration to (A-14) multiplied by \( i \omega \) to obtain the equation for the asymptotic solution of the equation for the downgoing one-way wave:

\[ -\omega^2 \left[ -\left( \varphi_x - \frac{1}{v} \right) A(D) \right] \]

\[ -\frac{1}{v \pi} \int_{-1}^{1} \sqrt{1-s^2} \frac{v^2 \varphi_x^2 ds}{1-s^2 v^2 \varphi_x^2} A(D) \]

\[ + i \omega \left[ A(D)_x + \frac{v}{2v} (A(D) + A(q(1; \cdot))) \right] \]

\[ + \frac{1}{v \pi} \int_{-1}^{1} \sqrt{1-s^2} \frac{B(A(D)) + s^2 B(A(q_d(s; \cdot))) ds}{1-s^2 v^2 \varphi_x^2} \]

\[ + [\cdots] + \cdots = 0 \]

The following integrals are needed

\[ J_n(b^2) = \frac{1}{v \pi} \int_{-1}^{1} \sqrt{1-s^2} \frac{ds}{1-b^2 s^2} \varphi. \]

We have

\[ J_0(b^2) = \frac{1}{2}, \quad J_1(b^2) = \frac{1 - (1 - b^2)^{1/2}}{b^2}, \]

\[ J_2(b^2) = \frac{1}{2(1 - b^2)^{1/2}} \quad \text{and} \quad J_3(b^2) = \frac{3(1 - b^2) + 1}{8(1 - b^2)^{3/2}}. \]

Then we can obtain the following integrals

\[ I_0 = \frac{1}{v \pi} \int_{-1}^{1} \sqrt{1-s^2} e(s) ds \]

\[ = \frac{1}{v \pi} \int_{-1}^{1} \sqrt{1-s^2} \frac{v^2 \varphi_x^2 ds}{1-s^2 v^2 \varphi_x^2} \]  

\[ = v^2 \varphi_x^2 J_1(v^2 \varphi_x^2), \]

\[ I_1 = \frac{1}{v \pi} \int_{-1}^{1} \sqrt{1-s^2} \frac{s^2 e(s) ds}{1-s^2 v^2 \varphi_x^2} \]

\[ = J_2(v^2 \varphi_x^2) - J_1(v^2 \varphi_x^2), \]

and

\[ I_2 = \frac{1}{v \pi} \int_{-1}^{1} \sqrt{1-s^2} \frac{s^2 e(s) ds}{1-s^2 v^2 \varphi_x^2} \]

\[ = \left( \frac{v^2 \varphi_x^2}{v^2 \varphi_x^2} \right)_x \left[ \frac{1}{v \pi} \int_{-1}^{1} \sqrt{1-s^2} \frac{v^2 \varphi_x^2 ds}{(1-s^2 v^2 \varphi_x^2)^{3/2}} \right] \]

\[ = \left( \frac{v^2 \varphi_x^2}{v^2 \varphi_x^2} \right)_x \left[ J_3(v^2 \varphi_x^2) - J_2(v^2 \varphi_x^2) \right] \]

\[ = \frac{(v^2 \varphi_x^2)}{8(1-v^2 \varphi_x^2)^{3/2}}, \]

with \( e(s) \) defined by (A-10).

Now we can proceed to simplify the various terms in (A-15). According to the definition of \( B(A) \) in (A-16) and using (A-11), we have

\[ B(A(D)) = 2v^2 \varphi_x A(D)_x + A(D) \left( \frac{\partial}{\partial x} \right)^2 \varphi \]

\[ = 2v^2 \varphi_x A(D)_x + A(D) \left( \frac{\partial}{\partial x} \right)^2 \varphi + E(D(A)p)_x \]

\[ = E(D(B(A(p))) + 2v^2 \varphi_x E(D)_x A(p). \]

Similarly, using (A-2), we find that

\[ B(A(q_d(s; \cdot))) = e(s) E(D(B(A(p)) \]

\[ + 2v^2 \varphi_x e(s) E(D)_x A(p). \]

So, for the order-i\( \omega \) term in (A-15) we conclude that

\[ B(A(D)) + s^2 B(A(q_d)) \]

\[ = E(D(B(A(p))),(1+s^2 e(s)) \]

\[ + 2v^2 \varphi_x E(D)_x A(p). \]

Now we can carry out the integration in the order-i\( \omega \) term in (A-17), using all of the results (A-19) - (A-22). The result is

\[ \frac{1}{v \pi} \int_{-1}^{1} \sqrt{1-s^2} \frac{B(A(D)) + s^2 B(A(q_d(s; \cdot))) ds}{1-s^2 v^2 \varphi_x^2} \]

\[ = \frac{1}{v} \left\{ E(D) \left[ B(A(p)) \left( I_0 + J_1 \right) \right] \right. \]

\[ + 2v^2 \varphi_x A(p) I_2 \]

\[ \left. + 2v^2 \varphi_x E(D)_x A(p) \right\} \]

\[ = \frac{1}{v} \left\{ E(D) \left[ B(A(p)) J_3(v^2 \varphi_x^2) + 2v^2 \varphi_x A(p) I_2 \right] \right. \]

\[ + 2v^2 \varphi_x E(D)_x A(p) J_3(v^2 \varphi_x^2) \} \].
A-3  Eikonal equation and transport equation for restored downgoing wave

From the coefficient of $\omega^2$ in (A-17) we have

$$[-\varphi_x + \frac{1}{v} (1 - \frac{1}{\pi} \int_{-1}^{1} \sqrt{1 - s^2} \frac{v^2 \varphi_x^2}{1 - s^2 v^2 \varphi_x^2} ds)] A(D) = 0.$$  \hspace{1cm} \text{(A-26)}$$

We seek a nontrivial asymptotic solution, so $A(D) \neq 0$. Therefore we conclude that

$$[-\varphi_x + \frac{1}{v} (1 - \frac{1}{\pi} \int_{-1}^{1} \sqrt{1 - s^2} \frac{v^2 \varphi_x^2}{1 - s^2 v^2 \varphi_x^2} ds)] = [-\varphi_x + \frac{1}{v} (1 - v^2 \varphi_x^2) ]^{1/2} = 0$$

This is the eikonal equation of the downgoing one-way wave equation (A-1). We can see this eikonal equation (A-27) is just the one of two branches of eikonal equation for the full wave equation (1) in 2D, namely,

$$\varphi_x^2 + \varphi_z^2 = \frac{1}{v^2}$$  \hspace{1cm} \text{(A-28)}$$

Clearly, except for details of computation, the derivation in 3D would follow along the same lines.

By setting the coefficient of the order-$i\omega$ term in (A-17) we obtain the transport equation of the restored downgoing wave $A(p)$. We consider two parts in this coefficient. One part is an integral (A-25), restated and expanded upon here:

$$I = \frac{1}{v \pi} \int_{-1}^{1} \sqrt{1 - s^2}$$

$$B(A(D)) + s^2 B(A(qD(s; s))) ds$$

$$= \frac{E(D)}{v} \left\{ \left[ 2v^2 \varphi_x A(p)x + A(p) \left( \frac{\partial}{\partial x} \right)^2 \varphi \right] \cdot J_2(v^2 \varphi_x^2) + 2v^2 \varphi_x A(p) I_2 \right\}$$  \hspace{1cm} \text{(A-29)}$$

$$+ 2v \varphi_x E(D)A(p)J_2(v^2 \varphi_x^2)$$

$$= 2v \varphi_x E(D)A(p)J_2(v^2 \varphi_x^2)$$

$$+ \left\{ \frac{E(D)}{v} \left[ \left( \frac{\partial}{\partial x} \right)^2 \varphi \cdot J_2(v^2 \varphi_x^2) + 2v^2 \varphi_x I_2 \right] \right\} A(p)$$

From (A-27), (A-12) and (A-21)

$$2v E(D)J_2(v^2 \varphi_x^2) = 2\sqrt{1 - v^2 \varphi_x^2} J_2(v^2 \varphi_x^2) = 1,$$  \hspace{1cm} \text{(A-30)}$$

$$\frac{E(D)}{v} 2v^2 \varphi_x I_2 = 2v E(D) \varphi_x \left( \frac{v^2 \varphi_x^2}{8(1 - v^2 \varphi_x^2)^{3/2}} \right)$$  \hspace{1cm} \text{(A-31)}$$

$$= \frac{\varphi_x (v^2 \varphi_x^2)}{4(1 - v^2 \varphi_x^2)}$$

$$2v \varphi_x E(D)J_2(v^2 \varphi_x^2)$$

$$= \frac{v \varphi_x E(D)}{\sqrt{1 - v^2 \varphi_x^2}} = \frac{v \varphi_x}{2 \sqrt{1 - v^2 \varphi_x^2}}$$  \hspace{1cm} \text{(A-32)}$$

Here, we derive $E(D)_x$ from (A-6):

$$E(D)_x = \frac{1}{2} \left[ \varphi_x + \frac{1}{v} \sqrt{1 - v^2 \varphi_x^2} \right]_x$$

$$= \frac{1}{2} \left[ \varphi_{xx} - \frac{v \varphi_x \varphi_{xx} + v_x/v^2}{\sqrt{1 - v^2 \varphi_x^2}} \right].$$

By using these results, we can now simplify $I$ in (A-29) as follows.
From (A-5), (A-6), (A-27) and (A-2) we have:

\[ I = \varphi_x A(p) + A(p) \left\{ \frac{1}{2v^2} \left( v \varphi_x + (v^2 \varphi_x)_x \right) \right\} + \varphi_x A(p) + \frac{v \varphi_x}{2 \sqrt{1 - v^2 \varphi_x^2}} \left( \varphi_{xx} - \frac{v \varphi_x \varphi_{xx} + v \varphi_x}{\sqrt{1 - v^2 \varphi_x^2}} \right) \]

Another part in the coefficient of term \( i \omega \) in (A-17) is:

\[ II = E(D) + \frac{v_x}{2v} (A(D) + A(q(1;1))). \]  

From (A-5), (A-6), (A-27) and (A-2) we have:

\[ I + II = \varphi_x A(p) + \frac{v_x}{2v} E(D) \left( 1 - v^2 \varphi_x^2 \right) \]

Consequently the coefficient of term \( i \omega \) in (A-17) is:

\[ I + II = \varphi_x A(p) + \frac{v_x}{2v} E(D) \left( 1 - v^2 \varphi_x^2 \right) \]

This last expression, then, is the coefficient of the order-\( i \omega \) term in (A-17) and, therefore, must be equal to zero. This leads to the first transport equation for the amplitude of the Downgoing wave; that is:

\[ 2 \nabla \varphi \cdot \nabla A(p) + A(p) \Delta \varphi = 0. \]  

This is just the transport equation for the full wave equation (1). Because \( \varphi \) is the downgoing traveltime,
the solution will be the amplitude for the downgoing wave.

A similar result can be derived for the upgoing one-way wave equation, starting again from (29), but using the lower signs. In that case, we would find that

$$\varphi_2 + \frac{1}{v} \sqrt{1 - v^2 \varphi_2^2} = 0. \quad (A-38)$$

It is another branch of the eikonal equation (A-28) of the full wave equation (1). In the same manner, we can obtain the same transport equation for the first amplitude of the restored upcoming wave $A(\rho U)$:

$$2\nabla \varphi \cdot \nabla A(\rho U) + A(\rho U) \Delta \varphi = 0. \quad (A-39)$$

This completes the proof.

**Appendix B**

In this appendix, we verify the integral identity (24) for $\lambda = ik_z$. More to the point, let us consider the integral

$$I = \frac{1}{\pi} \int_{-1}^{1} \sqrt{1 - s^2} \frac{(vk)^2}{\omega^2 - s^2 (vk)^2} ds \quad (B-1)$$

We view $s$ as a complex variable and replace the “contour” from $-1$ to $1$ by the “barbell” contour of Figure B-1 extending from $-1 + \epsilon$ to $1 - \epsilon$, then passing around the branch point at $s = 1$ in a clockwise direction, returning to $-1 + \epsilon$ encircling the branch point at $s = -1$ in a clockwise direction to complete a closed path of integration. The square-root changes sign when the path passes around the branch point at either end. Thus, when passing around both branch points, the integrand returns to its original value; this justifies the claim that the contour of integration is closed. On the other hand, after one change of sign passing around the branch point at $1 - \epsilon$, the integrand has changed sign compared to its previous value at each $s$. However, the direction of the path of integration has reversed as well. Thus, the integral on the path approximately between $1 - \epsilon$ and $-1 + \epsilon$ after that first circumnavigation is the same as the integral before. Further, it is standard in complex integration methodology to confirm that the integrals on the circles of radius $\epsilon$ shrink to zero as $\epsilon \rightarrow 0$. Thus, calling this new contour of integration, $C_1$, we need only introduce a factor of $1/2$ to equate the integral on $C_1$ to the original real integral on the interval $(-1, 1)$.

$$I = \frac{1}{2\pi} \int_{C_2} \sqrt{1 - s^2} \frac{(vk)^2}{\omega^2 - s^2 (vk)^2} ds. \quad (B-2)$$

We note that the integrand has two poles at $s_{\pm} = \pm |\omega|/(v|\bar{k}|) > 1$. We propose to recast the integral as a sum of residues at these poles plus an integral on a circle of radius $r > |\omega|/(v|\bar{k}|)$ Since we will now be concerned with the region where $|s| > 1$, we prefer to rewrite the integrand as

$$\sqrt{1 - s^2} \frac{(vk)^2}{\omega^2 - s^2 (vk)^2} = -i \sqrt{s^2 - 1} \frac{(vk)^2}{\omega^2 - s^2 (vk)^2} \quad (B-3)$$

with

$$\sqrt{s^2 - 1} \approx s$$

for $|s|$ “large.” To confirm this, note that we passed over the branch point at $s = 1$ to pass to the region of $|s| > 1$. In doing so, arg$(1 - s)$ passed from zero to $-\pi$, in which case the argument of this square-root passed from zero to $-\pi/2$, which is the argument of $-i$; hence the choice of sign in the redefined square-root.

We are now prepared to recast $I$ as an integral on the contour of radius $r$ plus residues, namely, the integral on the path $C_2$ of Figure B-1:

$$I = -\frac{i}{2\pi} \int_{C_2} \sqrt{s^2 - 1} \frac{(vk)^2}{\omega^2 - s^2 (vk)^2} ds$$

$$= \frac{1}{2\pi i} \int_{C_2} \sqrt{s^2 - 1} \frac{(vk)^2}{\omega^2 - s^2 (vk)^2} ds$$

$$= \sum_{\pm} (\text{Residues, } s = s_{\pm})$$

$$- \frac{1}{2\pi i} \int_{|s|=r} \sqrt{s^2 - 1} \frac{(vk)^2}{\omega^2 - s^2 (vk)^2} ds.$$

In the last line here, we have observed that the deformation of the contour onto the circle leads to two counter-clockwise contours around the poles. Further, in the last integral, we have reverted to the default that an integral on a contour of prescribed radius is understood to be a counter-clockwise contour; hence, the change in sign on the integral from the sign in the previous line, where the direction of $C_1$ was clockwise and the direction of $C_2$ is clockwise.

**Figure B-1.** Contours of integration. $C_1$ is the barbell contour. Deformation outward leads to the contour $C_2$. True amplitude wave equation migration
\[
\sum_{\pm} (\text{Residues, } s = s_{\pm}) = \sum_{\pm} \sqrt{s^2 - 1} \frac{(vk)^2}{-2s (vk)^2} \left|_{s = s_{\pm}} \right. \tag{B-4}
\]

Both the square-root here and \( s \) itself change sign in the evaluation at \( s_{\pm} \). Consequently, these two terms add to yield

\[
\sum_{\pm} (\text{Residues, } s = s_{\pm}) = -\sqrt{1 - \frac{(vk)^2}{\omega^2}} \tag{B-5}
\]

Next, we must evaluate the integral over \( |s| = r \) in the limit as \( r \to \infty \). For large \( r \), use

\[
\sqrt{s^2 - 1} \approx s \quad \text{and} \quad \frac{(vk)^2}{\omega^2 - s^2 (vk)^2} \approx -\frac{1}{s^2}
\]

to obtain

\[
-\frac{1}{2\pi i} \int_{|s|=r} \sqrt{s^2 - 1} \frac{(vk)^2}{\omega^2 - s^2 (vk)^2} ds \approx \frac{1}{2\pi i} \int_{|s|=r} ds = 1 \tag{B-6}
\]

\[
= \frac{1}{2\pi i} \int_{0}^{2\pi} i\theta = 1,
\]

\[
s = re^{i\theta}, \quad ds = ire^{i\theta} d\theta.
\]

Combining this result with (B-5) in (B-4) we conclude that

\[
I = 1 - \sqrt{1 - \frac{(vk)^2}{\omega^2}}. \tag{B-7}
\]

We only need to substitute this result into (24) to confirm that identity, which is what we set out to do in this appendix.

Appendix C

In this appendix, we derive the representation (41) for \( p_U \). Because of the proof of Appendix A, we can use the full wave equation to find \( p_U \). That is, \( p_U \) must satisfy the frequency domain equation

\[
\mathcal{L}_\omega p_U(\vec{x}; \omega) = \nabla^2 p_U + \frac{\omega^2}{v^2} p_U = 0, \tag{C-1}
\]

\[
p_U(x, y, 0; \omega) = Q(x, y; \omega).
\]

Furthermore, \( p_U \) is an upward propagating wave and therefore it satisfies the inward propagating Sommerfeld radiation conditions [Bleistein, 1984],

\[
 rp_U(\vec{x}; \omega) \text{ bounded, } \quad r \left[ \frac{\partial p_U}{\partial r} - \frac{i\omega}{v} p_U \right] \to 0, \tag{C-2}
\]

\[
r \to \infty, \quad r = |\vec{x}|.
\]

We introduce a Green’s function, \( G(\vec{x}, \vec{x}'; \omega) \) satisfying the following problem,

\[
\mathcal{L}_\omega G = -\delta(\vec{x} - \vec{x}'), \quad G(x, y, 0, \vec{x}') = 0. \tag{C-3}
\]

We also require that \( G \) satisfy the same radiation condition that \( p_U \) does, namely

\[
rG(\vec{x}, \vec{x}'; \omega) \text{ bounded, } \quad r \left[ \frac{\partial G}{\partial r} - \frac{i\omega}{v} G \right] \to 0. \tag{C-4}
\]

Now, we consider the integral

\[
I = \int_{D} \left\{ p_U(\vec{x}; \omega) \mathcal{L}_\omega G(\vec{x}, \vec{x}'; \omega) - G(\vec{x}, \vec{x}'; \omega) \mathcal{L}_\omega p_U(\vec{x}; \omega) \right\} d^3x. \tag{C-5}
\]

Here, \( D \) is a hemisphere of radius \( R \) centered at the origin, planar boundary at \( z = 0 \) and extending down into the domain \( z > 0 \).

We use the wave equations in (C-1) and (C-3) to replace the differential operators in the integral in (C-5) by the source terms and then conclude that

\[
I = -p_U(\vec{x}'; \omega). \tag{C-6}
\]

Now, we note that

\[
p_U(\vec{x}; \omega) \mathcal{L}_\omega G(\vec{x}, \vec{x}'; \omega) - G(\vec{x}, \vec{x}'; \omega) \mathcal{L}_\omega p_U(\vec{x}; \omega)
\]

\[
= p_U(\vec{x}; \omega) \nabla^2 G(\vec{x}, \vec{x}'; \omega) - G(\vec{x}, \vec{x}'; \omega) \nabla^2 p_U(\vec{x}; \omega)
\]

Next, we use Green’s theorem [Bleistein, 1984] to recast the integral in (C-5) as one over the boundary \( \partial D \) of the domain \( D \):

\[
I = \int_{\partial D} \left\{ \left. p_U(\vec{x}; \omega) \frac{\partial G(\vec{x}, \vec{x}'; \omega)}{\partial n} \right| - \frac{\partial G(\vec{x}, \vec{x}'; \omega)}{\partial n} p_U(\vec{x}; \omega) \right\} dS. \tag{C-7}
\]

Here, \( \partial / \partial n \) is the normal derivative in the outward direction from \( \partial D \). This surface consists of two pieces.

First, there is the hemisphere of radius \( R \) on which \( \partial / \partial n = -\partial / \partial r \). The second part of \( \partial D \) is the disk on the plane at \( z = 0 \) and of radius \( R \), centered at the origin. On this disk, \( \partial / \partial n = -\partial / \partial z \). On the hemisphere, we write
\[ \Delta = p_v(\vec{x}; \omega) \frac{\partial G(\vec{x}, \vec{x}; \omega)}{\partial n} - G(\vec{x}, \vec{x}; \omega) \frac{\partial p_v(\vec{x}; \omega)}{\partial n} \]

\[ = p_v(\vec{x}; \omega) \left\{ \frac{\partial G(\vec{x}, \vec{x}; \omega)}{\partial r} - \frac{i \omega}{v} G(\vec{x}, \vec{x}; \omega) \right\} \]

\[ - G(\vec{x}, \vec{x}; \omega) \left\{ \frac{\partial p_v(\vec{x}; \omega)}{\partial r} - \frac{i \omega}{v} p_v \right\}. \]

Note that we can write \( dS = R^2 d\Omega \) in (C-7) for this hemisphere. Here, \( d\Omega \) is the differential solid angle on the unit sphere. We need to evaluate the last expression at \( r = R \). We can pair up a multiplier of \( R \) from the \( dS \) multiplication with each of the factors in this last expression for \( \Delta \). By using the Sommerfeld conditions in (C-2) we conclude that

\[ R p_v(\vec{x}; \omega) R \left\{ \frac{\partial G(\vec{x}, \vec{x}; \omega)}{\partial r} - \frac{i \omega}{v} G(\vec{x}, \vec{x}; \omega) \right\} \bigg|_{r=R} \to 0, \]

\[ R \to \infty. \]

\[ R G(\vec{x}, \vec{x}; \omega) R \left\{ \frac{\partial p_v(\vec{x}; \omega)}{\partial r} - \frac{i \omega}{v} p_v \right\} \bigg|_{r=R} \to 0, \]

\[ R \to \infty. \]

Since the domain of integration is bounded (by \( 2\pi \)), we conclude that the surface integral in (C-7) over the hemisphere must approach zero as the radius of the hemisphere approaches infinity. Consequently, we are left with the surface integral over the plane \( z = 0 \) in that integral.

Here, we use the boundary data in (C-1) and (C-3) plus the fact that \( \partial / \partial n = - \partial / \partial z \) to conclude that

\[ p_v(\vec{x}; \omega) = \int_{z=0} Q(x, y; \omega) \frac{\partial G(\vec{x}, \vec{x}; \omega)}{\partial z} dxdy. \quad (C-8) \]

Now, we have to examine the Green’s function \( G \) more closely. Let us introduce the free space Green’s function, \( G_s(\vec{x}, \vec{x}; \omega) \) satisfying the same radiation conditions as \( G \). Further, let us introduce the image point \( \vec{x}^* = (x, y, z) \). We claim that the solution \( G \) can then be constructed by the “method of images” as

\[ G(\vec{x}, \vec{x}; \omega) = G_s(\vec{x}, \vec{x}; \omega) - G_s(\vec{x}^*, \vec{x}; \omega). \quad (C-9) \]

At \( z = 0 \), this solution is zero as required. Now, using ray theory, we can write

\[ G(\vec{x}, \vec{x}; \omega) = A(\vec{x}, \vec{x}) e^{i \omega \varphi(\vec{x}, \vec{x})} - A(\vec{x}^*, \vec{x}) e^{i \omega \varphi(\vec{x}^*, \vec{x})}. \]

To leading order the \( z \)-derivative requires differentiation of the phases, only. In those phases, the \( z \)-derivatives at \( z = 0 \) have opposite sign but are otherwise equal. Therefore, we conclude that

\[ \frac{\partial G(\vec{x}, \vec{x}; \omega)}{\partial z} \bigg|_{z=0} \sim 2i \omega \frac{\partial G}{\partial z} A(\vec{x}, \vec{x}) e^{i \omega \varphi(\vec{x}, \vec{x})} \bigg|_{z=0}. \quad (C-10) \]

Now we use this result and (C-9) in (C-8) to conclude that

\[ p_v(\vec{x}; \omega) \]

\[ = 2i \omega \int_{z=0} Q(x, y; \omega) \frac{\partial G}{\partial z} A(\vec{x}, \vec{x}) e^{i \omega \varphi(\vec{x}, \vec{x})} \bigg|_{z=0} dxdy. \quad (C-11) \]

To complete the story, we need to replace \( \vec{x}^* \) by \( \vec{x} \) and replace \( \vec{x} \) at \( z = 0 \) by \( \vec{x}_r \). Furthermore, we use the fact that \( \partial \varphi / \partial z \) is the \( z \)-component of the traveltime \( \varphi \), which is a vector making the angle \( \alpha_r \) with the vertical and having magnitude \( 1/v(\vec{x}_r) \). The result of these substitutions is equation (41). This completes the verification of that equation.
Closed-form expressions for map time-migration in VTI media and the applicability of map depth-migration in the presence of caustics

Huub Douma and Maarten V. de Hoop
Center for Wave Phenomena, Colorado School of Mines, Golden CO, USA

ABSTRACT
Provided the velocity of the medium is known and the medium does not allow different reflectors to have identical surface seismic measurements that persist under small perturbations of the medium, map migration achieves a one-to-one mapping from surface seismic data to the subsurface seismic image by using the slopes in addition to the location (and time) of the events in the data. In this paper we present 3D pre-stack map time-migration in closed form for P-waves in homogeneous isotropic and qP-waves in VTI media, and discuss the condition for applicability of pre-stack map depth-migration and demigration in the presence of caustics. As far as pre-stack time-demigration is concerned, we present closed-form expressions for the mapping in isotropic homogeneous media, while for homogeneous VTI media we derive a system of four nonlinear equations with four unknowns that needs to be solved numerically. In addition we present closed-form expressions for both pre-stack map time-migration and demigration in the common-offset domain for homogeneous isotropic media that use only the slopes in the common-offset domain as opposed to slopes in both the common-shot and common-receiver (or equivalently the common-offset and common-midpoint) domain. All time-migration and demigration equations presented can be used in media with mild lateral and vertical velocity variations, provided the velocity is replaced with the local RMS velocity. The expressions for pre-stack map time-migration in VTI homogeneous media can be used for anisotropic parameter estimation (i.e., the anellipticity parameter $\eta$) in the context of time-migration velocity analysis.

Key words: pre-stack map migration, closed-form, homogeneous, VTI, canonical relation, caustics

1 INTRODUCTION
The kinematics and geometry of seismic migration can be described in terms of surfaces of equal traveltime, i.e., isochrons. Migration encompasses the integration of signal processed data along diffraction surfaces corresponding to these isochrons. In terms of linear filter theory, the image in constant media is a convolution of the impulse response of the migration operator shaped in accordance with the isochrons. This approach uses the positions, traveltimes, and amplitudes of the events in the data, and thus uses the information given by the reflection slopes in common-shot or common-receiver gathers only implicitly.

In the high-frequency approximation, seismic waves (or singularities) propagate along rays through the subsurface. Provided the velocity in the earth is known, reflection slopes in the data determine the directions of rays at the recording surface (or the singular direction of the wavefront set of the recorded wavefield). Therefore, once the traveltime and the slopes at the source
and receiver are known along with the velocity, the location and local dip of a reflector in the subsurface (i.e., singularity) can in principle be determined with the aid of ray-tracing (Cerveny, 2000). The determination of the reflector position and orientation from the times and slopes of a reflection in the data at the source and receiver locations is generally referred to as map migration. In a mathematical context, provided that the velocity is known and the medium does not allow different reflectors to have identical surface seismic measurements that persist under small perturbations of the medium, the use of the slope information in map migration results in a one-to-one mapping from the unmigrated quantities associated with a reflection in the data to the migrated quantities associated with a reflector. Collecting the migrated and unmigrated quantities in a ‘table’, leads to the notion of canonical relation.

Here we aim to elucidate that pre-stack map depth-migration is closely related to the canonical relation of the single scattering modeling or imaging operators in complex media. Guillemin (1985), ten Kroode et al. (1998), de Hoop & Brandsberg-Dahl (2000), and Stolk & de Hoop (2002b) have shown that that imaging artefacts (or imaging phantoms) are avoided if the projection of this canonical relation on the unmigrated quantities is one-to-one. We explain this, and make clear that this condition provides the applicability condition that allows map depth-migration in the presence of caustics.

The concept of map migration is certainly not new. Weber (1955) gives an early account of map migration, wherein the zero-offset 3D map migration equations are derived for a constant-velocity medium and arbitrary recording surface. Independently Graeser et al. (1957) and Haas & Viallix (1976) derive the position of a reflector in 3D from zero-offset data using straight rays by using the slope information. In an early attempt at the use of numerical ray-tracing, Musgrave (1961) uses the slope information to calculate wavefront charts and migration lists, and Sattlegger (1964) derives a series expansion for the geometry of pre-migration
data to the migrated quantities associated with a reflection in the
migration results in a one-to-one mapping from the unmigrated quantities associated with a reflector. Collecting the migrated and unmigrated quantities in a ‘table’, leads to the notion of canonical relation.

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Map migration has been used for velocity estimation in several different approaches (e.g., Gjøystdal & Ursin (1981), Gray & Golden (1983), and Maher et al. (1987)). To improve horizon-based velocity model building, map migration has been used in seismic event-picking schemes consisting of map migration, followed by picking, map demigration, and remigration in the updated velocity model. The initial map-migration step in such a scheme attempts to reduce mispositioning of the velocity picks. The idea to use map migration for velocity analysis stems from the sensitivity of pre-stack map migration to errors in the migration velocity, as pointed out by Sattlegger et al. (1980). Sword (1987, p.22) develops a controlled directional reception (CDR) tomographic inversion technique, first suggested by Harlan & Burridge (1983), to find interval velocities from pre-stack seismic data. In that method the slopes (or horizontal slownesses) are picked automatically using the CDR picking technique — slant-stack over a short range of offsets with subsequent picking — developed in the former Soviet Union [e.g., Zavalishin (1981) and Riabinkin (1991)] but first introduced by (Rieber, 1936) and later reintroduced by Hermont (1979). Subsequently the estimates of the ray parameters are used to trace rays through the initial estimate of the velocity model, and a depth is found wherein the sum of the traveltime along the downgoing (source) and upgoing (receiver) rays equals the observed traveltime. Then, at this depth, the horizontal distance between the endpoints of two rays in the subsurface is minimized using a modified Gauss-Newton method to yield the velocity model.

Recently Iversen & Gjøystdal (1996) performed 2D map migration in arbitrarily complex media using a layer-stripping approach similar to that of Gray & Golden (1983) to achieve simultaneous inversion of velocity and reflector structure; they later extended this method for 2D anisotropic media (Iversen et al., 2000). Their linearized inversion scheme, which minimizes the projected difference along the reflector normal between events from different offsets, uses derivatives of reflection-point coordinates with respect to model parameters as introduced by van Trier (1990) rather than derivatives of traveltimes with respect to model parameters as used in classical tomographic inversion [e.g., Bishop et al. (1985)]. Such an approach allows for more consistent event picking because reflectors can be identified in a geological structure. In addition, an initial imaging step generally improves the signal-to-noise ratio allowing for more accurate event picking. Finally, Billette & Lambare (1998) most recently reiterated the importance of slope information in macro-velocity model estimation while validating that precision in measured slopes, traveltimes, and positions in seismic reflection data is sufficient to recover velocity fields using stereomorphology.

The outline of this paper is as follows. First, we develop closed-form expressions for the geometry of pre-stack map time-migration and demigration in 3D for a homogeneous isotropic medium. These equations assume, in addition to the velocity (as is common in seismic imaging), that only the location and the slopes in unmigrated common-offset gathers are known. This is in contrast to the migration equations in 2D presented by Sword (1987, p.22), which also require the slope within the common-midpoint gathers (or, alternatively, the slopes in the common-source and common-receiver gatherers). Time migration, which uses the assumption of a homogeneous model and the root-mean-square (RMS) velocity remains in large use in practice. In this context
our expressions have current applicability, provided the constant velocity in them is replaced by the local RMS velocity. To complement the work of Alkhalifah & Tsvankin (1995) and Alkhalifah (1996) on velocity analysis in transversely isotropic media, we derive closed-form expressions for 3D pre-stack map time-migration for qP waves in VTI media and present a system of four nonlinear equations with four unknowns for the demigration problem, which needs to be solved numerically. These map migration expressions can be used for anisotropic parameter estimation (i.e., the anellipticity parameter $\eta$) in the context of time-migration velocity analysis. We then proceed to explain the applicability of map depth-migration in the presence of caustics, and revisit pre-stack map time-migration in homogeneous isotropic media to show that the pre-stack map time-migration and demigration equations define the canonical relation of the single scattering modeling and imaging operators in such media. For practical issues such as slope estimation, accuracy and stability of the algorithm, and sampling and grid distortion, we refer to existing literature [e.g., Kleyn (1977) and Maher & Hadley (1985)].

2 ISOTROPIC HOMOGENEOUS MEDIA

2.1 The DSR equation for common-offset and common-azimuth

To illustrate the concept of map migration, we derive explicit 3D pre-stack map migration and demigration equations for isotropic homogeneous media. We assume that preprocessing has already compensated data for any topography on the acquisition surface, and deal with only the kinematics of migration. The results, however, could be extended to take geometrical spreading effects into account. For media with mild lateral and vertical velocity variations, these equations can be used provided the velocity is replaced with the local RMS velocity.

For 2D pre-stack map time-migration, Sword (1987p.22) derived closed-form expressions for the migrated location and reflector dip using the horizontal slownesses at both the source and the receiver. Since the velocity for the isotropic case does not depend on the angle, however, the actual angles at the source and the receiver given by the horizontal slownesses need not be known. We derive closed-form expressions for pre-stack map time-migration and demigration in 3D, using only the slopes in common-offset gathers, which prove it unnecessary to use both horizontal slownesses. These expressions provide a practical advantage over existing closed-form solutions that use both slownesses, since only one slope needs to be measured instead of two. Such a reduction, unfortunately, no longer holds in heterogeneous or anisotropic media.
The double square root (DSR) equation, governing traveltime in a homogeneous medium, in 3D is given by

\[
t_u = \frac{1}{v} \left( \sqrt{(\bar{x}_u - \bar{x}_m - h \sin \alpha)^2 + (\bar{y}_u - \bar{y}_m - h \cos \alpha)^2 + \left( \frac{vt_m}{2} \right)^2} + \sqrt{(\bar{x}_u - \bar{x}_m + h \sin \alpha)^2 + (\bar{y}_u - \bar{y}_m + h \cos \alpha)^2 + \left( \frac{vt_m}{2} \right)^2} \right),
\]

where \(\bar{x}_u\) and \(\bar{y}_u\) are the common-midpoint (CMP) coordinates, \(\bar{x}_m\) and \(\bar{y}_m\) are the reflection-point coordinates, \(t_m\) is the two-way migrated traveltime, \(\alpha\) is the acquisition azimuth measured positive in the direction of the positive \(\bar{x}\) axis, \(v\) is the velocity, and \(h\) is the half-offset (see Figure 1). Rotating the positive \(\bar{y}\) direction to the source-to-receiver direction, the DSR equation becomes

\[
t_u = \frac{1}{v} \left( \sqrt{(x_u - x_m)^2 + (y_u - y_m - h)^2 + \left( \frac{vt_m}{2} \right)^2} + \sqrt{(x_u - x_m)^2 + (y_u - y_m + h)^2 + \left( \frac{vt_m}{2} \right)^2} \right) .
\]

To find \(x_u, y_u, t_u, p_x^u,\) and \(p_y^u\) from \(\bar{x}_u, \bar{y}_u, t_m, p_x^\bar{u}\) and \(p_y^\bar{u}\), where \(p_x^\bar{u}\) and \(p_y^\bar{u}\) are the horizontal slownesses of the unmigrated reflection in the rotated and unrotated coordinate systems, respectively, we calculate

\[
\begin{pmatrix}
  x_u \\
  y_u \\
  t_u \\
  p_x^u \\
  p_y^u
\end{pmatrix} = \begin{pmatrix}
cos \alpha & \sin \alpha & 0 & 0 & 0 \\
-sin \alpha & cos \alpha & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & cos \alpha & -sin \alpha \\
0 & 0 & 0 & sin \alpha & cos \alpha
\end{pmatrix} \begin{pmatrix}
\bar{x}_u \\
\bar{y}_u \\
t_m \\
p_x^{\bar{u}} \\
p_y^{\bar{u}}
\end{pmatrix}.
\]

To be consistent with the general treatment of time-migration using common-midpoint coordinates and offset, we derive our results in this reference frame. Since we align the positive \(y\)-axis with the source-receiver direction, we develop our equations in the common-offset, common-azimuth domain. In the remaining text we assume the velocity to be known and equal to the RMS velocity; i.e., we develop the pre-stack map migration and demigration equations and their solutions in the context of time-migration. Table 1 summarizes our notation throughout the remaining text.

### 2.2 Pre-stack common-offset migration

Equation (2) has three unknowns — \(x_m, y_m,\) and \(t_m\). We obtain two additional equations by calculating the partial derivatives of \(t_u\) with respect to \(x_u\) and \(y_u\) while keeping the reflector location and offset constant, i.e., \(p_x^u = \frac{1}{2} \frac{\partial t_u}{\partial x_u}\) and \(p_y^u = \frac{1}{2} \frac{\partial t_u}{\partial y_u}\):

\[
p_x^u = \frac{1}{2v} \left( \frac{x_u - x_m}{\sqrt{(x_u - x_m)^2 + (y_u - y_m - h)^2 + \left( \frac{vt_m}{2} \right)^2}} + \frac{x_u - x_m}{\sqrt{(x_u - x_m)^2 + (y_u - y_m + h)^2 + \left( \frac{vt_m}{2} \right)^2}} \right),
\]

\[
p_y^u = \frac{1}{2v} \left( \frac{y_u - y_m - h}{\sqrt{(x_u - x_m)^2 + (y_u - y_m - h)^2 + \left( \frac{vt_m}{2} \right)^2}} + \frac{y_u - y_m + h}{\sqrt{(x_u - x_m)^2 + (y_u - y_m + h)^2 + \left( \frac{vt_m}{2} \right)^2}} \right),
\]

where the horizontal slownesses \(p_x^\bar{u}\) and \(p_y^\bar{u}\) can be measured. With these additional equations we arrive at a system of three equations with three unknowns. To derive equations (4) and (5) we used the property that on the pre-stack migration isochrone, defined by the DSR equation, \(\frac{\partial x_m}{\partial y_m} = \frac{\partial y_m}{\partial x_m} = 0\) for constant \(h\).

Solving equations (2), (4), and (5) for \(x_m, y_m\) and \(t_m\) results in

\[
x_m = x_u - \frac{v^2 p_y^\bar{u} t_m}{2} \left( 1 - \Lambda_u^2 \right),
\]

\[
y_m = y_u - \left( \frac{vt_m}{2} \right)^2 \Lambda_u \frac{h}{h}
\]

\[
t_m = 2 \left\{ \left( \frac{t_m}{2} \right)^2 \left( 1 - (vp_x^u)^2 \right) - \left( \frac{h}{v} \right)^2 + \left( \frac{vt_m \Lambda_u}{4h} \right)^2 \right\} \right)^{\frac{1}{2}},
\]
Table 1. Summary of notation.

<table>
<thead>
<tr>
<th>parameter</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x, y, z$</td>
<td>horizontal location $x$ and $y$, and depth $z$</td>
</tr>
<tr>
<td>$t$</td>
<td>two-way traveltime</td>
</tr>
<tr>
<td>$h$</td>
<td>half-offset</td>
</tr>
<tr>
<td>$v$</td>
<td>group velocity</td>
</tr>
<tr>
<td>$V$</td>
<td>phase velocity</td>
</tr>
<tr>
<td>$\psi$</td>
<td>phase angle</td>
</tr>
<tr>
<td>$\theta$</td>
<td>phase angle</td>
</tr>
<tr>
<td>$p_{m}^{x,y}$</td>
<td>horizontal slowness in $x$ or $y$ direction</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>acquisition azimuth (measured anti-clockwise with positive $y$-axis)</td>
</tr>
<tr>
<td>$u, m$</td>
<td>subscripts denoting migrated ($m$) and unmigrated ($u$) variables</td>
</tr>
<tr>
<td>$s, r$</td>
<td>subscripts denoting source ($s$) and receiver ($r$)</td>
</tr>
<tr>
<td>$\phi_{x,y}$</td>
<td>reflector dip angle with horizontal in $x$ or $y$ direction</td>
</tr>
<tr>
<td>$V_{P0, S0}$</td>
<td>vertical phase velocity $qP$ and $qSV$ waves respectively</td>
</tr>
<tr>
<td>$V_{NMO(0)}$</td>
<td>zero-dip NMO velocity</td>
</tr>
<tr>
<td>$\xi_{m}$</td>
<td>dip covector</td>
</tr>
<tr>
<td>$\epsilon, \delta$</td>
<td>Thomson parameters</td>
</tr>
<tr>
<td>$\eta$</td>
<td>anellipticity parameter</td>
</tr>
<tr>
<td>$(s, r)<em>{\gamma</em>{u}}$</td>
<td>$(\sin \gamma_{s}, \sin \gamma_{r})$</td>
</tr>
<tr>
<td>$(s, r)_{\theta}$</td>
<td>$(\sin \theta_{s}, \sin \theta_{r})$</td>
</tr>
</tbody>
</table>

in which

$$\Lambda_{u} = \Lambda_{u}(p_{m}^{x}, \Theta_{u}, h) \equiv \frac{1}{2 \sqrt{2p_{m}^{x}h}} \Theta_{u} \left(1 - \sqrt{1 - \frac{64 (p_{m}^{x}h)^{2}}{\Theta_{u}^{2}}}\right),$$

(9)

with

$$\Theta_{u} = \Theta_{u}(t_{u}, p_{m}^{y}, h) \equiv t_{u}^{2} + \left(\frac{2h}{v}\right)^{4} \frac{1}{t_{u}^{2}} - 2 \left(\frac{2h}{v}\right)^{2} \left(1 - (vp_{m}^{y})^{2}\right),$$

(10)

where the signs of the roots are chosen such that migration moves energy up-dip. The local dip of the reflector $p_{m}^{x,y} = \frac{1}{2} \frac{\partial v_{u}}{\partial (x,y)_{m}}$ can be found by calculating the partial derivatives $\frac{\partial}{\partial x_{m}}$, $\frac{\partial}{\partial y_{m}}$ of equation (2), using that $\frac{\partial v_{u}}{\partial x_{m}} = \frac{\partial u}{\partial x_{m}} = 0$ for constant $h$, and using equations (6)-(8) for $x_{m}$, $y_{m}$ and $t_{m}$. This yields

$$p_{m}^{x,y} = p_{m}^{x,y} t_{u} \left(\frac{|A_{u} - 1|}{|A_{u} - 1|} + \frac{|A_{u} + 1|}{|A_{u} + 1|}\right)$$

$$\times \left\{ \left(\frac{t_{u}}{2}\right)^{2} \left(1 - (vp_{m}^{x})^{2}\right) + \left(\frac{h}{v}\right)^{2} + \left(\frac{v t_{u} A_{u}}{4 h}\right)^{2} \left(8 (p_{m}^{x}h)^{2} - t_{u}^{2} + \left(\frac{2h}{v}\right)^{2} \left(1 - (vp_{m}^{x}A_{u})^{2}\right)^{\frac{1}{2}}\right)\right\}^{2}. \quad (11)$$

Equations (6)-(8) and (11) thus are explicit and exact expressions that determine the migrated reflector coordinates $(x_{m}, y_{m}, t_{m}, p_{m}^{x}, p_{m}^{y})$ from the specular reflection coordinates $(x_{u}, y_{u}, t_{u}, p_{u}^{x}, p_{u}^{y})$ given $h$. The 2D case is readily obtained by setting $x_{m} = x_{u} = 0$ and $p_{m}^{x} = p_{u}^{x} = 0$, which reduces equations (7), (8) and (11) to the 2D equivalent expressions. Note that equations (6)-(8) and (11) do not use the offset horizontal slowness $p_{h} = \frac{1}{2} \frac{\partial h}{\partial t_{m}}$. This means
that, in practice, only $p_u^x$ and $p_u^y$ need to be estimated, and the slope in a common-midpoint gather can be ignored. Usually expressions or algorithms for map migration use the slopes in both the common-offset and midpoint gathers, or, alternatively, the slopes in source and receiver gathers.

2.3 Zero-offset migration

Equations (6)-(8) and (11) are strictly valid when $h \neq 0$ and $p_u^h \neq 0$. For small $h$ or $p_u^h$, i.e.,

$$\frac{64(p_u^h)^4}{\Theta_u^2} \ll 1,$$

we use a first-order Taylor expansion for $\Lambda_u$ such that

$$\Lambda_u \approx \frac{2p_u^h}{\sqrt{\Theta_u}}.$$  

Substituting this approximation for $\Lambda_u$ into equations (6)-(8) gives

$$x_m \approx x_u - \frac{v^2 p_u^h t_u}{2} \left(1 - \frac{4(p_u^h)^2}{\Theta_u}\right),$$

$$y_m \approx y_u - \frac{(vt_u)^2 p_u^h}{2\sqrt{\Theta_u}},$$

$$t_m \approx 2 \left\{ \frac{v^2}{4} \left(1 - (vp_u^h)^2\right) - \left(\frac{h}{v}\right)^2 + \frac{1}{\Theta_u} \left(\frac{vt_u p_u^h}{2}\right) \left(8(p_u^h)^2 - t_u^2 + \left(\frac{2h}{v}\right)^2 \left[1 - \frac{1}{\Theta_u} (2vp_u^h p_u^h)^2\right]\right) \right\}^{\frac{1}{2}},$$

$$p_m^{x,y} \approx p_u^{x,y} t_u \left\{ \Theta_u \left(\frac{v^2}{4} \left(1 - (vp_u^h)^2\right) - \left(\frac{h}{v}\right)^2 + \left(\frac{vt_u p_u^h}{2}\right) \left(8(p_u^h)^2 - t_u^2 + \left(\frac{2h}{v}\right)^2 \left[1 - \frac{1}{\Theta_u} (2vp_u^h p_u^h)^2\right]\right) \right\}^{-\frac{1}{2}}.$$

For the special case when $h = 0$, these equations reduce to their zero-offset (or post-stack) counterparts, i.e.,

$$x_m = x_u - \frac{v^2 p_u^h t_u}{2},$$

$$y_m = y_u - \frac{v^2 p_u^h t_u}{2},$$

$$t_m = t_u \sqrt{1 - v^2 p_u^h},$$

$$p_m^{x,y} = \frac{p_u^{x,y}}{\sqrt{1 - v^2 p_u^h}},$$

where we have defined

$$p_u \equiv \sqrt{(p_u^x)^2 + (p_u^y)^2}.$$  

Setting $x_m = x_u = 0$ and $p_m^{x,y} = p_u^{x,y} = 0$ gives the expressions in 2D.

2.4 Pre-stack common-offset demigration

The demigration equations for $x_u$ and $y_u$ can be found by first solving equation (2) for $t_m$ and evaluating the partial derivatives $\frac{\partial t_u}{\partial x_u}$ and $\frac{\partial t_u}{\partial y_u}$. Using the resulting expressions for $x_u$ and $y_u$ in equation (2) then give the explicit expression for $t_u$. To find the slopes $p_u^x$ and $p_u^y$, we then simply substitute the expressions for $x_u$, $y_u$, and $t_u$ in equations (4) and
The resulting equations are

\[ x_u = x_m + \frac{v^2 p_m^u m_t}{2}, \]  
\[ y_u = y_m + \frac{v^2 p_m^y m_t}{2} + h\Lambda_m, \]  
\[ t_u = \sqrt{\frac{4h^2}{v^2} + \frac{2p_m^y h t_m}{\Lambda_m}}, \]  
\[ p_u^{x,y} = \frac{p_m^{x,y} t_m}{2} \left( \frac{|\Lambda_m - 1| + |\Lambda_m + 1|}{|\Lambda_m - 1||\Lambda_m + 1|} \right), \]

in which

\[ \Lambda_m = \Lambda_m (p_m^y, \Theta_m, h) \equiv \frac{4p_m^y h}{\Theta_m \left( 1 + \sqrt{1 + \frac{4(p_m^y h)^2}{\Theta_m^2}} \right)}, \]  
\[ \Theta_m = \Theta_m (t_m, p_m^x, p_m^y) \equiv t_m \left( 1 + v^2 \left[ (p_m^x)^2 + (p_m^y)^2 \right] \right). \]

Equations (23)-(26) determine the specular reflection \((x_u, y_u, t_u, p_u^x, p_u^y)\) from the migrated reflector \((x_m, y_m, t_m, p_m^x, p_m^y)\). Note that \(p_m^{x,y}\) can be estimated from the dip of the imaged reflector using that

\[ \tan \phi_{x,y} = \frac{v}{2} \frac{\partial t_m}{\partial (x,y)_m} = vp_m^{x,y}, \]

where \(\phi_{x,y}\) is the reflector dip angle with the horizontal in the \(x-\) or \(y\)-direction (measured positive clockwise). Again, the 2D case follows by setting \(x_m = x_u = 0\) and \(p_m^x = p_u^x = 0\).

### 2.5 Zero-offset demigration

The demigration mapping given by equations (23)-(26) indeed reduces to its zero-offset counterpart if \(h = 0\). The resulting expressions are

\[ x_u = x_m + \frac{v^2 p_m^u m_t}{2}, \]  
\[ y_u = y_m + \frac{v^2 p_m^y m_t}{2}, \]  
\[ t_u = t_m \sqrt{1 + v^2 p_m^2}, \]  
\[ p_u^{x,y} = \frac{2p_m^{x,y}}{\sqrt{1 + v^2 p_m^2}}, \]

where

\[ p_m = \sqrt{(p_m^x)^2 + (p_m^y)^2}. \]

With \(x_m = x_u = 0\) and \(p_m^x = p_u^x = 0\) these expressions reduce to the 2D case.

### 3 TANGSVERELY ISOTROPIC MEDIA WITH A VERTICAL SYMMETRY AXIS

#### 3.1 The DSR equation for common-offset and common-azimuth

Consider the case of homogeneous transversely isotropic media with a vertical symmetry axis (VTI). In general, the group velocity vector is perpendicular to the slowness surface, whereas the slowness vector is perpendicular to the wavefront. In transversely isotropic (TI) media, the group velocity depends only on the phase angle \(\theta\) with the axis of rotational symmetry, and for qP (and qSV) waves is given by (Tsankin, 2001 p.29)

\[ v = V(\theta) \sqrt{1 + \left( \frac{1}{V(\theta)} \frac{dV}{d\theta} \right)^2}, \]
where \( v \) is the group velocity, and \( V \) is the phase velocity. The group angle \( \psi \) in such media is given by
\[
\tan \psi = \frac{\tan \theta + \frac{1}{V(\theta)} \frac{dV}{d\theta}}{1 - \frac{\tan \theta \frac{dV}{d\theta}}{V(\theta)}}.
\]
(36)

The group angle \( \psi \) is defined as the angle of the ray with the rotational symmetry axis, and the phase angle \( \theta \) is the angle of the normal to the wavefront with the symmetry axis. Since the group velocity points in the direction of the ray, the DSR equation for a homogeneous VTI medium is given by
\[
t_u = \frac{\sqrt{(x_u - x_s, r)^2 + (y_u - y_m)^2 + z_m^2}}{v_u} + \frac{\sqrt{(x_u - x_s, r)^2 + (y_r - y_m)^2 + z_m^2}}{v_r},
\]
(37)

where \( v_{s,r} \) are the group velocities (which depend on the phase angles) along the source and receiver legs, respectively. Note that the positive \( y \)-direction here is in the source-receiver direction, and the coordinate system is right-handed as before.

3.2 Parametrization

For general transversely isotropic (TI) media, the phase velocity for \( qP \) waves using the parameterization introduced by Tomson (1986) is given by (Tsvankin, 2001, p. 22)
\[
V(\theta) = V_{P0} \left[ 1 + \epsilon \sin^2 \theta - \frac{f}{2} \left( 1 - \frac{2 \epsilon \sin^2 \theta}{f} \right) - \frac{2 (\epsilon - \delta) \sin^2 2\theta}{f} \right],
\]
(38)

where \( V_{P0} \) is the phase velocity for the \( qP \) wave at \( \theta = 0 \), \( f \equiv 1 - \frac{V_{S0}^2}{V_{P0}^2} \) with \( V_{S0} \) the phase velocity of the \( qSV \) wave at \( \theta = 0 \), and \( \epsilon \) and \( \delta \) are the Thomsen anisotropy parameters. Taking the derivative with respect to the phase angle \( \theta \) gives
\[
\frac{dV}{d\theta} = \frac{V_{P0}^2}{V(\theta)} \left( \epsilon s \sqrt{1 - s^2} + \frac{1 + 2 \epsilon \sin^2 \theta}{f} \frac{\epsilon s (1 - s^2 - 2 (\epsilon - \delta) s \sqrt{1 - 5 s^2 - 4 s^2})}{\sqrt{1 + 2 \epsilon \sin^2 \theta}} \right),
\]
(39)

where \( s \equiv \sin \theta \). Note that we have \( 0 \leq \theta < \pi/2 \), since rays cannot turn in homogeneous media.

The P-wave phase velocity, however, depends only weakly on the vertical shear-wave velocity \( V_{S0} \) [e.g., Tsvankin (1996) and Alkhaijifah (1998); a precise analysis is contained in Schoenberg & de Hoop (2000)], such that influence of \( V_{S0} \) in all kinematic problems involving P-waves can be ignored. Because in this paper we are only dealing with the geometry of map migration, we can for most practical purposes set \( f = 1 \) in equations (38), (39), and (56); for small \( V_{P0}/V_{S0} \) ratios a better choice is to use a typical \( V_{P0}/V_{S0} \) ratio. If \( V_{S0} \) is known, \( f \) can be calculated and subsequently used in equations (38), (39) and (56) to find the phase velocity, its derivative, and the phase angle.

Alkhaijifah and Tsvankin (1995) showed that the time-signatures (e.g., reflection move-out, DMO, and time-migration operators) of \( qP \)-waves in homogeneous VTI media are characterized mainly by the zero-dip normal-moveout (NMO) velocity \( V_{NMO}(0) = V_{P0}\sqrt{T + 2\delta} \) and the anellipticity parameter \( \eta = (\epsilon - \delta)/(1 + 2\delta) \), with an almost negligible influence of \( V_{P0} \). Using these expressions for \( \eta \) and \( V_{NMO}(0) \), equations (38) and (39) can be rewritten in terms of \( \eta, V_{NMO}(0) \), and \( V_{P0} \). Since in practice \( V_{NMO}(0) \) is estimated from move-out velocity analysis, the expressions we derive for map migration in VTI media can be used to estimate the anellipticity parameter \( \eta \), by using the slope information of one event at two (or more) different offsets and calculating the migrated times for these offsets for an assumed value of \( \eta \) (given arbitrary \( V_{P0} \)). The correct value of \( \eta \) and \( V_{NMO}(0) \) should yield the same migrated time for all offsets, since the data have one common reflection point.

3.3 Pre-stack or finite-offset migration

For VTI (and isotropic) media, all vertical planes are medium mirror symmetry planes. Both vertical planes, the one defined by the source position and the reflector position, and the one defined by the receiver position and the
Pre-stack map time-migration in VTI media

Figure 2. Definition of $\gamma_{s,r}$ as the angles of the horizontal projections of the slowness vectors with the positive $x$-axis.

Reflector position, thus also are symmetry planes. Throughout the remainder, we refer to these planes as the source and receiver planes (see Figure 1). In the source plane,

$$\tan \psi_s = \frac{\sqrt{(x_s - x_m)^2 + (y_s - y_m)^2}}{z_m},$$

while in the receiver plane, we have

$$\tan \psi_r = \frac{\sqrt{(x_r - x_m)^2 + (y_r - y_m)^2}}{z_m},$$

where $\psi_{s,r}$ are the group angles at the source and receiver, and $z_m = V_{p0} t_m/2$ is the migrated depth. The horizontal slownesses satisfy the relation

$$p_{s,r} = \sin \theta_{s,r} V_{s,r},$$

where we defined

$$V_{s,r} \equiv V(\theta_{s,r}),$$

$$p_{s,r} \equiv \sqrt{(p_{x,s,r})^2 + (p_{y,s,r})^2},$$

with $p_{x,y}^{s,r}$ denoting the horizontal slownesses at the source or receiver in the $x$ or $y$ direction, and $\theta_{s,r}$ the phase angle at the source or receiver. Then, using equation (42) in equation (36) and substituting the result in equations (40) and (41), we get

$$\sqrt{(x_{s,r} - x_m)^2 + (y_{s,r} - y_m)^2} = \frac{V_{s,r} p_{s,r}}{1 - \frac{p_{s,r}}{\sqrt{1 - V_{s,r}^2 p_{x,s,r}^2}} \frac{dV}{d\theta} \bigg|_{s,r}} + \frac{1}{V_{s,r} \frac{dV}{d\theta} \bigg|_{s,r}},$$

Here $\frac{dV}{d\theta} \bigg|_{s,r}$ is the derivative of the phase velocity with respect to the phase angle $\theta$ with the vertical symmetry axis, evaluated at the phase angle at the source ($\theta_s$) or receiver ($\theta_r$).

Using equation (45) in (37) then results in an expression for the migrated time

$$t_m = 2 t_u \frac{V_{p0}}{V_{s,r}} \left( \frac{1}{V_s \left( \sqrt{1 - V_s^2 p_{s,r}^2} - p_s \frac{dV}{d\theta} \bigg|_{s} \right)^2} + \frac{1}{V_r \left( \sqrt{1 - V_r^2 p_{r}^2} - p_r \frac{dV}{d\theta} \bigg|_{r} \right)^2} \right)^{-1}.$$

Then, defining $\gamma_{s,r}$ as the angles of the horizontal projection of the slowness vector at the source and receiver with the positive $x$-axis (see Figure 2), we find

$$\sin \gamma_{s,r} = \frac{y_{s,r} - y_m}{\sqrt{(x_u - x_m)^2 + (y_{s,r} - y_m)^2}} = \frac{p_{y,s,r}^2}{p_{s,r}}.$$
Then, using equation (45) and (46) in equation (47) gives

\[
y_m = y_s, r - \frac{t_u p^2_{s, r}}{V_{s, r} + \sqrt{\frac{1}{V_{s, r}^2 p^2_{s, r}} - 1}} \left( \frac{dV}{dt} \right)_{\xi_s, r} \left( \sqrt{1 - V_{r, s}^2 p^2_{r, s} - p_s} \frac{dV}{dt} \right)_{r, s}.
\]

(48)

Note the order of the subscripts \( s, r \) and \( r, s \).

To find \( x_m \), we first calculate (see Figure 2)

\[
\cos \gamma_{s, r} = \frac{x_{s, r} - x_m}{\sqrt{(x_{s, r} - x_m)^2 + (y_{s, r} - y_m)^2}} = \frac{p^x_{s, r}}{p_{s, r}}.
\]

(49)

Then, using equations (45) and (46) in this expression, gives

\[
x_m = x_{s, r} - \frac{t_u p^2_{s, r}}{V_{s, r} + \sqrt{\frac{1}{V_{s, r}^2 p^2_{s, r}} - 1}} \left( \frac{dV}{dt} \right)_{\xi_s, r} \left( \sqrt{1 - V_{r, s}^2 p^2_{r, s} - p_s} \frac{dV}{dt} \right)_{r, s}.
\]

(50)

Again note the order of the subscripts \( s, r \) and \( r, s \). Thus, equations (46), (48), and (50) are closed-form expressions for the migrated location. Setting \( x_m = x_u = 0 \) and \( p^y_s = p^y_r = 0 \) gives the expressions in 2D.

To find the reflector dip covector \( \xi_m \) (i.e., the wave-vector associated with the reflector in the image), we use that the slowness vectors \( \mathbf{p}_{s, r} \) at the source and receiver locations obey Snell’s law upon reflection at the reflector. Since we define the slowness vectors to point in the negative \( z \)-direction (i.e., upwards), we have

\[
\xi_m = \xi_x + \xi_y = \omega \mathbf{p}_s + \omega \mathbf{p}_r,
\]

(51)

where \( \omega \) is the angular frequency, and \( \xi_{x, r} \) are the wave-vectors associated with the source and receiver rays (see Figure 1). The slowness vectors at the source and receiver are given by

\[
\mathbf{p}_{s, r} = \left( \begin{array}{c}
-p^x_{s, r} \\
-p^y_{s, r} \\
\sqrt{\frac{1}{V_{s, r}^2} - p^2_{s, r}}
\end{array} \right).
\]

(52)

Therefore, the dip covector \( \xi_m \) is given by

\[
\xi_m = \omega \left( \begin{array}{c}
-\frac{p^x_s + p^x_r}{\sqrt{\frac{1}{V^2} - p^2_s}} \\
-\frac{p^y_s + p^y_r}{\sqrt{\frac{1}{V^2} - p^2_r}}
\end{array} \right).
\]

(53)

To translate \( \xi_m \) to the migrated horizontal slowness components \( p^{x, y}_m \), we use that (defining \( \nu_{x, y} \))

\[
-\nu_{x, y} \equiv \frac{\xi^{x, y}_m}{\xi_m} = \tan \phi_{x, y} = \frac{V_{p_0}}{2} \frac{\partial 
}{\partial (x, y)}_{m} = V_{p_0} p^{x, y}_m,
\]

(54)

where \( \xi^{x, y}_m \) are the components of the dip covector, and \( \phi_{x, y} \) is again the reflector dip with the horizontal in the \( x \)- or \( y \)-direction (measured positive clockwise). By using equation (53) in (54) it follows that

\[
p^{x, y}_m = \frac{p^{x, y}_s + p^{x, y}_r}{V_{p_0} \left( \frac{1}{V^2} - p^2_s + \frac{1}{V^2} - p^2_r \right)}.
\]

(55)

Again, setting \( x_m = x_u = 0 \) and \( p^y_s = p^y_r = 0 \) gives the 2D expressions.
3.4 Finding the phase angles

Equations (38) and (39) can be used to calculate the phase velocity and its derivative if the angle \( \theta \), and thus \( s = \sin \theta \), is known. To find \( s \), we need to solve equation (42) for \( s \), using equation (38) for the phase velocity. This gives

\[
\sin \theta_{s,r} = \left\{ p_{s,r} [ (2 - f) - 2P_{s,r} (\epsilon - \delta f) ] \right\}^{1/2} \left[ 1 + \frac{4 (1 - f) (1 - 2 \epsilon P_{s,r} - 2P_{s,r}^2 f (\epsilon - \delta))}{(f - 2 + 2P_{s,r} (\epsilon - \delta f))^2} \right]^{1/2},
\]

where

\[
P_{s,r} \equiv p_{s,r} V_{p0}^2,
\]

with \( p_{s,r} \) defined in equation (44).

Therefore, given \( V_{p0} \) and the anisotropic parameters \( \epsilon \) and \( \delta \) (or \( \eta \) and \( V_{NMO}(0) \)), we can use equation (56) with the measured \( p_{s,r} \) to calculate the phase angles \( \theta_{s,r} \). The resulting values can then be used in equations (38) and (39) to find the phase velocity and its derivative at both the source and receiver location.

Note that if the horizontal slowness is used to parametrize for the phase velocity and its derivative, we need not solve for the phase angles. To find the phase velocity as a function of the horizontal slowness, we simply replace \( \sin \theta_{s,r} \) with \( V_{s,r} p_{s,r} \) in equation (38) and solve for \( V_{s,r} \). In Appendix A, we give the resulting expressions for the phase velocity and its derivative as functions of the horizontal slowness are given, using that \( f = 1 \) for most practical purposes.

3.5 Zero-offset migration

By setting \( p_u^{x,y} = p_u^{x,y} = p_u^{x,y} \) and \( \theta_s = \theta_r = \theta \), the pre-stack map migration equations reduce to their zero-offset counterparts. Doing this for equations (46), (48), (50), (53), and (55) gives

\[
t_m = \frac{V(\theta) t_u}{V_{p0}} \left( \sqrt{1 - V^2(\theta) p_u^2} - p_u \frac{dV}{d\theta} \right),
\]

\[
(x, y)_m = (x, y)_u - \frac{V^2(\theta) p_u^{x,y} t_u}{2} - \frac{V(\theta) p_u^{x,y} t_u}{2} \sqrt{1 - V^2(\theta) p_u^2} - \frac{1}{2} \frac{dV}{d\theta},
\]

\[
\xi_m = 2 \omega \left( \frac{f_u}{\sqrt{1 - V^2(\theta) p_u^2}} - p_u \right),
\]

\[
p_m^{x,y} = \frac{V(\theta) p_u^{x,y}}{V_{p0} \sqrt{1 - V^2(\theta) p_u^2}},
\]

where \( p_u \) is defined in equation (22). Again, when we set \( x_m = x_u = 0 \) and \( p_m^x = p_m^y = 0 \), the 2D expressions follow from their 3D counterparts. Note that by setting \( \frac{dV}{d\theta} = 0 \) and replacing \( V(\theta) \) and \( V_{p0} \) with \( v \), these expressions for VTI media reduce to their counterparts for isotropic media.

3.6 Pre-stack or finite-offset demigration

For the demigration problem, we assume the migrated location \( (x_m, y_m, z_m) \) and migrated dips \( \phi_{x,y} \) are given. To find the unmigrated midpoint location, we need to find the phase angles with the (vertical) symmetry axis and the azimuths of the rays from both the source and the receiver to the reflection point. If we know the angles with the symmetry axis, we can use equations (38), (39), and (42) to find the phase velocity, its derivative and \( p_{s,r} \). The projections \( p_{s,r}^{x,y} \) are then calculated using

\[
p_{s,r}^x = -\text{sgn}(\nu_e) p_{s,r} \sqrt{1 - (s, r)_\gamma^2},
\]

\[
p_{s,r}^y = p_{s,r} (s, r)_\gamma,
\]
with \((s, r)_\gamma = \sin \gamma_{s, r}\) the azimuth angles \(\gamma_{s, r}\) defined in Figure 2, and \(\nu_z\) defined in equation (54). The unmigrated location then follows from solving equations (46), (48), and (50) for \(t_u, y_u,\) and \(x_u,\) using the values for the phase velocity, its derivative, and \(p^z_{u, y}.

To find the azimuth angles, \(\gamma_{s, r}\), and the angles with the vertical symmetry axis, \(\theta_{s, r}\), we use the offset and azimuth information, and the dips \(\phi_{x, y}\) or \(\nu_{x, y} = -\tan \phi_{x, y};\) see equation (54). Since in the rotated coordinate system (the positive \(y\)-axis in the source-receiver direction) the projection onto the \(x\)-axis of the rays connecting the source and the reflector equals that connecting the receiver and the reflector, we must have

\[
\sqrt{1 - s^2} \left( s\theta + \frac{\sqrt{1 - s^2}}{V_s} \frac{dV}{d\theta} \right) = \sqrt{1 - r^2} \left( r\theta + \frac{\sqrt{1 - r^2}}{V_r} \frac{dV}{d\theta} \right),
\]

(64)

where \((s, r)_\gamma \equiv \sin \theta_{s, r}\) with \(\theta_{s, r}\) the phase angles.

Note that since \(0 \leq \theta_{s, r} < \pi/2\) and \(0 \leq \gamma_{s, r} < 2\pi,\) we have \(0 \leq (s, r)_\gamma < 1\) and \(-1 \leq (s, r)_\gamma \leq 1.\) Because the azimuth angles vary between 0 and 2\(\pi\), we need to keep track of the sign of \(\cos \gamma_{s, r}.\) In our rotated coordinate system, the sign of \(\nu_z\) determines the sign of \(\cos \gamma_{s, r}.\) Therefore, for demigration,

\[
\cos \gamma_{s, r} = -\text{sgn}(\nu_z) \sqrt{1 - (s, r)_\gamma^2},
\]

(65)

This explains the form of equation (62).

Furthermore, the difference of the projections onto the \(y\)-axis should equal the offset \(2h,\) i.e.,

\[
\begin{pmatrix}
\left( r\theta + \frac{\sqrt{1 - r^2}}{V_r} \frac{dV}{d\theta} \right) \\
\left( s\theta + \frac{\sqrt{1 - s^2}}{V_s} \frac{dV}{d\theta} \right)
\end{pmatrix}
- s_\gamma
\begin{pmatrix}
\left( r\theta + \frac{\sqrt{1 - r^2}}{V_r} \frac{dV}{d\theta} \right) \\
\left( s\theta + \frac{\sqrt{1 - s^2}}{V_s} \frac{dV}{d\theta} \right)
\end{pmatrix}
= \frac{4h}{V_{Po} t_m}.
\]

(66)

From equation (53) for \(\xi_m\) and the definitions of \(p^z_{u, y}\) and \(\nu_{x, y},\) it follows that

\[
\nu_z = \text{sgn}(\nu_z) \left( \frac{V_r s_\gamma \sqrt{1 - s^2} + V_s \theta \sqrt{1 - r^2}}{V_r \sqrt{1 - s^2} + V_s \sqrt{1 - r^2}} \right),
\]

(67)

\[
\nu_y = - \left( \frac{V_r s_\gamma s_z + V_s \theta r_\gamma}{V_r \sqrt{1 - s^2} + V_s \sqrt{1 - r^2}} \right),
\]

(68)

with \(\nu_{x, y}\) defined in equation (54).

Equation (64) and equations (66)-(68) are four nonlinear equations with four unknowns: \((s, r)_\gamma\) and \((s, r)_\gamma.\) Attempts to eliminate, for example, \(r_\theta, s_\gamma,\) and \(r_\gamma\) to get one equation in \(s_\theta\) lead to a high order polynomial equation in \(s_\theta.\) Therefore, to find the unknown angles, a numerical scheme such as Gauss-Newton (Dennis, 1977) or Levenberg-Marquardt (Moré, 1977) can be used, with the isotropic solution as initial value. It goes beyond the purpose of this paper, however, to treat the details of the proper choice of numerical scheme. In Appendix A, we show that the system of equation (64) and equations (66)-(68) can be rewritten into a somewhat simpler system using the horizontal slowness instead of the phase angles, under the assumption that for most practical purposes we can set \(f = 1.\) Note that for the 2D problem, \((s, r)_\gamma = 0\) and \(-1 < (s, r)_\gamma < 1,\) so equations (66) and (68) form a set of two nonlinear equations with two unknowns, \((s, r)_\gamma.\)

### 3.7 Zero-offset demigration

For zero-offset demigration the source and receiver coincide, which means that the phase angle with the vertical symmetry axis equals the dip of the reflector. Since in the demigration problem this dip is known, we know the phase angle and thus the group angle. Therefore, once the position and orientation of the reflector are known, we can calculate the unmigrated location. From equation (61) we directly find

\[
p^z_{u, y} = \frac{V_{Po} p^z_{u, y}}{V(\theta) \sqrt{1 + V_{Po}^2}},
\]

(69)
with \( p_m \) defined in equation (34). From this expression we find

\[
p_u^2 = \frac{V_{0p}^2 p_m^2}{V^2(\theta)(1 + V_{0p}^2 p_m^2)},
\]  

(70)

which, when used in equation (58), gives

\[
t_u = \frac{t_m \sqrt{1 + V_{0p}^2 p_m^2}}{\left(\frac{V(\theta)}{V_{0p}} - p_m \frac{dV}{d\theta} \bigg|_0\right)}.
\]  

(71)

Then, using equations (58), (69), and (70) in equation (59), gives

\[
(x, y)_u = (x, y)_m + \frac{V_{0p}^2 p_m^2 \tau_m}{2} \left(\frac{V(\theta) + \frac{1}{V_{0p} p_m} \left.\frac{dV}{d\theta}\right|_0}{V(\theta) - V_{0p} p_m \frac{dV}{d\theta} \bigg|_0}\right).
\]  

(72)

Note that by setting \( \frac{dV}{d\theta} = 0 \) and replacing \( V(\theta) \) and \( V_{0p} \) with the constant velocity \( v \), the equations for zero-offset demigration in VTI media reduce to their isotropic equivalents. Also, setting \( x_m = x_u = 0 \) and \( p_m = p_u = 0 \) gives the expressions in 2D.

To find the angle \( \theta \), we use that the reflector dip equals the phase angle for zero-offset:

\[
\theta = \arctan \left(\frac{\nu^2}{\nu^2 + \nu'^2}\right).
\]  

(73)

The calculated value of \( \theta \) can subsequently be used in equation (38) and (39) (with \( f = 1 \) or some realistic estimate of \( f \)) to find the phase velocity and its derivative.

4 THE APPLICABILITY OF MAP DEPTH-MIGRATION IN THE PRESENCE OF CAUSTICS

The closed-form expressions derived for map time-migration in homogeneous isotropic and VTI media thus far explicitly show that the mapping from the surface measurements (i.e., source and receiver positions, travel times and slopes) to the subsurface (i.e., reflector position and orientation), or vice-versa, is one-to-one for such media. In this section we explain that this mapping remains one-to-one for arbitrary complex media in which caustics can develop, provided that the medium does not allow different reflectors to have identical surface seismic measurements that persist under small perturbations of the medium [this is the Bolker condition (Guillemin, 1985)]. This section thus treats map depth-migration as opposed to map time-migration which has been treated up to this point. If we assume large frequency, in arbitrary complex media the relation between the reflections measured at the surface and the reflectors in the subsurface is governed by ray-tracing. We capture this relation schematically with the symbol \( \Lambda \).

Since also we assume single-scattering only, this relation is by definition the canonical relation of the single scattering modeling or imaging operators that relates the surface seismic measurements to the subsurface reflectors. Throughout the remainder we will refer to this relation as the canonical relation.

4.1 Canonical relation

The canonical relation is formed by collecting the unmigrated and migrated quantities in a table, viz.

\[
\Lambda = \{(x^h_{s,r}, x^h_{s}, t_u, \omega p^h_{s}, \omega p^h_{r}, \omega; x_m, \xi_m)\},
\]  

(74)

where \( x^h_{s,r} \equiv (x_s, y_s, z_r) \) are the source and receiver locations, \( t_u = t_s + t_r \) is the two way traveltime, \( p^h_{s,r} \equiv (p^h_{s}, p^h_{r}) \) are the horizontal slownesses at the sources and receivers, \( x_m = (x_m, y_m, z_m) \) is the reflector subsurface position, \( \xi_m \) is the dip covector (i.e. the wave-vector associated with the reflector), and \( \omega \) is the angular frequency; here \( h \) denotes horizontal components only. In general media, this table is evaluated with the aid of ray-tracing equations.

Writing the solution to the ray-tracing equations, subject to initial conditions \( (x_0, \xi_0) \) at time 0, in the general form \( (x(x_0, \xi_0, t), \xi(x_0, \xi_0, t)) \), the canonical relation becomes, for a horizontal surface,

\[
\Lambda = \left\{(x^h(s), x^h(r), t_u, \xi^h(s), \xi^h(r), \omega; x_m, \xi_m) \mid \text{such that } [z(s) = 0, z(r) = 0]\right\}.
\]  

(75)
where we defined \( s \equiv (x_m, \xi_s, t_s) \), \( r \equiv (x_m, \xi_r, t_r) \), and \( \xi^h \equiv (\xi_s, \xi_y) \). This symbolically represents a table evaluated through upward ray-tracing from the reflector at location \( x_m \) with normal direction along \( \xi_m \), towards the source and receiver in the directions \( \xi_s \) and \( \xi_r \) respectively, such that \( \xi_s + \xi_r = \xi_m \).

Defining \( u \equiv (x_s, y_s, x_r, y_r, t_u) \) and \( v \equiv (\omega p^r_x, \omega p^r_y, \omega p^r_p, \omega p^r_r, \omega) \), \( (u, v) \) is a surface seismic measurement characterized by the source and receiver locations, two-way traveltimes, and slopes in common source and receiver gathers; this vector is an element of phase space \( U \rightarrow \) a mathematical precise notation would be that \( (u, v) \in T^*Y \setminus \emptyset \) when \( u \) is contained in a set \( Y \); with our notation we sacrifice mathematical precision to gain physical clarity. Similarly, defining \( m \equiv x_m \) and \( \mu \equiv \xi_m \), we see that the vector \( (m, \mu) \) is a subsurface reflector defined by its subsurface location and orientation; it is an element of phase space \( M \). Let \( \pi_U \) now denote the projection of \( \Lambda \) on \( U \) and \( \pi_M \) denote the projection of \( \Lambda \) on \( M \) so that

\[
\pi_M \circ \pi_U^{-1} : (u, v) \mapsto (m, \mu),
\]

which is precisely map depth-migration: the traveltimes and slopes at the source and receiver determine the reflector location and orientation. The Bolker condition is thus the condition of applicability of map depth-migration in complex media (i.e., in the presence of caustics).

We can always use \( (m, \mu) \) as the first four (2D) or six (3D) local coordinates on \( \Lambda \) (Stolk & De Hoop, 2002a). To find the surface seismic measurements from a subsurface reflector through map depth-demigration, however, we need to specify the scattering angle and azimuth at the reflector. By parameterizing the subsets \( \{(m, \mu)\} = \text{const} \) on \( \Lambda \) we can introduce these coordinates and denote them as \( e \). In the absence of caustics, \( e \) can be chosen to be acquisition offset and azimuth (as was done above in the section on pre-stack map time-demigration in homogeneous VTI media). Map depth-demigration then follows from the mapping

\[
(m, \mu, e) \mapsto (u, v) \quad \text{for given } e.
\]

### 4.2 The isotropic problem revisited

To make explicit the connection between the closed-form expressions for pre-stack map time-migration and demigration derived in the previous sections and the canonical relation, we revisit the isotropic case. We next show that for this case, the map time-migration and demigration equations define the canonical relation.

Since we derived all expressions for map time-migration in the rotated coordinate system with the \( y \)-axis positive in the source-receiver direction, these expressions implicitly assume common azimuth. Subjecting the canonical relation in equation (74) to the restriction \( x_r = x_s = x_u \), it attains the form

\[
\Lambda = \{(x_u, y_s, y_r, t_u, \omega(p^s_x + p^r_x), \omega p^s_y, \omega p^r_y, \omega; x_m, \xi_m)\}.
\]

For isotropic homogeneous media, the equations for the migrated location in the canonical relation follow directly from the equations for homogeneous VTI media by setting \( \frac{dV}{dt} = 0 \) and \( V_s = V_r = V_{p_{x}} = v \). Under this restriction, equations (46), (48) and (50) thus determine \( x_m = (x_m, y_m, vt_m/2) \) for isotropic homogeneous media. Because the expressions for the reflector dip in homogeneous VTI media do not contain the derivative of the phase velocity, the equations for the dip covector \( \xi_m \) in the canonical relation (76) are given by equation (53), with \( V_s = V_r = v \).

To find the remaining parameters of the canonical relation, we need to find \( x_u, y_s, r, t_u, p_{s_{x}}^{y_{p}} \) for homogeneous isotropic media. By setting \( \frac{dV}{dt} = 0 \) and \( V = V_t = v \) in equations (64), (66)-(68), we get the system of equations that need to be solved to find the scattering angles \( \theta_s, \gamma_r \) and azimuths \( \gamma_s, \gamma_r \) at the reflector. The resulting system of equations and its solution are given in appendix B. Once we find \( (s, r)_{b} = (\sin \theta_s, \sin \theta_r) \) and \( (s, r)_{a} = (\sin \gamma_r, \sin \gamma_s) \), we calculate the source and receiver locations \( (x_u, y_s, r) \), two-way traveltimes \( t_u \), and the horizontal slownesses \( p_{s_{x}}^{y_{p}} \)
Pre-stack map time-migration in VTI media

from simple geometry considerations. This gives

\[ x_u = x_{s,r} = x_m + \text{sgn}(\nu_x) \frac{vt_m(s,r)\zeta_0}{2} \sqrt{1 - (s,r)^2_\theta}, \]  

\[ y_{s,r} = y_m + \frac{vt_m(s,r)\gamma_0}{2} \sqrt{1 - (s,r)^2_\theta}, \]  

\[ t_u = t_m \sqrt{1 - s^2_\theta} + t_m \sqrt{1 - r^2_\theta}, \]  

\[ p_{x,r} = (s,r)_{\gamma_0} \frac{(s,r)_{\varphi}}{v}, \]  

\[ p_{y,r} = \text{sgn}(\nu_x) \sqrt{1 - (s,r)^2_\gamma} \frac{(s,r)_{\varphi}}{v}. \]  

For homogeneous isotropic media, therefore, our map time-migration and demigration equations define the canonical relation of the single scattering modeling or imaging operators in such media.

5 DISCUSSION

We have presented closed-form 3D pre-stack map time-migration expressions for qP-waves in homogeneous isotropic and VTI media. Our equations for the isotropic case do not need the slope in the common-midpoint domain (i.e., \( p_h \)); only the slope in the common-offset domain needs to be determined. This provides an additional advantage over methods where both \( p_u \) and \( p_h \) (or equivalently both \( p_s \) and \( p_r \), the slopes at the source and receiver position) are required, especially since estimating slopes can be cumbersome in the presence of noise. All the derived pre-stack expressions reduce to their zero-offset equivalents. Since map migration is currently used primarily for velocity estimation, and since such methods are typically iterative, our closed-form expressions for pre-stack map time-migration in homogeneous isotropic and VTI media allow for a significant speed-up of existing velocity-inversion algorithms that use map migration in such media. In addition, our expressions for pre-stack map time-migration in homogeneous VTI media can be used to determine the anellipticity parameter \( \eta \) for such media in a time-migration velocity analysis context. Note that for media with mild lateral and vertical velocity variations, our equations can be used provided the velocity is replaced by the local RMS velocity.

Not surprising, our closed-form expressions for pre-stack map time-migration and demigration in homogeneous isotropic and VTI media, exemplify that, for such media, the mapping from the surface seismic measurements (i.e., the source and receiver locations, two-way traveltime, and slopes in common source and receiver gathers) to the subsurface reflectors (i.e., location and orientation), is one-to-one. We have explained that this mapping remains one-to-one for pre-stack map depth-migration in arbitrary complex media in which caustics can develop, provided the medium does not allow different reflectors to have identical surface seismic measurements that persist under small perturbations of the medium [this is the Bolker condition (Guillemin, 1985)]. This condition is thus the condition of applicability of pre-stack map depth-migration in the presence of caustics. In addition, we have shown that for homogeneous isotropic media, our pre-stack map time-migration and demigration expressions define the canonical relation of the single scattering modeling or imaging operators.

Map migration and demigration provide a mapping between the wavefront sets of seismic data and the medium perturbations. The tangential directions to the wavefronts recorded at the acquisition surface directly give us the directions locally in which the data are smooth; imaging these wavefronts gives the directions in which the medium perturbations are smooth. For the purpose of data and image compression, the highest compression rate will be accomplished in the smooth directions, i.e., along the wavefronts. Since map migration provides a one-to-one mapping from the singular directions in the data (i.e., the directions normal to the wavefronts) to the singularities in the image (i.e., the normals to the reflectors), one can thus think of map migration and demigration as mappings between optimal compression directions.

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APPENDIX A: FROM PHASE ANGLE TO HORIZONTAL SLOWNESS WHEN $V_{S0} = 0$

The slowness surface for qP-waves is convex, which in combination with a VTI medium, assures that the only branch points occur at $\theta = \pm \pi/2$. In homogeneous media, these branch-points are never reached, since turning waves do not occur in such media. Thus, for qP-waves in such media we can parametrize the phase velocity and its derivative uniquely in terms of the horizontal slowness $p$.

Substituting $\sin \theta = pV$ in equation (38) and solving for $V$, leads to

$$V(p) = V_{P0} \sqrt{\frac{1 + 2p^2V_{P0}^2(\delta - \epsilon)}{1 - 2p^2V_{P0}^2(\delta - \epsilon)}}.$$  \hspace{1cm} (A1)

where we used $V_{S0} = 0$ (i.e., $f = 1$) and that the kinematics of qP-waves in anisotropic media are independent of $V_{S0}$ within the limits of seismic accuracy (Alkhalifah, 1998). The derivative $\frac{dV}{dp}$ is then

$$\frac{dV}{dp} = \frac{2p^2V_{P0}^3(\delta - \epsilon) - 4p^3V_{P0}^2(\delta - \epsilon) - 4p^5V_{P0}^2(\delta - \epsilon)^2}{(1 - 2p^2V_{P0}^2(\delta - \epsilon))^3/2 \sqrt{1 + 2p^2V_{P0}^2(\delta - \epsilon)}}.$$  \hspace{1cm} (A2)

Note that both expressions can be readily rewritten in terms of $\eta$, $V_{NMO}(0)$, and $V_{P0}$.

Using these expressions, we can write the vertical slowness $q = \sqrt{1/V^2 - p^2}$ as

$$q = \frac{1}{V_{P0}} \sqrt{1 - p^2V_{P0}^2(1 + 2\epsilon) \left( \frac{1}{1 + 2p^2V_{P0}^2(\delta - \epsilon)} \right)},$$  \hspace{1cm} (A3)

or in terms of $\eta$, $V_{NMO}(0)$, and $V_{P0}$ [see also Alkhalifah (1998 equation A-10)],

$$q = \frac{1}{V_{P0}} \sqrt{1 - \frac{p^2V_{NMO}^2(0)}{1 - 2p^2V_{NMO}^2(0)\eta}},$$  \hspace{1cm} (A4)

In order to calculate the group angle, equation (36), we use the chain rule,

$$\frac{dV}{d\theta} = \frac{dV}{dp} \frac{dp}{d\theta}.$$  \hspace{1cm} (A5)

Using $p = \sin \theta/V$ this becomes

$$\frac{dV}{d\theta} = \left( \frac{dV}{dp} \sqrt{1 - p^2V^2} \right) / \left( V + p \frac{dV}{dp} \right).$$  \hspace{1cm} (A6)

Then, using this expression in equation (36) we find

$$\tan \psi = \frac{pV_{P0}(1 + 2\delta)}{(1 + 2p^2V_{P0}^2(\delta - \epsilon))^{3/2} \sqrt{1 - p^2V_{P0}^2(1 + 2\epsilon)}},$$  \hspace{1cm} (A7)

or in terms of $\eta$, $V_{NMO}(0)$, and $V_{P0}$,$^,$

$$\tan \psi = \frac{pV_{NMO}^2(0)}{V_{P0}(1 - 2p^2V_{NMO}^2(0)\eta)^{3/2} \sqrt{1 - p^2V_{NMO}^2(0)(1 + 2\eta)}},$$  \hspace{1cm} (A8)

Using the simplified expressions (A4) and (A8) for the group angle and vertical slowness in equations (64) and
(66)-(68) then leads to the following system of equations for pre-stack map demigration in homogeneous VTI media,

\[
\frac{p_r\sqrt{1-r_s^2}}{(1-2\nu^2p^2_p NMO(0))^{1/2}} \frac{1-2
u^2p^2_p NMO(0)(1+2\nu)}{V^2_{NMO}(0) t_m s_r p_r} = \frac{p_s\sqrt{1-s_r^2}}{(1-2\nu^2p^2_p NMO(0))^{1/2}} \frac{1-2\nu^2 p^2_p NMO(0)(1+2\nu)}{V^2_{NMO}(0)t_m s_r p_r} \tag{A9}
\]

\[
4h = \sqrt{1-2\nu^2p^2_p NMO(0)(1+2\nu)} \frac{V^2_{NMO}(0)t_m s_r p_r}{(1-2\nu^2p^2_p NMO(0))^{1/2}} \tag{A10}
\]

\[
\sqrt{1-\nu^2} V_{p0}(p_s\sqrt{1-s_r^2} + p_r\sqrt{1-r_s^2}) \frac{1-2\nu^2p^2_p NMO(0)}{1-2\nu^2p^2_p NMO(0)} + \sqrt{1-\nu^2} V_{p0}(s_r p_s + r_s p_r) \frac{1-2\nu^2p^2_p NMO(0)}{1-2\nu^2p^2_p NMO(0)} = -\nu_y \tag{A11}
\]

\[
\sqrt{1-\nu^2} V_{p0}(p_s\sqrt{1-s_r^2} + p_r\sqrt{1-r_s^2}) \frac{1-2\nu^2p^2_p NMO(0)}{1-2\nu^2p^2_p NMO(0)} + \sqrt{1-\nu^2} V_{p0}(s_r p_s + r_s p_r) \frac{1-2\nu^2p^2_p NMO(0)}{1-2\nu^2p^2_p NMO(0)} = -\nu_x \tag{A12}
\]

Note, that for the 2D problem we have \((s, r) = 0\) and \(-1 < V_{s, r} p_{s, r} < 1\), and that equations (A10) and (A12) then form a nonlinear system of two equations with two unknowns, viz. \(p_{s, r}\).

**APPENDIX B: SOLVING FOR SCATTERING ANGLES AND AZIMUTHS FOR PRE-STACK DEMIGRATION IN HOMOGENEOUS ISOTROPIC MEDIA**

To find the scattering angles \(\theta_{s, r}\) (see Figure 1) and azimuths \(\gamma_{s, r}\) (see Figure 2) for pre-stack demigration in homogeneous isotropic media, we set \(\frac{dV}{dt} = 0\) and \(V_s = V_r = v\) in equations (64), (66)-(68). The resulting system of equations is

\[
0 = \sqrt{1-r_s^2} - \sqrt{1-s_s^2} - \frac{s_s}{\sqrt{1-s_s^2}}, \tag{B1}
\]

\[
\frac{4h}{vt_m} = \left( \frac{r_s - s_s\sqrt{1-r_s^2}}{\sqrt{1-s_s^2}} \right), \tag{B2}
\]

\[
\nu_x = sgn(\nu_x) \left( \frac{s_s\sqrt{1-s_s^2} + r_s\sqrt{1-r_s^2}}{\sqrt{1-s_s^2} + \sqrt{1-r_s^2}} \right), \tag{B3}
\]

\[
\nu_y = -\left( \frac{s_s s_{s\gamma} + r_s r_{r\gamma} - s_s r_{r\gamma}}{\sqrt{1-s_s^2} - \sqrt{1-r_s^2}} \right), \tag{B4}
\]

where \((s, r) = (\sin \theta_s, \sin \theta_r)\) and \((s, r) = (\sin \gamma_s, \sin \gamma_r)\) are unknown. (From these equations it is clear that the pathological cases \((s, r) = 1\), i.e. 90 degree dipping reflectors, are not included in this system of equations, and that by choosing \(0 < \theta < \pi/2\) we exclude these impractical cases.) In appendix C it is shown that this system of equations reduces to a quadratic equation in \(s_\theta\). The 2D problem is in this appendix treated as a special case of the 3D problem.

Once \(s_\theta\) is found, we need to solve for the remaining parameters \(r_\theta\) and \((s, r)_{\gamma\gamma}\). To find \(r_\theta\) we first use equation (B4) in (B2) to eliminate \(r_\theta r_{\gamma\gamma}\) to give

\[
-\frac{4h}{vt_m} = \nu_y + \nu_y \frac{1-s_s^2}{s_s^2 - r_s^2} + s_s s_{s\gamma} \left( \frac{1}{s_s^2 - r_s^2} + \frac{1}{1-r_s^2} \right), \tag{B5}
\]

Then, to eliminate \(s_s\) we use equation (C3) and subsequently solve for \(r_\theta\) to give

\[
r_\theta = \sqrt{1 - \frac{h v t_m \sqrt{s_s^2 - r_s^2}(1-s_s^2)^2 - \sqrt{1-s_s^2} \nu_y h v t_m + \Delta}{4 h^2 + 2 \nu_y h v t_m + \Delta}} \tag{B6}
\]

where we defined

\[
\Delta \equiv \frac{\nu_x^2 + \nu_y^2}{4} \left( \frac{s_s^2 - r_s^2}{1-s_s^2} \right), \tag{B7}
\]
Once \((s,r)\) are found, we solve equation (B5) for \(s\gamma\) to get

\[
\frac{\nu_y}{s\theta} \sqrt{1 - s\gamma} + \frac{\nu_y}{1 - s\gamma} \left( \frac{4h}{vt_m} + \nu_y \right)
\]

provided \(s\theta \neq 0\). Then, using equation (B4) in (B2) to eliminate \(s\gamma s\theta\) and solving for \(r\gamma\), we find

\[
\frac{\nu_y}{\theta} \left( \sqrt{1 - s\gamma} \left( \frac{4h}{vt_m} - \nu_y \right) - \nu_y \sqrt{1 - s\gamma} \right)
\]

provided \(r\theta \neq 0\).

### APPENDIX C: THE QUADRATIC EQUATION IN \(s\gamma\) FOR PRE-STACK DEMIGRATION IN HOMOGENEOUS ISOTROPIC MEDIA

To find \(s\theta\), we first use equations (B2) and (B4) to eliminate \(r\theta r\gamma\). Then we eliminate \(s\theta \sqrt{1 - s\gamma^2}\) from equations (B1) and (B3), and combine the results to give an equation with only \(s\gamma\), \(s\gamma\), and \(r\gamma\):

\[
\left( \frac{\nu_y vt_m}{2} \right)^2 \left\{ \nu_y \sqrt{1 - s\gamma^2} + s\theta s\gamma \right\}^2 = (1 - r\gamma^2) \left\{ 4h^2 + 2hvt_m \left( \nu_y + \frac{s\theta s\gamma}{\sqrt{1 - s\gamma^2}} \right) + \left( \frac{s\theta vt_m}{2} \right)^2 \left( \nu_y + \frac{s\theta s\gamma}{\sqrt{1 - s\gamma^2}} \right)^2 \right\}.
\]

In order to get \(r\gamma^2\) as a function of \(s\gamma\) and \(s\gamma\), we use equations (B1) and (B2) to eliminate the term \(r\gamma/\sqrt{1 - r\gamma^2}\), and subsequently solve for \(r\gamma^2\). This gives

\[
r\gamma^2 = \frac{\left( 2h \sqrt{1 - s\gamma^2} + \frac{s\theta s\gamma vt_m}{2} \right)^2}{4h^2 (1 - s\gamma^2) + 2h s\theta s\gamma vt_m \sqrt{1 - s\gamma^2} + \left( \frac{s\theta vt_m}{2} \right)^2}.
\]

Before we substitute this expression for \(r\gamma^2\) into equation (C1), we first eliminate \(r\gamma\sqrt{1 - r\gamma^2}\) from equations (B1) and (B3), and solve for \(s\gamma^2\), which gives

\[
s\gamma^2 = 1 + \frac{\nu_y^2}{s\theta^2} \nonumber.
\]

where \(s\theta \neq 0\). Then, using equations (C2) and (C3) in (C1), gives

\[
(1 - s\gamma^2) \left\{ h^2 \left( \beta (1 - s\gamma^2) - 4\nu_y^2 \right) + 2h\nu_y \alpha \right\} = \frac{\nu_y^2}{\sqrt{\frac{vm}{2}}} \left\{ \beta \left[ (1 + 2\nu_y^2) s\gamma^2 - (1 + \nu_y^2) s\theta^2 - \nu_y^2 \right] - \nu_y^2 \right\}.
\]

where we defined

\[
\alpha \equiv \frac{vt_m}{2} \left( 1 + \nu_y^2 - \nu_y^2 \right), \quad \beta \equiv (1 + \nu_y^2 + \nu_y^2)^2.
\]

Equation (C4) is a quadratic equation in \(s\gamma\) that can be solved for \(s\gamma\) to give

\[
s\gamma = \sqrt{\frac{2h^2 \left( \beta (1 - \nu_y^2) + 2h\nu_y (\alpha + \gamma) + \beta \left( \frac{vt_m}{2} \right)^2 (1 + 2\nu_y^2) + \alpha \beta \right)}{2\beta \left( h^2 + (1 + \nu_y^2) \left( \frac{vt_m}{2} \right)^2 \right)}},
\]

with

\[
\gamma \equiv \sqrt{4h^2 \nu_y^2 + \beta^2 \left( \frac{vt_m}{2} \right)^2}.
\]

The proper root in equation (C7) can be found through substitution in the original system of equations (B1)-(B4).
SPECIAL CASES

In 2D, the system of equations (B1)-(B4) reduces to two equations with two unknowns:

\[
\begin{align*}
\left( \frac{r_\theta}{\sqrt{1 - r_\theta^2}} - \frac{s_\theta}{\sqrt{1 - s_\theta^2}} \right) &= \frac{4h}{v_t m}, \\
\left( \frac{s_\theta + r_\theta}{\sqrt{1 - s_\theta^2} + \sqrt{1 - r_\theta^2}} \right) &= \nu_y,
\end{align*}
\]  

(C9)  

(C10)

with \(-1 < (s, r)_\theta < 1\). In order to solve this system for the unknowns \((s, r)_\theta\), we first rewrite equation (C9) to get

\[
\sqrt{1 - r_\theta^2} = \frac{r_\theta v_t m}{\sqrt{1 - s_\theta^2}} \sqrt{1 - s_\theta^2}.
\]  

(C11)

Using this expression in equation (C10) to eliminate \(\sqrt{1 - r_\theta^2}\) then gives

\[
r_\theta = -\left( s_\theta + \frac{4h \nu_y (1 - s_\theta^2)}{s_\theta v_t m + \sqrt{1 - s_\theta^2} (4h + \nu_y v_t m)} \right).
\]  

(C12)

Then, squaring both sides of this expression, and using the result in equation (C11) to eliminate \(r_\theta^2\), gives a quadratic equation in \(\tan \theta_s = s_\theta/\sqrt{1 - s_\theta^2} \equiv \tau_{\theta s}\), viz.

\[
\nu_y v_t m \tau_{\theta s}^2 + (4h \nu_y + v_t m (\nu_y^2 - 1)) \tau_{\theta s} + 2h (\nu_y^2 - 1) - \nu_y v_t m = 0,
\]  

(C13)

with roots

\[
\tau_{\theta s} = \frac{1}{2\nu_y} - \frac{2h}{v_t m} \pm \frac{1}{v_t m} \pm \frac{1}{2} \sqrt{\frac{4h^2}{v_t m} + \frac{(1 + \nu_y^2)^2}{4\nu_y^2}}.
\]  

(C14)

Then, by using this expression in equation (C9) we find

\[
\tau_{\theta r} = \frac{1}{2\nu_y} - \frac{2h}{v_t m} \pm \frac{1}{v_t m} \pm \frac{1}{2} \sqrt{\frac{4h^2}{v_t m} + \frac{(1 + \nu_y^2)^2}{4\nu_y^2}}.
\]  

(C15)

Therefore, \((s, r)_\theta\) are then given by

\[
(s, r)_\theta = \sin \left( \arctan \left( \tau_{\theta s}, \tau_{\theta r} \right) \right).
\]  

(C16)

The proper roots in equation (C14) and (C15) can be chosen through substitution in the original system of equations (C9) and (C10). Note that the pathological cases \((s, r)_\theta = 0\) mentioned in the previous subsection, are included in this solution.

For the special case \(\nu_y = 0\), i.e., the zero dip case, the solution for \((s, r)_\theta\) is given simply by

\[
s_\theta = \sin \left( \arctan \left( \frac{-2h}{v_t m} \right) \right) = -r_\theta.
\]  

(C17)
Explicit expressions for map time-migration in weakly anisotropic VTI media

Huub Douma
Center for Wave Phenomena, Colorado School of Mines, Golden CO, USA

ABSTRACT
If the velocity of the medium is known and the medium does not incorporate different reflectors that have identical surface seismic measurements that remain identical under small perturbations of the medium, map migration achieves a one-to-one correspondence between the reflections recorded at the surface, and the reflectors in the subsurface, through explicit use of the slopes in the data. Because of this property, it has had many applications in velocity estimation. In this paper, I apply the weak anisotropy approximation to the 3D post-stack and pre-stack map time-migration equations for qP-waves in transversely isotropic homogeneous media with a vertical symmetry axis. In this way the dependence of qP-wave time imaging (in this approximation) on the anellipticity parameter η and the zero-dip NMO velocity $V_{NMO}(0)$ only, is analytically confirmed in the context of map time-migration. The accuracy of the equations is verified with numerical examples. The developed equations allow for the estimation of the anellipticity parameter η in the context of time-migration velocity analysis.

Key words: map time-migration, homogeneous, VTI, weak anisotropy

1 INTRODUCTION
Imaging the subsurface involves measuring seismic waves at the earth-surface and migrating the data down to true subsurface positions. It is known (Guillemin, 1985; ten Kroode et al., 1998; De Hoop & Brandsberg-Dahl, 2000; Stolk & De Hoop, 2002; Douma & De Hoop, 2002) that for arbitrary complex media, the slope information together with the position (space and time) of the reflections are related through a one-to-one mapping to the subsurface position and orientation of the reflectors, provided the medium does not allow different reflectors to have identical surface seismic measurements that persist to be identical under small perturbations of the medium (this is the Bolker condition (Guillemin, 1985)). Most imaging algorithms use the slope information in the data only implicitly, whereas map migration uses this information explicitly.

Map migration is certainly not new. Many authors explicitly use the slope information in data to find the reflection points (Weber, 1955; Graesser et al., 1957; Haas & Viallix, 1976; Musgrave, 1961; Sattlegger, 1964; Cohen, 1998), although they do not always use the term map migration. Also, map migration has had many applications in velocity estimation (Gjoystdal & Ursin, 1981; Gray & Golden, 1983; Maher et al., 1987; Sword, 1987; Iversen & Gjoystdal, 1996; Billette & Lambare, 1998; Iversen et al., 2000).

Douma and de Hoop (2002) derived closed-form expressions for map time-migration in homogeneous VTI media. Alkhalifah & Tsvankin (1995) and Tsvankin (2001) have shown that the time-signatures (e.g. reflection move-out, DMO and post-stack and pre-stack time-migration operators) of qP-waves in VTI media are governed mainly by the anellipticity parameter η and the zero-dip NMO-velocity $V_{NMO}(0)$. Alkhalifah and Tsvankin (1995) numerically confirm this dependence for transversely isotropic media with a vertical symmetry axis. In addition, Tsvankin (1996) shows numerically that the influence of the vertical shear-wave velocity $V_{S0}$ on the kinematics of qP-waves in transversely isotropic media is negligible, while Alkhalifah (1998) shows analytically that all time-related processing depends exactly on η and $V_{NMO}(0)$ only, when $V_{S0} = 0$. The expressions in Douma and de Hoop (2002) for pre-stack map time-migration in homogeneous VTI media,
however, depend on the four parameters \(\eta, V_{NMO}(0), V_{P0}, \) and \(V_{S0}\). Here, through application of the weak anisotropy approximation (Thomsen, 1986), I derive closed-form expressions for map time-migration in such media, explicit in the anellipticity parameter \(\eta\) and the zero-dip NMO-velocity \(V_{NMO}(0)\) only, thus confirming Alkhalifah and Tsvankin (1995). Although the obtained expressions merely confirm our current knowledge about the time-signatures of qP-waves in VTI media, the equations presented are new and can be used to invert for the anellipticity parameter \(\eta\) in the context of time-migration velocity analysis.

The outline of this paper is as follows. First, to establish a connection with earlier work on time-signatures of qP-waves in VTI media, I prove that the expression for zero-offset migrated traveltime of Douma and de Hoop (2002) is equivalent to the expression derived by Cohen (1998) for the vertical two-way traveltime \(t_m\). Then, I proceed to derive the weak-anisotropy approximations for post-stack (or, equivalently, zero-offset) time-migration and demigration respectively, followed by the weak anisotropy approximations for pre-stack (or finite-offset) map migration.

2 POST-STACK MAP MIGRATION

In arbitrary complex media a reflection is kinematically fully determined by the two-way traveltime \((t_u)\), the source and receiver positions \((x_s, y_s, z_s)\), and the horizontal slownesses at the source and receiver \((p^x_s, p^y_s)\). Similarly, in a time-migration context, the image of a reflector (ignoring the amplitudes in the image) is determined by its horizontal position \(x_m\) and \(y_m\), its vertical two-way traveltime \(t_m\), and the wave-vector associated with the imaged reflector (i.e. the dip vector \(\xi\)) which can be related to the horizontal slowness components \(p^x_m\) and \(p^y_m\) in the note. That the subscripts \(u\) and \(m\) denote unmigrated and migrated variables respectively. Douma and de Hoop (2002) show that map migration establishes a one-to-one mapping from the unmigrated quantities, associated with reflections in the data, to the migrated quantities associated with reflectors in the subsurface, and make clear that the one-to-one mapping holds even in the presence of caustics, provided the Bolker condition is satisfied. In addition they derive closed-form expressions for pre-stack map time-migration and demigration in homogeneous transversely isotropic media with a vertical symmetry axis (VTI). Here I apply the weak anisotropy approximation to the post-stack (or zero-offset) expressions for map migration and demigration from (Douma & De Hoop, 2002). Note that all equations from (Douma & De Hoop, 2002) are derived in a right handed reference frame with the \(x\)-axis positive in the source-receiver direction, and depth (and thus time) positive downwards.

From Douma and de Hoop (2002) the zero-offset expressions for 3D map migration in homogeneous VTI media are

\[
t_m = \frac{V(\theta) t_u}{V_{P0}} \left( \sqrt{1 - V^2(\theta) p^2_u} - p_u \frac{dV}{d\theta} \right) ,
\]

\[
(x, y)_m = \frac{(x, y)_u - \frac{V^2(\theta) p^x_u t_u}{2} - \frac{V(\theta) p^x_u t_u}{2} \frac{dV}{d\theta} \sqrt{1 - \frac{1}{V^2(\theta) p^2_u} - 1} ,
\]

\[
p^x_m = \frac{V(\theta) p^x_u}{V_{P0} \sqrt{1 - V^2(\theta) p^2_u}} ,
\]

where the notation \((x, y)\) means \(x\) or \(y\), \(V_{P0}\) is the vertical P-wave velocity, \(V(\theta)\) is the phase velocity at angle \(\theta\) from vertical, \(p_u \equiv \sqrt{(p^x_u)^2 + (p^y_u)^2}\), and \(p^x_u, p^y_u\) are the horizontal slownesses in the seismic image. Eqs. (1) - (3) thus determine the reflector position and orientation from the unmigrated quantities associated with a reflection in the data. Note that the expressions for 2D map migration are found from the 3D expressions by setting \(x_m = x_u = 0\) and \(p^x_m = p^y_m = 0\).

To establish the connection with earlier work on the time signature of qP-waves in homogeneous VTI media, appendix A shows that eq.(1) is equivalent to the expression derived by Cohen (1998) for the vertical two-way traveltime \(t_m\). Recalling that Alkhalifah & Tsvankin (1995) and Tsvankin (2001) have shown that qP-wave time signatures in VTI media are governed mainly by the anellipticity parameter \(\eta\) and the zero-dip NMO-velocity \(V_{NMO}(0)\), and largely independent of \(V_{P0}\). I confirm this independence for the 2D expressions for post-stack map time-migration in Figure 1. The values \(\eta = 0.0833, V_{NMO}(0) = 3.29 \text{ km/s}\) and the horizontal slowness components in this figure are the same values used by Alkhalifah & Tsvankin (1995)Figure 17); since the used value of the vertical shear-wave velocity \(V_{S0}\) is however omitted in their paper, my Figure 1a closely resembles Alkhalifah and Tsvankin’s Figure 17, but is not identical. Figure 1 shows results for \(V_{S0} = 0\) (grey) and \(V_{P0}/V_{S0} = 1.5\) (black), which in practice can be treated as a lower bound on the ratio of P and S wave velocity. Thus the map time-migration equations are governed mainly by \(\eta\) and \(V_{NMO}(0)\) and are largely independent of \(V_{P0}\). In addition, this figure confirms that qP-wave time signatures are completely independent of \(V_{P0}\) if \(V_{S0}\) is set to zero (Alkhalifah, 1998).

Although the numerical results in Figure 1 suggest that post-stack map time-migration is governed mainly by \(\eta\) and \(V_{NMO}(0)\), this dependence is not obvious analytically. In terms of the Thomsen parameters \(\delta\) and \(\epsilon\) (Tsvankin, 2001p. 22), the phase velocity \(V(\theta)\) is given
Figure 1. Dependence on $V_{P0}$ of (a) the ratio $t_u/t_m$, (b) the ratio $V_{NMO}(0)t_u/(2s_m)$ with $s_m \equiv x_u - x_m$, and (c) the ratio $p_u/p_m$. All ratios are calculated for the values $p_u = 0.10$ s/km (drawn lines), $p_u = 0.18$ s/km (dashed lines), $p_u = 0.22$ s/km (dotted lines) and $p_u = 0.25$ s/km (dashed-dotted lines), with the values $V_{NMO}(0) = 3.29$ km/s, $\eta = 0.0833$, and $V_{P0}/V_{S0} = \frac{3}{2}$ (i.e., $f = \frac{9}{16}$).

by

$$V(\theta) = V_{P0} \left\{ 1 + \epsilon \sin^2 \theta - \frac{f}{2} \right\} \left( 1 - \sqrt{\left( 1 + \frac{2 \sin^2 \theta}{\kappa} \right)^2 - \frac{2(1-\delta)\sin^2 \theta}{\kappa}} \right)^\frac{1}{2},$$

with $f \equiv 1 - \frac{V_{S0}^2}{V_{P0}^2}$. This equation can be rewritten in terms of $\eta$, $V_{NMO}(0)$ and $V_{P0}$, using that $\eta = (\epsilon - \delta)/(1 + 2\delta)$ and $V_{NMO}(0) = V_{P0}\sqrt{1 + 2\delta}$. Doing this we get

$$V(\theta) = \frac{V_{P0}}{2} \left\{ 1 + (\kappa - 1) \sin^2 \theta \right\} + \sqrt{\frac{\kappa}{1+2\eta}} (\cos 4\theta - 1) + \left\{ 1 + [\kappa - 1] \sin^2 \theta \right\},$$

where I defined

$$\kappa(V_{P0}, V_{NMO}(0), \eta) \equiv \frac{V_{NMO}(0)^2}{V_{P0}^2} (1 + 2\eta),$$

and given that the qP-wave phase velocity is largely independent of the vertical S-wave velocity $V_{S0}$, for simplicity I set $f = 1$. (In practice, however, the influence of $V_{S0}$ is not zero, and it is better to calculate $f$ with a reasonable choice of $V_{P0}/V_{S0}$ ratio, especially when the true $V_{P0}/V_{S0}$ ratio is relatively small, i.e. close to 1.5.) Substituting eq.(5) in the post-stack map time-migration equations (1) - (3), show a functional dependence of the reflector position and orientation on $V_{P0}$. Thus although the numerical results indicate that the dependence on $V_{P0}$ is very weak, the analytic expressions so far do not indicate this behaviour.

To show analytically that the dependence on $V_{P0}$ is weak, I apply the weak anisotropy approximation to the map time-migration equations in appendix B. The accuracy of these approximations is verified by calculating the approximated values for the 2D migrated quantities shown in Figure 2a. From the top downwards, the figure shows the migrated time, horizontal displacement (i.e. $s_m \equiv x_u - x_m$) and migrated horizontal slowness, normalized by their elliptic counter-parts $[t_m/(t_m|_{\eta=0}), s_m/(s_m|_{\eta=0})$ and $p_m/(p_m|_{\eta=0})$ respectively]. The grey lines show the approximated values, which are indeed close to their true values (black lines) for $V_{NMO}(0) = 3.29$ km/s, $V_{P0} = 3$ km/s, $f = \frac{9}{16}$ (i.e., $V_{P0}/V_{S0} = \frac{3}{2}$) and $\eta = 0.0833$ (drawn lines), but deviate from the exact values for larger values of $\eta$ [i.e., $\eta = 0.1666$ (dashed) or $\eta = 0.25$ (dotted)]. Note that overall the weak anisotropy approximation becomes worse for larger angles, but is highly accurate for the position (horizontal and in time) and orientation up to about 60 degree dips for $\eta = 0.0833$. This decrease in accuracy of the weak anisotropy approximations with increasing dip is due to the fact that higher orders of $\sin \theta$ usually go together with higher orders in the Thomsen parameters, which are ignored in the weak anisotropy approximation. Also, ignoring the anellipticity parameter $\eta$ in post-stack imaging leads to significant errors in the migrated location (at the surface and in time) and orientation of the reflector, since the ratios of the migrated quantities $x_m$, $t_m$ and $p_m$ to their elliptic equivalents are substantially different from unity, even for moderate dips. The mispositioning in time and orientation of the reflector, as a result of ignoring $\eta$, however, is negligible for dips up to about 20 degrees for $\eta = 0.0833$. In contrast, the mispositioning in horizontal position is in this case negligible only to about 10 degrees. Mis-positioning in post-stack imaging as a result of ignoring subsurface anisotropy has been studied by Larner & Cohen (1993) and Alkhalifah & Larner (1994).

3 POST-STACK DEMIGRATION

The post-stack map time-demigration equations in homogeneous VTI media are given by [see (Douma &
De Hoop, 2002)]

\[ t_u = \frac{t_m \sqrt{1 + V_{P0}^2 p_m^2}}{V(\theta) - p_m \frac{dV}{d\theta}|_{\theta}} \]  

(6)

\[ (x, y)_u = (x, y)_m + V_{P0}^{\sigma \nu} t_m \left( V(\theta) + \frac{1}{V_{P0} p_m} \frac{dV}{d\theta}|_{\theta} \right) \]  

(7)

\[ p_u^{\sigma \nu} = \frac{V_{P0} p_m^{\sigma \nu}}{V(\theta) \sqrt{1 + V_{P0}^2 p_m^2}} \]  

(8)

with \( p_m \equiv \sqrt{(p_m^\sigma)^2 + (p_m^\nu)^2} \). Then, using eqs.(B1) and (B2) and again consistently linearizing in \( \delta \) and \( \epsilon \), I find

\[ t_u \approx t_m \left( 1 + \frac{p_m^2 V_{NMO}^2(0)}{1 + 4 \eta \frac{p_m V_{NMO}^2(0)}{\eta p_m^2 V_{NMO}^2(0)}} \right)^{-\frac{1}{2}} \]  

(9)

\[ (x, y)_u \approx (x, y)_m + V_{NMO}^2(0) p_u^{\sigma \nu} t_m \]  

\[ \times \left( 1 + 4 \eta \frac{p_m V_{NMO}^2(0)}{1 + 4 \eta \frac{p_m V_{NMO}^2(0)}{\eta p_m^2 V_{NMO}^2(0)}} \right) \]  

(10)

\[ p_u^{\sigma \nu} \approx p_u^{\sigma \nu} \left( 1 + \frac{p_m^2 V_{NMO}^2(0)}{1 + 4 \eta \frac{p_m V_{NMO}^2(0)}{\eta p_m^2 V_{NMO}^2(0)}} \right)^{-\frac{1}{2}} \]  

(11)
Explicit expressions for map time-migration in weakly anisotropic VTI media

4 PRE-STACK MAP MIGRATION

In appendix C the weak anisotropy approximation for pre-stack (or finite-offset) map time-migration in homogeneous VTI media are derived. Figure 3 shows the accuracy of the approximations as a function of the reflector dip and the offset-to-depth ratio. The values were calculated by first solving the system for 2D pre-stack demigration in VTI media (Douma & De Hoop, 2002) with two non-linear equations with two unknowns for the phase angles at the source and the receiver, viz.

$$\frac{4h}{vt_m} = \left( \frac{r_\theta + \sqrt{1 - r_\theta^2} \frac{dV}{d\theta}}{r_\theta - \frac{dV}{d\theta}} \right) \left( \frac{\frac{V_\theta + \sqrt{1 - \frac{V_\theta^2}{V_s^2} \frac{dV}{d\theta}}}{V_r \sqrt{1 - \frac{V_\theta^2}{V_s^2}}}}{\frac{s_\theta + \sqrt{1 - s_\theta^2} \frac{dV}{d\theta}}{s_\theta - \frac{dV}{d\theta}}} \right),$$}

(12)

$$\nu = -\left( \frac{V_s s_\theta + V_r r_\theta}{V_r \sqrt{1 - s_\theta^2} - V_s \sqrt{1 - r_\theta^2}} \right).$$

(13)

with $h$ the half-offset, $\nu \equiv -\tan \phi$, with $\phi$ the reflector dip (measured clockwise positive with the horizontal), and $(s, r)_d \equiv \sin \theta_{s, r}$ with $\theta_{s, r}$ the phase angles at the source and receiver, respectively. Note that in 2D, $-\pi/2 < \theta_{s, r} < \pi/2$. The angles were found using the Gauss-Newton method (Dennis, 1977). Then, these values were used in the exact migration equations and in their weak anisotropy approximations with $\eta = 0.0833$, $V_{NMO}(0) = 3.29 \text{ km/s}$, $V_{PO} = 3 \text{ km/s}$, $f = 0.5$ and the migrated time $t_m = 2 \text{ s}$. The differences (relative to the exact values) between the approximated values and the exact ones are $(t_m^{\text{weak}} - t_m^{\text{exact}})/t_m^{\text{exact}}$, $(s_m^{\text{weak}} - s_m^{\text{exact}})/s_m^{\text{exact}}$, and $(p_m^{\text{weak}} - p_m^{\text{exact}})/p_m^{\text{exact}}$, with $s_m \equiv \psi - \rho_m$. Figure 3 shows that indeed for all offset-to-depth ratios used, the approximations are highly accurate up to about 60 degrees for $\eta = 0.0833$.

In addition, Figure 4 shows contoured values of the exact equations normalized by their elliptic ($\eta = 0$) counterparts, for the same values of $\eta$, $V_{NMO}(0)$, $V_{PO}$, $f$ and $t_m$ as used in Figure 3. Clearly, ignoring $\eta$ leads to significant error in the migrated time, lateral location, and orientation of the reflector. The errors are
more substantial for larger offset-to-depth ratios and reflector dips. Note that errors caused by ignoring \( \eta \) for the migrated time and slope show a similar dependence on reflector dip and offset-to-depth ratio, whereas the migrated lateral location shows a maximum error as a function of reflector dip for larger offset-to-depth ratios. Pre-stack migration error as a result of ignoring subsurface anisotropy has been studied by Jaramillo & Larner (1995).

4.1 Pre-stack demigration

While pre-stack map migration in homogeneous VTI media is a straightforward generalization of the post-stack problem, Douma and de Hoop (2002) show that the 3D pre-stack map demigration problem involves the solution of a nonlinear set of four equations with four unknowns, i.e. two azimuth angles and two dips, that cannot be solved in closed-form. Even applying the weak anisotropy approximation to this system does not allow for a solution in closed-form. Therefore, I refrain from treating the pre-stack demigration problem under the weak anisotropy approximation.

5 CONCLUSIONS AND DISCUSSION

Although the post-stack and pre-stack map migration equations for qP-waves in homogeneous VTI media in general depend on \( V_{NMO}(0) \), \( \eta \), \( V_P \) and \( V_S \), I show analytically that (at least) under the weak anisotropy approximation the dependence on \( V_P \) and \( V_S \) disappears, and reconfirm this independence numerically. This is consistent with the result that the time signatures of qP waves in VTI media depend mainly on \( V_{NMO}(0) \) and \( \eta \), showing this in the context of map time-migration. Although this treatment merely confirms this published result, the equations presented are new. The numerical examples presented show that the approximations are accurate (within 5%) up to about 60 degrees for offset-to-depth ratios up to three, at least for \( \eta \) as large as 0.0833 and \( V_{NMO}(0) = 3.29 \) km/s, and that the errors in the imaged position and orientation caused by ignoring \( \eta \) in homogeneous VTI media can be substantial even for such weak anisotropy.

The equations derived here allow for inversion for the anellipticity parameter \( \eta \) in the context of time-migration velocity analysis. By using data from two (or more) different offsets, we can use the difference between the migrated times, lateral positions or slopes, or all simultaneously, as an objective function to be minimized, by changing \( \eta \). The optimum value for \( \eta \) then results in the minimum objective function. Such a scheme would be similar to the approach of Iversen et al. (2000) who use qP and qSV data to invert for the Thomsen anisotropy parameters \( \epsilon \) and \( \delta \) from map migration via ray tracing in heterogeneous anisotropic media. Using the expressions presented, qP data could be used to invert for \( \eta \) instead of \( \epsilon \) and \( \delta \), albeit using straight rays only. In such a scheme, the inverted value for \( \eta \) would be related to a subsurface position, rather than a CMP position as in conventional non-hyperbolic move-out analysis. In most practical situations, however, the lateral variation of \( \eta \) is small, thus rendering such a scheme at first sight obsolete. Nevertheless, it remains to be seen how the sensitivity of the estimation of \( \eta \) from non-hyperbolic move-out analysis compares to that of migration.

Although the equations presented are valid only in VTI media that are homogeneous (i.e., straight rays only), for media with mild lateral and vertical velocity variations, the equations can be used provided the velocity is replaced with the local RMS velocity. Moreover, they can be extended to include vertical heterogeneity.

**Figure 4.** Ratios (a) \( t_m/(t_m|_{\eta=0}) \), (b) \( s_m/(s_m|_{\eta=0}) \) and (c) \( p_m/(p_m|_{\eta=0}) \) for 2D pre-stack map migration, as a function of reflector dip and offset-to-depth ratio. All values used are as in Figure 3.
through the use of Dix-type averaging (Dix, 1955). Then the
value of \( \eta \) obtained from the map time-migration
equations presented here, can be treated as an effective
\( \eta \), and the interval values of \( \eta \) can be obtained through
layer stripping approach.

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APPENDIX A: COMPARISON WITH COHEN’S (1998) EXPRESSION FOR 2D ZERO-OFFSET MAP MIGRATION

From eqs.(27) and (28) in Cohen (1998) we can deduce, by substituting \( z \equiv z_m = V_{p0} t_m / 2 \) and \( \tau \equiv t_u \), that

\[
t_m = \frac{t_u}{V_{p0} \left( q - \frac{2V\delta}{dp} \right)},
\]

(A1)

where \( t_m \) is the migrated two-way traveltime, \( t_u \) is the unmigrated two-way traveltime, and \( p \) and \( q \) are the horizontal and vertical components of the slowness vector. This equation is valid in any vertical symmetry plane of the medium. Although Cohen does not explicitly call this equation an equation for map time-migration, it indeed is, since it makes explicit use of the slope information in the data to find the migrated position in time.

Since the vertical slowness can be written as \( q = \sqrt{1/V(\theta)^2} - p^2 \) with \( V(\theta) \) the phase velocity, we have

\[
\frac{dq}{dp} = \frac{V(\theta)}{\sqrt{1 - p^2 V(\theta)^2}} \left( \frac{1}{V(\theta)^2} \frac{dV}{dp} + p \right),
\]

(A2)

To evaluate this expression, we need to find the derivative of the phase-velocity with respect to the horizontal slowness component. Using the chain rule we have

\[
\frac{dV}{dp} = \frac{dV}{d\theta} \left( \frac{dp}{d\theta} \right)^{-1}.
\]

(A3)

Since in general we have \( p = \sin \theta/V \), we find that

\[
\frac{dV}{d\theta} = \frac{V(\theta)}{\sqrt{1 - p^2 V(\theta)^2}}.
\]

(A4)

Using this expression in eq.(A2) and substituting the result in eq.(A1) then gives

\[
t_m = \frac{t_u}{V_{p0}} \left( \sqrt{1 - p^2 V(\theta)^2} - p \frac{dV}{d\theta} \right),
\]

(A5)

which is identical to eq.(1) in the main text, when we substitute \( p_u \) for \( p \).

APPENDIX B: WEAK-ANISOTROPY APPROXIMATIONS TO THE POST-STACK MAP TIME-MIGRATION EQUATIONS

Under the weak anisotropy approximation (i.e. \( \delta, \epsilon \ll 1 \)) we have [e.g. (Tsvankin, 2001, p. 23)]

\[
V(\theta) \approx V_{p0} \left( 1 + \delta \sin^2 \theta + (\epsilon - \delta) \sin^4 \theta \right),
\]

(B1)

and

\[
\frac{dV}{d\theta} \approx 2V_{p0} \sin \theta \cos \theta [\epsilon - (\epsilon - \delta) (1 - 2\sin^2 \theta)].
\]

(B2)

Using eqs.(B1) and (B2) in (1) and linearizing in \( \delta \) and \( \epsilon \), I find

\[
t_m \approx t_u \sqrt{1 - p_u^2 V_{NMO}^2 (1 + 2\delta) - 6(\epsilon - \delta) p_u^2 V_{NMO}^4 + 4(\epsilon - \delta) p_u^4 V_{NMO}^6},
\]

(B3)

where I used the approximation \( \sin \theta \approx p_u V_{p0} \) whenever \( \sin \theta \) was multiplied with \( \delta \) or \( \epsilon \), in order to consistently linearize in the Thomsen parameters. Since in the weak anisotropy approximation we have \( \eta = (\epsilon - \delta)/(1 + 2\delta) \approx \epsilon - \delta \) and since in general we have \( V_{NMO}(0) = V_{p0} \sqrt{1 + 2\delta} \), we can rewrite eq.(B3) to find

\[
t_m \approx t_u \sqrt{1 - p_u^2 V_{NMO}^2 (0) - 4\eta p_u^2 V_{NMO}^4 (0) \left[ \frac{3}{2} - p_u^2 V_{NMO}^2 (0) \right]}.
\]

(B4)

Similarly, by using eqs.(B1) and (B2) in eqs.(2) and (3) and further linearizing in the Thomsen parameters, I find

\[
(x, y)_m \approx \frac{x, y}_u + \frac{V_{NMO}^2 (0) p_u^x t_u}{2} \left( 1 + 4\eta p_u^2 V_{NMO}^2 (0) \left[ 1 - p_u^2 V_{NMO}^2 (0) / 2 \right] \right),
\]

(B5)

\[
p_{m}^{x, y} \approx p_{u}^{x, y} \sqrt{1 - p_u^2 V_{NMO}^2 (0) (1 + 2\eta p_u^2 V_{NMO}^2 (0))}.
\]

(B6)
Thus, indeed in the weak anisotropy approximation the dependence of the map time-migration equations on $V_{p0}$ disappears; also for $\eta = 0$ and $V_{pNMO(0)} = V_{p0} = v$ the equations indeed reduce to their isotropic equivalents [see Douma and de Hoop (2002) for the isotropic equations].

APPENDIX C: WEAK-ANISOTROPY APPROXIMATIONS TO THE PRE-STACK MAP TIME-MIGRATION EQUATIONS

For pre-stack (or finite-offset) map migration in homogeneous VTI media, the equations for the reflector location and orientation in the subsurface (Douma & De Hoop, 2002) are

$$t_m = \frac{2u_s}{V_{p0}} \left( \frac{1}{V_x \left( \sqrt{1 - V_x^2 p_x^2 - p_s \frac{dV}{d\theta}} \right)} + \frac{1}{V_r \left( \sqrt{1 - V_r^2 p_r^2 - p_r \frac{dV}{d\theta}} \right)} \right)^{-1} \quad (C1)$$

$$(x, y)_m = (x, y)_{s, r} - t_u p_{s, r}^x \left( V_{s, r} + \sqrt{\frac{1}{V_{s, r}^2 p_{s, r}^2 - p_{s, r} \frac{dV}{d\theta}}} \right) \left( \sqrt{1 - V_{s, r}^2 p_{s, r}^2 - p_{s, r} \frac{dV}{d\theta}} \right) \quad (C2)$$

$$p_{m, y}^x = \frac{1}{V_{p0}} \frac{p_{s, y}^x + p_{r, y}^x}{\sqrt{\frac{1}{p_s^2} - \frac{1}{p_r^2} \left( \sqrt{1 - V_{s, r}^2 p_{s, r}^2 - p_{s, r} \frac{dV}{d\theta}} \right) \left( \sqrt{1 - V_{s, r}^2 p_{s, r}^2 - p_{s, r} \frac{dV}{d\theta}} \right) \left( \sqrt{1 - V_{s, r}^2 p_{s, r}^2 - p_{s, r} \frac{dV}{d\theta}} \right)}} \quad (C3)$$

where $V_{s, r}$ is the phase-velocity at the source or receiver, $p_{s, r}$ is the horizontal slowness at the source or receiver, $h$ is the half-offset, and $p_{s, r} = \sqrt{(p_{s, r}^x)^2 + (p_{s, r}^y)^2}$. Note the order of the subscripts $s, r$ and $r, s$ in eq.(C2).

Comparison of eq.(C1) with eq.(1) reveals that the right hand side of eq.(C1) involves the sum of two fractions with denominators that have the same form as $V_{p0} t_m / t_u$ in eq.(1). Using the weak anisotropy approximation to this form (cf. eq.(B4)) to approximate both fractions in eq.(C1), it follows that the weak anisotropy approximation to $t_m$ for pre-stack map migration is given by

$$t_m \approx 2u_s \left( \sum_{s, r} \frac{1}{1 - p_s^2 V_{NMO(0)}^2} - 4p_s^4 V_{NMO(0)}^4 \left\{ \frac{1}{2} - p_s^2 V_{NMO(0)}^2 \right\} \right)^{-1} \quad (C4)$$

Inspection of eqs.(C1) and (C2) reveals that eq.(C2) can be rewritten as

$$(x, y)_m = (x, y)_{s, r} - \frac{V_{p0} p_{s, r}^x t_m}{2} \frac{V_{s, r}^2 + \sqrt{\frac{1}{p_s^2} - \frac{V_{s, r}^2}{p_s^2} \left( \sqrt{1 - V_{s, r}^2 p_{s, r}^2 - p_{s, r} \frac{dV}{d\theta}} \right) \left( \sqrt{1 - V_{s, r}^2 p_{s, r}^2 - p_{s, r} \frac{dV}{d\theta}} \right)}} {V_{s, r} \left( \sqrt{1 - V_{s, r}^2 p_{s, r}^2 - p_{s, r} \frac{dV}{d\theta}} \right)} \quad (C5)$$

Since we already have a weak anisotropy approximation for $t_m$ (cf. eq.(C4)), to find the approximated expression for $(x, y)_m$, we need only consider the approximation of the fraction

$$A \equiv \frac{V_{s, r}^2 + \sqrt{\frac{1}{p_s^2} - \frac{V_{s, r}^2}{p_s^2} \left( \sqrt{1 - V_{s, r}^2 p_{s, r}^2 - p_{s, r} \frac{dV}{d\theta}} \right) \left( \sqrt{1 - V_{s, r}^2 p_{s, r}^2 - p_{s, r} \frac{dV}{d\theta}} \right)}} {V_{s, r} \left( \sqrt{1 - V_{s, r}^2 p_{s, r}^2 - p_{s, r} \frac{dV}{d\theta}} \right)} \quad (C6)$$

The denominator in this fraction is again of the same form as $V_{p0} t_m / t_u$ in eq.(1) with its weak anisotropy approximation given in eq.(B4). We thus have

$$V_{s, r} \left( \sqrt{1 - V_{s, r}^2 p_{s, r}^2 - p_{s, r} \frac{dV}{d\theta}} \right) \approx V_{p0} \left( 1 - p_s^2 V_{NMO(0)}^2 - 4p_s^4 V_{NMO(0)}^4 \left\{ \frac{1}{2} - p_s^2 V_{NMO(0)}^2 \right\} \right) \quad (C7)$$
Hence we only need look for a weak anisotropy approximation to the numerator of $A$. Using eqs.(B1) and (B2) and consistently linearizing in the Thomsen parameters gives

$$V_{s,r}^2 + \sqrt{\frac{1}{p_{s,r}^2} - V_{s,r}^2} \frac{dV}{dp} \bigg|_{s,r} \approx V_{s,r}^2 NMO(0) \left\{ 1 + 4\eta p_{s,r}^2 V_{s,r}^2 NMO(0) \left[ 1 - \frac{p_{s,r}^2 V_{s,r}^2 NMO(0)}{2} \right] \right\}. \quad (C8)$$

Then, using this approximation in eq.(C6) together with eq.(C7) we get

$$A \approx \frac{V_{s,r}^2 NMO(0) \left\{ 1 + 4\eta p_{s,r}^2 V_{s,r}^2 NMO(0) \left[ 1 - \frac{p_{s,r}^2 V_{s,r}^2 NMO(0)}{2} \right] \right\}}{V_P \sqrt{1 - p_{s,r}^2 V_{s,r}^2 NMO(0) - 4\eta p_{s,r}^2 V_{s,r}^2 NMO(0) \left\{ \frac{1}{2} - p_{s,r}^2 V_{s,r}^2 NMO(0) \right\}}}. \quad (C9)$$

Finally, using this expression together with eq.(C4) in eq.(C5) then gives

$$\frac{(x,y)_s - p_{x,y}^s t_i V_{s,r}^2 NMO(0) \left\{ 1 + 4\eta p_{s,r}^2 V_{s,r}^2 NMO(0) \left[ 1 - \frac{p_{s,r}^2 V_{s,r}^2 NMO(0)}{2} \right] \right\}}{ \sum_{i=s,r} \sqrt{1 - p_{i}^2 V_{i}^2 NMO(0) - 4\eta p_{i}^2 V_{i}^2 NMO(0) \left\{ \frac{1}{2} - p_{i}^2 V_{i}^2 NMO(0) \right\}}}. \quad (C10)$$

Note the order of the subscripts $s, r$ and $r, s$.

Using the weak anisotropy approximation in eq.(C3) and further linearizing in $\delta$ and $\epsilon$, I find for the horizontal slowness components in the subsurface

$$p_{m}^{x,y} \approx \frac{p_{x,y}^s + p_{x,y}^r}{\sum_{i=s,r} \sqrt{1 - p_{i}^2 V_{i}^2 NMO(0) (1 + 2\eta p_{i}^2 V_{i}^2 NMO(0))}}. \quad (C11)$$

Note that indeed all weak anisotropy approximations to the migrated quantities for pre-stack migration, reduce to their post-stack equivalent expressions if we set $p_{x,y}^s = p_{x,y}^r = p_{u}^{x,y}$. 

Migration velocity analysis in factorized VTI media

Debashish Sarkar & Ilya Tsvankin
Center for Wave Phenomena, Dept. of Geophysics, Colorado School of Mines

ABSTRACT
One of the main challenges in anisotropic velocity analysis and imaging is reliable estimation of both velocity gradients and anisotropic parameters from reflection data. Approximating the subsurface by a factorized VTI (transversely isotropic with a vertical symmetry axis) medium provides a convenient way of building vertically and laterally heterogeneous anisotropic models for prestack migration. The algorithm for P-wave migration velocity analysis (MVA) introduced here is designed for 2-D models composed of factorized VTI layers or blocks with constant vertical and lateral gradients in the vertical velocity \( V_p \).

The anisotropic MVA method is implemented as an iterative two-step procedure that includes prestack depth migration (imaging step) followed by an update of the medium parameters (velocity-analysis step). The iterations for a particular block continue until the corresponding reflection events in image gathers are sufficiently flat. The residual moveout of the migrated events, which is needed to compute parameter updates, is described by a nonhyperbolic equation governed by the moveout parameters determined from semblance analysis.

For piecewise-factorized VTI media treated here, the residual moveout of P-wave events in image gathers is governed by four effective quantities in each block: (1) the normal-moveout (NMO) velocity \( V_{nmo} \) at a certain point within the block, (2) the vertical velocity gradient \( k_z \), (3) the combination \( \bar{k}_x = k_x \sqrt{1+2\delta} \) of the lateral velocity gradient \( k_x \) and the anisotropic parameter \( \delta \), and (4) the anellipticity parameter \( \eta \). We show that all four parameters can be estimated from the residual moveout for at least two reflectors within a block and establish the minimum depth separation between the reflectors and the minimum lateral distance to be covered by the image gathers. Stable inversion for the parameter \( \eta \) also requires using either long-spread data (with the maximum offset-to-depth ratio no less than two) from horizontal interfaces or reflections from dipping interfaces.

To find the depth scale of the section and build a model for prestack depth migration using the MVA results, the vertical velocity \( V_p \) needs to be specified for at least a single point in each block. When no borehole information about \( V_p \) is available, a well-focused image can often be obtained by assuming that the vertical-velocity field is continuous across layer boundaries. A synthetic test for a three-layer model with a syncline structure confirms the accuracy of our MVA algorithm in estimating the interval parameters \( V_{nmo}, k_z, \bar{k}_x, \) and \( \eta \) and illustrates the influence of errors in the vertical velocity on the image quality.

1 INTRODUCTION
Most existing velocity-analysis methods for VTI (transversely isotropic with a vertical symmetry axis) media approximate the subsurface with homogeneous or vertically heterogeneous layers or blocks (e.g., Alkhalifah and Tsvankin, 1995; Le Stunff and Jeannot, 1998; Tsvankin, 2001; Grechka et al., 2002). Anisotropic layers, however, are often characterized by non-negligible lateral velocity gradients that may distort the shape of underlying reflectors and cause errors in the anisotropic parameters. Since lateral homogeneity is an inherent assumption in time imaging, whether isotropic or anisotropic, it is justified to ignore lateral gradients in the time-domain velocity analysis of P-waves in VTI media (e.g., Alkhalifah, 1997; Han et al., 2000). In contrast, anisotropic
depth imaging has to account properly for both vertical and lateral variations of the velocity field.

An analytic correction of normal-moveout (NMO) ellipses for lateral velocity variation in anisotropic media was developed by Grechka and Tsvankin (1999). Their method, however, is limited to horizontal layers, small lateral velocity gradients, and the hyperbolic portion of reflection moveout. Also, for purposes of depth imaging, we are interested in estimating lateral velocity variation rather than just removing its influence on anisotropic inversion. The main problem in reconstructing a spatially varying anisotropic velocity field is caused by the trade-offs between the velocity gradients, anisotropic parameters, and the shapes of the reflecting interfaces. Even in isotropic media, some trade-offs between the velocity field and reflector shapes cannot be resolved even in isotropic media without a priori information. A practical way to incorporate vertical and lateral velocity variations into anisotropic velocity analysis without excessively compromising the uniqueness of the solution is to adopt the so-called factorized anisotropic model in which the ratios of the stiffness coefficients (and, therefore, the anisotropic parameters) are constant.

Here, we consider a model composed of factorized VTI layers, where each block is bounded by plane or irregular interfaces. The problem is treated in two dimensions, which implies that the vertical incidence plane that contains sources and receivers should coincide with the dip plane of the subsurface structure. The vertical P-wave velocity \( V_{P0} \) is assumed to vary linearly within each block, so the vertical \( k_z \) and lateral \( k_x \) gradients in \( V_{P0} \) are constant. The kinematics of P-wave propagation in each block can be described by five parameters: the velocity \( V_{P0} \) defined at a certain spatial location, the gradients \( k_z \) and \( k_x \), and Thomsen (1986) anisotropic parameters \( \epsilon \) and \( \delta \). Although it is possible to introduce jumps in velocity across the boundaries of the blocks, this model can be conveniently used to generate smooth velocity fields required by many migration algorithms (in particular, those based on ray tracing).

Since our goal is to estimate the relevant VTI parameters and carry out depth imaging for models with significant lateral and vertical velocity variation and considerable structural complexity, velocity model building is conveniently implemented in the prestack depth-migrated domain (e.g., Stork, 1991; Liu, 1997). Parameter estimation in the post-migrated domain, usually referred to as migration velocity analysis (MVA), consists of two main steps: (1) parameter update designed to minimize the residual moveout of events in the image gathers, and (2) prestack depth migration that creates an image of the subsurface using the updated parameters estimated in step (1). These two steps are iterated until the events in the image gathers are sufficiently flat. Note that MVA is quite robust in the presence of random noise because migration improves the signal-to-noise ratio (Gardner et. al., 1974; Liu, 1997).

The parameter-estimation methodology employed here is based on the results of Sarkar and Tsvankin (2002), hereafter referred to as Paper I. Although depth imaging of P-wave data requires knowledge of the five parameters listed above \( (\epsilon, k_z, k_x, \epsilon, \delta) \), each block, Paper I shows that the moveout of events in image gathers is governed by the following four combinations of these parameters:

1. The NMO velocity at a certain point on the surface of the factorized layer or block: \( V_{nmo} \equiv V_{P0} \sqrt{1 + 2\delta} \).
2. The vertical velocity gradient \( k_z \).
3. The lateral velocity gradient combined with the parameter \( \delta: k_x = k_x \sqrt{1 + 2\delta} \).
4. The anellipticity parameter \( \eta \equiv (\epsilon - \delta)/(1 + 2\delta) \).

If prestack migration is performed with the correct values of \( V_{nmo}, k_z, k_x, \) and \( \eta \), the image gathers for reflections from both horizontal and dipping interfaces are flat. To decouple the horizontal gradient \( k_x \) from the coefficient \( \delta \) and determine the other anisotropic coefficient \( \epsilon \), the velocity \( V_{P0} \) has to be known at a certain point within the factorized block (Paper I).

The paper starts with a discussion of the minimum information required to estimate the model parameters from P-wave moveout data. Then we give a description of the MVA methodology including nonhyperbolic moveout analysis on image gathers needed to constrain the anisotropic velocity field. The accuracy of the algorithm and its robustness in the presence of random noise are assessed by synthetic tests for a single layer and a multilayered factorized VTI medium. We also discuss different ways to specify the vertical velocity \( V_{P0} \) and the influence of errors in \( V_{P0} \) on the inverted values of the other parameters and on the quality of the migrated image.

## 2 PARAMETER ESTIMATION IN A FACTORIZED VTI LAYER

Here, we use the results of Paper I to evaluate the feasibility of estimating the parameters of a factorized VTI layer from P-wave reflection data. By replacing the actual factorized \( v(x, z) \) model with narrow vertical strips of factorized \( v(z) \) media, Paper I demonstrates that the moveout of a single horizontal event in an image gather is governed by the effective values of the NMO velocity and the parameter \( \eta \):

\[
\eta(x, t_0) = \frac{1}{8} \left( \frac{1 + 8\eta}{2(\epsilon + \delta)} + 1 \right) \quad (2)
\]

\[
V(x) \equiv V_{P0} + k_z x \quad \text{is the vertical P-wave velocity at the surface, and } t_0 \equiv t_0(x, z) \quad \text{is the zero-offset time at location } x \text{ from a horizontal reflector at depth } z.
\]

If long-offset data needed to constrain \( \eta \) (Grechka...
and Tsvankin, 1998) have been acquired, moveout analysis of a single event can yield estimates of both \( v_{nmo}(x, t_0) \) and \( \eta(x, t_0) \). Next, suppose that P-wave traveltimes from two horizontal reflectors sufficiently separated in depth are available. Then the ratio of the NMO velocities for these two events \( \left( \frac{v_{nmo,1}}{v_{nmo,2}} \right) \) can be used to find \[ \text{equation (1)} \]

\[
\frac{v^2_{nmo,1}(x, t_0,1)}{v^2_{nmo,2}(x, t_0,2)} = \frac{t_{0,2} (e^{2k_z t_{0,1}} - 1)}{t_{0,1} (e^{2k_z t_{0,2}} - 1)},
\]

where \( t_{0,1} \) and \( t_{0,2} \) are the zero-offset times for the two events. According to equation (3), conventional hyperbolic moveout analysis of two horizontal events located in the same factorized block can provide an estimate of the vertical gradient \( k_z \). Knowledge of \( k_z \) and the zero-offset time \( t_0 \) is sufficient for obtaining the anellipticity parameter \( \eta \) from equation (2) applied to one or both reflection events. The remaining two key quantities, \( V_{nmo} = V_{p0} \sqrt{1 + 2\eta} \) and \( k_x = k_x \sqrt{1 + 2\eta} \), can then be computed from equation (1), if the effective NMO velocities are determined at two or more locations \( x \).

We conclude that the moveout of horizontal events at two different depths and two image locations provides enough information to estimate the parameters \( V_{nmo}, k_z, k_x, \) and \( \eta \). For the special case of a factorized \( v(z) \) medium with a constant vertical gradient \( k_z \), the moveouts of two horizontal events at a single image location can be inverted for the parameters \( V_{nmo}, k_z, \) and \( \eta \). As shown in Paper I, reflection moveout of dipping events in factorized \( v(x, z) \) VTI media is controlled by the same parameters \( (V_{nmo}, k_z, k_x, \) and \( \eta \)) as that of horizontal events. Most importantly, NMO velocity of events dipping at 25-30° or more is highly sensitive to the parameter \( \eta \) (Alkhalilah and Tsvankin, 1995; Tsvankin, 2001), whereas the inversion of nonhyperbolic moveout from horizontal reflectors for \( \eta \) may suffer from instability (Grechka and Tsvankin, 1998). Therefore, the inclusion of dipping events in velocity analysis is helpful in obtaining accurate estimates of \( \eta \); also, dip-dependent reflection moveout provides additional information about the parameters \( V_{nmo}, k_z, \) and \( k_x \).

Still, even if both horizontal and dipping events are available, the parameters \( V_{p0}, k_z, \) and \( \delta \) remain generally unconstrained by P-wave reflection traveltimes. In particular, the vertical velocity \( V_{p0} \) is needed to define the depth scale of the VTI model in the migration of P-wave data. Hence, to build an anisotropic model for depth imaging, at least one medium parameter must be known a priori. Unless specified otherwise in the synthetic data examples discussed below, the velocity \( V_{p0} \) is assumed known at some location on the surface of each factorized layer. Given this information about \( V_{p0} \), we can use velocity analysis of P-wave data to estimate the parameters \( k_z, k_x, \) and \( \delta \).

### 3 ALGORITHM FOR MIGRATION VELOCITY ANALYSIS

Inversion of seismic data is a nonlinear problem that can be solved through an iterative application of migration and velocity updating. Migration creates an image of the subsurface for trial values of the medium parameters, and then velocity analysis is used to update the model for the next run of the migration code. This iterative procedure, conventionally called migration velocity analysis (MVA), is continued until a certain criterion (e.g., small residual moveout of events in image gathers) is satisfied.

Here, we apply anisotropic prestack depth migration (the migration algorithm is described in detail in Paper I) and tomographic velocity update to P-wave data acquired over the subsurface composed of factorized \( v(x, z) \) VTI blocks. The iterations are stopped when the residual moveout for at least two reflectors in each factorized block is close to zero (i.e., the migrated depth stays the same to within a specified fraction of the wavelength for different offsets). The overall organization of our MVA algorithm is similar to that developed by Liu (1997) for isotropic media, but the VTI model is characterized, for P-waves, by two additional parameters – \( \epsilon \) and \( \delta \).

The tomographic update of the medium parameters is based entirely on the residual moveout of events in image gathers. For horizontal reflectors embedded in a weakly anisotropic homogeneous VTI medium, the migrated depth \( z_M \) in image gathers can be written as (Paper I):

\[
z^2_M(h) \approx z^2_M(0) + h^2 V^2_{p0,M} \left( \frac{1}{V^2_{nmo,M}} - \frac{1}{V^2_{nmo,T}} \right) + \frac{2h^4 \left( \eta_M \frac{V^2_{nmo,M}}{V^2_{nmo,T}} - \eta_T \frac{V^2_{nmo,T}}{V^2_{nmo,M}} \right)}{h^2 + z^2_p},
\]

where the subscripts \( T \) and \( M \) denotes the true and migration medium parameters, respectively, \( h \) is the half-offset, and \( z_p \) is the true zero-offset depth of the reflector. Equation (4) is nonhyperbolic and governed by both independent parameters – \( V_{nmo} \) and \( \eta \). The NMO velocity \( V_{nmo} \) controls the hyperbolic (described by the \( h^2 \)-term) part of the moveout curve and also contributes to the nonhyperbolic \( (h^n) \) term, while \( \eta \) influences nonhyperbolic moveout only. A similar closed-form expression is not available for dipping reflectors, but both the hyperbolic and nonhyperbolic portions of the residual moveout curve for dipping events also depend on \( V_{nmo} \) and \( \eta \) (Paper I).

As discussed above, the residual moveout of P-waves in factorized \( v(x, z) \) VTI media is a function of the parameters \( V_{nmo}, k_z, k_x, \) and \( \eta \). Although it is difficult to express the migrated depth \( z_M \) in laterally heterogeneous media analytically in terms of these parameters, the residual moveout equation can be cast in a form
similar to that in equation (4):
\[ z_m^2(h) \approx z_m^2(0) + A h^2 + B \frac{2h^4}{h^2 + z_m^2(0)}. \] (5)

A and B are dimensionless constants that describe the hyperbolic and nonhyperbolic portions of the moveout curve, respectively. Numerical tests (see below) confirm that the functional form in equation (5) with fitted coefficients A and B provides a good approximation for P-wave moveout in long-spread image gathers.

To apply equation (5) in velocity analysis, we first pick an approximate value of the zero-offset reflector depth \( z_m(0) \) on the migrated stacked section. The parameters A and B are obtained by a 2-D semblance scan on image gathers at each migrated zero-offset depth point. The best-fit combination of A and B that maximizes the semblance value is substituted into equation (4) to update the medium parameters. The superscript \( T \) denotes the transpose, and \( b \) contains the migrated depths that define the residual moveout. The full definitions of the matrix \( A \) and vector \( b \) are given in Appendix A.

For all examples described below, each iteration of the MVA consists of the following four steps:
1. prestack depth migration with a given estimate of the medium parameters;
2. picking along two reflectors in each VTI block to delineate the reflector shapes;
3. semblance scanning using equation (5) to estimate A and B for image points along each reflector;
4. application of equation (6) to update the medium parameters in such a way that the variance of the migrated depths as a function of offset is minimized (see Appendix A for more details about the minimization procedure).

Steps 1–4 are repeated until the magnitude of residual moveout of events in image gathers becomes sufficiently small.

\[ A^T A \Delta \lambda = A^T b. \] (6)

Here \( A \) is a matrix with \( M \times P \) rows (\( P \) is the total number of image gathers used in the velocity analysis and \( M \) is the number of offsets) and \( N \) columns that includes the derivatives of the migrated depth with respect to the medium parameters. The superscript \( T \) denotes the transpose, and \( b \) contains the migrated depths that define the residual moveout. The full definitions of the matrix \( A \) and vector \( b \) are given in Appendix A.

Figure 1. True image of two reflectors embedded in a factorized \( v(x, z) \) VTI medium with the parameters \( V_{P0}(x = 3 \text{ km}, z = 0) = 2600 \text{ m/s}, k_z = 0.6 \text{ s}^{-1}, k_x = 0.2 \text{ s}^{-1}, \epsilon = 0.1, \) and \( \delta = -0.1. \) The corresponding effective parameters are \( V_{P0}(x = 3 \text{ km}, z = 0) = 2326 \text{ m/s}, k_z = 0.6 \text{ s}^{-1}, k_x = 0.18 \text{ s}^{-1}, \) and \( \eta = 0.25. \)

Figure 2. (a) Image of the model from Figure 1 obtained using a homogeneous isotropic velocity field with \( V_{P0} = 2600 \text{ m/s}. \) (b) Semblance contour plot computed from equation (5) for the shallow reflector at the surface location 3 km.

4 EXAMPLE WITH A SINGLE FACTORIZED LAYER

First, we consider two irregular reflectors embedded in a factorized \( v(x, z) \) VTI medium with \( k_z > k_x > 0 \) and a positive value of \( \eta \) typical for shale formations (Figure 1). For the first application of prestack depth migration, we choose a homogeneous, isotropic medium \( (V_{P0} = 2600 \text{ m/s}, k_z = k_x = \epsilon = \delta = 0) \) as the initial velocity model. The migrated stacked image in Figure 2a is clearly inferior to the true image in Figure 1. We start the velocity-updating process by manually picking along both imaged reflectors to outline their shapes. Then
A and B is similar to that between the NMO velocity of the migrated depths. Note that the interplay between most semblance contour gives almost the same variance location. Although a certain degree of trade-off exists vide an accurate description of residual moveout at this process. The maximum semblance coefficient in Figure 2b pro-

Figure 2b. The values of A and B that correspond to the re
ector at the surface coordinate 3 km is displayed in Figure 4. Stacked image after (a) four iterations; (b) eight iterations. The residual moveout is minimized during the velocity-updating process.

Figure 3. Stacked image after (a) four iterations; (b) eight iterations. Residual moveout in image gathers for both re
ectors at the surface location 3 km: (a) for the initial model; (b) after two, (c) four, (d) six, and (e) eight iterations. The magnitude of the residual moveout for both re
ectors during the velocity update. The magnitude of the residual moveout for both re
ectors decreases as the model parameters converge toward their actual values (Figure 4). The velocity-updating procedure is stopped after eight iterations because events in all analyzed image gathers are practically "flat."

For purposes of velocity analysis, we use the image gathers at 12 equally spaced surface locations between 3 km and 4.2 km. The maximum offset-to-depth ratio for the selected image gathers at the shallow re
ector is close to two, which is marginally suitable for estimating the parameter $\eta$. Tighter constraints on $\eta$ are provided by the NMO velocities of reflections from the dipping segments of the shallow re
ector (the dips exceed 30° in the middle of the section).

After the residual moveout has been evaluated, we fix the vertical velocity $V_{P0}(x = 3000 \text{ m}, z = 0) = 2600 \text{ m/s}$ at the correct value and update the parameters $k_1$, $k_2$, $\epsilon$, and $\delta$ using equation (6). The stacked images after four (Figure 3a) and eight (Figure 3b) iterations illustrate the improvements in the focusing and positioning of both re
ectors during the velocity update. The accurate results of the above test were obtained with the correct value of the vertical velocity at a given point on the surface of the factorized layer. Next, we apply the MVA method with an erroneous value of $V_{P0}(x = 3 \text{ km}, z = 0) = 2000 \text{ m/s}$, which is 23% smaller than the true velocity (2600 m/s). The stacked images of both re
ectors obtained after the velocity analysis (Figure 5a) are well focused, which indicates that the im-
signal-to-noise ratio, measured as the ratio of the peak amplitude of the signal to the root-mean-square (rms) amplitude of the background noise, is about 1.5, and the frequency bands of the noise and signal are identical (Figure 6a). The estimates of the medium parameters obtained after the migration velocity analysis with the correct value of $V_{P0}$ at the surface location 3 km are as follows: $k_x = 0.56 \pm 0.04 \text{ s}^{-1}$, $k_z = 0.2 \pm 0.0 \text{ s}^{-1}$, $\epsilon = 0.12 \pm 0.02$, and $\delta = -0.09 \pm 0.02$. The error bars were computed in the same way as those for the noise-free synthetic example above (Figure 3), but the depth-picking error for all offsets and image locations was assumed to be 15 m instead of 5 m. Clearly, the noise contamination did not cause measurable errors in the medium parameters or noticeable distortions in the stacked image (Figure 6b).

Even for the much more severely contaminated data set in Figure 7, the inverted medium parameters are close to the actual values: $k_x = 0.52 \pm 0.07 \text{ s}^{-1}$, $k_z = 0.2 \pm 0.01 \text{ s}^{-1}$, $\epsilon = 0.13 \pm 0.03$, and $\delta = -0.07 \pm 0.03$. Here the error bars were computed under the assumption that the noise increased the depth-picking error to 20 m. (Since the dominant wavelength in this example was about 80 m, picking errors are unlikely to exceed 20 m, even for a substantial level of noise.) Also, despite the low signal-to-noise ratio, the migrated stacked section in Figure 7b has a sufficiently high quality, comparable to that of the true image in Figure 1.

We conclude that, the migration velocity analysis employed here gives reliable estimates of the anisotropic parameters and velocity gradients in the presence of random noise. One aiding factor is that the MVA operates on migrated data, which have a higher signal-to-noise ratio than do those of the original records because of partial stacking applied to the data during the migration step. The semblance (coherency) operator used to evaluate the residual moveout on image gathers also contributes to the robustness of the parameter estimation by suppressing remaining random noise in the migrated data.

6 SENSITIVITY STUDY

The above results demonstrate that, in principle, the residual moveout from two reflectors in a factorized layer is sufficient to estimate the four key parameters $V_{mono}$, $k_x$, $k_z$, and $\eta$. This section is devoted to two important practical issues related to the implementation of our algorithm. By performing a series of numerical tests, we establish the minimum depth separation between the two reflectors and the minimum lateral spread of the image gathers (i.e., the difference between the largest and smallest surface coordinates of the image locations) needed for stable parameter estimation.

Consider two horizontal reflectors embedded in the factorized $v(x, z)$ medium with the parameters listed in

5 MVA IN THE PRESENCE OF NOISE

To evaluate the influence of noise on the estimation of the medium parameters and the quality of imaging, we added Gaussian noise to the data set from Figure 1. The
Figure 8. Influence of the vertical distance $d$ between the two horizontal reflectors used in the velocity analysis on the absolute error in the vertical gradient $k_z$. The depth of the shallow reflector is 1 km; the maximum offset is 2 km for the upper curve and 1.5 km for the lower curve. The model parameters are $V_{P0}(x = 3 \text{ km}, z = 0) = 2000 \text{ m/s}$, $k_z = 0.6 \text{ m/s}$, $k_x = 0.2 \text{ m/s}$, $\epsilon = 0.2$, and $\delta = 0.1$.

Figure 9. Influence of the distance between the two horizontal reflectors on the absolute error in the horizontal gradient $k_x$. The parameters are the same as in Figure 8.

Figure 10. Influence of the distance between the two horizontal reflectors on the absolute error in the parameter $\epsilon$. The parameters are the same as in Figure 8.

Figure 11. Influence of the distance between the two horizontal reflectors on the absolute error in the parameter $\delta$. The parameters are the same as in Figure 8.

Figure 12. Influence of the lateral spread of the image gathers on the absolute error in $k_x$. The reflector depths are 1 km and 1.2 km; the other parameters are the same as in Figure 10. The velocity analysis is performed on 12 image gathers, each with 20 offsets.

The error initially decreases rapidly with increasing $d$ and then becomes almost constant as $d$ approaches 500 m. For a maximum offset-to-depth ratio (at the shallow reflector) of two, the error curves flatten out for $d \approx 250$ m, which is equal to 1/5 of the depth of the bottom reflector. The depth of the shallow reflector is fixed at 1000 m, while the depth of the second reflector varies from 1050 m to 2000 m. Figures 8–9 illustrate the dependence of the error in the estimated parameters $k_z$, $k_x$, $\epsilon$, and $\delta$ on the distance between the reflectors. The errors in the parameters were computed from equation (6) assuming that the error in picking the migrated depths is ±5 m. The velocity analysis operated with the residual moveout on 12 image gathers (with 20 offsets each) whose horizontal coordinates span a distance of 1200 m. For all tests, the vertical velocity at one location on the surface was held at the correct value.

For the parameters $k_z$, $\epsilon$ and $\delta$, the dependence of the estimated error on the distance $d$ between the reflectors has a similar character (Figures 8, 10, and 11). The error initially decreases rapidly with increasing $d$ and then becomes almost constant as $d$ approaches 500 m. For a maximum offset-to-depth ratio (at the shallow reflector) of two, the error curves flatten out for $d \approx 250$ m, which is equal to 1/5 of the depth of the bottom reflector.
d. The curve flattens out for a larger depth $d \approx 350$ m ($\approx 1/4$ of the depth of the bottom reflector).

This behavior of the error curves is in good agreement with the analysis of the effective NMO velocity and parameter $\eta$ in Paper I. Accurate estimation of the vertical gradient $k_z$, and then the NMO velocity at the surface of the factorized layer, requires a sufficiently large difference between the NMO velocities of the two events used in the velocity analysis [see equation (3)]. In other words, the reflectors should be sufficiently separated in depth to resolve the interval NMO velocity, which carries information about the gradient $k_z$. An accurate estimate of $k_z$ makes it possible to obtain $V_{nmo}$ at the surface and then, using the nonhyperbolic portion of the moveout curve, the parameter $\eta$. The minimum suitable vertical distance $d$ found here is close to the minimum layer thickness conventionally assumed in interval velocity estimation based on the Dix equation.

In contrast, the error in the horizontal gradient $k_x$ is practically insensitive to variations in the distance between the two reflectors (Figure 9) because the lateral spread of the coordinates of the image gathers is kept constant at 1.2 km. The influence of the maximum horizontal distance between the image gathers on the error in $k_x$ is shown in Figure 12. As expected, the gradient $k_x$ becomes better constrained with increasing lateral spread of the image gathers, with the error curve flattening out for spreads exceeding 300-400 m.

Note that the errors in all parameters reduce with increasing number of offsets in the image gathers, which can influence the sensitivity estimates. Although the results of the error analysis also depend on the anisotropic coefficients $\epsilon$ and $\delta$ and the velocity gradients, this dependence is not significant if the velocity update is performed with reasonable constraints on the model parameters.

7 TEST FOR A MULTILAYERED MODEL

After performing a series of tests for a single factorized layer, we apply the algorithm to a three-layer model shown in Figure 13. Each layer contains two reflecting interfaces, as required in our method, with every second reflector serving as the boundary between layers. The first and third layers are vertically heterogeneous $[v(z)]$ and isotropic, while the second layer is a factorized, laterally heterogeneous $[v(x, z)]$ medium. All interfaces are quasi-horizontal, with the largest dips (at the flanks of the syncline) of $10^\circ$ or less. The model is designed to represent a typical depositional environment in the Gulf of Mexico, where anisotropic shale layers (the middle layer in Figure 13) are often embedded between isotropic sands.

For the velocity analysis we use image gathers located along the left flank of the syncline with surface coordinates ranging from 4400 m to 5600 m; the maximum offset-to-depth ratio for the image gathers is close to two. The medium parameters are estimated in the layer-stripping mode starting at the surface. For the first (top) layer, the vertical velocity is assumed to be known at a single surface location [$V_{p0}(x = 4000$ m, $z = 0$ m) = 1500 m/s]. The chosen value of $V_{p0}$ corresponds to that for water-bottom sediments; on land, $V_{p0}$ at the

![Figure 13. True image of a three-layer factorized medium. Every second reflector (indicated here with arrows) represents the bottom of a layer. The parameters of the first subsurface layer are $V_{p0}(x = 4000$ m, $z = 0$ m) = 1500 m/s, $k_x = 1.0$ s$^{-1}$, and $k_z = \epsilon = \delta = 0$; for the second layer, $V_{p0}(x = 4000$ m, $z = 800$ m) = 2300 m/s, $k_x = 0.6$ s$^{-1}$, $k_z = 0.1$ s$^{-1}$, $\epsilon = 0.1$, and $\delta = -0.1$; for the third layer, $V_{p0}(x = 4000$ m, $z = 1162$ m) = 2718 m/s, $k_x = 0.3$ s$^{-1}$, and $k_z = \epsilon = \delta = 0$.](image)

![Figure 14. Estimated (○) and true (⋆) parameters of the first layer obtained using the correct $V_{p0}(x = 4000$ m, $z = 0$ m) = 1500 m/s on the surface.](image)

![Figure 15. Estimated (○) and true (⋆) parameters of the second layer obtained using the correct $V_{p0}(x = 4000$ m, $z = 800$ m) = 2300 m/s at the layer’s top.](image)
Migration velocity analysis and anisotropy

Figure 16. Estimated (o) and true (\(\ast\)) parameters of the third layer obtained using the correct \(V_P0(x = 4000\ m, z = 1162\ m) = 2718\ m/s\) at the layer’s top.

Figure 17. Stacked image obtained after prestack depth migration using the estimated parameters from Figures 14–16. The vertical velocity \(V_P0\) at the top of each layer was known.

The top of the model may be estimated from near-surface velocity measurements. Starting with a homogeneous isotropic model \((V_P0 = 1500\ m/s)\) the parameters \(k_z, k_x, \epsilon, \) and \(\delta\) in the first layer, obtained from the migration velocity analysis with the correct vertical velocity \(V_P0(x = 4000\ m, z = 0\ m)\), are close to the true values (Figure 14).

To estimate the medium parameters in the second and third layers, we need to fix the vertical velocity at a certain spatial location in each layer. Three different scenarios for choosing \(V_P0\) in the second and third layers are examined below.

Figure 18. Estimated (o) and true (\(\ast\)) parameters of the second layer obtained with an inaccurate value of the vertical velocity at the top of the second layer \([V_P0(x = 4000\ m, z = 800\ m) = 2600\ m/s]\) but the correct \(V_P0(x = 4000\ m, z = 1208\ m) = 2732\ m/s\) at the top of the third layer.

Figure 19. Estimated (o) and the true (\(\ast\)) parameters of the third layer obtained assuming that \(V_P0\) is continuous between the first and second layers at the point \((x = 3900\ m, z = 800\ m)\).

Figure 20. Stacked image obtained after prestack depth migration using the estimated parameters from Figures 14, 18, and 19. The vertical velocity \(V_P0\) at the top of the second layer was inaccurate.

7.1 \(V_P0\) at the top of each layer is known

Suppose a vertical borehole was drilled at the surface location 4000 m, and the vertical velocity at the top of the second and third layers was measured from sonic logs or check shots. Prestack depth migration with the estimated parameters of the first layer yields the depth of the top of the second layer at the surface location 4000 m. Using the correct value of the vertical velocity at this point \(V_P0(x = 4000\ m, z = 800\ m) = 2300\ m/s,\)

\[\begin{align*}
\text{surface coordinate (km)} & \quad \text{depth (m)} \\
0 & \quad 2000 \\
5 & \quad 2000 \\
10 & \quad 2000
\end{align*}\]
we carry out the velocity analysis for the second layer, which results in good estimates of all four parameters (Figure 15). Repeating the same procedure for the third layer with the velocity \( V_{P0}(x = 4000 \text{ m}, z = 1162 \text{ m}) = 2718 \text{ m/s} \), we obtain the interval parameters close to the true values (Figure 16).

The shapes and depths of the reflectors imaged for the reconstructed velocity model (Figure 17) closely resemble those on the true image (Figure 13). This test confirms that migration velocity analysis in layered factorized VTI \( v(x, z) \) media can be used to invert for the velocity gradients \( k_x \) and \( k_z \) and the anisotropic coefficients \( \varepsilon \) and \( \delta \) if the vertical velocity is known at a single point in each layer.

### 7.2 \( V_{P0} \) in the second layer is incorrect

Now suppose that the vertical velocity \( V_{P0}(x = 4000 \text{ m}, z = 800 \text{ m}) \) used for the top of the second layer has error (2600 m/s instead of 2300 m/s). Although this error in \( V_{P0} \) causes distortions in the inverted values of the other parameters (Figure 18), the effective quantities \( V_{true}(x = 4000 \text{ m}, z = 800 \text{ m}) = 2080 \text{ m/s}, k_x = 0.56 \text{ s}^{-1}, k_z = 0.09 \text{ s}^{-1} \), and \( \eta = 0.23 \) do not significantly differ from the true values, which corroborates our results for a single layer (Figure 5). Since the assumed value of \( V_{P0}(x = 4000 \text{ m}, z = 800 \text{ m}) \) is higher than the correct velocity, the second layer is stretched in depth by about 13%, and the bottom of the syncline is imaged at a depth that is 80 m too large (Figure 20). This depth stretch in the second layer also causes a tilt of the syncline’s flanks whose dips in Figure 20 exceed the true values.

To continue the velocity analysis, we fix the vertical velocity at the imaged top of the third layer at the correct value. Despite the depth shift of the top of the third layer, the algorithm yields accurate values of all four interval parameters (Figure 19). Because of the depth and dip distortions in the second layer, however, the two bottom reflectors are imaged at somewhat greater depths and are slightly deformed (Figure 20). In particular, on the left side of the section the fifth and sixth reflectors are no longer horizontal; they have acquired mild dips to conform to the stretched synclinal structure above.

### 7.3 \( V_{P0} \) is continuous across the boundaries

If no borehole information is available, one assumption that might be made is that the velocity \( V_{P0} \) is a continuous function of depth at a certain horizontal coordinate. To identify this point of continuity at the boundary between the first and second layers, we examine the moveout along the third and fourth reflectors (only for offsets smaller than 1000 m) after migration with an isotropic homogeneous velocity field in the second layer. The migration velocity was chosen to be equal to the estimated velocity at the bottom of the first layer (i.e., at the second reflector). To select the point of continuity, we pick the surface coordinate with the smallest residual moveout on the image gathers at the third and fourth reflectors. This criterion yielded \( x = 3900 \text{ m} \), which is sufficiently close to the true point of continuity for the second reflector \( (x = 4000 \text{ m}) \). Using the estimated vertical velocity at \( x = 3900 \text{ m} \) \( [V_{P0}(x = 3900 \text{ m}, z = 800 \text{ m}) = 2316 \text{ m/s}] \), we estimate the parameters of the second layer with high accuracy (Figure 21).

To find the point of continuity between the second and third layers, we again perform prestack depth migrations assuming that the third layer is homogeneous and isotropic. Since the second layer is laterally heterogeneous, the migration velocities range from 2400 m/s to 3400 m/s. Applying the criterion of minimum residual moveout for the fifth and the sixth reflectors, the point of continuity was found at \( (x = 5037 \text{ m}, z = 1483 \text{ m}) \), where the vertical velocity is \( V_{P0} = 2900 \text{ m/s} \). Although
the location \((x = 5937 \text{ m}, z = 1483 \text{ m})\) is shifted by almost 1000 m from the true continuity point between the second and third layers, the results of the velocity analysis (Figure 22) and imaging (Figure 23) are quite satisfactory.

In the absence of borehole data, the assumption of continuous vertical velocity provides a practical way to build an anisotropic heterogeneous model for prestack migration. Depending on the complexity of the model, however, the point of continuity may be estimated with a substantial lateral shift or may not exist at all. Still, our tests show that for models without steep dips or strong lateral heterogeneity, an error in identifying the point of continuity does not distort the effective parameters \(V_{\text{nmo}}, k_x, k_z, \text{ and } \eta\). Therefore, the migrated section would still be well focused, although the imaged reflectors would be subject to a depth stretch.

8 DISCUSSION AND CONCLUSIONS

Approximating heterogeneous VTI models by factorized blocks or layers with linear velocity variation, provides a convenient way to reconstruct anisotropic velocity fields for P-wave prestack imaging. The migration velocity analysis (MVA) algorithm introduced here estimates the anisotropic parameters and velocity gradients in each block by minimizing the residual moveout of P-wave reflection events in image gathers.

The residual moveout of both horizontal and dipping events in factorized VTI media is governed by four effective parameters – the NMO velocity \(V_{\text{nmo}}\) at the surface of the factorized block, the vertical gradient \(k_z\), the quantity \(\hat{k}_\varepsilon = k_x \sqrt{1 + 2 \delta}\) that contains the lateral velocity gradient \(k_x\) and the anisotropic parameter \(\delta\), and the anellipticity parameter \(\eta\). Application of our MVA method confirms the conclusion of Sarkar and Tsvankin (2002; Paper I) that stable recovery of the parameters \(V_{\text{nmo}}, k_z, \hat{k}_\varepsilon, \text{ and } \eta\) requires reflection moveout from at least two interfaces within each block sufficiently separated in depth.

Numerical tests indicate that the velocity-analysis algorithm yields robust estimates of the four parameters if the vertical distance between the two interfaces exceeds 1/4 of the depth of the bottom reflector. For a specific model, which may be typical of the subsurface, we also determined the minimum lateral spread in the image gathers for a stable recovery of the lateral gradient \(k_x\). Another essential condition for stable estimation of the parameter \(\eta\) is either the presence of dipping interfaces (dips should exceed 25°) or acquisition of long-spread data from subhorizontal reflectors providing maximum offset-to-depth ratios of at least two.

The residual moveout on image gathers for large offset-to-depth ratios was described by a nonhyperbolic function that depends on two independent moveout parameters. Although these parameters are not directly used in the velocity analysis, their best-fit values found from semblance search give an accurate approximation for the residual moveout. The MVA is implemented in an iterative fashion, with the residual moveout minimized at each iteration step by solving a system of linear equations for the parameter updates. Since the parameter estimation is performed in the post-migrated domain, the algorithm is robust in the presence of random noise and does not lose accuracy for models with significant lateral heterogeneity and dipping structures.

The main problem in the application of P-wave velocity analysis for VTI media is that the vertical velocity \(V_{p0}\), needed to build velocity models for depth migration, is generally unconstrained by P-wave reflection moveout (Alkhalifah and Tsvankin, 1995; Grechka et. al., 2002; Tsvankin, 2001). Also, the lateral gradient \(k_x\) is always coupled to the anisotropy coefficient \(\delta\) through the parameter \(\hat{k}_\varepsilon = k_x \sqrt{1 + 2 \delta}\). A priori knowledge of \(V_{p0}\) at any single point in the factorized block, however, is sufficient for estimating the true lateral gradient \(k_x\) and, therefore, reconstructing the spatially varying vertical-velocity field, as well as the Thomsen anisotropic parameters \(\epsilon\) and \(\delta\).

The vertical velocity can often be estimated from borehole data using either check shots or sonic logs. If no borehole information is available, a suitable model for depth imaging can be constructed by assuming that \(V_{p0}\) is continuous across layer boundaries. Then, given the value of the vertical velocity at a single point on the surface, the entire velocity model in depth can be estimated from the residual moveout of P-wave reflection events. The examples presented in the paper demonstrate that the assumption of continuity of \(V_{p0}\) offers a practical way to build reasonably accurate anisotropic velocity models that are particularly suitable for migration codes that require a smooth velocity field. As the level of structural complexity increases, however, the migration result becomes more dependent on the lateral location of the assumed continuity point, and the adopted continuous velocity field may cause errors in the final image.

For relatively simple models with subhorizontal interfaces, the distortions related to an error in the vertical velocity are limited to a depth stretch that can vary from one layer to another. In the presence of dipping interfaces, an overstated value of \(V_{p0}\) causes the imaged ones to be larger than the true dips; if \(V_{p0}\) is understated, the imaged dips are too small. In multilayered media, a depth stretch for dipping interfaces in the overburden can distort the shape of the underlying reflectors, even if the parameters immediately above these reflectors are estimated correctly.

Still, the examples given above show that the moveout of events in image gathers is not influenced by an incorrect choice of \(V_{p0}\), and the migrated image remains well focused as long as the algorithm yields accurate values of \(V_{\text{nmo}}, k_z, \hat{k}_\varepsilon, \text{ and } \eta\). This conclusion, however, may break down if the subsurface contains interfaces
with significant dip or curvature. Then P-wave reflection moveout and, therefore, residual moveout on image gathers become dependent on the vertical velocity and the parameters $\epsilon$ and $\delta$ (Le Stunff et al., 2001; Grechka et al., 2002). For models of this type, the layer-stripping approach adopted in our MVA algorithm is not always adequate because the parameters of a given layer may remain unconstrained in the absence of reflection data from deeper interfaces (Le Stunff et al., 2001).

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**References**


**APPENDIX A: ALGORITHM FOR VELOCITY UPDATE**

Following the approach suggested by Liu (1997), we design the velocity-updating algorithm to minimize the variance in the migrated depths of events in image gathers. To simplify a generally nonlinear inverse (minimization) problem, we perform the velocity analysis iteratively, with a set of linear equations being solved at each iteration. Below we discuss the velocity update performed at a single ($l^{th}$) step of the iterative process.

Suppose that prestack migration after the ($l - 1$)th iteration of the velocity analysis resulted in the migrated depths $z_0(x_j, h_k)$ ($x_j$ is the surface coordinate of the $j^{th}$ image gather, and $h_k$ is the half-offset). Then the migrated depths $z(x_j, h_k)$ after the $l^{th}$ iteration can be represented as a linear perturbation of $z_0(x_j, h_k)$:

$$z(x_j, h_k) = z_0(x_j, h_k) + \sum_{i=1}^{N} \frac{\partial z_0(x_j, h_k)}{\partial \lambda_i} \Delta \lambda_i,$$  \hspace{1cm} (A1)

where $\partial z_0(x_j, h_k)/\partial \lambda_i$ are the derivatives of the migrated depths with respect to the medium parameters $\lambda_i$ ($i = 1, 2, 3, \ldots, N$), and $\Delta \lambda_i = \lambda'_i - \lambda_i$ are the desired parameter updates. The goal of the updating procedure is to estimate $\Delta \lambda_i$ and, therefore, find the parameters $\lambda_i'$ to be used for the migration after the $l^{th}$ iteration.

The variance $V$ of the migrated depths for a single reflection event at all offsets and image gathers is

$$V = \Sigma_{j=1}^{P} \Sigma_{k=1}^{M} [z(x_j, h_k) - \tilde{z}(x_j)]^2,$$  \hspace{1cm} (A2)

where $\tilde{z}(x_j) = (1/M) \Sigma_{k=1}^{M} z(x_j, h_k)$ is the average migrated depth of the event at surface coordinate $x_j$, $P$ is the number of image gathers used in the velocity update, and $M$ is the number of offsets in each image gather. The minimization at each iteration step is accomplished by searching for the parameter updates that satisfy the condition $\partial V/\partial (\Delta \lambda_r) = 0$ ($r = 1, 2, 3, \ldots, N$). Substituting equation (A1) in equation (A2), differentiating the variance with respect to the parameter updates, and
setting $\partial V/\partial (\Delta \lambda_i) = 0$ yields
\[ - \Sigma_j^p \Sigma_k^M \Sigma_l^N (g_{j,k,i} - \hat{g}_{j,k,i})(g_{j,k,i} - \hat{g}_{j,k,i}) \Delta \lambda_i \\
= \Sigma_j^p \Sigma_k^M [z_0(x_j, h_k) - \hat{z}_0(x_j)](g_{j,k,i} - \hat{g}_{j,k,i}), \]  \tag{A3}
where $g_{j,k,i} \equiv \partial z_0(x_j, h_k)/\partial \lambda_i$, $g_{j,k,i} \equiv \partial z_0(x_j, h_k)/\partial \lambda_i$, and $\hat{g}_{j,k,i} \equiv (1/M) \Sigma_k^M g_{j,k,i}$; all derivatives are evaluated for the medium parameters $\lambda_i$.

Equation (A3) can be rewritten in matrix form as
\[ A^T A \Delta \lambda = A^T b, \]  \tag{A4}
where $A$ is a matrix with $M \cdot P$ rows and $N$ columns whose elements are $g_{j,k,r} - \hat{g}_{j,k,r}$, and $b$ is a vector with $M \cdot P$ elements defined as $z_0(x_j, h_k) - \hat{z}_0(x_j)$. $A^T A$ is a square $N \times N$ matrix, and the vector $A^T b$ has $N$ elements, so the problem has been reduced to a system of $N$ linear equations with $N$ unknowns $\Delta \lambda$. We solve the system (A4) using a linear conjugate gradient scheme to obtain $\Delta \lambda$ and the updated parameters $\lambda' = \Delta \lambda + \lambda$.

The derivatives of the depths $z(x_j, h_k)$ with respect to the medium parameters $\lambda_i$ (and, therefore, the matrix $A$) can be determined from the imaging equations (e.g., Liu, 1997; Paper I):
\[ \tau_s(y, h, x, z, \lambda) + \tau_r(y, h, x, z, \lambda) = t(y, h), \]  \tag{A5}
\[ \frac{\partial \tau_s(y, h, x, z, \lambda)}{\partial y} + \frac{\partial \tau_r(y, h, x, z, \lambda)}{\partial y} = \frac{\partial t(y, h)}{\partial y}. \]  \tag{A6}

Here $y$ is the common-midpoint location at the surface, $h$ is the half-offset, $\tau_s$ is the traveltime from the source location $x_s$ ($x_s = y + h$) to the diffractor location ($x, z$) that was obtained after prestack depth migration with the medium parameters $\lambda_i$, $\tau_r$ is the traveltime from the receiver location $x_r$ ($x_r = y - h$) to the point ($x, z$), and $t(y, h)$ is the observed reflection traveltime. Note that $y$, $x$, and $z$ depend on the medium parameters $\lambda_i$, while $h$ is an independent variable. Because $x$ is fixed at the surface location where a particular image gather is analyzed, however, the derivative of $x$ with respect to $\lambda_i$ is set to zero.

Differentiating equation (A5) with respect to $\lambda_i$ gives
\[ \left[ \frac{\partial \tau_s}{\partial y} + \frac{\partial \tau_r}{\partial y} \right] \frac{dy}{d\lambda_i} + \left[ \frac{\partial \tau_s}{\partial z} + \frac{\partial \tau_r}{\partial z} \right] \frac{dz}{d\lambda_i} + \left[ \frac{\partial \tau_s}{\partial \lambda_i} + \frac{\partial \tau_r}{\partial \lambda_i} \right] = \frac{\partial t}{\partial y} \frac{dy}{d\lambda_i}. \]  \tag{A7}

Taking equation (A6) into account simplifies equation (A7) to
\[ \left[ \frac{\partial \tau_s}{\partial z} + \frac{\partial \tau_r}{\partial z} \right] \frac{dz}{d\lambda_i} = - \frac{\partial \tau_s}{\partial \lambda_i} - \frac{\partial \tau_r}{\partial \lambda_i}, \]  \tag{A8}
or
\[ \frac{dz}{d\lambda_i} = - \frac{\partial \tau_s}{\partial \lambda_i} + \frac{\partial \tau_r}{\partial \lambda_i} \frac{1}{q_s + q_r}, \]  \tag{A9}
where $q_s = \partial \tau_s/\partial z$ and $q_r = \partial \tau_r/\partial z$ are the vertical slownesses evaluated at the diffractor for the specular rays connecting the diffractor with the source and the receiver, respectively.

To find the derivatives $dz/d\lambda_i$, we perform ray tracing using the prestack-migrated image after the $(l - 1)^{th}$ iteration. First, the dip of the reflector needed to define the specular reflected rays is estimated by manual picking on the image. Then, for a given diffraction point on the reflector and a fixed source-receiver offset, the specular ray is traced through two models, one of which is defined by the parameters $\lambda_i$ and the other by parameters slightly deviating from $\lambda_i$ (i.e., $\lambda_i$ are slightly perturbed). The corresponding perturbation of the traveltime between the source and the diffractor is divided by the perturbation in $\lambda_i$ to obtain $\partial \tau_s/\partial \lambda_i$, while the same quantity for the traveltime leg between the diffractor and the receiver gives $\partial \tau_r/\partial \lambda_i$. The slownesses $q_r$ and $q_s$ at the diffraction point are part of the output of the ray-tracing algorithm (Červený, 1972).
Quartic moveout coefficient for dipping azimuthally anisotropic layers

Andrés Pech & Ilya Tsvankin
Center for Wave Phenomena, Dept. of Geophysics, Colorado School of Mines, Golden, CO 80401

ABSTRACT
Interpretation and inversion of azimuthally varying nonhyperbolic reflection moveout requires accounting for both velocity anisotropy and subsurface structure. Here, our previously derived general expression for the quartic moveout coefficient $A_4$ is applied to P-wave reflections in orthorhombic models typical for fractured reservoirs.

The weak-anisotropy approximation for the coefficient $A_4$ in a homogeneous orthorhombic layer above a dipping reflector is controlled by the anellipticity parameters $\eta^{(1)}$, $\eta^{(2)}$ and $\eta^{(3)}$, which are responsible for time processing of P-wave data. The azimuthal variation of the quartic coefficient depends on reflector dip and is quite sensitive to the signs and the relative magnitudes of $\eta^{(1)}$, $\eta^{(2)}$ and $\eta^{(3)}$. The dip-line $A_4$ is proportional to the parameter $\eta^{(2)}$ and rapidly decreases with dip, always going to zero for a dip of 30° where it changes sign. In contrast, the value of $A_4$ on the strike line depends on all three $\eta$ coefficients, but for mild dips it is mostly governed by $\eta^{(1)}$. The contribution of the parameter $\eta^{(3)}$ increases with dip and may lead to a complicated azimuthal signature of the quartic coefficient, with two azimuths of vanishing $A_4$ between the dip and strike directions.

The strong influence of the magnitudes and signs of the anellipticity parameters on the azimuthal pattern of the quartic coefficient suggests that nonhyperbolic moveout recorded in wide-azimuth surveys can help to constrain the anisotropic velocity field. Since for fracture-induced orthorhombic media the parameters $\eta^{(1,2,3)}$ are closely related to the fracture density and infill, the results of azimuthal nonhyperbolic moveout analysis can be also used in fracture characterization.

Key words: reflection moveout, azimuthal anisotropy, long spreads, wide-azimuth data, P-waves

1 INTRODUCTION

Although conventional seismic processing algorithms are largely limited to analysis of hyperbolic moveout on moderate-length spreads, acquisition of long-offset data becomes more and more common. In particular, the technology of ocean-bottom cable is well-suited for recording long-offset reflections for a wide range of source-receiver azimuths. The azimuthal dependence of nonhyperbolic reflection moveout at large offsets may be strongly influenced by elastic anisotropy (Sayers and Ebrom, 1997; Al-Dajani and Tsvankin, 1998) and, therefore, can help in estimating the anisotropic parameters.

In velocity analysis of pure (non-converted) waves, nonhyperbolic moveout is conventionally described using the quartic moveout coefficient $A_4$. Tsvankin and Thomsen (1994) combined the coefficient $A_4$ with the normal-moveout (NMO) velocity $V_{nmo}$ in a nonhyperbolic moveout equation that proved to be accurate for P-waves and converted PS-waves in anisotropic media (Tsvankin, 2001; Al-Dajani and Tsvankin, 1998). While the formalism for modeling NMO velocity in arbitrarily anisotropic, heterogeneous media has been developed
by Grechka and Tsvankin (1998b, 2002) and Grechka, Tsvankin and Cohen (1999), derivation of the quartic moveout coefficient proved to be much more involved because $A_4$ depends on reflection-point dispersal. Analytic expressions for $A_4$ in horizontally layered VTI (transversely isotropic media with a vertical symmetry axis) media and symmetry planes of azimuthally anisotropic media are given in Tsvankin (2001). Al-Dajani and Tsvankin (1998) obtained the azimuthally varying coefficient $A_4$ in layer-cake HTI (transversely isotropic media with a horizontal symmetry axis) media; their results were extended to a horizontal orthorhombic layer by Al-Dajani et al. (1998).

In our previous paper (Pech et al., 2002; hereafter referred to as Paper I), we presented a general expression for the coefficient $A_4$ of pure (non-converted) modes that takes into account reflection-point dispersal on irregular interfaces and is valid for arbitrary anisotropy and heterogeneity. It was emphasized that this result can be used to model long-spread moveout without time-consuming multi-offset, multi-azimuth ray tracing because all needed quantities can be computed during the tracing of the zero-offset ray. The equation for $A_4$ was applied in Paper I to study azimuthally varying nonhyperbolic moveout of P-waves in a dipping transversely isotropic (TI) layer with a tilted symmetry axis. For weak anisotropy, the quartic coefficient proved to be proportional to the anellipticity parameter $\eta$ defined as (Alkhalifah and Tsvankin, 1995)

$$\eta \equiv \frac{\epsilon - \delta}{1 + 2\delta},$$

with the azimuthal variation of $A_4$ being highly sensitive to the tilt of the symmetry axis.

Here, we use the expression for the quartic coefficient from Paper I to analyze nonhyperbolic moveout for the more complicated azimuthally anisotropic models with orthorhombic symmetry often used to describe naturally fractured reservoirs (Bakulin et al., 2000). Valuable insight is provided by the weak-anisotropy approximation that represents $A_4$ as a function of three anellipticity coefficients $\eta^{(1)}$, $\eta^{(2)}$, and $\eta^{(3)}$ defined in the symmetry planes of the model. Numerical examples illustrate the high sensitivity of the azimuthally varying quartic coefficient to the signs and relative magnitudes of the parameters $\eta^{(1,2,3)}$.

2 NONHYPERBOLIC MOVEOUT AND THE QUARTIC MOVEOUT COEFFICIENT

A detailed analytic description of nonhyperbolic moveout in anisotropic media can be found in Tsvankin (2001). The nonhyperbolic portion of the moveout curve is governed by the quartic moveout coefficient $A_4$ defined by expanding the squared reflection traveltime $t^2$ in a Taylor series in the squared source-receiver offset $X^2$:

$$A_4 = \frac{1}{2} \left. \frac{d^2}{dX^2} \left[ \frac{d(t^2)}{dX^2} \right] \right|_{X=0},$$

(2)

Long-spread moveout of P-waves (and, for some models, PS-waves) in both isotropic and anisotropic media can be well-approximated by the following equation suggested by Tsvankin and Thomsen (1994):

$$t^2 = t_0^2 + \frac{X^2}{V_{nmo}^2} + \frac{A_4 X^4}{1 + A X^2},$$

(3)

where $t_0$ is the zero-offset time, $V_{nmo}$ is the NMO velocity, and the denominator of the nonhyperbolic term is designed to make the equation convergent at infinitely large offsets. The coefficient $A$ is expressed through $V_{nmo}$, $A_4$, and the horizontal group velocity $V_{hor}$ as

$$A = \frac{A_4}{V_{hor}^2 - V_{nmo}^2}. $$

(4)

NMO velocity for anisotropic, heterogeneous media can be found from the Dix-type averaging equations developed by Grechka, Tsvankin and Cohen (1999) and Grechka and Tsvankin (2002). An analytic expression for the quartic coefficient $A_4$ that has the same level of generality was introduced in Paper I. Since both $V_{nmo}$ and $A_4$ can be computed by tracing only one (zero-offset) ray, equation (3) can serve as a computationally efficient replacement for anisotropic ray tracing in modeling and inversion algorithms.

Paper I shows that the quartic moveout coefficient $A_4$ can be found as a function of the spatial derivatives of the one-way traveltime $\tau$ between the common-midpoint $y$ and point $x$ on the reflector (Figure 1):

$$A_4(L) = \frac{1}{16} \left[ \frac{\partial^2 \tau}{\partial y_k \partial y_l} \frac{\partial^2 \tau}{\partial y_m \partial y_n} + \frac{\tau_0}{3} \frac{\partial^4 \tau}{\partial y_k \partial y_l \partial y_m \partial y_n} - \frac{\tau_0}{3} \frac{\partial^3 \tau}{\partial y_k \partial y_l \partial x_i} \frac{\partial^2 \tau}{\partial x_j \partial y_m \partial y_n} \right] L_k L_i L_m L_n. $$

(5)

Here $L = [\cos \alpha, \sin \alpha, 0]$ is a unit vector parallel to the CMP line with the azimuth $\alpha$ and $\tau_0$ is the one-zero-offset traveltime; summation over repeated indices from one to two is implied. All derivatives in equation (5) are evaluated for the zero-offset reflection ray at the CMP location $y$. Equation (5) is valid for arbitrarily anisotropic, heterogeneous media and reflectors of irregular shape, as long as reflection traveltime can be expanded in a Taylor series near the common midpoint.

Simplifying the spatial derivatives of the traveltime $\tau$ in equation (5) under the assumption of weak anisotropy provides valuable insight into the dependence of the coefficient $A_4$ on the medium parameters. Below, we obtain the weak-anisotropy approximation for the
coefficient $A_4$ of P-waves in an orthorhombic layer and analyze its dependence on the anisotropic parameters and reflector dip.

3 BRIEF DESCRIPTION OF ORTHORHOMBIC MEDIA

Models with orthorhombic symmetry are often used to describe naturally fractured reservoirs that contain, for example, two orthogonal fracture systems or a single system of penny-shaped cracks embedded in a VTI matrix (e.g., Bakulin et al., 2000; Tsvankin, 2001). Orthorhombic media have three mutually orthogonal planes of mirror symmetry, one of which we assume to be horizontal.

Apart from the wavefront distortions near point shear-wave singularities (such as point A in Figure 2), velocities and traveltimes in the symmetry planes of orthorhombic media can be described by the corresponding VTI equations. Therefore, reflection moveout and other seismic signatures in orthorhombic media can be described by the corresponding VTI equations. The quartic moveout coefficient below is played by the parameters $\eta^{(1)}, \eta^{(2)}, \eta^{(3)}$, which quantify deviations from the elliptically anisotropic model in the symmetry planes (Grechka and Tsvankin, 1999):

$$
\eta^{(1)} = \frac{\epsilon^{(1)} - \delta^{(1)}}{1 + 2 \delta^{(1)}} \approx \epsilon^{(1)} - \delta^{(1)},
$$

$$
\eta^{(2)} = \frac{\epsilon^{(2)} - \delta^{(2)}}{1 + 2 \delta^{(2)}} \approx \epsilon^{(2)} - \delta^{(2)},
$$

$$
\eta^{(3)} = \frac{\epsilon^{(1)} - \epsilon^{(2)} - \delta^{(3)}(1 + 2 \epsilon^{(2)})}{(1 + 2 \epsilon^{(2)}) (1 + 2 \delta^{(3)})} \approx \epsilon^{(1)} - \epsilon^{(2)} - \delta^{(3)}.
$$

The approximate expressions for $\eta^{(1,2,3)}$ in equations (6)–(8) are obtained by linearizing the exact definitions in the anisotropic parameters.

4 P-WAVE QUARTIC COEFFICIENT IN A DIPPING ORTHORHOMBIC LAYER

The model studied here includes a homogeneous orthorhombic layer with a horizontal symmetry plane above a plane dipping reflector (Figure 3). The other two symmetry planes of the layer are vertical and coincide with the coordinate planes $[x_1,x_3]$ and $[x_2,x_3]$. For simplicity, we assume that the plane $[x_1,x_3]$ is also parallel to the dip plane of the reflector, which makes it the only symmetry plane for the model as a whole. Therefore, the zero-offset reflected ray has to be confined to the $[x_1,x_3]$-plane (Figure 3).
4.1 Weak-anisotropy approximation

The weak-anisotropy approximation for the quartic coefficient $A_4$ of P-waves in this model is derived in Appendix A by linearizing the exact equation (5) in the anisotropic parameters:

$$A_4 = \frac{-1}{2 t_{P0} V_{P0}^4} \left\{ \eta^{(2)} \cos^2 \phi \left[ 2 \cos 2\phi (1 + \cos 2\alpha \cos 2\phi) + \cos 4\phi - 1 \right] + 4\eta^{(1)} \cos^2 \phi \sin^2 \alpha + 2\eta^{(3)} \sin^2 \alpha \cos 2\alpha \cos 2\phi \right\}. \quad (9)$$

where $t_{P0}$ is the two-way zero-offset traveltime, and $\eta^{(1)}$, $\eta^{(2)}$, and $\eta^{(3)}$ are the anellipticity parameters defined in equations (6)–(8). Equation (9) shows that the azimuthal variation $A_4(\alpha)$ is controlled just by $\eta^{(1,2,3)}$ and reflector dip. This result is not surprising because the parameters $\eta^{(1,2,3)}$ in combination with the NMO velocities from a horizontal reflector in the vertical symmetry planes, fully describe P-wave time-domain signatures for orthorhombic media (Grechka and Tsvankin, 1999).

Note that the quartic coefficient is symmetric not only with respect to the dip direction of the reflector (the dip plane $\alpha = 0^\circ$ is a symmetry plane for the whole model), but also with respect to the reflector strike. Indeed, since $A_4(\alpha) = A_4(-\alpha)$ and $A_4(\alpha) = A_4(\pi + \alpha)$, it follows that $A_4(\alpha) = A_4(\pi - \alpha)$, which means that the quartic coefficient is symmetric with respect to the strike direction $\alpha = \pm 90^\circ$.

Figure 4 confirms that equation (9) is suitable for at least a qualitative description of the quartic moveout coefficient for small and moderate values of the anisotropic parameters. Note that the magnitude of the $\eta$ coefficients in Figure 4 is quite substantial for fracture-induced orthorhombic media (Bakulin et al., 2000). Application of the quartic coefficient in velocity analysis, however, should not be based on the weak-anisotropy approximation, as was shown by Tsvankin and Thomsen (1994) for the simpler VTI model.

4.2 Special cases

If the reflector is horizontal ($\phi = 0^\circ$), equation (9) yields the linearized version of the exact solution for $A_4$ in a horizontal orthorhombic layer given by Al-Dajani et al. (1998):

$$A_4(\phi = 0^\circ) = \frac{-2}{t_{P0} V_{P0}^4} \left[ \eta^{(2)} \cos^2 \alpha - \eta^{(3)} \cos^2 \alpha \sin^2 \alpha + \eta^{(1)} \sin^2 \alpha \right]. \quad (10)$$

Although equation (10) is an approximation valid for small values of the anisotropic coefficients, the term in brackets accurately reproduces the azimuthal variation of the quartic coefficient. For a horizontal layer, $A_4$ in both symmetry-plane directions ($\alpha = 0^\circ$ and $90^\circ$) depends just on the corresponding parameter $\eta$ ($\eta^{(2)}$ in the $[x_1, x_3]$-plane and $\eta^{(1)}$ in the $[x_2, x_3]$-plane). The coefficient $\eta^{(3)}$ contributes only to the cross-term that reaches its maximum at the azimuth $\alpha = 45^\circ$. 
Another special case is that of a dipping VTI layer since transversely isotropic models can be treated as a subset of the more general orthorhombic media. The quartic coefficient in VTI media, which was derived and analyzed in Paper I, can be found from equation (9) by setting $\eta^{(1)} = \eta^{(2)} = \eta$ and $\eta^{(3)} = 0$ (Tsvankin, 1997):

$$A_{4,\text{VTI}}^{\text{VTI}} = -\frac{2\eta}{t_{P0}^2 V_{P0}^4} \cos^4 \phi \left(1 - 4 \sin^2 \phi \cos^2 \alpha\right).$$  \hspace{1cm} (11)

The dip-line ($\alpha = 0$) and strike-line ($\alpha = 90^\circ$) coefficients $A_4$ for VTI media, which we will need below for comparison with the expressions for orthorhombic media, are given by

$$A_{4,\text{dip}}^{\text{VTI}} = -\frac{2\eta}{t_{P0}^2 V_{P0}^4} \cos^4 \phi \left(1 - 4 \sin^2 \phi\right),$$  \hspace{1cm} (12)

$$A_{4,\text{strike}}^{\text{VTI}} = -\frac{2\eta}{t_{P0}^2 V_{P0}^4} \cos^4 \phi.$$  \hspace{1cm} (13)

### 4.3 Dip-line and strike-line expressions

As discussed above, the quartic coefficient in our model is symmetric with respect to the dip and strike directions of the reflector. On the dip line ($\alpha = 0$°), the coefficient $A_4$ from equation (9) takes the form

$$A_{4,dip} = -\frac{2\eta^{(2)}}{t_{P0}^2 V_{P0}^4} \cos^4 \phi \left(1 - 4 \sin^2 \phi\right).$$  \hspace{1cm} (14)

Equation (14) becomes identical to the corresponding VTI equation (12) if the parameter $\eta^{(2)}$ is replaced with $\eta$. Indeed, reflected rays (and their phase-velocity vectors) on the dip line are confined to the symmetry plane $[x_1, x_3]$ where the kinematics of wave propagation is the same as that in the VTI model with the vertical velocities $V_{P0}$ and $V_{S0}$ and the anisotropic coefficients $\epsilon = \epsilon^{(2)}$, $\delta = \delta^{(2)}$, $\gamma = \gamma^{(2)}$, and $\eta = \eta^{(2)}$ (Tsvankin, 1997, 2001).

The dip-line quartic coefficient from equation (14) vanishes for the vertical reflector ($\phi = 90^\circ$) and for the dip $\phi = 30^\circ$: If $\eta^{(2)} > 0$, $A_{4,dip}$ is negative for mild dips $\phi < 30^\circ$ and becomes positive for $\phi > 30^\circ$, as discussed in Paper I for VTI media (also, see numerical results in Tsvankin, 1995, 2001).

Substitution of $\alpha = 90^\circ$ into equation (9) yields the strike-line quartic coefficient:

$$A_{4,\text{strike}} = -\frac{2}{t_{P0}^2 V_{P0}^4} \begin{bmatrix} \eta^{(1)} \cos^2 \phi \\
\eta^{(2)} \cos^2 \phi \sin^2 \phi + \eta^{(3)} \sin^2 \phi \end{bmatrix}.$$  \hspace{1cm} (15)

It is interesting that the functional dependence of $A_{4,\text{strike}}$ on the dip $\phi$ is similar to that of the quartic coefficient in a horizontal layer on the azimuth $\alpha$ [equation (10)]. In contrast to the coefficient $A_{4,dip}$ from equation (14), $A_{4,\text{strike}}$ depends on all three parameters $\eta^{(1,2,3)}$ because reflected rays recorded on the strike line $x_2$ deviate from the vertical symmetry plane $[x_2, x_3]$.

For the same reason, $A_{4,\text{strike}}$ differs from the strike-line coefficient in VTI media [equation (13)] with $\eta = \eta^{(1)}$ (the parameter $\eta^{(1)}$ is defined in the $[x_2, x_3]$-plane). However, equation (15) indicates that the parameter $\eta^{(1)}$ does dominate the quartic coefficient for mild reflector dips $\phi$.

For a vertical interface ($\phi = 90^\circ$), reflected rays travel in the horizontal symmetry plane, and the strike-line quartic coefficient is governed just by the parameter $\eta^{(3)}$:

$$A_{4,\text{strike}}(\phi = 90^\circ) = -\frac{2\eta^{(3)}}{t_{P0}^2 V_{P0}^4}.$$  \hspace{1cm} (16)

Equation (16) coincides with the weak-anisotropy approximation for $A_4$ in a horizontal VTI layer with $\eta = \eta^{(3)}$.

### 4.4 Azimuthal signature

The azimuthal dependence of the quartic coefficient from equation (9) strongly depends on reflector dip as well as the magnitudes and signs of the anellipticity parameters $\eta^{(1,2,3)}$. For mild dips, $A_4(\alpha)$ is largely controlled by the parameters $\eta^{(1)}$ (near the $[x_2, x_3]$-plane) and $\eta^{(2)}$ (near the $[x_1, x_3]$-plane). As illustrated by Figure 5, if both $\eta^{(1)}$ and $\eta^{(2)}$ are positive and greater by absolute value than $\eta^{(3)}$, $A_4$ typically stays negative for dips smaller than $30^\circ$; the same result was obtained in Paper I for a dipping VTI layer [see equation (11)]. Also, as in VTI media, for a dip of $30^\circ$ $A_4$ goes to zero only in the single (dip) direction (Figure 6). For the $\eta$ parameters from Figures 5 and 6 and dips between $30^\circ$ and $90^\circ$, equation (9) yields two azimuths $\pm \alpha$ for which $A_4 = 0$. If the dip $\phi = 45^\circ$ (Figure 7), the quartic coefficient is positive near the dip plane, vanishes at $\alpha \approx 60^\circ$ and becomes negative close to the strike direction.

The azimuthal variation of the quartic coefficient has a different character if $\eta^{(1)}$ and $\eta^{(2)}$ have opposite signs. In this case, for mild dips $A_4$ changes sign between the dip and strike directions [see equations (14) and (15)], with the azimuthal direction of vanishing $A_4$ dependent on the relative magnitudes of $\eta^{(1)}$ and $\eta^{(2)}$ (Figure 8). Since the quartic coefficient decreases with the dip $\phi$ more rapidly in the dip plane than in the strike direction, the direction where $A_4 = 0$ rotates towards the dip plane as $\phi$ increases. For the example in Figure 8, $A_4$ goes to zero at an angle of about $\pm 35^\circ$ from the dip plane. When the dip reaches $30^\circ$ (Figure 9), the quartic coefficient goes to zero only in the dip direction, and the azimuthal variation of $A_4$ is similar to that in Figure 6. However, the sign of $A_4$ away from the dip plane in Figure 9 is positive because $\eta^{(1)} < 0$. Finally, for dips larger than $30^\circ$, the quartic coefficient is positive for all azimuths (Figure 10).

To analyze the influence of the parameter $\eta^{(3)}$ on the quartic coefficient, in the next two examples we set $\eta^{(1)}$ and $\eta^{(2)}$ to zero (Figures 11 and 12). Clearly, the

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Figure 5. Magnitude of the azimuthally-varying quartic moveout coefficient $A_4$ for a dipping orthorhombic layer computed from equation (9). The dip plane of the reflector is at zero azimuth (the azimuth is marked on the perimeter). The anellipticity parameters are $\eta^{(1)} = 0.05$, $\eta^{(2)} = 0.1$, and $\eta^{(3)} = 0.03$; the reflector dip $\phi = 15^\circ$. The other parameters ($t_{P0}$ and $V_{P0}$) change only the scale of the plot (intentionally undefined here). For this model, $A_4 < 0$ for all azimuthal directions.

Figure 6. Same as Figure 5, but the reflector dip is $30^\circ$. The minus signs inside the lobes indicate negative values of $A_4$.

term involving $\eta^{(3)}$ creates a rather complicated azimuthal signature of $A_4$. Since $A_{4,dip}$ [equation (14)] is proportional to $\eta^{(2)}$, it vanishes for all dips when $\eta^{(2)} = 0$. In addition, the quartic coefficient goes to zero for another azimuth near the strike direction, even for a dip of just $15^\circ$ (Figure 11). The signature of $A_4$ for a dip of $30^\circ$ has a similar character, but with different relative magnitudes of the lobes (Figure 12).

For nonzero values of $\eta^{(1)}$ and $\eta^{(2)}$, the contribution of the term proportional to $\eta^{(3)}$ generally increases with reflector dip (Figures 13–15). While the dip-line coefficient $A_4$ depends just on $\eta^{(2)}$, $A_{4,strike}$ is significantly influenced by $\eta^{(3)}$, in particular for the dips exceeding $30^\circ$. If $\eta^{(3)}$ is larger by absolute value than $\eta^{(1)}$ and $\eta^{(2)}$, the quartic coefficient typically goes to zero in at least one off-symmetry direction for a wide range of dips. For example, if the dip $\phi = 30^\circ$, the influence of $\eta^{(3)}$ produces an additional direction of vanishing $A_4$ near the
reflector strike, and the azimuthal variation of the quartic coefficient is described by six lobes with alternating signs from one lobe to the next (Figure 14).

5 DISCUSSION AND CONCLUSIONS

The general expression for the quartic moveout coefficient $A_4$ derived by Pech et al. (2002; Paper I) was used here to study P-wave nonhyperbolic moveout in a dipping orthorhombic layer. Similar to the NMO ellipse, the coefficient $A_4$ can be computed by tracing a single (zero-offset) ray and then used in the nonhyperbolic moveout equation of Tsvankin and Thomsen (1994) to model reflection traveltimes without time-consuming ray tracing for each source-receiver pair.

The main emphasis of this paper, however, is on the analysis of the azimuthal variation of the P-wave quartic moveout coefficient as a function of the anisotropic parameters and reflector dip. The model consisted of a homogeneous orthorhombic layer with a horizontal symmetry plane and a dipping lower boundary (reflector). It was assumed that the dip plane of the reflector coincides with a vertical symmetry plane of the layer and, therefore, represents a symmetry plane for the whole model. Another symmetry direction for reflection moveout in
Figure 13. Magnitude of the quartic moveout coefficient $A_4$ for an orthorhombic layer computed from equation (9). The anellipticity parameters are $\eta^{(1)} = -0.025$, $\eta^{(2)} = 0.1$, and $\eta^{(3)} = 0.2$; the reflector dip $\phi = 15^\circ$.

Figure 14. Same as Figure 8, but the reflector dip is $30^\circ$.

Figure 15. Same as Figure 8, but the reflector dip is $45^\circ$.

dia including NMO correction, DMO removal, and time migration.

The dip-line quartic coefficient $A_{4,dip}$ is described by the same equation as in VTI media and depends on a single anisotropic parameter $-\eta^{(2)}$. $A_{4,dip}$ vanishes for a vertical reflector (dip $\phi = 90^\circ$) and for the dip $\phi = 30^\circ$; the sign of $A_{4,dip}$ for $\phi < 30^\circ$ is opposite to that of $\eta^{(2)}$. The strike-line $A_4$ depends on all three $\eta$ coefficients but for mild dips is largely governed by $\eta^{(1)}$.

Hence, for dips smaller than $15$–$20^\circ$, the azimuthally varying quartic coefficient mostly depends on the parameters $\eta^{(1)}$ and $\eta^{(2)}$. If $\eta^{(1)}$ and $\eta^{(2)}$ have opposite signs, $A_4$ for mildly dipping reflectors changes sign between the dip and strike directions. The influence of $\eta^{(3)}$ generally increases with dip and may create a rather complicated azimuthal signature of the quartic coefficient, sometimes with two azimuths of vanishing $A_4$ between the dip and strike directions.

The high variability of the azimuthal signature of the quartic coefficient and its sensitivity to the time-processing parameters $\eta^{(1,2,3)}$ can be exploited in the inversion of P-wave data for orthorhombic media. Despite the known instability in estimating the quartic moveout coefficient (Grechka and Tsvankin, 1998a), wide-azimuth long-offset reflection data may be used to determine the sign of $A_4$ and the azimuthal directions of its minimum values. This information can help in constraining the parameters $\eta^{(1,2,3)}$, which are not only needed in velocity analysis, but can also be used in fracture characterization. For example, if the effective orthorhombic anisotropy is caused by two orthogonal systems of penny-shaped cracks embedded in isotropic host rock, all three $\eta$ coefficients vanish for dry (gas-filled) cracks and are positive for cracks filled with fluid (Bakulin et al., 2000).

The exact equation for the quartic moveout coeffi-
cient from Paper I can be applied to model nonhyperbolic moveout in more complicated layered orthorhombic media. The analytic results of this work can still provide useful insight into the behavior of $A_4$ in layered media because, at least for mild dips, the effective quartic coefficient at the surface can be approximated by an average of the interval values of $A_4$ computed for the same azimuth (Al-Dajani and Tsvankin, 1998; Al-Dajani et al., 1998).

6 ACKNOWLEDGMENTS

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7 REFERENCES


APPENDIX A: WEAK-ANISOTROPY APPROXIMATION FOR THE QUARTIC MOVEOUT COEFFICIENT IN ORTHORHOMBIC MEDIA

Here we apply the approach discussed in Paper I to obtain a linearized approximation for the P-wave quartic coefficient $A_4$ valid for small values of the anisotropic parameters. The model consists of a homogeneous orthorhombic layer with a horizontal symmetry plane above a plane dipping reflector (Figure 3). It is assumed that the dip plane of the reflector coincides with one of the mutually orthogonal vertical symmetry planes of the overburden.

The one-way traveltime between the common-midpoint (CMP) $y$ and the plane reflector $z(x_1,x_2)$ is

$$
\tau = \frac{\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + z^2(x_1,x_2)}}{V_G},
$$

where $V_G$ is the group velocity. The orientation of the ray that connects the CMP with the reflector can be characterized by the polar angle $a$ and the azimuthal angle $b$:

$$
\sin a = \frac{\sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}}{\sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + z^2}},
$$

$$
\cos a = \frac{z}{\sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + z^2}},
$$

$$
\sin b = \frac{(y_2 - x_2)}{\sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}},
$$

$$
\cos b = \frac{(y_1 - x_1)}{\sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}}.
$$

The P-wave group velocity $V_G$ linearized in the anisotropic parameters can be determined from the weak-
anisotropy approximation for the phase velocity given in Tsvankin (1997b, 2001):

\[ V_G = V_{P0} \left\{ 1 + (\epsilon^{(2)} - \delta^{(2)}) (\sin a \cos b)^4 \\
+ \delta^{(1)} (\sin a \sin b)^2 + (\epsilon^{(1)} - \delta^{(1)}) (\sin a \sin b)^4 \\
+ \cos^2 b \left[ (\delta^{(2)})^2 a - (\delta^{(1)} + \delta^{(2)} - \delta^{(3)} - 2 \epsilon^{(2)}) \sin^4 a \sin^2 b \right] \right\}; \quad (A6) \]

the angles \( a \) and \( b \) are defined in equations (A-2)–(A-5).

Next, equation (A1) with the velocity \( V_G \) from equation (A6) is substituted into the general equation (5) to evaluate the spatial derivatives of the traveltime, which yields the quartic coefficient \( A_4 \) as a function of the coordinates of both common midpoint and the zero-offset reflection point. Since the zero-offset ray is confined to the dip plane where all kinematic signatures are described by the corresponding VTI equations, we can relate the horizontal coordinates \( x_{1(0)} \) and \( x_{2(0)} \) of the zero-offset reflection point to the CMP coordinates \( y_1 \) and \( y_2 \) by adapting the results of Paper I:

\[ y_1 = 2z \tan \phi [0.5 + \epsilon^{(2)} - (\epsilon^{(2)} - \delta^{(2)}) \cos 2\phi ] + x_{1(0)} , \quad (A7) \]
\[ y_2 = x_{2(0)} . \quad (A8) \]

Using equations (A7) and (A8) and applying further linearization in the anisotropic parameters, we obtain the following approximation for the \( P \)-wave quartic moveout coefficient in orthorhombic media:

\[ A_4 = - \frac{1}{2 t_{P0} V_{P0}^4} \left\{ \eta^{(2)} \cos^2 \phi \left[ 2 \cos 2\phi (1 \\
+ \cos 2\alpha \cos 2\phi) + \cos 4\phi - 1 \right] \\
+ 4\eta^{(1)} \cos^2 \phi \sin^2 \alpha \\
- 2\eta^{(3)} \sin^2 \alpha (\cos 2\alpha + \cos 2\phi) \right\} , \quad (A9) \]

where \( t_{P0} \) is the two-way zero-offset traveltime, and the coefficients \( \eta^{(1,2,3)} \) are defined in equations (6)–(8).
Global wave-equation reflection tomography

Maarten V. de Hoop∗ & Robert D. van der Hilst†
∗Center for Wave Phenomena, Colorado School of Mines, Golden, CO 80401-1887
†Department of Earth, Atmospheric and Planetary Sciences, Massachusetts Institute of Technology, Cambridge MA 02139-4307

ABSTRACT
In tomography it has been recognized that the finiteness of the frequency content of seismic data leads to interference effects in the process of medium reconstruction, that need to be accounted for. Various ways of looking at these effects in the framework of transmission tomography can be found in the literature. Here, we consider single-scattered body waves and develop from our earlier work on inverse scattering a method of wave-equation reflection tomography – which in exploration seismics is identified as a method of wave-equation migration velocity analysis – admitting band-limited data. In the transition from transmission tomography to reflection tomography the usual cross correlation between modelled and observed data is replaced by modelled annihilators applied to the observed data. Using the generalized screen expansion for one-way wave propagation, we derive the Fréchet or sensitivity kernels and explain how they can be evaluated with an adjoint state method.

INTRODUCTION
In tomography the finiteness of the bandwidth of seismic data leads to interference effects in the process of medium reconstruction that need to be accounted for. Various ways of investigating these effects in the framework of transmitted body waves can be found in the literature (Luo & Schuster, 1991; Woodward, 1992; Dahlen et al., 2000). The implied approaches fall in the category of wave-equation tomography: We distinguish the one based on the idea of backprojecting phase residuals over a Fresnel-like volume rather than backprojecting residual traveltimes along rays, and the one based on the wave equation combined with the Born approximation formulated as an adjoint state method (Vasco et al., 1995). Here, we consider single-scattered body waves and from our earlier work on imaging-inversion develop a method of wave-equation reflection tomography – which in exploration seismics would be a method of wave-equation migration velocity analysis – admitting finite-frequency data. In the transition from transmission tomography to reflection tomography the usual cross correlation between modelled and observed data is replaced by modelled annihilators applied to the observed data.

In the context of optimization, the key quantity to be found is the Fréchet kernel derived from the error criterion. In wave-equation tomography, the wavefields in the kernel can be computed directly from the time-domain wave equation, using a normal mode summation (Zhao & Jordan, 1998), or using the frequency-domain one-way wave equation, which is the method exploited here.

As with the high-frequency, ray-based approaches to reflection tomography and migration velocity analysis, the idea is to utilize scattered phases in conjunction with the inherent ‘redundancy’ in their observation. (In the absence of caustics, the redundancy is arises from observations at multiple offsets, as in exploration seismology, or at multiple angular epicentral distances, as in global seismology, and azimuths.) In the framework of reflection seismology in sedimentary basins, the waves scattered from the many reflectors and faults in the subsurface can be used for this purpose, while in the framework of global seismology in the study of the mantle, scattered phases such as PnP and ScS can be identified and used for this purpose. The idea of using scattered phases in global tomography is certainly not new: The ScS travel times, for example, were used by Grand (1994) to build a mantle model. Scattering includes the possibility of mode conversions.

The components of the approach presented in this paper are as follows. The data are downward continued (Clayton, 1978) and subjected to a wave-equation angle transform (De Bruin et al., 1990; Sava et al., 1999). Such a transform generates multiple images of the same part of Earth’s reflecting structure for waves scattered over a range of angles (Stolk & De Hoop, 2003); these images are without artifacts (false reflectors) even in the presence of caustics unlike their counterparts generated by a generalized Radon transform (Brandsberg-Dahl et al., 2003a). However, if the velocity model is incorrect, the images will differ for different scattering angles and azimuths. From the angle transform, annihilators of the data are derived. Whether the velocity model is acceptable is based upon whether the data are in the range of, i.e. can be predicted by, the modelling operator underlying-
ing our approach; annihilators detect this particular property of the data (Stolk & De Hoop, 2002). (In conventional tomography, one would detect whether the travel times in the data are in the range of the modelled travel times by differencing.) Tomography is then formulated as the problem of minimizing the action of these annihilators on the data. We develop the theory here up the expression for the Fréchet kernel following this approach. The evaluation of this kernel is formulated as an adjoint state method (see, for example, Tarantola, 1987)). The key assumption invoked in the application of one-way wave theory is that the rays connected to data points are nowhere horizontal in the subsurface (Stolk & De Hoop, 2003). This does not exclude the formation of caustics with multipathing.

Downward continuation and its use in seismic imaging dates back to Claerbout (1985). The principle of characterizing the range of operators of the type encountered in the modelling underlying the presented procedure can be found in Guillemin and Uhlmann (1981) and has been connected to seismic body-wave scattering by De Hoop and Uhlmann (2003) and the generalized screen wave-equation angle transform (Stolk & De Hoop, 2003).

Our annihilator-based approach to velocity analysis is of the differential semblance type. Systematic methods for updating velocity models by optimization using the differential semblance criterion have been introduced by Symes and et al. (2001), Mulder and Ten Kroode (2002), and others. The wave-equation angle transform occurs in the work of De Bruin et al. (1990) and of Sava et al. (1999) for the purpose of generating and analyzing common-image-point gathers.

For simplicity of formulation, we consider here P waves, possibly in transversely isotropic media with vertical symmetry axis described, approximately, by a scalar wave equation (see, e.g., Schoenberg and De Hoop (2000)). We also make the flattened Earth assumption to avoid contributions from the metric on a manifold in the equations.

The approach presented here is a synthesis of various developments: The generalized Bremmer coupling series to model seismic reflection data (De Hoop, 1996), inverse scattering based upon the first-order term of this series and the wave-equation angle transform (Stolk & De Hoop, 2003), the relation between wavefield reciprocity and optimization (De Hoop & De Hoop, 2000), and the generalized screen expansion for one-way wave propagation (Le Rousseau & De Hoop, 2001), which constitutes the downward continuation.

1 DIRECTIONAL WAVEFIELD DECOMPOSITION

We identify the $z$ coordinate with depth, which coincides with the direction of continuation. The remaining lateral coordinates are collected in $x$. Time is indicated by $t$. The partial derivatives are denoted by

$$D_{x_1, \ldots, x_{n-1}} = -i\partial_{x_1, \ldots, x_{n-1}}, \quad D_z = -i\partial_z, \quad D_t = -i\partial_t$$

so that their Fourier domain counterparts become multiplications by $\xi$, $\zeta$ (the wave vector) and $\omega$. Here, $n = 2$ for 2D seismics while $n = 3$ for 3D seismics.

We consider the scalar wave equation for pressure $u$ rewritten as a first-order system

$$\frac{\partial}{\partial z} \left( \begin{array}{c} u \\ \frac{\partial u}{\partial z} \end{array} \right) = \left( \begin{array}{cc} -A(x, z, D_z, D_t) & 1 \\ 0 & 0 \end{array} \right) \left( \begin{array}{c} u \\ \frac{\partial u}{\partial z} \end{array} \right) + \left( \begin{array}{c} 0 \\ f \end{array} \right), \tag{1}$$

where $A(x, z, \xi, \omega) = c_0(x, z)^{-2}\omega^2 - |\xi|^2$ is the principal (‘high-frequency’) part of the symbol of $A$. In a stratified medium that is translationally invariant in the horizontal directions, the principal symbol equals the full symbol; in more general media, a symbol calculus develops the subprincipal part of the symbol. In fact, $(x, z, t, \xi, \zeta, \omega)$ are coordinates on phase space; the symbol of $A$ is defined on this phase space. Localization in phase space is called microlocalization in the analysis literature.

1.1 The system of one-way wave equations

Microlocally, away from the zeroes of $A(x, z, \xi, \omega)$, system (1) can be transformed into diagonal form modulo a smoothing operator. A family of pseudodifferential operator matrices $Q(z) = Q(x, z, D_z, D_t)$ exists such that

$$\left( \begin{array}{c} u_+ \\ u_- \end{array} \right) = Q(z) \left( \begin{array}{c} u \\ \frac{\partial u}{\partial z} \end{array} \right), \quad \left( \begin{array}{c} f_+ \\ f_- \end{array} \right) = Q(z) \left( \begin{array}{c} 0 \\ f \end{array} \right),$$

satisfy the one-way wave or single-square-root equations

$$\left( \frac{\partial}{\partial z} ± iB_{\pm}(x, z, D_z, D_t) \right) u_± = f_±. \tag{2}$$

Any coupling between $+$ and $-$ constituents is absorbed in $f_±$, but is of higher order. The principal symbol $b$ of the $B_{\pm}$ is given by $b(x, z, \xi, \omega) = \sqrt{A(x, z, \xi, \omega)} = \omega \sqrt{\frac{1}{c_0(x, z)^2} - \omega^{-2}|\xi|^2}$; $\omega^{-1}b$ has the appearance of vertical wave slowness. For $(x, t, \xi, \omega)$ such that the symbol $B_{\pm}$ is real, the equation is of hyperbolic type, corresponding microlocally to propagating waves.

The subprincipal part of the $B_{\pm}$ depends on the normalization of $Q(z)$. We choose this normalization such that (2) is selfadjoint microlocally where the symbol of $B_{\pm}$ is real; then

$$u = Q_+^* u_+ + Q_-^* u_-, \quad f_+ = ± \frac{i}{2} Q_+ f, \quad f_- = ± \frac{i}{2} Q_- f,$$

where $Q_{\pm} = Q_\pm(z) = Q_\pm(x, z, D_z, D_t)$ are z-families of pseudodifferential operators with principal symbols $\omega^{-1/2}\left(\frac{1}{c_0(x, z)^2} - \omega^{-2}|\xi|^2\right)^{-1/4}$. The physical meaning of this choice of $Q_{\pm}$ is that the down- and upgoing fields are normalized in vertical-acoustic-power flux. The operators $Q_{\pm}$ form the first column of $Q$. We observe that $Q_-^* u_-$ represents the upgoing and $Q_+^* u_+$ represents the downgoing constituent of the wavefield $u$.

Upon regularization (for a precise treatment, see (Stolk & De Hoop, 2003 Appendix A)), there is a well defined solution operator $G_-(z, z_0)$, $z < z_0$, of the initial value problem for $u_-$ given by (2) with $f_- = 0$; this operator describes propagation from $z_0$ to $z$, in the upward direction (of decreasing $z$). The adjoint $G_-^*(z, z_0)^* describes propagation from $z$ to $z_0$ of (2) or from $z_0$ to $z$ in reverse time. The operator $G_-$ is a Fourier integral operator with complex phase
(Melin & Sjostrand, 1975), (Hörmander, 1985 chapter XXV), (Treves, 1980 chapters X and XI). By Duhamel’s principle, a solution operator for the inhomogeneous equation (2) is given by

$$u_-(\cdot, z) = \int_0^\infty G_-(z, z_0) f_-(\cdot, z_0) \, dz_0 .$$  \hspace{1cm} (5)

Note that the ‘−’ or upgoing constituent of the original Green’s function is generated by \(-\frac{i}{2}iQ_-(z)G_-(z, z_0)Q_-(z_0)\) (cf. (3)-(4)).

\subsection{Generalized screen expansion}

\subsubsection{The single-square-root operator}

The generalized screen expansion of the principal symbol of the single-square-root operator up to order \(N\) is of the form

$$b(x, z, \xi, \omega) = \sum_{j=0}^N A_j(\xi, \omega, z) \, U_j[\delta c_0](x, z) ,$$  \hspace{1cm} (6)

with \(U_0[\delta c_0](x, z) = 1\) (cf. (Le Rousseau & De Hoop, 2001(16))) \(^*\). The factors \(A_j\) mostly control the shape (bending) of the local slowness surface while the factors \(U_j\) account for the change in the slowness surface due to the lateral medium fluctuations with respect to some average background \(\bar{c}_0 = c_0(z)\) varying with depth only. The factors \(A_j\) depend upon \(\bar{c}_0\) but not upon the lateral medium fluctuations. This expansion is valid away from \(||\omega^{-1}\xi|| = \bar{c}_0^{-1}\) at each depth \(z\).

Up to principal parts, interpreting the symbol as a right or dual symbol, the operator \(B_-\) acts on the one-way wavefield \(u_-\) as

$$B_- u_- \sim F_{\omega \to \xi}^{-1} \sum_{j=0}^N F_{\xi \to x}^{-1} A_j(\xi, \omega, z) F_{x' \to \xi} U_j[\delta c_0](x', z) F_{\xi' \to -\omega} u_- ,$$

rewritten in the time-Fourier or frequency \((\omega)\) domain as

$$\hat{B}_- \hat{u}_- \sim \sum_{j=0}^N F_{\xi \to x}^{-1} A_j(\xi, \omega, z) F_{x' \to \xi} U_j[\delta c_0](x', z) \hat{u}_-$$  \hspace{1cm} (7)

(cf. (Le Rousseau & De Hoop, 2001(31))), where \(F\) denotes the Fourier transform and \(\hat{\cdot}\) indicates the time-Fourier domain. Thus the dependency of the operator on the laterally varying component of the background medium is completely contained in the factors \(U_j\).

\subsubsection{The perturbed single-square-root operator}

Here, for the later application of tomography, we consider how the single-square-root operator is perturbed under a smooth perturbation \(\delta c_0(x, z)\) of \(c_0\) subject to the constraint that \(c_0(z)\) is kept fixed. (In fact, any perturbation in \(c_0(z)\) is absorbed in \(\delta c_0(x, z)\).)

In view of (7) we have

$$(\delta \hat{B}_-) \hat{u}_- \sim \sum_{j=0}^N F_{\xi \to x}^{-1} A_j(\xi, \omega, z) F_{x' \to \xi} \delta U_j(x', z) \hat{u}_- .$$  \hspace{1cm} (8)

The perturbation \(\delta U_j\) in \(U_j\) is expressed in terms of Fréchet derivatives \(U_j'\) as

$$\delta U_j(x, z) = U_j'[\delta c_0](x, z) \, \delta c_0(x, z) ;$$  \hspace{1cm} (9)

this is a directional derivative, along a curve in \(c_0\) space \(^\dagger\). The curves follow a parametric representation of \(c_0\), for example, in terms of cubic \(B\)-splines. The operation in (9) is multiplicative.

Substituting (9) into (8), we find an operator

$$\hat{B}'_-(\hat{u}_-) \sim \sum_{j=0}^N F_{\xi \to x}^{-1} A_j(\xi, \omega, z) F_{x' \to \xi} \underbrace{U_j'[\delta c_0](x', z)}_{\text{'screen'}} \hat{u}_- .$$  \hspace{1cm} (10)

\(^*\)The generalized screen expansion implies \(A_j(\xi, \omega, z) = \omega a_j[c_0(z)^{-2} - \omega^{-2}||\xi||^2](-2j+1)/2\) with \(a_j = (-1)^j+1 \frac{3j+1}{j+1} \frac{2j-1}{(2j-1)}\), and \(U_j(x, z) = [c_0(x, z)^{-2} - \hat{c}_0(x, z)^{-2}]^{-j}(-2)c_0(x, z)^{-3}\).

\(^\dagger\)Here, \(U_j'(x, z) = j[c_0(x, z)^{-2} - \hat{c}_0(x, z)^{-2}]^{j-1}(-2)c_0(x, z)^{-3}\), \(j = 1, 2, \ldots\).
inhomogeneous double-square-root (DSR) equation, receivers (stations) where
\( \delta c \) is contained in the half space \( \{(x, z) \mid z > 0\} \). For (18) to be valid we assume that the rays are nowhere horizontal. In equation (18) we recognize the time convolution of an incoming Green’s function connecting the source at \( s \) (and zero depth) with the scattering point at \( r_0 = s_0 \) and depth \( z_0 \) with the outgoing Green’s function connecting this scattering point with the receiver at \( r \) (and zero depth).
2.2 Downward continuation of seismic data

We assume that the original data, \( d = d(s, r, t) \), are taken on the level surface \( z_0 = 0 \). (This may require a source depth correction.) The adjoint operator \( H(0, z)^\ast \) is used to propagate the data backward in time. We consider

\[
H(0, z)^\ast : d \mapsto H(0, z)^\ast \psi d = H(z, 0)\psi d ,
\]

where \( \psi \) is an appropriate tapered mute. The tapered mute is applied to guarantee that the seismic events corresponding to rays in the scattering process that do not satisfy the various conditions are eliminated from the data. \( H(0, z)^\ast \psi d \) is a function of \((s, r, t, z)\).

3 WAVE-EQUATION ANGLE TRANSFORM AND ANNIHILATORS

We discuss here how redundancy in the (downward-continued) data can be understood and exploited for the purpose of tomography. The redundancy arises in the data because the wavefield in the subsurface scatters over different angles from the same scattering (bounce) point. The reflection point becomes the image point in the imaging process, and the redundancy in the data becomes manifest in the multiple images that are generated by extracting the different scattering angles and azimuths in this process. This is accomplished by the angle transform.

The test whether the velocity model is acceptable is based on the test whether the data are in the range of the single scattering operator (Stolk & De Hoop, 2003). The notation in (21) is such that

\[
\xi(s, z, \rho, \omega) = \frac{1}{2\pi} \int \int \int \int \int u(x - \frac{h}{2}, x + \frac{h}{2}, ph, z) \chi(x, z, h) \ dx \ dh
\]

This is basically an example of double beamforming with the downward continued data, in sources and receivers (for the surface counterpart, see Scherbaum et al. (1997)). Clearly

\[
(Ad)(x, z, p) = \int \int \int \int \int \int (H(0, z)^\ast \psi d)(x - \frac{h}{2}, x + \frac{h}{2}, ph) \chi(x, z, h) \ dx \ dh
\]

The ‘true-amplitude’ counterpart of transform \( A \) follows from replacing

\[
d \mapsto u = (H(0, z)^\ast \psi d)
\]

by

\[
d \mapsto u = 2c_0(x, z)^3 \Xi(z)Q_{-s}(z)^{-1}Q_{-r}^{-1}H(0, z)^\ast Q_{-s}(0)^{-1}Q_{-r}(0)^{-1}D_{x}^{-2}\psi d ,
\]

where \( \Xi(z) \) is a pseudodifferential operator with principal symbol

\[
\Xi(s, r, t, \sigma, \rho, \omega) = \frac{1}{c_0(s, z)^3} \left( \frac{1}{c_0(s, z)^2} - \omega^{-2}\|\sigma\|^2 \right)^{-1/2} + \frac{1}{c_0(r, z)^3} \left( \frac{1}{c_0(r, z)^2} - \omega^{-2}\|\rho\|^2 \right)^{-1/2}
\]

(Schul & De Hoop, 2003(34),(41)). Using the true-amplitude version of the angle transform, the outcome is related to \( \delta\epsilon \) as an operator accounting for illumination effects only. We suppress the true-amplitude component operators in our formulas to keep presentation the simple.
3.2 Annihilators

Since the outcome \( (Ad)(x, z, p) \) should be independent of \( p \), annihilators of the data follow to be

\[
W_i := \langle A^{-1} \frac{\partial}{\partial p_i} A \rangle, \quad i = 1, \ldots, n - 1,
\]

where \( \langle A^{-1} \rangle \) indicates a regularized inverse of \( A \). Thus \( W_i d = 0 \). Rather than considering the annihilators in the data domain, we can consider the operators

\[
(R'_i u)(x, z) = \frac{\partial}{\partial p_i} AD_i^{-1} d
\]

\[
= \frac{1}{2\pi} \int \int_{\mathbb{R}^{n-1}} \hat{u}(x - \frac{\theta}{2}, x + \frac{\theta}{2}, \omega, z) \exp(-i\omega p h_i) h_i \chi(x, z, h) dh d\omega
\]

in the image domain. The annihilation of the data is replaced by an annihilation of the set of images \( AD_i^{-1} d \). This approach was followed in the framework of the ray geometrical generalized Radon transform (or Kirchhoff-style) migration velocity analysis by Brandsberg-Dahl, Ursin and De Hoop (2003b).

For the purpose of tomography, we also need the adjoint \( (R')^* \) of \( R' \). Let \( I \) denote a test image as a function of \( (x, z, p) \), then the adjoint follows from

\[
\langle R'u, I \rangle_{(x, p)} = \int_{\mathbb{R}^{2n-2}} \int_{\mathbb{R}^{n-1}} \hat{u}(x - \frac{\theta}{2}, x + \frac{\theta}{2}, \omega, z) \exp(-i\omega p h_i) h_i \chi(x, z, h) dh d\omega I(x, z, p)^* dx dp
\]

\[
= \langle \hat{u}, (R')^* I \rangle_{(s, r, \omega)} \text{ for given } z,
\]

from which we identify \( (R')^* \).

4 AN OPTIMIZATION PROCEDURE FOR REFLECTION TOMOGRAPHY

The model estimation in our approach to reflection tomography is based upon minimizing the functional

\[
\mathcal{J}[u] = \frac{1}{2} \int |(R'u)(x, z)|^2 dx dz dp.
\]

Note that the measure of traveltime mismatch has been replaced by the operator \( R' \); picking of traveltimes has been avoided. The integration is over all scattering or image points and angles.

The method for evaluating the Fréchet kernel or gradient of a functional derived from the solution of a partial (or pseudo)differential equation is known as the adjoint state method. It has been introduced and used in seismology by Tarantola (1987) and many others.

4.1 Perturbing the functional

Under a smooth perturbation \( \delta c_0 \) of \( c_0 \) subject to the constraint that \( \bar{c}_0 \) is kept the same, the perturbation of the downward-continued data \( u \) is \( \delta u \) while the perturbation of the functional follows to be

\[
\delta \mathcal{J} = \int (R'u) (R' \delta u) d(x, z) dp,
\]

because \( R' \) is independent of \( c_0 \). Hence,

\[
\delta \mathcal{J} = \frac{1}{2\pi} \int \int_{\mathbb{R}^{n-1}} (\delta \hat{u})^* \frac{\text{field}}{\text{source}} (R'u)^* d(s, r, z) d\omega.
\]
by definition of adjoint in (25). In preparation of applying the reciprocity theorem of the time-correlation type in the framework of one-way wave theory, we interpret \((R^*)^* R'u\) as a source, viz., in the adjoint field equation for \(v\), say,
\[
\left(-\frac{\partial}{\partial z} + iB_-(s, z, D_s, D_t) + iB_-(r, z, D_r, D_t)\right)\delta u = (R^*)^* R'u .
\]  
(29)
This equation is solved in the direction of decreasing \(z\) ("upward") with vanishing initial condition for some large \(z\) at the bottom of the model. Its right-hand side is interpreted as a penalty force, which vanishes if \(\delta_0\) were an acceptable background medium. The field \(\delta u\) satisfies the scattered field equation
\[
\left(-\frac{\partial}{\partial z} - iB_-(s, z, D_s, D_t) - iB_-(r, z, D_r, D_t)\right)\delta u = i\delta C(s, r, z, D_s, D_r, D_t) u
\]  
(30)
in the single-scattering approximation, where \(\delta C\) is perturbation of operator \(C\) with \(\delta_0\), see (8) and (13). This equation is solved in the direction of increasing \(z\) (downward) with homogeneous initial conditions at \(z = 0\). Its right-hand side is interpreted as a contrast source.

The reciprocity theorem of the time-correlation type now implies that we can write (28) in the equivalent form
\[
\delta J = \frac{1}{2\pi} \int \int \text{field} \left(\frac{1}{v} (i\delta \hat{C}u)^* d(s, r, z)\right) d\omega .
\]  
(31)

4.2 The kernel

To arrive at the sensitivity kernel, we write the perturbation \(\delta \hat{C}\) in expression (31) in the form \(\delta \hat{C}u = \tilde{C}'(\hat{u}) \delta \delta_0\), with
\[
\tilde{C}'(\hat{u}) \sim \sum_{j=0}^{N} F_{\rho \to \sigma}^{-1} A_j(\sigma, \omega, z) F_{\rho \to \sigma} U_j^*(s, z) \hat{u}(s, r, \omega, z) + \sum_{j=0}^{N} F_{\rho \to \sigma}^{-1} A_j(\rho, \omega, z) F_{\rho \to \sigma} U_j^*(r, z) \hat{u}(s, r, \omega, z) .
\]
Substituting this expression into (31) and taking the adjoint, we obtain
\[
\delta J = \frac{1}{2\pi} \int \int \hat{v} (i\delta \hat{C}u)^* d(s, r, z) d\omega = \int (i C'(u))^* v \delta \delta_0 d(x, z) .
\]  
(32)
To extract the kernel of the (directional) derivative of \(J\) out of expression (31), we make use of relation (12) with \(\hat{x}'\) playing the role of \(s\) or \(r\). Thus, the kernel attains the form
\[
C'(u)^* v = \frac{1}{2\pi} \int d\omega \left[ \int \hat{B}'_{\rho \to \sigma}(\hat{u})^* \hat{v} d\tau + \int \hat{B}'_{\rho \to \sigma}(\hat{u})^* \hat{v} d\tau \right]
\sim \frac{1}{4\pi} \text{Re} \int_0^{\infty} d\omega \sum_{j=0}^{N} \left[ \int \hat{u}(x, r, \omega, z) U_j^*(x, z) F_{\rho \to \sigma}^{-1} A_j(\sigma, \omega, z) F_{\rho \to \sigma} \hat{v}(\tau, r, \omega, z) d\tau \right]
+ \int \hat{u}(s, x, \omega, z) U_j^*(x, z) F_{\rho \to \sigma}^{-1} A_j(\rho, \omega, z) F_{\rho \to \sigma} \hat{v}(\tau, s, \omega, z) d\tau .
\]  
(33)
We used the symmetry in frequency to restrict the evaluation to positive values. This expression is of the form of a time cross correlation of the downward continued data \(u\) (cf. (19)) and the adjoint field \(v\) excited by a penalty force nested in a generalized screen operation. It differs from the standard imaging condition in particular through the integrations over \(s\) and \(r\).

4.3 Discretization in depth and generalized screen propagators

Discretization of the integral over \(z\) in (31) into a sum over a finite set of depths \(\{\Delta z\}\) admits the incorporation of a DSR wave propagators. Let the depth step be \(\Delta z = \Delta z_n\). The downward propagator, \(h(z_{n+1}, z_n)\), is derived from (17) and satisfies the relation
\[
u_{n+1} = h(z_{n+1}, z_n) u_n .
\]
First, the contrast source \(i\delta Cu\) evaluated at depth \(z_n\) in (31) is identified with the discretization of the left-hand side of (30),
\[
(\Delta z)^{-1} [\delta u_{n+1} - (1 + i\Delta z C(z_n)) \delta u_n] .
\]
where the subscript \( n \) indicates the evaluation at \( z = z_n \). Secondly, the operator \( 1 + i \Delta z \mathcal{C}(z_n) \) is viewed as an approximation to the propagator \( h(z_{n+1}, z_n) \). Thirdly, substituting such a replacement in the expression above implies that upon discretization the contrast source \( i \delta \mathcal{C} \check{u} \) inside the \( z \)-integral can be viewed as an approximation to
\[
ih(z_{n+1}, z_n)(\delta \mathcal{C}(z_n) \check{u}_n)
\]
inside a summation over \( n \). This expression is of the form of the difference of two residual propagators (Le Rousseau & De Hoop, 2001(27)) acting on \( \check{u}_n \).

5 DISCUSSION

We derived an explicit expression for the kernel driving wave-equation reflection tomography, using the generalized screen expansion for the downward continuation of the seismic data. Our criterion is based upon the condition that a velocity model is acceptable if the data are in the range of our modelling operator, here developed in the single scattering approximation. The evaluation of the kernel has a few unusual aspects: (i) the ‘penalty’ occurs and is evaluated in the subsurface rather than at the surface, (ii) the adjoint state method, derived from the reciprocity theorem of the time correlation type, leads to a procedure of propagating the adjoint field upwards and the data downwards and taking their cross correlation at the depths where the fields ‘meet’, (iii) a generalized screen operator is nested in the cross correlation in time. The penalty force can attain non-vanishing values only in regions that contain reflectors, where \( \delta \check{c} \neq 0 \).

The presented procedure admits bandlimited data. Any effects related to ‘wavefront healing’ are accounted for in the downward continuation restricted to the frequency band of the data irrespective of the formation of caustics. The full waveform is used, not the phase of the data.

The current method is developed for single scattering, i.e. primary reflections, and qP waves. Replacing the acoustic wavefield decomposition and generalized screen expansion by their elastic counterparts (Le Rousseau & De Hoop, 2003), in principal, the method can be extended to shear waves and mode conversions.

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Generation and processing of pseudo shear-wave data: Theory and case study

Vladimir Grechka* & Pawan Dewangan†

*Shell International Exploration and Production, 3737 Bellaire Blvd., P.O.Box 481, Houston, TX 77001-0481
†Center for Wave Phenomena, Department of Geophysics, Colorado School of Mines, Golden, CO 80401-1887

ABSTRACT

Processing of converted (PS) waves currently adopted by the exploration industry is essentially based on resorting the PS data into common conversion point gathers and using them for velocity analysis. Here we explore an alternative procedure. Our key idea is to generate the so-called pseudo-shear (ΨS) seismograms from the recorded PP and PS traces and run conventional velocity analysis on the reconstructed ΨS data. This results in an effective S-wave velocity model because our method creates data that possess kinematics of pure shear-wave primaries. We never deal with such complexities of converted waves as moveout asymmetry, reflection point dispersal, and polarity reversal, therefore, these generally troublesome features become irrelevant.

We describe the details of our methodology and examine its behavior both analytically and numerically. We apply the developed processing flow to a four-component ocean bottom cable line acquired in the Gulf of Mexico. Since the obtained stacking velocities of P- and ΨS-waves indicate the presence of effective anisotropy, we proceed with estimating a family of kinematically equivalent vertical transversely isotropic (VTI) velocity models of the subsurface.

1 INTRODUCTION

Due to a relatively high cost and usually poor quality of S-wave data excited on land and the absence of shear sources for marine surveys, converted (PS) waves are often used to infer shear-wave velocities in the subsurface. Processing of reflection PS data, however, is not straightforward. The main obstacle that precludes applying conventional velocity-analysis techniques to converted waves is the asymmetry of PS-wave reflection moveout on common-midpoint (CMP) gathers (e.g., Thomsen, 1999). As a consequence, neither a hyperbolic nor nonhyperbolic moveout equation routinely used for moveout correction of P-waves is generally expected to flatten PS data. Therefore, any technique designed for velocity estimation from reflected PS-waves should take into account the presence of some linear moveout at zero offset. This invariably complicates the moveout-estimation procedure for converted waves compared to that for pure modes and consequently decreases its robustness in the presence of noise. PS-wave processing is further compounded by polarity reversal and reflection (or conversion) point dispersal — phenomena that are almost nonexistent for conventional P-waves.

Recently, Grechka and Tsvankin (2002b) proposed a solution to the above outlined problems. Their \( PP + PS = SS \) method uses traveltime picks of reflected PP and PS primaries from selected horizons to reconstruct traveltimes of the corresponding pure shear (SS) waves. Since the obtained S-wave moveouts are symmetric on CMP gathers, conventional velocity analysis can be applied to them. Implementation of the original, kinematic version of the method has two key elements: identifying PP and PS events reflected from the same interfaces (this is the only indirect velocity information required) and picking traveltimes from PP and PS prestack data.

In this paper, we make the \( PP + PS = SS \) method more practical. We show that although an interpretive step of establishing the correspondence of PP and PS events cannot be avoided, the travelt ime picking can. We develop a procedure that replaces the direct picking with a specially designed convolution of the original PP and PS traces. The result, which we call pseudo-shear (ΨS) data, has kinematics of pure S-wave primaries and, therefore, represents an appropriate input for conventional velocity analysis.
We give the formulation of our procedure, explain why it works, and examine its performance on synthetic and field data. In particular, we demonstrate that our method is robust in the presence of random noise. Application of our technique to a 2-D multicomponent line acquired in the Gulf of Mexico indicates anisotropy and, therefore, requires building of at least a VTI (transversely isotropic with a vertical symmetry axis) velocity model. Although the narrow-azimuth nature of our data and the absence of substantial dip leads to a family of kinematically equivalent subsurface models (Grechka et al., 2002a), we show that none of them is isotropic.

2 OVERVIEW OF THE PP + PS = SS METHOD

A natural point of departure for our development is the kinematic conditions of PP + PS = SS method (Grechka and Tsvankin, 2002b). Figure 1 shows PP and PS ray trajectories that the method finds from 2-D split-spread PP and PS reflection data. Three rays $x^{(1)}$, $x^{(2)}$, and $x^{(3)}$ have exactly the same reflection point $R$ if the pairs of PP and PS reflection slopes coincide at the $P$-wave source and receiver locations $x^{(1)}$ and $x^{(3)}$. Since by definition the slope $p(x^{(s)}, x^{(r)})$ measured at source location $x^{(s)}$ on the common-receiver gather located at $x^{(r)}$ is

$$ p(x^{(s)}, x^{(r)}) = \frac{\partial t_{PP}(x^{(s)}, x^{(r)})}{\partial x^{(s)}} = \frac{\partial t_{PS}(x^{(s)}, x^{(r)})}{\partial x^{(s)}} $$

for any reflection mode, the requirement of equal PP and PS slopes yields

$$ \frac{\partial t_{PP}(x^{(1)}, x^{(2)})}{\partial x^{(1)}} = \frac{\partial t_{PS}(x^{(1)}, x^{(3)})}{\partial x^{(1)}} $$

and

$$ \frac{\partial t_{PP}(x^{(2)}, x^{(3)})}{\partial x^{(2)}} = \frac{\partial t_{PS}(x^{(2)}, x^{(4)})}{\partial x^{(2)}}. $$

Here $t_{PP}$ and $t_{PS}$ are the traveltimes of $PP$- and $PS$-waves, $x^{(j)}$ ($j = 1, 2, 3, 4$) denote the source and receiver coordinates, and

$$ t_{PP}(x^{(1)}, x^{(2)}) = t_{PP}(x^{(2)}, x^{(1)}) $$

due to reciprocity. The geometry in Figure 1 produces the pure-$S$ reflected ray $x^{(3)}R x^{(4)}$. The travelt ime $t_{SS}$ along it is given by

$$ t_{SS}(x^{(3)}, x^{(4)}) = t_{PP}(x^{(1)}, x^{(3)}) + t_{PS}(x^{(2)}, x^{(4)}) $$

Clearly, one needs to pick prestack reflection traveltimes $t_{PS}$ and $t_{PP}$ along a selected horizon to calculate $t_{SS}$ from equations (2)–(4). Although this is feasible in principle (Grechka et al., 2002a), prestack travelt ime picking is known to be tedious, labor-intensive, and noise prone. Below, we describe a technique that not only makes traveltime picking unnecessary but also produces seismograms that resemble pure shear-wave reflection data for all horizons.

3 THEORETICAL ASPECTS OF GENERATING $\Psi S$ DATA

3.1 Statement of the result

It turns out that direct travelt ime picking can be replaced by computing the integral

$$ w_{PS}(t, x^{(3)}, x^{(4)}) = \iint \left[ w_{PS}(t, x^{(1)}, x^{(3)}) 
+ w_{PP}(-t, x^{(1)}, x^{(2)}) \ast w_{PS}(t, x^{(2)}, x^{(4)}) \right] dx^{(1)} dx^{(2)}. $$

Here $w_{PP}$ and $w_{PS}$ are the $PP$ and $PS$ seismic traces, $t$ is time, and asterisks denote convolutions in time. The $PP$ traces are taken in reverse time because the $P$-wave time gets subtracted in equation (4) to produce the pure-shear time. Integration is performed over $P$-wave source and receiver coordinates $x^{(1)}$ and $x^{(2)}$. The result of integration (5) is the $\Psi S$ trace for the source and receiver located at $x^{(3)}$ and $x^{(4)}$.

Evaluation of integral (5) using the stationary phase method (Appendix A) shows that if the traces $w_{PP}$ and $w_{PS}$ consist of $PP$ and $PS$ primaries corresponding to a selected reflector, the trace $w_{PS}$ contains the pure $S$-wave primary from the same reflector. The fact that the proof in Appendix A is possible only for an isolated interface indicates the necessity for windowing the input $w_{PP}$ and $w_{PS}$ traces around events of interest. This in turn requires establishing the correspondence of $PP$ and $PS$ events prior to evaluating integral (5).
3.2 Amplitudes of $\Psi S$-waves

Even though integral (5) produces $\Psi S$ data that ideally have traveltimes of pure $S$-waves, their amplitudes do not correspond to those of shear-wave primaries. This is immediately clear from equation (5). Indeed, the $\Psi S$-wave amplitudes are inherited from those of the $PP$ and $PS$ events and, therefore, generating $\Psi S$ data is not a true-amplitude procedure. This is not surprising because, as follows from the reciprocity theorem, $PP$ and $PS$ data generally do not contain enough information for reconstructing the true $S$-wave amplitudes. The last statement can be illustrated by considering zero-offset $PP$ and $PS$ reflections from a single, sufficiently thick horizontal layer. The amplitude of reflected $P$-wave in this model is proportional to the $P$-wave impedance contrast at the interface, while the amplitude of converted wave is zero. Clearly, no information about the contrast of shear-wave impedance that governs amplitude of the pure $S$-wave can be extracted from the zero-offset $PP$ and $PS$ traces.

On the other hand, $\Psi S$-wave amplitudes are obtained in a deterministic way from those of $PP$ and $PS$ reflections, therefore, they might be an attractive subject for amplitude-versus-offset (AVO) analysis. The benefits of using $\Psi S$-waves instead of $PS$ ones include the simplicity of their moveouts and usually higher signal-to-noise ratio than that of the original $PP$ and $PS$ data (see Figure 7 as an example). Below, however, we mostly ignore amplitude issues and concentrate primarily on velocity analysis of $\Psi S$-waves. We will see that the described amplitude behavior does not cause any problem for obtaining accurate shear-wave stacking velocities.

3.3 Extension to 3-D

Similarly to the original $PP + PS = SS$ method of Grechka and Tsvankin (2002b), integral (5) can be extended to 3-D multiazimuth reflection data. Since the sources and receivers are now allowed to cover a certain area, their coordinates $x \equiv [x_1, x_2]$ become two-dimensional vectors. The $\Psi S$ traces are formally given by the same equation (5)

$$w_{\Psi S}(t, x^{(3)}, x^{(4)}) = \iiint \left[ w_{PS}(t, x^{(1)}, x^{(3)})ight. \
* w_{PP}(-t, x^{(1)}, x^{(2)}) * w_{PS}(t, x^{(2)}, x^{(4)}) \left.] dx^{(1)} dx^{(2)}, \right.$$ \hspace{1cm} (6)

where all $x^{(j)}$ ($j = 1, 2, 3, 4$) are 2-D vectors and, therefore, integration becomes four-fold.

4 SYNTHETIC EXAMPLES

Here we present numerical tests that illustrate the above outlined theory in 2-D. We begin with examining the time function $\tau$ [equation (A5)] whose extrema determine kinematics of $\Psi S$-waves, then compute integral (5) to study features of $\Psi S$-data, and finally analyze the performance of our procedure in the presence of random noise.

4.1 Time function $\tau$

As follows from Appendix A, the ability of integral (5) to represent kinematics of $S$-waves depends upon the presence of roots $x^{(1)}$ and $x^{(2)}$ of equations (A6). Clearly, the $PP$ and $PS$ data are supposed to be physically recorded at the source and receiver locations $x^{(1)}$ and $x^{(2)}$ to contribute to the integral. Since the number of roots $x^{(1)}$ and $x^{(2)}$ and their very existence are unknown in advance, we present two examples that demonstrate what normally might be expected.

Defining the $P$-wave half-offset $h_{PP}$ and the midpoint $y_{PP}$ as

$$h_{PP} = \frac{x^{(2)} - x^{(1)}}{2} \quad \text{and} \quad y_{PP} = \frac{x^{(2)} + x^{(1)}}{2},$$

we plot the time functions $\tau(h_{PP}, y_{PP})$ in a plane homogeneous isotropic layer (Figure 2). The $PP$- and $\Psi S$-wave midpoints, $y_{PP}$ and $y_{PS}$, coincide because the layer is horizontal. As a consequence, the curves in Figure 2 were computed for the correct, i.e., stationary values $y_{PP} = y_{PS}$. Figure 2 dis-
plays two clear maxima* of $\tau(h_{PP})$ for $\Psi S$ half-offsets $h_{\Psi S} = (x^{(3)} - x^{(4)})/2 = 0$ and $h_{\Psi S} = 0.25$ km (solid and dashed lines) and a poorly defined one for $h_{\Psi S} = 0.5$ km (dotted line). Those maxima, marked with large dots, produce stationary points that yield pure shear-wave reflection traveltimes. This statement can easily be verified by computing the $S$-wave moveout in our model (the parameters are given in the caption to Figure 2).

Note that the curves in Figure 2 flatten out at large $P$-wave half-offsets $h_{PP}$ suggesting that the time function $\tau$ has extrema at $h_{PP} \rightarrow \pm \infty$. Although these extrema might seem irrelevant for the problem at hand because data are never acquired at infinite offsets, we will see that both the flatness of time function $\tau$ and finite frequency bandwidth of seismic data lead to noticeable contributions associated with those distant extrema even at relatively moderate $h_{PP}$.

The extrema at $h_{PP} \rightarrow \pm \infty$ relate to critical offsets of shear-waves (Figure 3). The $P$-wave incident and $S$-wave reflection angles, $\theta_P$ and $\theta_S$, respectively, satisfy Snell’s law

$$\sin \theta_P = \frac{V_P}{V_S} \sin \theta_S,$$

where $V_P$ and $V_S$ are the $P$- and $S$-wave velocities. Therefore, the maximum reflection angle of shear waves

$$\theta_S^{\text{crit}} = \sin^{-1}(V_S/V_P).$$

This yields the critical half-offset

$$h_{\Psi S}^{\text{crit}} = D \tan \left[ \sin^{-1} \left( \frac{V_S}{V_P} \right) \right]$$

in our simple model in Figure 3. Here $D$ is the layer thickness. Since the $P$-wave half-offset corresponding to $h_{\Psi S}^{\text{crit}}$ is infinite, equation (7) establishes a physical limit of the maximum $\Psi S$ offset that can be obtained from $PP$ and $PS$ reflection data regardless their original offsets. Clearly, a low $V_S/V_P$ velocity ratio results in a small spread of the $\Psi S$ data, thus, increasing the uncertainty of shear-wave velocity analysis. Our method explicitly shows the dependence of accuracy of the estimated shear-wave velocities on the $V_S/V_P$ ratio, an issue that does not obviously follows from direct analysis of converted waves.

Figure 2 indicates that waves corresponding to the contributions of the flat areas of the time function $\tau(h_{PP})$ at $h_{PP} \rightarrow \pm \infty$ propagate with the $P$- rather than shear-wave velocity. For this reason, we will call them the $\Psi P$-waves. Figure 2 also shows that the extrema of $\tau(h_{PP})$ corresponding to $\Psi S$- and $\Psi P$-waves approach each other in a continuous fashion and become indistinguishable as $h_{\Psi S} \rightarrow h_{\Psi S}^{\text{crit}}$ (dotted line). Since the $\Psi S$ and $\Psi P$ arrivals have different kinematics, mixing them together (under the improper name $\Psi S$) will invariably lead to errors in the estimated $S$-wave velocities. The only remedy against that is to restrict integration in (5) to the half-offsets $h_{\Psi S} \leq h_{\Psi S}^{\text{crit}}$.

Figure 4 gives an example of the time function $\tau(h_{PP}, y_{PP})$ for a dipping layer. The stationary (saddle) point corresponding to the $\Psi S$ arrival is located at $h_{PP} = 0.5$ km and $y_{PP} = -0.1$ km (cross in Figure 4); the common midpoints $y_{\Psi S}$ and $y_{PP}$ are different due to the reflector dip. The flattening of the time function in the bottom corners of Figure 4 indicates approaching the extrema associated with $\Psi S$-waves.

### 4.2 $\Psi S$ CMP gathers

Having learned that the integration limits in equation (5) need to be chosen properly to avoid mutual contamination of $\Psi S$ and $\Psi P$ data and also that convolutions have to be performed within time gates that enforce correlation of $PP$ and $PS$ events, we proceed with actual computation of $\Psi S$ seismograms.

Figures 5a and 5b show the input $PP$ and $PS$ traces, while Figure 5c demonstrates the result, a com-

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*They are actually the saddle points. To show this, one needs to plot $\tau(h_{PP}, y_{PP})$ in the $[h_{PP}, y_{PP}]$ coordinates as done in Figure 4 below.
computed \( \Psi S \) CMP gather. The \( PP \) and \( PS \) data have the following features.

- \( PP \) and \( PS \) wavelets differ in both the shape and the frequency content; the ratio of dominant frequencies of \( PP \)- and \( PS \)-waves is 1.5.
- The polarity of reflected \( PS \)-waves flips at the zero offset.
- The converted-wave moveouts corresponding to dipping reflectors are asymmetric with respect to interchange of the source and receiver positions (Figure 5b). This phenomenon precludes applying conventional velocity analysis to \( PS \) data.

Figure 5c illustrates that the above features, which are usually troublesome for conventional converted-wave processing, do not prevent us from reconstructing meaningful \( \Psi S \) data. The observation of primary importance in Figure 5c is that the reconstructed \( \Psi S \) events follow correct (i.e., those computed by ray tracing) \( S \)-wave moveouts. The other features of \( \Psi S \) are also worthwhile noting.

- As expected from the nature of convolution in equation (5), the \( \Psi S \) wavelets are longer and have more lobes than those of either \( PP \) or \( PS \) ones. Their dominant frequency lies between those of \( PP \)- and \( PS \)-waves.
- The \( \Psi S \) CMP gather does not have polarity reversal. This follows from our formulation (5), which uses \( PS \) data twice so that the sign of \( PS \) polarity gets cancelled out.
- Weak amplitudes of \( \Psi S \)-waves reconstructed at small offsets relate to the correspondingly weak \( PS \) signals. In particular, the \( PS \) amplitude is zero at \( x^{(1)} = x^{(2)} = 0 \). Since this is the stationary point for the output offset \( X_{\Psi S} \equiv 2 h_{\Psi S} x^{(3)} - x^{(4)} = 0 \), the zero-order term of the stationary phase method predicts a vanishing \( \Psi S \)-wave amplitude. The weak \( PS \) arrivals observed at \( h_{\Psi S} = 0 \) in Figure 5c correspond to the contributions of higher-order terms in the stationary-phase expansion.
- The absence of \( \Psi S \)-waves at large offsets is a result of limiting integration (5) to pre-critical offsets only. Specifically, our algorithm checks the value of \( h_{\Psi S} \) and evaluates integral (5) only if \( h_{\Psi S} \leq h_{\Psi S}^{\text{crit}} \); otherwise it fills the output \( \Psi S \) trace with zeros. Termination points for two shallow reflections in Figure 5c suggest that \( \Psi S \)-waves can be reconstructed only for offset-to-depth ratios smaller than \( X_{\Psi S}/D \approx 1 \). This observation is in agreement with our theoretical result (7), which yields \( X_{\Psi S}^{\text{crit}}/D = 2 h_{\Psi S}^{\text{crit}}/D = 1.15 \).

Following the discussion above, one might wonder what happens if we try to perform the same computation at half-offsets \( h_{\Psi S} > h_{\Psi S}^{\text{crit}} \). Figure 6 addresses this question. We observe the events whose moveouts are approximately linear with the slopes \( dt/dh_{\Psi S} \approx 2/V_P \) or \( dt/dX_{\Psi S} \approx 1/V_P \). Clearly, their kinematics has nothing to do with that of reflected \( S \)-waves. This explains why we have chosen to call these arrivals the \( \Psi P \)-waves. Only the changes in lengths of \( P \)-wave segments of converted-wave ray trajectories contribute to the moveouts in Figure 6 because the \( S \)-wave segments remain constant at any half-offset \( h_{\Psi S} > h_{\Psi S}^{\text{crit}} \).

Since the \( \Psi P \)-wave traveltimes are smaller than those associated with \( \Psi S \)-waves (compare Figures 5c and 6), mixing the two events together and performing conventional velocity analysis will bias the estimated \( S \)-wave velocities towards higher values. As was already

**Figure 5.** Input (a) \( PP \), (b) \( PS \), and (c) generated \( \Psi S \) CMP gathers computed for midpoint at the origin. The model is homogeneous and isotropic with \( V_P = 4 \) km/s and \( V_S = 2 \) km/s. Three planar reflectors have the depths \( D = 0.7, 1.2, \) and 1.8 km beneath the origin; their dips are \( 0^\circ, 10^\circ, \) and \( 20^\circ \). Dots indicate correct \( S \)-wave reflection traveltimes.
mentioned, we suggest to remove this bias by restricting integration in (5) to half-offsets $h_{SS} \leq h_{SS}^{\text{crit}}$.

4.3 Influence of random noise

Since we intend to apply the discussed procedure to field data, it is important to examine its robustness with respect to noise. Figures 7a and 7b show $PP$ and $PS$ traces similar to those in Figures 5a and 5b but contaminated with Gaussian noise that has standard deviation equal to 1/2 of the maximum amplitude in the data. Even though the $PP$ and $PS$ reflections are hardly recognizable in Figures 7a and 7b, Figure 7c displays remarkably clean $\Psi$ traces. Such a result is a direct consequence of applying convolutions, which efficiently attenuate random noise.

5 GULF OF MEXICO CASE STUDY

We tested our methodology on a multicomponent (hydrophone, vertical geophone, and inline horizontal geophone) 2-D line acquired in the Gulf of Mexico. First, the $PP$-to-$PS$ event correspondence was established. This gave us the relationship between $PP$ and $PS$ zero-offset times, $t_{PP0}$ and $t_{PS0}$, which were used to compute the shear-wave traveltimes $t_{SS0}$ and the ratios $g_0$ as

$$g_0(t_{PS0}) \equiv \frac{t_{PP0}}{2t_{PS0} - t_{PP0}} = \frac{t_{PP0}}{t_{SS0}}.$$  (8)

The result of our computations is shown in Figure 8. Due to mild lateral heterogeneity of the subsurface, both functions $t_{SS0}(t_{PP0}, Y)$ and $g_0(t_{PP0}, Y)$ have a relatively weak dependence on the CMP coordinate $Y$.

An important observation that can be made from Figure 8b is that the ratio $g_0$ is quite small (about 0.15) at shallow depths. If we ignore the possible presence of anisotropy, reflector dip, and lateral heterogeneity, and use the value of $g_0 = V_S/V_P = 0.15$ to estimate the critical $\Psi$S offset-to-depth ratio from equation (7), we find $X_{\Psi S}^{\text{crit}}/D = 2h_{PS}^{\text{crit}}/D \approx 0.3$. Clearly, shear-wave stacking velocities cannot be picked accurately from such short-spread moveouts. The ratio $g_0$, however, rapidly increases as we go deeper, reaching the value of $g_0 = 0.35$ at $t_{PS0}$ of around 5.5 s. This yields the ratio $X_{\Psi S}^{\text{crit}}/D \approx 0.75$ which makes the results of $\Psi$S-wave velocity analysis substantially more accurate despite the influence of growing $V_S(t_{SS0})$ that flattens $\Psi$S moveouts and, therefore, tends to reduce the accuracy of velocity picking.

Figure 9, which displays the generated $\Psi$S data, corroborates the above discussion. Indeed, we observe a rapid increase of the maximum $\Psi$-wave offset as the time $t_{SS0}$ grows. Velocity analysis performed on $\Psi$S CMP gathers such as those in Figure 9 produces poorly resolved semblance maxima at shear times $t_{SS0}$ smaller than about 5-6 s (not shown). This, however, is a consequence of low $V_S/V_P$ ratio in the shallow layers rather than a deficiency of the applied procedure. The semblance maxima of $\Psi$-waves become better focused as $t_{SS0}$ exceeds 6-7 s, which corresponds to $P$-wave times $t_{PP0}$ between 2 and 3 s. The results of velocity analysis on $PP$ and $\Psi$S CMP gathers (Figure 10) are used below to estimate the interval velocities.

5.1 Evidence for effective anisotropy

Prior to estimating the interval-velocity model, one usually needs to understand which its features are constrained by the available data. Here, we show that the event correlation (Figure 8) and the picked normal-moveout (NMO) velocities (Figure 10) unambiguously indicate non-negligible anisotropy. To draw such a conclusion, we essentially rely on the fact that the subsurface structure, being characterized by weak lateral heterogeneity and mild dip (see Figure 13 below), is approximately one-dimensional.

Following Grechka et al. (2002a), let us examine two velocity ratios: $g_0$, which is equal to the ratio of the vertical velocities $V_S$ and $V_P$ for a horizontally layered media,

$$g_0 \equiv \frac{t_{PP0}}{t_{SS0}} = \frac{V_S}{V_P},$$  (9)

and

$$g_{nmo} \equiv \frac{V_{S,nmo}}{V_{P,nmo}}.$$  (10)

The ratio $g_0$ derived from the $PP$ and $PS$ event correlation is shown in Figure 8b, while the ratio of $\Psi$S- and $P$-wave NMO velocities is given in Figure 11. Comparing the two, we observe that $g_{nmo}$ is consistently greater.
Figure 7. Same as Figure 5 but for PP and PS data contaminated with Gaussian noise that has signal-to-noise ratio equal to 2. Dots indicate correct S-wave reflection traveltimes.

Figure 8. Event correlation displayed (a) as function $t_{SS0}(t_{PP0}, Y)$ and (b) as ratio $g_0(t_{PP0}, Y)$. Here $t_{SS0}$ and $t_{PP0}$ are the S- and P-wave two-way zero-offset times (in s), respectively, and $Y$ is the distance (in km) along the line or the CMP coordinate. The color bars refer to the values of $t_{SS0}$ (a) and $g_0$ (b).

than $g_0$. For the P-wave times $t_{PP0} > 2.5$ s, where we expect the $\Psi S$-wave NMO velocities to be sufficiently accurate, $g_0 \approx 0.3$ whereas $g_{nmo} \approx 0.4$.

In isotropic layered media, $g_{nmo}$ can be greater than $g_0$ only because of larger vertical variability of S-wave velocities compared to that of the P-wave ones. According to Grechka et al. (2002a), relative vertical changes of $V_S$ are supposed to be factors of magnitude greater than those of $V_P$ to explain the observed values of $g_0$ and $g_{nmo}$. Since such a high vertical heterogeneity of shear-waves is not supported by the data, only the presence of anisotropy can give plausible explanation for the difference between the two velocity ratios. Therefore, the reconstructed subsurface model has to be anisotropic. Next, we need to choose the type of anisotropy. This choice is essentially governed by our ability to estimate relevant anisotropic parameters from given 2-D narrow-azimuth data and a relatively simple (close to 1-D) subsurface structure. Taking into account that the data were acquired in a sedimentary basin, transversely isotropic model with a vertical symmetry axis (VTI) is a proper choice. Also, we assume that we are dealing with $SV$-waves because converted waves recorded on inline horizontal component were used to generate the $\Psi S$ data.

The ratios $g_0$ and $g_{nmo}$ defined by equations (9) and (10) can be directly linked to anisotropic coefficients $\delta$ (Thomsen, 1986) and $\sigma$ (Tsvankin and Thom-
Figure 9. Typical ΨS CMP gathers.

Figure 10. Picked (a) PP- and (b) ΨS-wave NMO velocities (in km/s).

Figure 11. Ratio \(g_{nmo} \) of the picked ΨS- and P-wave NMO velocities. For comparison, the color scale in this figure is the same as that in Figure 8b.

\[ g_{nmo} = \sqrt{1 + 2\delta} \]

\[ g_{0} = \sqrt{1 + 2\sigma} \]

\[ \chi = \frac{\sigma - \delta}{1 + 2\delta} \]

5.2 Estimation of interval parameters

The feasibility of estimating anisotropy from reflection seismic data is governed by the amount of angular information that can be used in the inversion. Both acquisition design and the presence or absence of dipping
structures influence angular coverage of the subsurface. While 3-D, wide-azimuth, long-spread data generally constrain anisotropic parameters better than do 2-D, narrow-azimuth, conventional-offset data, all pertinent quantities still cannot be estimated uniquely in horizontally layered VTI media unless check shots or well logs are available (e.g., Grechka and Tsvankin, 2002a; Grechka et al., 2002b). Therefore, it is important to examine whether or not our data contain enough dip information for the inversion.

Figure 13 addresses this issue. It shows a histogram of more than thousand dips $\psi$ automatically picked from the stacked $PP$ data. Even though we estimated the dips using the equation

$$\psi = \tan^{-1}\left(\frac{dt_{PP0}}{dY} \cdot \frac{V_{P,nmo}^{1/2}}{2}\right)$$

which ignores anisotropy and heterogeneity, the message communicated by Figure 13 is clear. The subsurface can be treated as horizontally layered for the purpose of anisotropic inversion because the mean absolute dip is about 2.5° and 50% of our picks fall below 1.7°.

As Grechka and Tsvankin (2002a) show, the true depths in horizontally layered VTI media cannot be estimated from reflected $PP$- and $PS$-waves even when long-spread data are available. Therefore, we essentially have the following two choices. We could set either anisotropic coefficient $\delta$ or $\sigma$ to any chosen constant value or any predetermined function and proceed with anisotropic stacking velocity tomography as was done by Grechka et al. (2002a). The arbitrariness in $\delta$ or $\sigma$ will produce a family of kinematically equivalent VTI depth models. Alternatively, we can perform parameter estimation in time domain that targets the interval NMO velocities $V_{P,nmo}$, $V_{S,nmo}$, and the quantity $\chi$ treated as functions of the $P$-wave vertical time $t_{PP0}$. This yields a unique VTI time model, with the above mentioned ambiguity hidden in the time-to-depth conversion. We select this option here.

Figures 14a and 14b show the interval $V_{P,nmo}$ and $V_{S,nmo}$ that were obtained by performing conventional Dix (1955) differentiation of the NMO velocities displayed Figure 10. The interval anisotropy $\chi$ in Figure 14c was computed from the interval ratios $g_{nmo}$ and the event correspondence (Figure 8). Even though the higher $\chi$ values for $t_{PP0} < 2.5$ s seem to relate to inaccuracies in shear-wave velocity picking, we again observe non-negligible anisotropy.

Note that the inversion produces similarly dipping features in the middle of the $V_{P,nmo}$, $V_{S,nmo}$, and $\chi$ sections. Such kind of general similarity is expected because the three fields correspond to the same area of the subsurface. The sections of $V_{P,nmo}$, $V_{S,nmo}$, and $\chi$, however, differ in many details. This also could have been predicted because they represent different physical properties obtained from different components of seismic data which went through different processing sequences. Still, in some sense, the three sections in Figure 14 complement each other, thus, providing information that cannot be extracted from $P$-wave data alone.

Even though Figure 14 displays our final output, it hides significant depth ambiguity of the estimated anisotropic models. Since the pairs $\{t_{PP0}, V_{P,nmo}\}$ and $\{t_{SS0}, V_{S,nmo}\}$ have been obtained, we would still need to fix either anisotropic coefficient $\delta$ or $\sigma$ in order to perform time-to-depth conversion. Neither of these coefficients, however, can be estimated from the data. Instead, we can constrain only their combination $\chi = (\sigma - \delta)/(1 + 2\delta)$ which does not allow us to resolve $\delta$ and $\sigma$ individually, as a consequence, the true depth remains unknown. One might notice an analogy between this conclusion and nonhyperbolic velocity analysis of $P$-waves, where the estimated $V_{P,nmo}$ and the Alkhalifah-Tsvankin (1995) anellipticity coefficient $\eta$ also tell us nothing about the depth.
shear-wave primaries. As a result, conventional velocity analysis performed on $\Psi S$ CMP gathers yields $S$-wave NMO velocities. These velocities, along with those of the $P$-waves and the corresponding reflection dips, can be used for building elastic (usually anisotropic) interval-velocity models. Analyzing this final step, however, is outside the scope of our paper.

To construct the $\Psi S$ data, one has to integrate specially designed convolutions of $PP$ and $PS$ traces [equations (5) and (6) in 2-D and 3-D, respectively]. We found, however, that successful implementation of the technique requires selecting time gates that enforce the correspondence of $PP$ and $PS$ events and restricting the integration limits. While the time gates are needed to avoid generating artificial multiples and unphysical events (Grechka and Tsvankin, 2002b), choosing the limits of integration around proper stationary points ensures that the output traces are not contaminated with $\Psi P$-waves and other unwanted arrivals. One conclusion we drew along the way was that the spread of $\Psi S$ data is always limited by the critical offset. Although this result directly follows from Snell’s law, it has the following important practical implication. When the $S$-to $P$-wave velocity ratio is small, the correspondingly small ratio of maximum $\Psi S$ offset to reflector depth will invariably compromise the quality of shear-wave velocity analysis.

We tested our methodology on both synthetic and field data. While synthetic examples helped us to establish some data requirements and characteristics (e.g., $PP$ and $PS$ wavelets do not have to be the same, and random noise is not a problem), the presented case study also demonstrated that our processing flow can produce useful results.

A relevant issue we have not discussed in detail relates to the errors in establishing event correspondence. Substantial errors in relating the events to the same reflector lead to convolving wrong $PP$ and $PS$ arrivals and, therefore, produce $S$-wave NMO velocities that can be significantly incorrect. Sometimes those errors are relatively easy to recognize. We have examined the influence of small deviations from correct event correspondence on the output shear-wave zero-offset times and NMO velocities. We found that the ratio of errors in $t_{SS0}$ and $V_{S, nmo}$ is always positive. Therefore, when the velocity $V_{S, nmo}(t_{SS0})$ grows (which is often the case), it is not going to be severely distorted because erroneous values of $V_{S, nmo}$ and $t_{SS0}$ will fall relatively close to the correct trend $V_{S, nmo}(t_{SS0})$.

We have left a few issues almost completely undeveloped. First, almost horizontally propagating $\Psi P$-waves generated by our procedure might be used to get more accurate estimates of anisotropic coefficient $\eta$ than those obtained from $P$-wave nonhyperbolic moveout. A similar idea, although for $PS$-rather than $\Psi P$-waves, was proposed by Grechka and Tsvankin (2002a). They noticed that longer and closer to horizontal $P$-ray segments in $PS$ data compared to those in $PP$ ones more
Finally, generating Ψ in making use of Ψ data includes their simpler moves and usually higher signal-to-noise ratio. Generating Ψ waves and estimating shear-wave velocities provides solid ground for performing migration of Ψ data. This has not been done so far, therefore, it remains to be seen what can be gained from the Ψ images.

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APPENDIX A: KINEMATICS OF Ψ-S WAVES

Here we prove that integral (5) produces waves that have kinematics of pure-S primaries when the input PP and PS traces contain primaries only. Applying the Fourier transform to equation (5) yields

\[ W_{PS}(\omega, x, y) = \int \left[ W_{PS}(\omega, x', y') \right] \times \left[ W_{PP}(\omega, x', y') \right] d\omega d\omega \] (A1)

where \( \omega \) is radial frequency, \( W_{PP}, W_{PS} \), and \( W_{PS} \) are the spectra of PP, PS, and S waves, respectively, and star denotes complex conjugate.

If we assume that the input PP and PS traces consist of primaries reflected from a given interface, their spectra have the form

\[ W_{PP}(\omega, x, y) = F_{PP}(\omega) A_{PP}(x, y) \]

and

\[ W_{PS}(\omega, x, y) = F_{PS}(\omega) A_{PS}(x, y) \]

Here \( A_{PP} \) and \( A_{PS} \) represent the amplitudes of reflected PP- and PS-waves, \( F_{PP} \) and \( F_{PS} \) denote the spectra of their wavelets, and indexes \( s \) and \( r \) take the pairs of values \( \{ s = 1, r = 3 \} \) and \( \{ s = 2, r = 4 \} \). Substituting equations (A2) and (A3) into integral (A1), we obtain

\[ W_{PS}(\omega, x, y) = \int \left[ A_{PS}(x', y') \right] \times \left[ A_{PP}(x', y') \right] e^{i\omega t_{PP}(x', y')} d\omega d\omega \] (A4)

The time \( t \) in this equation is given by

\[ t \equiv t(x', y') = t_{PS}(x', y') + t_{PP}(x', y') \] (A5)

Since our goal is to show that the Ψ event has kinematics of pure-S primary, we need to evaluate integral (A4) in the limit of \( \omega \to \infty \). This can be done with the stationary phase method. According to this method, the main contributions to the integral occur around points \( \{ x', y' \} \) that satisfy the conditions of stationarity

\[ \frac{\partial \tau}{\partial x} = 0 \] (A6)
Let us note that equations (A6) coincide with expressions (2) and (3) representing the requirements of matching reflection slopes of reflected $PP$- and $PS$-waves (Figure 1). If equations (A6) have a solution
\[ \{x^{(1)}, x^{(2)}\} \equiv \{(x^{(3)}, x^{(4)}), (x^{(3)}, x^{(4)})\}, \quad (A7) \]
the time function $\tau$ defined by equation (A5) becomes equal to the traveltine $t_{SS}$ given by formula (4). Therefore, kinematics of a pure-$S$ primary is, indeed, represented by integral (A4) in the high-frequency limit.
Measuring, imaging and suppressing multiply scattered surface waves

X. H. Campman¹, K. van Wijk², J. A. Scales² & G. C. Herman³

¹Centre for Technical Geoscience, Department of Applied Mathematics
Delft University of Technology, P.O. Box 5031, 2600 GA Delft, The Netherlands
²Center for Wave Phenomena and Physical Acoustics Laboratory, Colorado School of Mines
³Shell International E&P and Centre for Technical Geoscience, Department of Applied Mathematics
Delft University of Technology, P.O. Box 5031, 2600 GA Delft, The Netherlands

ABSTRACT
Near-surface scattering can contaminate the arrival of energy from target reflectors. We developed a 3D wave-theoretical method, as a multi-channel alternative for short-wavelength static corrections, that was successfully tested on laboratory data, excited and monitored with a computer controlled, non-contacting system.

1 INTRODUCTION
When a wave front travels through a complex overburden, it is disturbed by scattering from heterogeneities. For a detailed structural image of the deeper subsurface it is important to minimize these disturbances in arrival time and amplitude of upcoming reflections. Currently, residual static correction methods correct for rapid variations in arrival times of a reflector, but these techniques are based on a model that assigns the same uniform time shift to each trace from a distinct surface location (e.g., (Wiggins et al., 1976)), assuming vertical ray paths through the overburden. Such corrections are usually referred to as time- and surface-consistent corrections (Taner et al., 1974). Although statics techniques are based on this simple (transmission) model of the subsurface, they can be effective in many cases, but in a strongly heterogeneous shallow subsurface, this statics model breaks down (e.g., (Combee, 1994)). Neglecting (multiply) scattered waves, like in the static assumption, can degrade the high-frequency content of the data, due to destructive interference of rapidly varying traces during stacking.

We estimate a surface impedance distribution of the region directly under the receivers from one particular event and subsequently predict and subtract the scattered energy for the entire record, improving resolution of the target reflectors. This method is based on an integral-equation formulation of the scattering process near the surface. We present examples based on laboratory models, where we excite and measure wave fields that are scattered at the near surface. With our non-contacting data acquisition, receiver intervals are less than the dominant wavelength, allowing us to filter in the wavenumber domain, like the dense receiver arrays that are currently being tested in exploration geophysics (Baeten et al., 2000).

2 SCATTERED NOISE MODEL
The objective is to obtain an estimate of the complete wave field without scattered energy from near-surface heterogeneities. On account of linearity of the elastic wave field, the vertical velocity component $v(x, t)$, measured at position $x = (x, y, z)$ and due to a fixed vertical point source of force type can be written as

$$v(x, t) = v^0(x, t) + v^1(x, t).$$  \hspace{1cm} (1)

Here, $v$ is the measured field, $v^0$ is the field that would have been measured if the overburden were homogeneous and $v^1$ is the part of the wave field that accounts for scattering from heterogeneities close to the acquisition surface. Thus, we want an estimate of $v^0$. Our approach is to find the scattered noise, $v^1$, and then subtract it from the data (as expressed in Equation 1). From the elastic wave-equation for particle displacement, we can derive an approximate integral representation for the scattered noise in terms of the vertical particle velocity, measured at the surface $z_0$:

$$v^1(x_1, z_0, \omega) = \int_{x_1' \in \Sigma} u^G_z(x_1 - x_1', \Delta z, \omega) \sigma(x_1', z_1, \omega) \times v(x_1', z_1, \omega) dx_1',$$  \hspace{1cm} (2)

where $u^G_z$ is the vertical component of the Greens displacement tensor, due to a vertical point force (see Ap-
appendix B). Horizontal position is denoted by \( x \), \( y \), \( z \) is the scattering depth and \( \Delta z = z_0 - z_1 \). The impedance distribution is denoted by \( \sigma \) and \( \omega \) is angular frequency. The surface \( \Sigma \) is the area occupied by the receivers (i.e. the acquisition surface). If the scattering takes place close to the surface (\( z_1 \approx z_0 \)), we can approximate the field at \( z_1 \) by the field recorded field at depth \( z_0 \) and we can calculate the scattered field \( v \), once we know \( \sigma \). Note that the integral is over a surface and we thus expresses scattering by a scattering volume in terms of a surface impedance distribution. The validity of this assumption for scattering of surface waves close to the surface is investigated in exploration geophysics by (Blonk & Herman, 1994) and in global seismology by (Snieder, 1986). To account for variations in the actual depth of the scatterers we allow the impedance distribution to depend on frequency.

3 INVERSE SCATTERING

Suppose the data contain many reflections from deeper layers. All these events excite surface waves at the same heterogeneities close to the acquisition surface. This implies that we can use the scattered energy from a single event to estimate the impedance distribution and use it to predict the scattered energy on every reflection using Equation (2). In fact, this is comparable to residual statics methods, where one selects a strong reflection event from the data, to derive the time-shifts for each trace separately. Instead of this single-channel operation, we now select one event to derive an impedance distribution to estimate the scattered energy (i.e. a multi-channel operation).

First, we select an event:

\[
v(x, t) = d(x, t) + r(x, t),
\]

where \( v \) are the data, \( d \) is the selected event and \( r \) denotes the rest of the data. Selecting \( d \) can be done by time windowing. The window should be long enough to include scattering tails but it should not include other events. Next, we decompose the strong event \( d(x, t) \), in a similar way as in Equation (1):

\[
d(x, t) = d^0(x, t) + d^1(x, t).
\]

Here, \( d^0 \) is the field in the near surface that would exist without scattering and \( d^1 \) is the scattered field, excited by the incident field being scattered from heterogeneities in the near-surface. The impedance model is obtained from back-propagating the near-surface scattered energy with the Green’s function derived in Appendix B. The impedance distribution is determined by minimizing an \( L^2 \)-norm, using conjugate gradient. To set up the minimization scheme, we write Equation (2) for a single event, in the form

\[
d^1 = K\sigma,
\]

where \( \sigma \) is the surface impedance distribution and the operator \( K \) is defined as

\[
\{ K\sigma \}(x_i, z_0, \omega) = \int_{x_j \in \Sigma} u^0_l(x_i - x_j, \Delta z, \omega) \times \sigma(x_j, z_1, \omega) d(x_j, z_1, \omega) dx_j.
\]

We then minimize the squared difference between the observed scattered field and the reconstructed scattered field, regularized by the norm of the distribution of scatterers:

\[
F = \frac{||d^1 - K\sigma||^2}{||d^1||^2} + \lambda ||\sigma||^2,
\]

where the size of \( \lambda \) determines the penalty on the norm of the distribution of scatterers. By assumption, the scatterers are close to the surface so that in Equation (6), we can substitute the field at depth \( z_1 \) with the field at \( z_0 \), leaving \( \sigma \) the only unknown. In contrast to Born-type imaging methods, this method accounts for multiply scattered waves.

4 DESCRIPTION OF EXPERIMENT I

In the Physical Acoustics Laboratory at CSM we measure the wave field on the surface of an aluminum block, excited by a pulsed infrared laser (Scales & Malcolm, 2003). We focused the laser beam on a line to create a line surface wave source. This wave front is scattered by a cylindrical cavity with a diameter of 2 mm and a depth of 3 mm, which is also roughly the size of the dominant wavelength. The wave field is detected using a scanning laser interferometer that measures the vertical component of the particle velocity on the surface of the model via the Doppler shift (Scales & van Wijk, 1999). Traces are recorded at 0.25 mm intervals, which implies about 10 samples per wavelength. The left panel of Figure 1 is the top view of the experimental configuration, while the right panel is an in-line panel of the 3D data set, with several surface-wave events and a body wave (P-wave) identified. The quality of these data is such, that no pre-processing is required.

5 RESULTS OF EXPERIMENT I

We use data from Experiment 1 to validate the algorithm. First, we select an event by time windowing; in this case the direct Rayleigh wave with the energy scattered by the cavity, plotted in the left panel of Figure 2. We separate the incoming (\( d^0 \)) from the energy scattered by the cavity (\( d^1 \)). To do so, we exploit the near-planar character of the incoming wave. Since a two-dimensional spatial Fourier transformation maps a plane wave to a point in the wavenumber-frequency domain, we can use this to separate the incoming plane wave from its local perturbations. These perturbations...
are attributed to the presence of the cavity. The separated scattered field $d^1$ is shown in the middle panel of Figure 2. Next, we estimate the impedance distribution using Equation (7) and an independent estimate of the background velocity of the surface waves in aluminum. From the data, we estimate that $c_R \approx 3000$ m/s.

The impedance distribution for this in-line data set is shown in the right panel of Figure 2. It coincides with the actual location and in-line width of the cavity. Applying this sequence of steps for the entire 3D data volume, leads to the image of Figure 3. The circular shape of the impedance distribution, slightly to the left and down from center, represents the actual shape and location of the cavity. Anomalies in the right corners of the figure are due to local data quality issues.

Having obtained an estimate of the spatial impedance distribution from the direct Rayleigh wave, we calculate the scattered wave field $v^1$ for a different event. This event has not been used for determining the impedance distribution, and therefore prediction of the scattered field is a good test of the method.

Figure 1. Left: top view of the aluminum block with cavity. The light shaded area is the area covered by the receivers. The source-width (dark shade) is 0.5 mm. Right: in-line panel (see left panel for location) of the data, where strong reflectors have been identified.

Figure 2. Left: part of the direct Rayleigh wave (event $d$ in the text). This event is used to derive the scattered energy $d^1$. Middle: separated scattered energy, $d^1$, using a wavenumber frequency domain filter. Right: image of the cavity along the same line.

Figure 3. Cross section of the image at $t = 0$ for the first experiment, showing the right position and size of the cavity.
select the Rayleigh wave that is reflected by the end of the aluminum block behind the source. We call this event the “ghost,” shown in the left panel of Figure 4. The predicted scattered field is shown in the middle panel. Finally, we obtain the wave field minus the scattered energy from Equation (1), shown in the right panel of Figure 4. We observe that the scattering has been effectively removed and continuity of the reflector has increased.

6 DESCRIPTION OF EXPERIMENT 2

In order to simulate an upcoming reflection, Experiment 2 involves a transmission model. Body waves are excited at the bottom of a two-layered model, where an aluminum layer is topped by a Lucite layer, in which we drilled a 2-mm wide and 3-mm deep cavity. When the body waves reach the surface, energy is scattered at the cavity. We record the wave field in a 4 cm² region, at 0.1 mm intervals. Compared to Experiment 1, these data are further complicated by the fact that they contain multiples from the layer boundary and reflections from the sides of the aluminum block as depicted in the side-view in the left panel of Figure 5. Data through the cavity show the multiples between the layers, reflections from the sides, all scattered by the cavity (right panel of Figure 5). Pre-processing of the data consisted of tapering-off low frequencies (including a dc-component) and then stacking each trace with four adjacent traces to boost signal-to-noise.

7 RESULTS OF EXPERIMENT 2

The data in Experiment 2 present a more challenging test for the method, because of the multiples and the interfering reflections from the sides of the aluminum. Apart from the pre-processing of these data, the algorithm is applied in the same way as in Experiment 1. Again, we start by selecting a clear event. In this case we select the first upcoming reflection, shown in the left panel of Figure 6. We separate the incoming \((d^0)\) from the energy scattered by the cavity \((d^1)\), using a narrow wavenumber-frequency domain filter. The separated scattered field is shown in the middle panel. Using the surface wave velocity in Lucite \((c_R \approx 1000 \text{ m/s})\), we estimate the impedance distribution, shown in the right panel. Figure 7 is a top-view of the image at the surface for the entire 3D data volume. The dimensions and location of the image are in agreement with the actual cavity in the Lucite.

Finally, we predict the near-surface scattered field in the rest of the data. In the left panel of Figure 8 we show part of the data line crossing through the cavity, minus the first event used to construct the image. Thus, the data shown in the left panel of Figure 8 were not used to derive the impedance distribution. Because the surface wave velocity may not be accurately known, the desired result may still contain residual tails from surface waves, but these can be removed by dip filtering. In order to make a comparison between the data before and after applying the algorithm, we have used a dip filter on the input data \(v\) as well as in the output data \(v^0\). The filtered input data are shown in the middle panel of Figure 8. Obviously, the dip filter only
Figure 5. Top- (left) and side-view (middle) of the two-layered model with cavity. Right: seismogram of part of the data through the cavity.

Figure 6. Right: the first upcoming event from the data. (event $d$ in the text). This event is used to derive the scattered energy. Middle: separated scattered energy, $d^1$, using a narrow wavenumber frequency domain filter. Right: image of the cavity along the same receiver line.

Figure 8. Left: part of the rest of the record. Middle: same as in the left panel but after dip-filtering to attenuate surface waves. Right: rest of the record after subtracting near-surface scattered energy and after dip-filtering.
removed the flanks of the surface waves, but not the apices, which have higher apparent velocities and were therefore unaffected by the filter. Especially this part of the surface waves is important to remove, because it is the interference between surface wave scattering and the incident field that diminishes the quality of the target reflector. The output $v^D$ after applying the same dip-filter is shown in the right panel of Figure 8. We conclude that the algorithm has improved the continuity of the reflectors.

8 CONCLUSIONS

We present a robust prediction-and-removal algorithm to attenuate strong near-surface scattering from seismic data. Using data from a laboratory-scale scattering experiments, we are able to estimate the surface impedance distribution using a single event. This impedance distribution is then used to predict and remove the scattered field from other events, restoring the continuity of target reflectors. A similar test on a more challenging multiple scattering experiment also gives promising results. The success of the algorithm is aided by dense 3D data acquisition, allowing filtering in the wavenumber domain.

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APPENDIX A: FOURIER TRANSFORMATION CONVENTIONS

In this Appendix we define our Fourier transformation conventions. Let $g$ be a causal, real valued function of $t$, $t \in \mathbb{R}^+$. Then the temporal Fourier transformation is defined by

$$F_t\{g\}(\omega) = \int_0^{\infty} g(t) \exp(-i\omega t) dt,$$  \hspace{1cm} (A1)

and the inverse transformation can be written as

$$F_t^{-1}\{g\}(t) = \frac{1}{\pi} \text{Re} \int_0^{\infty} g(\omega) \exp(i\omega t) d\omega.$$  \hspace{1cm} (A2)

For the Fourier transformation in two lateral directions we have

$$F_s\{g\}(k) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x) \exp(ik_x x + ik_y y) dx dy,$$  \hspace{1cm} (A3)

and

$$F_s^{-1}\{g\}(x) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(k) \times \exp(-ik_x x - ik_y y) dk_x dk_y.$$  \hspace{1cm} (A4)

APPENDIX B: GREEN'S FUNCTION

Start with the equation for the vertical component of particle displacement, $u_x$, due to a vertex force source $f_x$, in an isotropic, (laterally) homogeneous, elastic solid, in the frequency domain:

$$(\lambda + \mu) \partial_z (\partial_z u_x + \partial_y u_y + \partial_x u_z) + \mu \nabla \cdot \nabla u_z(x, \omega) + \rho(x) \omega^2 u_z(x, \omega) = -f_x(x, \omega).$$  \hspace{1cm} (B1)

The operator $\nabla = (\partial_x, \partial_y, \partial_z)$ contains the spatial derivatives and $\omega$ denotes angular frequency. The lamé parameters, $\lambda$ and $\mu$, have been assumed constant here. The associated compressional wave velocity is $c_P = \sqrt{\frac{\lambda + 2\mu}{\rho}}$ and the shear wave velocity $c_S = \sqrt{\frac{\mu}{\rho}}$. For up-coming reflections, we assume that the horizontal displacements are negligibly small compared to the vertical displacement (Barends, private communication; for other examples of this approximation see (Los et al., 2001) or (De Hoop, 2002). By introducing the scaled vertical coordinate $\zeta = \frac{x}{c_P} z$, Equation (B1) can be reduced to the scalar Helmholtz equation:

$$L_0 u_x(x, \zeta, \omega) = (\nabla^2 + k^2) u_x(x, \zeta, \omega) = -\frac{1}{\mu} f_x(x, \zeta, \omega),$$  \hspace{1cm} (B2)

where $x = (x, y)$ is the horizontal position vector and we have introduced the wavenumber $k = \frac{\omega}{c_P}$, where $c = \sqrt{\frac{\mu}{\rho}}$. The Green’s function is defined as the solution to

$$L_0 u_x^G(x, \zeta, \omega, x', \zeta') = -\delta(x - x') \delta(\zeta - \zeta').$$  \hspace{1cm} (B3)

After a two dimensional spatial Fourier transformation Equation (B3) is

$$L_z u_x^G(k_z, k_y, \zeta, \omega, x', \zeta') =$$

$$(\partial_\zeta \partial_\zeta + k_z^2) u_x(k_z, k_y, \zeta, \omega, x', \zeta') = -\delta(\zeta - \zeta') \exp(-i(k_z x' + k_y y')),$$  \hspace{1cm} (B4)

where $k_z = (k^2 - k_y^2 - k_y^2)^{1/2}$ with $\text{Re} \ k_z > 0$ and $\text{Im} \ k_z < 0$. Equation (B4) is subject to boundary conditions. We consider a homogeneous half space with a stress free surface. This leads to the condition

$$\partial_\zeta u_x = 0, \quad \zeta = 0.$$  \hspace{1cm} (B5)

And, to allow only outgoing waves at infinity, we use an appropriate radiation condition (Bleistein et al., 2001, Ch. 2). The result is

$$u^O(k_z, k_y, \zeta, \omega, x') =$$
\[
\frac{1}{2ik_z} \left[ \exp(-i k_z |\zeta - \zeta'|) + \exp(-i k_z |\zeta + \zeta'|) \right] \times \\
\exp(-i(k_z x' + k_y y')).
\]

(B6)

The integral in Equation (2) can be efficiently calculated using the convolution theorem. Therefore, we calculate the Green’s function in the horizontal wavenumber-frequency domain as given in Equation (B6). In order not to complicate the argument in the main text, from Equation (2) on, we use \( z \) to denote the scaled depth instead of \( \zeta \).

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Fault-plane reflections as a diagnostic of pressure differences in reservoirs: A case study

Matthew Haney¹, Jon Sheiman², Roel Snieder¹, Steve Naruk² & Jay Busch²

¹Center for Wave Phenomena, Colorado School of Mines, Golden, CO 80401
²Shell International Exploration and Production Inc., Houston, TX 77025

ABSTRACT

Seismic data taken at Blocks 314, 315, 330, and 331 of the South Eugene Island field contain reflections from a major growth fault. Out of a number of possible causes, we find that differences in pore pressure across the fault give rise to the fault-plane reflections. The pressure differences are detectable since pore pressures that exceed the hydrostatic pressure, or overpressures, lower the seismic velocity. Thus, the presence of the reflections point to the fault providing a significant seal. We develop a processing scheme to highlight the fault-plane reflections while simultaneously removing the reflections from the layered structure. Using this processed data set, we extract the amplitude of the fault-plane reflections on the fault-plane itself. The areas of strong reflection amplitude correlate well with the geology and known areas of overpressure.

INTRODUCTION

The importance of faults as delimiters of compartments in hydrocarbon reservoirs cannot be stressed enough. The role of faults, however, is complicated by their dual nature as both fluid seals and conduits. Classifying a fault as one or the other typically demands extensive knowledge of a basin’s geologic history, core samples, and well logs. Only recently have geophysicists begun to incorporate conventional seismic data into the evaluation of fluid pressure near faults (Dutta, 2002a; Huffman, 2002). The method relies on seismic waves detecting the presence of pressure changes in the subsurface (Pennebaker, 1968; Reynolds, 1970), and, when successful, manages to predict regions of overpressure that affect drilling operations.

Evidence of faults can often be seen on migrated seismic data. Automated fault identification algorithms avoid the tedious picking of faults in 3D volumes of seismic data (Townsend et. al, 1998). According to Sheriff (1984), the imprints of faults on seismic data are: “(a) abrupt termination of events, (b) diffractions, (c) changes in dip, (d) distortions of dips seen through the fault, (e) deterioration of data beneath the fault producing a shadow zone, (f) changes in the pattern of events across the fault, and (g) occasionally a reflection from the fault plane.” Items (a), (c), and (f) provide indirect evidence of faults and form the basis of automated fault identification. Problems with how time migration treats lateral velocity variations lead to items (d) and (e). Here, we address the last point; namely, what are the occasional fault-plane reflections telling us about the nature of the fault itself?

Several fault-zone models describe different aspects of the seismic properties of a fault. One possibility is that the fault is a linear-slip interface (Worthington and Hudson, 2000; Coates and Schoenberg, 1995). Physically, this means that the fault is a zone of low shear modulus. Another model takes into account that the fault may be a barrier to lateral fluid flow. A high shale content in the fault gouge causes fluid pressures to build up on one side of the fault. As a result, the adjacent sediments are undercompacted, and subsequently have lower velocities (Dutta, 1997). Because of throw across the fault, lithological (sand/shale) contacts across a fault can also contribute to reflections from the fault-plane (Sheriff, 1984; Yielding et. al, 1991). A main difference in these models is that reflection from a low-shear zone acts as a high-pass filter; in essence, the fault zone is a thin bed. The magnitude of the reflection in this case depends on the seismic properties and thickness of the zone. Pore pressure and lithologic differences across a fault act as traditional seismic interfaces that preserve frequency in the reflection process. A true fault zone could be a combination of two or more of these models.

Here, we show that, at the South Eugene Island field, fault-plane reflections from a major growth fault,
known as the A-fault, arise from pore pressure differences across the fault. We focus on this fault since previous studies (Losh et al., 1999) suggest that it serves as a significant barrier to lateral fluid movement. The strength of the fault-plane reflections varies along strike and dip of the A-fault. We make the correlation between areas of fault-plane reflections and areas of strong pressure gradients based on existing well information and known zones of overpressure. No evidence yet suggests that the fault-plane reflections originate from low-shear zones or lithologic contacts.

**OVERPRESSURE AT SOUTH EUGENE ISLAND**

Growth faults, or syndepositional normal faults, divide the Pliocene-Pleistocene sediments at South Eugene Island into several compartments. Depending on whether or not the faults are significant barriers to fluid flow, the individual compartments can be overpressured. For instance, Losh et al. (1999) report an increase of more than 780 psi in pore pressure over a distance of 18 m while drilling through the A-fault in Block 330 of the South Eugene Island field.

A contour plot of the two-way reflection time from the A-fault is shown in map view in Fig. 1. The arrow in Fig. 1 points in the downdip direction that makes an angle of approximately 50° with the horizontal. The A20ST well, where Losh et al. observed a large pore pressure jump, intersects the fault on the lower right.

Several mechanisms effectively cause anomalously high pore pressures in the subsurface (Dutta, 1997). In the Gulf of Mexico, overpressure commonly results from dipping sands being bounded above and below by shales. The relatively high permeability of the sands allows fluid pressures from depth to move into high points of subsurface structure. Termed unloading, this mechanism results in a seismic velocity decrease due to undercompaction of the sediments. More specifically, the high pore pressure causes a decrease in the vertical effective stress and, as a result, a decrease in the area of grain-to-grain contacts.

We have identified anomalous decreases in velocity from a smoothed Dix-type inversion on the South Eugene Island data. Line C-C’ (see Fig. 1) from this velocity cube is displayed in Fig. 2. The velocity at marker ‘1’ monotonically increases with depth. This represents the velocity variation resulting from the normal compaction trend. On the left side of this plot, between markers ‘2’ and ‘3’, a noticeable decrease in velocity occurs with increasing depth. Experience has shown that, in the Gulf of Mexico, such a decrease in velocity with depth is due to high pore pressures rather than lithology (Dutta, 1997). The spatial pattern of the relatively higher velocity pocket at ‘2’ corresponds to the trend of the A-fault, providing evidence from the seismic data that the overpressures are related to the A-fault.

Zones of overpressure can be monitored while drilling from the rate of penetration of the drill bit into the formation (Jordan and Shirley, 1966). Such a method suffices for detecting the gradual build up of overpressure above a moderately sealing shale. By moderately sealing, we mean that the pressure front has, over geologic time, diffused somewhat through the shale. Any sort of overpressure prediction while drilling should fail when a highly overpressured zone is quickly and unexpectedly encountered. This has motivated the
Reflections from a sealing fault

3

A − fault B − fault

Figure 3. A typical fault-plane reflection from the A-fault along the A-A’ line. Shown is the post-stack time-migrated image. The H-sand is marked to show the amount of throw across the fault.

development of pre-drill pore pressure prediction from seismic data.

For a smooth increase in fluid pressure across a moderate seal, the velocity decreases gradually, and a transmitted wave passes through the overpressured region with almost no reflection. However, the most dangerous instances of overpressure occur over distances less than a typical seismic wavelength (∼ 200 m), and therefore, quick onsets of high fluid pressure across a sealing fault give rise to strongly reflected waves. By mapping out the amplitude of fault-plane reflections on the fault-plane itself, areas of sharp increase in pore pressure across the fault, as described in Losh et al., should stand out.

ISOLATING FAULT-PLANE REFLECTIONS

Even at their strongest, the fault-plane reflections at South Eugene Island are less prominent than the layer reflections. This may be due to either small reflectivity at the fault or the deterioration of the imaging procedure (Kirchhoff) for steep dips. As a result, the fault image contains “noise” from the horizontal layers terminating at the fault. We employ a simple slant stack along the fault to effectively remove the layers while at the same time accentuating the reflection from the fault plane.

Displayed in Fig. 3 is a post-stack time migrated section along the A-A’ line (see Fig. 1). Several growth faults stand out in this image. A particularly strong fault-plane reflection cuts through the center of Fig. 3 - this is the A-fault. To the right, another fault (the B-fault) can be made out from the mismatch of adjacent layers; however, the B-fault does not give rise to a reflection. This is likely because the throw on the A-fault is greater than that on the B-fault. We have highlighted

Figure 4. The slant stack technique used to accentuate the fault-plane reflections. Stack ‘1’ is zero since it intersects the upper horizontal reflection at an angle. The fault-plane reflection coherently adds into stack ‘2’. We also stacked in the opposite direction ‘3’ to capture any antithetic faults.

the H-sand as an indicator of the throw on these two faults. The greater the throw on a fault, the more developed its gouge, and, therefore, the more likely it is to be a barrier for lateral fluid flow.

To bring the fault-plane reflections out, we designed an adaptive local slant stack routine. Pictured in Fig. 4 is the basic procedure. Along each trace, we scan over a small angular window for the maximum coherence direction. The range of angles is selected to correspond to the dip of the fault. We then construct the slant stacks by summing over the five adjacent traces. In Fig. 4, we show two layer reflections with no dip and a dipping fault-plane reflection. Slant stack ‘1’ does not lie in the fault-plane and the contributions from the upper layer cancel. In contrast, ‘2’ lies in the fault plane and the fault-plane reflection coherently stacks. Since faults in an extensional regime typically have antithetic counterparts, we also stack in the opposite direction ‘3’.

A slant stack is shown as a wiggle-plot in Fig. 5 with the migrated image of Fig. 3 in the background. The horizontal layers effectively cancel in the stack, leaving the fault-plane reflections to stand out. Both the amplitude and phase of the fault-plane reflection from the A-fault vary along the fault. The phase seems to change at points where sandstones encounter the fault. For instance, moving up the fault-plane from the bottom, the wavelet changes shape and grows stronger as it moves past the JD-sand. It then vanishes between the downthrown and upthrown segments of the H-sand, only to continue again above the upthrown H-sand.

We cannot make quantitative use of the phase yet since the absolute phase of the reflections may be contaminated with stacking and migration errors. Future
numerical work will attempt to get a handle on this error. For example, the phase would be an excellent indicator of high fluid pressures inside a fault-zone. High fluid pressure in the A-fault has been reported by Losh et al. (1999) and leads to small shear velocities in the fault. Since scattering from such a linear slip interface is frequency dependent, the phase of the reflected wavelet could change. Though we cannot trust the absolute phase, we interpret the relative phase changes along the fault to be related to occurrences of the sands.

We extended the above procedure to 3D to gain a more extensive picture of the variations in fault-plane reflections. By breaking up the 3D seismic data volume into successive 2D planes, we could perform the slant stacking on each plane individually. The slant-stacked lines were then reassembled into a 3D data volume.

CORRELATION OF FAULT-PLANE REFLECTIONS WITH REGIONS OF OVERPRESSURE

The attributes of the slant-stacked 3D seismic data on the A-fault contain information about the fault seal. Using Shell’s Rosebud seismic attributes software, we extracted the maximum amplitude along the picked fault-plane (Fig. 1) within a small time gate. The reflection amplitudes are displayed in map view on the fault plane in Fig. 6. The view in this figure is identical to the view in Fig. 1. Higher amplitudes, shown as lighter colors in Fig. 6, come and go on the fault plane. We save discussing the details of the reflections except in a triangle in the upper left portion of the fault, labeled as points D, E, and F in Fig. 6.

The triangle DEF forms the most strikingly coherent feature on the amplitude map and its two sides DE and EF have geologic meaning. The time-migrated image and the slant stack along the B-B’ line (see Fig. 6) are shown in Fig. 7. The intersection of DE with the B-B’ line corresponds to the meeting of the A-fault and the JD-sand. The lack of strong amplitudes south of DE in Fig. 6 means that the A-fault does not reflect below the JD horizon. This is likely because the JD-sand is itself overpressured. Stump et al. (1998) have observed that at well 331 #1 (see Fig. 1) the JD-sand marks the onset of overpressure in the sedimentary column. Essentially, beneath the JD horizon, both sides of the A-fault are overpressured and, hence, no reflection occurs from the fault-plane. Below the JD-sand, the smaller fault to the left of the A-fault (Fig. 7) is reflecting, suggesting that this fault forms a seal. Line EF marks the intersection of the A-fault with the antithetic fault shown in Fig. 7. The lack of strong amplitudes north of EF in Fig. 6 means that the A-fault does not reflect above the top of the wedge formed by it and the antithetic fault. Since the antithetic fault is reflecting, this suggests that the seal transfers from the A-fault to the antithetic fault at the top of the wedge. The presence of lower velocities, consistent with higher pore pressures, on the upthrown side of the A-fault is supported by the results of the Dix-type velocity inversion (Fig. 2).

CONCLUSIONS

At the South Eugene Island field, observed fault-plane reflections from the A-fault arise due to pressure differences across the fault. By applying a technique to accentuate the fault-plane reflections, we are able to
**Reflections from a sealing fault**

JD antithetic fault A − fault top of wedge B − fault reflecting part of A-fault B

**Figure 7.** An overlay of the time-migrated seismic section along line B-B′ with the wiggle-trace slant stack. An antithetic fault exists at this location. The reflecting part of the A-fault corresponds to the part of it between the JD-sand and the top of the wedge formed by the antithetic fault.

map out the reflection amplitudes on the fault plane. The spatial distribution of the reflections has a geologic meaning and shows which part of the fault-plane is acting as a seal. Future work will focus on other faults in the South Eugene Island field and attempt to get quantitative estimates of fault-zone properties from the fault-plane reflections.

**ACKNOWLEDGMENTS**

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Quantifying the uncertainties in absorption estimates from VSP spectral ratios

Albena Mateeva

ABSTRACT

To make inferences about reservoir conditions such as saturation and permeability from absorption data, one must know how accurate the absorption estimates are. In seismic exploration, absorption is often assessed from VSP spectral ratios, and its uncertainty quantified by the variability of the fitted slope. This greatly underestimates the uncertainty, especially in media with strong scattering. In this paper, I propose ways of quantifying the absorption errors introduced by different factors. Special attention is given to the bias and variability caused by scattering from thin horizontal layers, because it is the largest source of error in stratified media. The next largest uncertainties are associated with receiver positioning and first arrival timing. Ambient noise and event windowing, if properly done, have a much smaller influence on the fitted spectral ratio slopes. When inverting for the quality factor of thick geological units, it may be advantageous to have receiver pairs with a common receiver. However, this introduces a correlation between the spectral ratios, which must be taken into account in the uncertainty analysis of the mean attenuation in the layer. I illustrate the error assessment and absorption estimation steps by a real data example.

1 INTRODUCTION

Absorption carries valuable information about lithology and reservoir conditions, such as saturation and permeability (Winkler & Nur, 1979; Batzle et al., 1996), but to infer them, we must know how accurate the absorption measurements are. In seismic exploration, most absorption estimates come from Vertical Seismic Profiles (VSP). Many people would quantify the reliability of the derived estimates by simply quoting the errors determined when fitting a straight line to the logarithmic spectral ratio between the first arrivals at two depths. At best, this is an optimistic estimate for the uncertainty of the effective attenuation caused by both stratigraphic filtering and absorption. The existence of apparent “attenuation” caused by scattering, and particularly by thin layering, is well known (O’Doherty & Anstey, 1971; Schoenberger & Levin, 1974). The quotation marks around attenuation are put because, as I showed previously (Mateeva, 2003), non-stationary reflectivity may cause apparent gain rather than loss of high frequencies through backscattering (reflections from the thin layers immediately beneath a VSP receiver). Using the VSP spectral ratios as an estimator of absorption is acceptable only when the scattering attenuation is small compared to the intrinsic attenuation. Often this is not the case, and the scattering effects must be subtracted from the effective attenuation to get a physically plausible absorption estimate (e.g., a positive $Q$). In doing so, the bias in the attenuation estimate is removed, but its variability is increased.

Characterizing the bias and variability caused by thin layering is part of the goal of this paper, which is to quantify the total uncertainty in the absorption estimates. Other factors to consider are: uncertainty of the measured traveltime between two VSP receivers, receiver positioning errors when modeling the scattering, spectral distortions due to windowing, and ambient noise. I propose simple ways of quantifying the different uncertainties in the context of a real data example. Eventually, an absorption profile with fair error estimates is obtained.

2 DATA

The data for this study are a VSP with known source and receiving instrumentation signatures, and well-logs acquired in the same borehole. The VSP is used to pro-
file the effective absorption, i.e., the combined action of anelasticity and scattering. Sonic and density logs are used to compute synthetic seismograms from which to assess the share of scattering in the effective attenuation. The known signatures of the VSP source and receiving instrumentation allow us to find the frequency band for most reliable absorption estimation, as well as to evaluate the errors caused by the windowing of first arrivals and ambient noise.

The VSP consists of 175 traces, starting at 150 m below the surface. The depth-coverage is not uniform. The first six receivers are 150 m apart, spanning the first kilometer of the section. The rest of the receivers are 15 m apart and span the 1-3.5 km interval. The source for the VSP is a vibrator, 70 m away from the borehole head. This offset is negligible compared to the receiver depths. The well-logs start at about 600 m depth and stop at the same depth as the VSP (Fig. 1).

\section{Model Parametrization}

Surface seismic images suggest the investigated area is horizontally layered. Thus, we can consider a 1D earth model and invert for the average intrinsic Q of the major geological units. Four main intervals with thicknesses on the order of a kilometer are evident on the well-logs (Fig. 1). It is a priori known that there is a thin sandstone layer in the near-surface, not captured by the well-logs. I assume that the top interval present in the well-logs extends up to the base of the thin sandstone layer (the interval appears quite uniform on the VSP data, which start above the well-logs). Thus, the preliminary earth model consists of five layers: a thin near-surface sandstone with a quality factor \( Q_0 \), and four thick subsurface layers, characterized by mean quality factors \( Q_1 - Q_4 \). Only the deep layer parameters \( Q_1 - Q_4 \) can be constrained by VSP spectral ratios because the VSP starts below the sandstone. In principle, \( Q_0 \) can be assessed from the signal in the shallowest VSP receiver if the source and receiving instrumentation signatures are known and if the source and receiver coupling with the ground is frequency-independent or known.

Initial estimates of \( Q_1 - Q_4 \) indicated that the quality factor of the top part of Layer 3 is substantially different from that of its lower part. Indeed, a closer inspection of the well-logs reveals a thin layer at about 2500 m depth that may separate the interval into two zones with different fluid contents that result in different Q-values. Thus, I denote them by \( Q_{3a} \) and \( Q_{3b} \) and assess them separately.

\section{Method of Estimating Q}

There are a number of approaches to estimating absorption from VSP experiments (Tonn, 1991). The most popular techniques are variations of the spectral-ratio method, developed by Hauge (1981) and Kan (1981). To make this study relevant to as many users as possible, I consider a generic spectral-ratio approach, in which the effective attenuation \( S_{\text{eff}} \) of a given depth interval \([z_1; z_2]\) is measured by the slope of the log-amplitude spectral ratio between the first arrivals at depths \( z_1 \) and \( z_2 \),

\[
\frac{20}{t_2 - t_1} \log \frac{A(f, z_2)}{A(f, z_1)} = const_f + S_{\text{eff}} f,
\]

where \( t_1 \) and \( t_2 \) are the first arrival traveltimes at the respective receivers, and \( f \) is frequency. The left hand side of eq. (1) is measurable from VSP data. The slope \( S_{\text{eff}} \) can be found by a linear regression and is related to the effective quality factor \( Q_{\text{eff}} \) by \( S_{\text{eff}} \approx -27/Q_{\text{eff}} \).

In a homogeneously absorbing medium, anelasticity and scattering contribute cumulatively to the effective attenuation, because arrivals with equal traveltimes have suffered the same amount of absorption regardless of their trajectory. Therefore,

\[
S_{\text{eff}} = S + S_{\text{sc}}
\]

(2)

where \( S \approx -27/Q \) characterizes the loss of high frequencies caused by absorption (i.e., \( Q \) is the intrinsic quality factor) and \( S_{\text{sc}} \approx -27/Q_{\text{sc}} \) characterizes the spectral change due to scattering (\( Q_{\text{sc}} \) is the apparent quality factor). After \( S_{\text{eff}} \) has been assessed from VSP data (eq. 1), the intrinsic attenuation \( S \) can be isolated by modeling and subtracting the scattering attenuation \( S_{\text{sc}} \) from \( S_{\text{eff}} \). Given a reflection coefficient log, and assuming the medium is horizontally layered, we can compute synthetic seismograms (absorption-free synthetic VSP) from which to get \( S_{\text{sc}} \) by fitting a line to the spectral ratio between the same two receivers from which \( S_{\text{eff}} \) was extracted. Note that, while the intrinsic \( Q \) is assumed to be frequency-independent, we do not have to assume that \( Q_{\text{sc}} \) is frequency independent (even though, over the narrow frequency band of the seismic source, it arguably is). By fitting the spectral ratio between synthetic traces by a straight line we do not aim at estimating the total scattering attenuation. We only aim to get its linear component \( S_{\text{sc}} \) which causes the bias in the effective attenuation (\( S_{\text{eff}} \) being fit by a linear regression, too).

The intrinsic slope \( S \) is always negative (the intrinsic \( Q \) is positive). In contrast, the slope \( S_{\text{sc}} \) can be positive if reflections from below make the signal in the deeper receiver relatively richer of high frequencies than the signal in the shallower receiver (Mateeva, 2003). In other words, contrary to popular belief, scattering does not necessarily lead to an overestimate of the intrinsic absorption. Ignored scattering (thin layering) is the most probable cause for the unphysical, negative Q-factors reported in VSP studies sometimes.

Eq. (2) is strictly valid in homogeneously absorbing media. Of course, in reality, the thin beds responsible for the scattering are likely to have different quality factors. Thus, the medium is not homogeneously ab-
Uncertainties in absorption estimates

Figure 1. Well-logs used to identify the main subsurface intervals, the mean quality factors $Q_1 - Q_4$ of which are to be determined. Shown on the left is the span of the VSP; dots represent the first 7 VSP receivers (with large non-uniform spacing; the rest of the receivers are close and uniformly spaced). The existence of a sandstone layer in the near surface is known a priori; its base with a reflection coefficient $r$ is drawn approximately.

sorbing. However, as long as the absorption is constant on the macro-scale (e.g., within each thick layer of our model), eq. (2) can still be used, with $S$ being an average characteristic of the region.

5 PREPARATIONS FOR SPECTRAL RATIO ESTIMATION

5.1 Choice of receiver pairs

Suppose a VSP is acquired at $n$ different depths in a given subsurface interval. The $n$ traces can be combined into $n-1$ non-redundant spectral ratios. There are many possible ways to pair the receivers. A reliable absorption estimate is obtained when the slope of the spectral ratio is large compared to its variability. Thus, I chose to maximize receiver separation. In every layer of the model, I paired the receivers from the top half of the layer with the receiver at the bottom of the layer, and the receivers from the bottom half of the layer with the receiver at the top of the layer (Fig. 2a). In this way all traces are used in a non-redundant manner, with minimum receiver separation of about half of the layer thickness.

As a by-product of the chosen pairing scheme, we get an indication of whether the model discretization is reasonable. For the mean quality factor to be a representative characteristic of a layer, it should not vary too much throughout the layer. One definition of “varying too much” would be the estimated intrinsic Q of the top half of the layer to be substantially different from the intrinsic Q of the lower half. Such instances are easy to spot if we plot a measure of absorption versus receiver separation, as in the cartoon in Fig. 2b. This is how Layer 3 was identified as a candidate for splitting into two sub-layers, as already mentioned in Section 3.

A potential drawback of the proposed scheme for receiver pairing is that anomalies* in the top or bottom receiver would propagate into many spectral ratios and cause systematic errors. Severe problems may be iden-

*An anomaly may be caused by coupling, source variations, noise outbursts, or inadequate scattering simulations (e.g., the source offset not being negligible for a shallow trace).
Figure 2. Cartoon: (a) Pairing the receivers in Layer $i$. Pairs containing the bottom-most receiver will be presented by a light color throughout the paper. Pairs containing the top-most receiver will be presented by a dark color. The distinction is made because the two sets sample different parts of the layer. (b) Indication for significant and systematic absorption variations in Layer $i$: small-separation pairs that sample predominantly the top or bottom halves of the layer (e.g., 3 and 4) give different estimates of the intrinsic attenuation, while pairs that span most of the layer (e.g., pairs 1, 2 and 6) show similar values for the intrinsic attenuation.

The existence of a correlation between the spectral ratios obtained from receiver pairs with a common receiver must be taken into account when computing the mean attenuation in a layer. Failing to do so would give an erroneous uncertainty estimate for the mean attenuation, even though the mean attenuation itself would not suffer much because the individual spectral ratios are consistent estimators of it. The covariance matrix needed for fair uncertainty analysis is derived in Appendix A.

5.2 Choice of frequency band

To get meaningful absorption estimates, it is important to identify the frequency band over which the signal to noise ratio is sufficiently high. Since scattering from thin layers will be explicitly taken into account in the absorption estimates, it does not represent “noise” (when not taken into account, this source-generated “noise” is a dominant cause of bias and uncertainty). The noise in our data is the ambient background that can be seen on the VSP traces before the first arrivals. Fig. 3 shows the power spectrum of the noise assessed from windows before the first arrivals together with a model of the signal spectrum, consisting of the known source function (Klauder wavelet), filtered by the known receiving instrumentation responses, and scaled to the first arrival amplitude of a representative VSP trace (a trace in the middle of the profile). As is seen from Fig. 3, only frequencies between 15 and 85 Hz can be used for absorption estimation; the rest of the spectrum is dominated by noise. On most traces the signal-to-noise ratio in the usable frequency band is about 20 dB.

6 ERRORS

The main sources of error in the effective and scattering attenuation estimates are discussed below together with strategies for quantifying them.
6.1 Error due to finite time windowing

Spectral-ratios are based on a time window around the first arrival. Suppose \( A_2(f)/A_1(f) \) is the true amplitude ratio between the early portions of two traces. What we measure is

\[
\frac{W * A_2}{W * A_1} \neq \frac{A_2}{A_1},
\]

where \( W(f) \) is the amplitude spectrum of the taper (the time window). The taper influence depends on the smoothness of \( A_1, A_2 \). A simplistic model of \( A_2(f) \) is the signal model in Eq. 3, multiplied by the spectrum of the transmission impulse response of the shallow sandstone.\(^\dagger\)

The latter is needed because the reverberations in the sandstone are strong—they roughen the trace spectra (introduce notches). To assess the tapering effects, we can construct “true” spectra \( A_2(f) \) by imposing an exponential decay with different \( Q \) values on \( A_1(f) \), and compare the true slope of \( A_2/A_1 \) to the slope fitted to \( W * A_2 / W * A_1 \) for a given taper. Unlike \( A_2/A_1 \), the tapered ratio does not fall on a perfect straight line; i.e., tapering not only biases the absorption estimates, it induces some uncertainty in the slope estimates as well. I call the difference between the slope fitted to \( W * A_2 / W * A_1 \) and the true slope the “tapering bias”. The residuals of the fit determine the “variability of the bias”, which is in fact the variability of the estimated attenuation introduced by the finite time window.

The tapering bias and its variability were measured for a 20% cosine taper with length 64, 128, or 256 samples. Qualitatively, the following was observed (Fig. 4):\(^\dagger\)

- The bias is positive, i.e., negative slopes appear less negative (\( Q \) appears higher), while positive slopes corresponding to fictitious negative \( Q \)-factors appear even more positive.
- The bias decreases as the true \( Q \) increases.
- Longer windows reduce both the bias and the variability of absorption estimates. The bias is reduced because the biases of the individual amplitude spectra in the spectral ratio are reduced. The variability is reduced mainly because of the larger number of frequency samples in the usable frequency band. A longer taper also preserves better the exponential relationship between \( A_1 \) and \( A_2 \) and allows less leakage of noise from outside the useful frequency band (next section). The increased stability of the spectral ratio slopes estimated from long time-windows has been noted by Goldberg et al. (1984) and Ingram et al. (1985) when studying spectral ratios between sonic log waveforms.
- For all windows and \( Q \)-values tested, the tapering bias was small compared to the other uncertainties in the absorption estimates (quantified later).

Given the latter, I decided to use the shortest 64-sample (128 ms) taper in order to localize the attenuation estimates as much as possible (a long time window would carry information about regions far away from a receiver pair, especially in a high-velocity medium).

The ordinate values in Fig. 4a and 4b show that the bias and its uncertainty are comparable. Thus, the true slope falls within the error bars of the measured slope. Moreover, the bias for the 64-point taper is only 1% of the measured slope (compare the vertical to the horizontal scale in the Fig. 4a). Thus, the widening effect is negligible, despite that the trace spectra are rough. This seems to contradict earlier findings (e.g. Sams & Goldberg, 1990\(^\dagger\)), and permits us to use relatively simple spectral estimation techniques (e.g. tapering) instead of, say, multi-tapering (Thomson, 1982; Walden, 1990) or data flipping (Pan, 1998). Such more sophisticated methods are needed when attenuation is estimated “point-wise” from individual frequency samples (e.g. Patton, 1988) rather than from the slope fit over many frequencies.

6.2 Ambient Noise

Since background noise is time-windowed together with the signal, it makes sense to consider the combined effect of tapering and ambient noise on the absorption estimates. The bias estimation procedure from the previous section was repeated after adding ambient noise (assessed from windows before the first arrivals) to the time-series corresponding to \( A_1 \) and \( A_2 \) (Fig. 5). Now the bias is larger than in the noise-free case; namely, it is about 4% of the measured slope for \( Q = 5 \), 13% of the measured slope for \( Q = 50 \), etc. (Fig. 5a). As the true \( Q \) increases, the absolute value of the bias decreases more slowly than in the noise-free case, and it never goes to zero. This reflects the fact that noise makes the records in two receivers different even if the medium is non-absorbing. Also, unlike in the noise-free case, the true slope is outside the error bars of the measured slope (compare the ordinates in Fig. 5a, 5b).

To quantify the combined effect of background noise and windowing on attenuation estimates, numerical models were derived from the data in Fig. 5. Now both the bias and its variability can be fit by quadratic functions of \( \hat{S} \), i.e.,

\[
b_S = a_0 + a_1 \hat{S} + a_2 \hat{S}^2
\]

\(^\dagger\)A likely explanation is that the notches in our trace spectra occur at the same frequencies at all receivers, and the spectral ratios near them do not fluctuate much more than at other frequencies. This is true even in the presence of noise, when tapering may stabilize the spectra near the notches by “leaking signal” into them from the neighboring regions.

\(^\dagger\)Here the sandstone layer is modeled as a homogeneous slab with one-way time-thickness of 15 ms (Appendix B), bounded by reflection coefficients -1 (top) and -0.45 (bottom).
Figure 4. Tapering effects in the absence of noise: (a) slope bias measured over the 15-85 Hz band for three window lengths; the data for the 64-sample window are fit by a linear regression. (b) variability of the bias estimate for a 64-point taper – measured (circles) and fit (solid line) by a quadratic function of the measured slope magnitude.

Figure 5. Analogous to Fig. 4 but in the presence of ambient noise: (a) bias for three window lengths; the data for 64-sample taper are fit by a quadratic model. (b) variability of the bias estimate for a 64-point taper – measured (circles) and fit (solid line) quadratic model.

The estimated coefficients \( \alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2 \) for the 64-point taper are used later to predict the tapering and ambient noise errors in \( S_{\text{eff}} \) and \( S_{\text{sc}} \).

6.3 Window positioning and traveltime uncertainties

Near the first arrival, seismic traces are not stationary, so the frequency content of an early window is sensitive to its exact position. For spectral ratios to measure the earth filtering, care should be taken to window the same signal on all traces. In the absence of significant dispersion (as in our data set), this can be done by adjusting the window position so that the first arrival peaks at the same instant relative to the beginning of the window on every trace. This is an important detail in the preparation for absorption estimation. Inconsistent windowing causes erratic behavior of the spectral ratios.

The time separation \( \Delta t \) between the receivers in a given pair \( [\Delta t = t_2 - t_1 \text{ in eq. (1)}] \) can be measured from the first arrival peaks with a precision on the order of the sampling interval, e.g., \( \sigma_{\Delta t} \approx \pm 2 \text{ ms} \). This uncertainty propagates in the spectral ratio slope as

\[
\sigma = \frac{\sigma_{\Delta t}}{\Delta t} \hat{S},
\]

where \( \hat{S} = S_{\text{eff}} \), for example.

6.4 Receiver positioning errors in the synthetic seismograms

The timing uncertainty described by eq. (6) is present only in ratios between real VSP traces, not in synthetic
traces (the time separation between them is known). However, since the receiver positions for the synthetic traces are determined by the first arrival traveltimes measured on the VSP traces, errors in VSP traveltimes translate into positioning errors in the synthetic data—the receivers in the scattering simulations and those in the real VSP are not identical positioned with respect to the fine structure of the subsurface. As a consequence, the spectra of the synthetic traces do not match wiggle by wiggle the VSP spectra. The slope of a spectral ratio is less sensitive to such positioning errors than the spectral ratio itself. That is why I chose to compensate for the scattering effects by first fitting the slopes of the synthetic ratios and then subtracting them from the slopes of the real VSP ratios, rather than first subtracting the synthetic ratios from the VSP ratios and then fitting a slope. The sensitivity of $S_{sc}$ to local interference (that changes with receiver position) depends strongly on the usable frequency band. In our case of a 64-point taper and 2 ms sampling, the usable frequency band has only 11 samples and the slope uncertainty can be significant.

Suppose $t$ is the first arrival travelt ime measured on a real VSP trace. Let $\sigma_t$ denote its uncertainty. This travelt ime uncertainty translates into a receiver positioning error in the synthetic VSP, which in turn, leads to a variability $\sigma^2_{pos}$ in $S_{sc}$. According to the error-propagation method,

$$\sigma^2_{pos} = \left( \frac{dS_{sc}}{dt} \right)^2 \sigma^2_s$$

(7)

The coefficient 2 is there because each of the two receivers in the pair from which $S_{sc}$ was estimated has a positioning uncertainty $\sigma_t$. For this particular data set I assume $\sigma_t = \pm 1$ ms. The squared derivative in eq. (7) can be assessed by differencing the estimated slopes $S_{sc}$ for each set of receiver pairs with a common receiver (only one receiver is moving), and taking the mean of the squared results, i.e.,

$$\left( \frac{dS_{sc}}{dt} \right)^2 = \frac{1}{n} \left( \frac{S_{sc}^{(i)} - S_{sc}^{(i-1)}}{t^{(i)} - t^{(i-1)}} \right)^2$$

(8)

where $t^{(i)}$ is the first arrival travelt ime at the moving receiver from pair $i$. I assign the same $\sigma^2_{pos}$ to all pairs with a common receiver.

6.5 Fitting uncertainties (local interference)

Now let us concentrate on the uncertainties that are inherent to the problem rather than caused by imperfect measurements.

Unlike the intrinsic attenuation which can be described by an exponential law at seismic frequencies, scattering attenuation can be described by a certain law only in a statistical sense. For a given realization of the medium, each frequency is modulated by local interference so that a spectral ratio never falls on a straight line even if the statistical average does. This has several implications to absorption estimation in heterogeneous media:

- A spectral ratio slope estimated from an error-free experiment has a finite uncertainty.
- Effective-attenuation estimates should be corrected for the scattering measured over the same frequency band—it can be quite different from that measured over a larger frequency band (Fig. 6). The stronger the scatterers, the larger the deviation of the locally fitted slope from the average can be.
- If $S_{eff}$ and $S_{sc}$ are assessed from the same frequency band, the additional linear trend in $S_{eff}$ caused by the particular realization of scattering over the target frequency band is modeled and removed—it is not “noise”. Only the residuals of the fit constitute noise in the spectral ratios (both in the real and synthetic VSP). Assuming those residuals are independent and normally distributed, the uncertainty of a spectral-ratio slope is well known (e.g., Johnson and Wichern, 2002):

$$\sigma_{fit} = \frac{\sigma_s}{\sqrt{n_f} \sigma_f}$$

(9)

where $\sigma_s$ is the standard deviation of the residuals of the least-square fit, $n_f$ is the number of the data points (frequency samples) in the usable frequency band and $\sigma_f$ is the standard deviation of the frequency samples (i.e., $\sigma_f$ characterizes the width of the usable frequency band). In a perfect world, the residuals of the fit for a given receiver pair would be the same for the real and the synthetic VSP. In reality, they are only on the same order of magnitude but are not identical mainly because of positioning errors in the synthetic VSP.

7 ESTIMATING ATTENUATION

Now we are ready to derive some attenuation estimates. First, the effective attenuation is evaluated from the VSP data. Then, thin layering contributions are assessed and subtracted to isolate absorption. Finally, the absorption estimates from different receiver pairs are appropriately weighted and averaged to get the mean absorption (mean intrinsic Q) in each layer.

7.1 Effective attenuation from VSP data

The first arrivals on all traces were windowed by a 64-point 20% cosine taper, positioned so that the main

\footnote{A Goupillaud model is used to generate the synthetic seismograms; thus, receiver “positions” are specified in terms of travelt ime from the earth surface.}

\footnote{In fact, the residual distribution seems sharper than a Gaussian, so eq. (9) may overestimate the slope uncertainty.}
event was not degraded\[\|\]. The receivers were paired as illustrated in Fig. 2, and a linear regression was used to fit the spectral ratios on a log-linear scale (eq. 1) over the 15-85 Hz band. The uncertainty of the obtained slope $S_{\text{eff}}$ has two main components. One comes from interference [fitting error – eq. (9)], the other comes from measuring the time-separation between the receivers (eq. 6). Thus,

$$\text{Var}(S_{\text{eff}}) = \frac{\sigma_{	ext{ef}}^2}{n_f \sigma_f^2} + \frac{\sigma_{\text{fit}}^2}{\Delta t^2} S_{\text{eff}}^2$$  \hspace{1cm} (10)

Typically, the first term is an order of magnitude larger than the second one.

The error caused by tapering and ambient noise, albeit small compared to the uncertainty (10), is also taken into account. For each receiver pair, the predicted bias (eq. 4) is subtracted from the estimated slope $S_{\text{eff}}$ and the slope uncertainty is adjusted according to

$$\text{Var}(S_{\text{eff}} - b_S) = (1 - 2\alpha_1) \text{Var}(S_{\text{eff}}) + \text{Var}(b_S).$$  \hspace{1cm} (11)

where $\text{Var}(b_S)$ is predicted from $S_{\text{eff}}$ by eq. (5), and $\alpha_1$ is the coefficient from the bias model (eq. 4). This is a very minor adjustment compared to the total uncertainty of $S_{\text{eff}}$.

The so obtained effective slope estimates are shown in the left column of Fig. 7.

### 7.2 Scattering effects

The synthetic VSP for assessing $S_{\text{sc}}$ was computed from the reflection coefficient log in Fig. 8 by a time-domain reflectivity code [augoupillaud from the free software package SU, (Cohen & Stockwell, 2002)], assuming the medium is horizontally layered and non-absorbing. Spectral ratio slopes were estimated in the same manner as from the real VSP. The only difference is that the slopes $S_{\text{sc}}$ contain positioning errors instead of timing errors, i.e., the equivalent of eq. (10) is

$$\text{Var}(S_{\text{sc}}) \approx \frac{\sigma_{\text{ef}}^2}{n_f \sigma_f^2} + \sigma_{\text{pos}}^2,$$  \hspace{1cm} (12)

where $\sigma_{\text{pos}}^2$ is given by eq. (7). Usually the positioning error $\sigma_{\text{pos}}^2$ is smaller than the fitting uncertainty (the first term), but larger than the timing error in eq. (10).

The results for $S_{\text{sc}}$ are shown in the central column of Fig. 7. Note that these are estimates from the narrow frequency band 15-85 Hz – they quantify the scattering effects as seen by the real VSP, not the scattering effects that would be measured over many realizations of the fine layering or over a larger frequency band. Being strongly influenced by local interference, these values of $S_{\text{sc}}$ are hard to predict even qualitatively by looking at the reflectivity log in Fig. 8. For example, $S_{\text{sc}}$ tends to be positive in Layer 2, while one would expect it to be negative, given the non-increasing reflection coefficient series in that layer (Mateeva, 2003). Such negative values are readily observable if the spectral ratio slope is fit over a larger frequency band (Fig. 9).

### 7.3 Intrinsic attenuation (absorption)

Finally, the intrinsic attenuation for each receiver pair is found as $S = S_{\text{eff}} - S_{\text{sc}}$. Its variance is

$$\text{Var}(S) = \text{Var}(S_{\text{eff}}) + \text{Var}(S_{\text{sc}}),$$  \hspace{1cm} (13)

because the effective and scattering attenuation estimators are independent. Since both $\text{Var}(S_{\text{eff}})$ and
Figure 7. Attenuation estimators: (left) $S_{\text{eff}}$ measured from VSP data, (center) $S_{\text{sc}}$ measured from synthetic traces in a horizontally layered non-absorbing medium, (right) computed intrinsic attenuation: $S = S_{\text{eff}} - S_{\text{sc}}$. Dark and light data points correspond to receiver pairs that contain, respectively, the top and bottom receiver in a layer. All plots are on the same scale.
Fig. 9. Scattering attenuation estimates in Layer 2 – local fit (left) versus global fit (right). The behavior of the global fit is in an excellent agreement with the theoretical predictions in Mateeva (2002). The local fit is quite erratic.

$Var(S_{sc})$ are dominated by fitting errors (local interference), and the fitting errors are on the same order of magnitude for the VSP and synthetic spectral ratios, the uncertainty of the intrinsic attenuation estimate is about twice as large as that of the effective attenuation (Fig. 7 right).

The results for Layers 3 and 4 call for a comment. The effective attenuation in Layer 3 is clearly larger in the bottom part of the layer than in the top part. The scattering correction has reduced, but not eliminated the trend. This is why I divided the layer in two sub-layers (3a and 3b). The attenuation estimates for these sub-layers are shown in Fig. 10. The intrinsic attenuation in them turns out to be substantially different, indeed.

In Layer 4, a number of receiver pairs (especially among those containing the deepest receiver) give positive $S_{sc}$ and $S_{sc}$; i.e., the signal appears to gain high frequencies with depth. This must be caused by reflections from the fine layering below the deepest receiver and indicates that the reflection coefficient series becomes substantially stronger beneath the borehole (Mateeva, 2003). Unfortunately, this reflectivity change is not observable on the well logs, and thus, is not present in the reflection coefficient log used to predict the scattering effects (Fig. 8). Therefore, scattering and intrinsic attenuation cannot be separated for the deepest VSP receivers which feel the medium beneath the borehole bottom. The intrinsic attenuation can be only assessed from the receivers that feel the “correct” fine layering captured by the well logs. There are ten such receivers in Layer 4, and the attenuation extracted from them is shown in Fig. 11. The reduced receiver separation leads to very large uncertainties. The ten usable receivers give three unphysical, though statistically plausible, intrinsic slopes (Fig. 11 right). I discarded the unphysical slopes when assessing the mean $Q$ of Layer 4.

7.4 Mean intrinsic $Q$ profile

As a last step in obtaining the mean absorption of the subsurface layers, the values of $S$ from different receiver pairs were averaged by a weighted-least-squares procedure within each layer. The covariance matrix for the procedure has diagonal elements $\sigma^2_{ij}$, given by eq. (13). The off-diagonal elements $\sigma^2_{ij}$ are non-zero for pairs $i$ and $j$ that have a common receiver, and are given by (Appendix A)

$$\sigma^2_{ij} \approx \frac{1}{\Delta t_i \Delta t_j} \frac{\sigma^2_{0i}}{n_i} \frac{\sigma^2_{0j}}{n_j},$$

where $\Delta t_i$ is the time-separation in the $i$-th receiver pair, and $\sigma^2_{0i}$ characterizes the uncertainty of the log-amplitude spectrum of the common receiver. It can be estimated by (Appendix A)

$$\sigma^2_{0i} = \frac{\text{median}(\Delta t^2 \sigma^2_{t0})_{eff} + \text{median}(\Delta t^2 \sigma^2_{t0})_{sc}}{2}$$

where the subscripts “eff” and “sc” indicate estimation from the real and synthetic VSP respectively, $\sigma_r$ is the residuals’ standard deviation [as in eqs. (10), (12)], and the median is taken over all pairs sharing the common receiver.

The mean-$Q$ profile ($\bar{Q} = -2\overline{S}/\overline{S}$) resulting from this averaging procedure is shown in Fig. 12. Note that the error bars in Fig. 12 refer to the mean quality factor of each layer. They depend both on the variability of the quality factor inside each layer and on the data acquisition and inversion.

Not all estimates in Fig. 12 are the same. Shown by circles are estimates based on all available receiver pairs – they are purely data driven. Such an estimate for Layer 4 (using only 10 receivers) is not feasible because one third of the results correspond to unphysical $Q$ values. I chose to discard them before computing the mean in Layer 4. The result is shown by a different symbol to indicate that this estimate of $\bar{Q}$ is not like the others – it is conditioned by a priori knowledge about absorption (i.e., the intrinsic $Q$ is positive). I also computed conditional estimates for Layers 3a and 3b by discarding outliers, even if physically plausible. The results (white crosses) turn out to be compatible with the unconditional estimates, but their error-bars (dashed in white) are smaller. In Layers 1 and 2 there were no obvious outliers.

8 DISCUSSION

The most remarkable feature of the intrinsic $Q$ profile in Fig. 12 is that the absorption of each layer is clearly resolved (outside the error bars of its neighbors), despite that the data set was challenging. A beneficial factor in obtaining such a result was the dense VSP coverage, providing many data points per layer. Another favorable

**Initial attempts to extract absorption from this particular VSP by feeding it to a commercial flow were unsuccessful.
factor is the geology, consisting of thick units with distinct properties and relatively low Q-factors (easier to assess than high Q-s). Last but not least, the reflection coefficient series was not very strong. The correlation between weak scattering and absorption resolution is clearly seen in Fig. 7. Compare, for example, the intrinsic attenuation obtained from pairs with separation 150-200 ms in Layers 3 and 2 (weak and strong reflectivity respectively) – the error bars are larger in the stronger reflectivity. The distinction between the absorption values in the top and bottom halves of Layer 3 would have been impossible in the presence of scattering as strong as that in Layer 2 – the absorption change would have been masked by the large variability of the absorption estimates.

The most uncertain slopes tend to come from pairs with a small separation. Again, this is largely due to scattering effects, rather than timing and positioning errors. If we had a purely transmissional experiment (no reflections from below the receivers), the longer a pulse propagated through the scattering medium, the better the self-averaging in its amplitude spectrum would be. Shapiro and Zien (1993) showed that the standard deviation of the estimated scattering attenuation $\alpha$ is...
\[
\sigma_\alpha \propto \sqrt{\frac{\pi}{L}},
\]
where \(L\) is the distance traveled. As \(L \to \infty\), \(\sigma_\alpha\) diminishes and the spectral ratio of the output to the input pulse approaches its expected value, e.g., a straight line over a limited frequency band. The inability of the downgoing pulse to stabilize over a short path of propagation, especially in a strong reflectivity, is one of the reasons for the large fitting uncertainties in VSP spectral ratios. An additional reason is that reflections from below cause deviations from linearity in the spectral ratios that do not diminish as the receiver separation increases (they do not self-average). One way to reduce this uncertainty is to fit the spectral ratio over a large frequency band. However, this option is limited by the frequency range of the VSP – we need to assess the scattering as “seen” by the VSP, i.e., over a narrow frequency band. Another way to reduce the uncertainty caused by reflections from below is to separate the up- and down-going wavefields and apply the spectral-ratio method only to the downgoing part (Harris et al., 1992).

To summarize, the uncertainties of all attenuation estimates are larger for pairs with a small separation, and in strong reflectivities. This could have been intuitively expected and has been noted in earlier studies (e.g. Spencer et al., 1982).

It should be pointed out, however, that the scattering in a weak reflectivity can also play an important role in absorption estimation. For example, look at the effective attenuation in the almost homogeneous Layer 3a (Fig. 10, top left). Many of the slope estimates are positive (\(Q_{\text{eff}}\) is negative). Synthetic seismograms show that is a scattering effect – after correcting for it, the intrinsic attenuation stands at about -0.25 dB/Hz/s (Fig. 10, top right). As an extra benefit from the thin-layering correction, the scatter of the attenuation estimates in Layer 3a has been reduced. This is easily seen for the set of dark data points – compare their alignment before and after the scattering was subtracted. The small scatter of the estimates suggests that, in terms of absorption, Layer 3a is quite homogeneous. The attenuation estimates from different receiver pairs, however, are not always made more consistent by the thin layering corrections – it depends both on the geology and the quality of the estimates. For instance, the scatter of the estimates is increased in Layer 3b, despite that the reflectivity strength in it is comparable to that in Layer 3a. Layer 3b is another illustration of how thin layering effects can be important even when the reflectivity is weak. The effective attenuation appears different for the top and bottom parts of Layer 3b (Fig. 10 left). However, the scattering corrections reconcile the results for the two sets of receiver pairs, and the intrinsic attenuation does not exhibit a systematic variation with depth (Fig. 10, bottom right).

The price of removing the bias caused by thin layering is increased uncertainty. The variance of \(S\) is essentially twice that of \(S_{\text{att}}\). Given the trade-off between bias and variability, is it worthy to correct for the scattering? The conventional way to answer this question is to look at the mean square error (sum of variance and squared bias) of the two absorption estimators. The mean square error (MSE) of the effective attenuation is

\[
\text{MSE}(S_{\text{att}}) = \text{Var}(S_{\text{att}}) + \sigma_{\text{sc}}^2,
\]

for the unbiased estimator \(S\) it is

\[
\text{MSE}(S) = \text{Var}(S) = \text{Var}(S_{\text{att}}) + \sigma_{\text{sc}}^2
\]

Since in most cases of slope fitting over a narrow frequency band \(\text{std}(S_{\text{sc}}) > |S_{\text{sc}}|\), eqs. (17) and (18) give \(\text{MSE}(S) > \text{MSE}(S_{\text{att}})\); i.e., in a mean-square-error sense, the corrected slope \(S\) is worse than \(S_{\text{att}}\), at least for an individual receiver pair. For the average attenuation in a layer, it may happen that \(\text{MSE}(\bar{S}) < \text{MSE}(\bar{S}_{\text{att}})\) if the scattering compensation makes the estimates of \(S\) from different receiver pairs more consistent. In our example, this happens only in Layers 1 and 3a. So it seems that, even in terms of layer averages, the effective attenuation has a smaller MSE than the intrinsic attenuation.

Unfortunately, this is not a “green light” to ignore the scattering. In some cases it is more important to have an unbiased estimate rather than a small variability. An obvious such case is when \(S_{\text{att}}\) is positive (i.e., \(Q_{\text{eff}} < 0\)). Another case is when the bias due to scattering is large compared to the intrinsic attenuation. Estimates of \(|S_{\text{sc}}/S|\) are shown in Table 1. Note that the layer with the highest fraction of scattering (highest albedo) happens to be the almost homogeneous but low-absorbing Layer 3a.

Absorption uncertainties depend on many factors, but, if we are to summarize in coarse figures, we could...
References

\[ \frac{|S_{sc}/S|}{\text{Layer}} \]

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<td>30%</td>
</tr>
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<td>3b</td>
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Table 1. Scattering versus intrinsic attenuation - a median estimate over all receiver pairs in a given layer.

say that an absorption estimate derived from a single receiver pair has an uncertainty \( \sim 50\% \) (median over all receiver pairs in this study). To reduce it to about 10\%, we have to average at least 25 independent estimates. With VSP receiver spacing of \( \sim 10^3 \) m, that corresponds to a typical absorption resolution of \( \sim 10^2 \) m.

9 CONCLUSION

To characterize lithology or reservoir conditions from attenuation data, one must separate absorption from scattering effects and have an objective estimate of the absorption uncertainties. The price for removing the scattering is increased variability of the absorption estimates. It is worthwhile to pay if the apparent attenuation is large compared to the intrinsic attenuation. This may happen even when the scattering is weak. Therefore, scattering should not be neglected just because “the medium seems homogeneous” before its share in the effective attenuation has been assessed.

Incoherent scattering is the largest source of uncertainty. A fundamental way to reduce its influence is to have a VSP with a broader frequency band; additional improvement may be sought through wavefield separation. The next largest uncertainties are associated with positioning and timing errors in the synthetic and real VSP respectively. Ambient noise and tapering have a much smaller impact on the fitted slopes. Finally, the correlation between attenuation estimates from pairs with a common receiver must be taken into account when estimating the uncertainty of the mean quality factors of thick geological units.

ACKNOWLEDGMENTS

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REFERENCES


APPENDIX A: COVARIANCE FOR PAIRS WITH A COMMON RECEIVER

Suppose two spectral ratios, \(y_1, y_2\), are based on a common receiver, i.e.,

\[
y_1(f) = \frac{1}{\Delta t_1}[P_1(f) - P(f)]
\]

\[
y_2(f) = \frac{1}{\Delta t_2}[P_2(f) - P(f)]
\]

where \(P, P_1\) and \(P_2\) are log-amplitude spectra, measured at frequencies \(f_1, \ldots, f_n\), and \(\Delta t_{1,2}\) is the time-separation in the corresponding receiver pair. The covariance between the two spectral ratios caused by the common receiver is

\[
\text{Cov}(y_1, y_2) = \frac{1}{\Delta t_1 \Delta t_2} \text{Var}[P(f)]
\]

Suppose that the amplitude spectra of all traces have equally large uncertainties. Then, as seen from (A1), the variability of the common receiver spectrum can be estimated, for example, by

\[
\text{Var}[P(f)] = \text{median}_i \frac{\sigma^2_{y_i} (\Delta t_i)^2}{2},
\]

where the median is taken over all ratios containing the spectrum \(P(f)\), and \(\sigma^2_{y_i}\) is the variance of the residuals of the best linear fit of spectral ratio \(y_i\) [i.e., \(\sigma^2_{y_i} \approx \text{Var}(y_i)\)].

The correlation between the spectral ratios caused by the common receiver propagates in the fitted slopes \(s_i\) of \(y_i(f)\). As is known from statistical analysis (e.g., Johnson and Wichern, 2002),

\[
\text{Cov}(s_1, s_2) \approx \frac{\text{Cov}(y_1, y_2)}{n \sigma^2_f}
\]

where \(\sigma_f\) characterizes the frequency range over which the spectral ratios were fit. Eq. (A4) is strictly valid for data with Gaussian noise, while the fitting residuals seems to have a distribution that is sharper than a Gaussian, hence the approximate sign.

Substituting (A2) in (A3), and (A3) in (A4), we get

\[
\text{Cov}(s_1, s_2) \approx \frac{1}{2 n \sigma^2_f} \text{median}_i \frac{\sigma^2_{y_i} (\Delta t_i)^2}{\Delta t_1 \Delta t_2}
\]

This equation is applied separately to the real and synthetic VSP-s to get the off-diagonal elements of the covariance matrix needed when averaging slopes within a macro-layer.

APPENDIX B: REFLECTIVITY LOG FOR MODELING THE SCATTERING EFFECTS

To compute synthetic seismograms, a Goupillaud model was used, i.e, an earth model, consisting of horizontal layers of equal time thickness. The reflection coefficient (RC) series defining such a model is computed from sonic and density logs. After converting it to the time domain and interpolating to the nearest uniform time-grid, one may anti-alias filter and resample to the VSP rate (e.g., 2 ms t.w.t.). I did not resample because the computation of the synthetic seismograms from the full reflectivity log was fast enough; later, I used only the low-frequency part of the synthetic spectra to evaluate scattering.

The well logs span only the 600-3500 m interval. To fill in the missing reflectivity of the upper 600 m, I assumed that the top sequence present in the reflectivity log (600-1000 m) extends up to the surface. I combined its amplitude spectrum with a random phase spectrum, drawn from a uniform distribution \(U[-\pi, \pi]\), and inverse-Fourier transformed to the time domain to create a synthetic RC with which to append the real log. The magnitudes of the synthetic reflection coefficients do not follow a mixed Laplace distribution as the real ones do (Walden & Hosken, 1986). However, this does not matter in the apparent attenuation estimation which, in my experience, depends mainly on the power spectrum of the RC series.

In the same manner the reflectivity log was extended below the borehole bottom using the power spectrum of the reflection coefficient log in Layer 4 (3100-3500 m). This was needed because the deepest VSP receivers feel the medium below the borehole; to predict the scattering effects in them, we need a model of the reflectivity below the borehole.

Finally, the near-surface sandstone layer was added by putting a reflection coefficient of -0.45 at 15 ms o.w.t. below the earth surface. The time thickness of the sandstone was determined from notches in the spectra of the VSP traces. The choice of the reflection coefficient magnitude was a bit arbitrary. The main consideration was that it should be large compared to the other coefficients in order to create such strong notches. An additional requirement was that it be consistent with the VSP and well-log data. Continuing the sonic log trend up to the sandstone base suggests a sub-sandstone velocity of roughly 3400 m/s. Then, a reflection coefficient of -0.45 can be explained by a 20 m thick sandstone with a velocity 1300 m/s, which is a plausible model. Tests with slightly different values led to virtually identical estimates of \(S_e\). Similarly, the synthetic spectral ratios are not sensitive to the earth’s surface reflection coefficient. For a free surface, it is appropriately set to -1 (as “seen” from above by the displacement field). However, given that the thin sandstone layer is expected to have a very low quality factor, it may be more appropriate to model the earth as bounded by a semi-absorbing surface with a smaller reflection coefficient. In general, it is very important to account for the earth’s surface in apparent attenuation studies (Mateeva, 2001). However, the spectral ratios between early windows on VSP traces are an exception in that they are not sensitive to the properties of the near surface (Mateeva, 2003).
Constraining relative source locations with the seismic coda

Roel Snieder

Center for Wave Phenomena and Department of Geophysics, Colorado School of Mines, Golden, CO 80401, USA

ABSTRACT
The relative location of seismic sources is of importance for the location of aftershocks on a fault, for the positioning of sources in repeat seismic surveys, and for monitoring induced seismicity. In this paper I show theoretically how the seismic coda can be used to infer a measure of the relative source location of two identical seismic sources from the correlation of the waveforms recorded at a single receiver. The theory is applicable to an explosive source in an acoustic or elastic medium, and for a point force or double couple in an elastic medium. For an explosive source the relative source location is constrained to be located on a sphere, while for a point force and a double couple the relative source location can be constrained to be located on an ellipsoid whose symmetry axis is determined by the point force or double couple.

1 INTRODUCTION
In a number of applications it is useful to determine the relative location of seismic sources. Aftershocks after a large earthquake help constrain the location and extent of the fault plane (Lay & Wallace, 1995). Earthquake clusters have been used to locate faults planes, both in a tectonic setting (Fuis et al., 2001) and in hydrocarbon reservoirs (Maxwell & Urbancic, 2001). Seismicity has also been used to monitor the fluid transport properties in reservoirs (Shapiro et al., 2002). In repeat seismic surveys with down hole sources, it is essential that the relative source locations in the two surveys are known with great accuracy in order to reduce the imprint of errors in the source location in time lapse measurements.

In principle, the relative position of two source locations can be found by locating each of the sources, and subsequently computing their relative location. The disadvantage of this approach is that errors in the velocity model along the whole path from the sources to the receivers are erroneously mapped into location errors. The relative position computed by comparing the absolute locations may be dominated by the location errors for the individual sources (Pavlis, 1992). For this reason it is advantageous to determine the relative location of the source directly from the recorded waveforms.

Earthquakes that occur within the same cluster of events often generate waveforms that are highly repeatable. Such highly repeatable waveforms have been used to constrain the relative positions of these events (Poupinet et al., 1984; Frémont & Malone, 1987; Got et al., 1994; Nadeau & McEvilly, 1997; Bokelmann & Harbes, 2000; Waldhauser & Ellsworth, 2000) by measuring the delay times between the arriving P and S waves of the different events. Such a measurement of the delay time of P and S waves has also been used to find the relative location between one master event and a number of smaller events (Ito, 1985; Scherbaum & Wendler, 1986; Frémont & Malone, 1987; VanDecar & Crosson, 1990; Deichmann & Garcia-Fernandez, 1992; Lees, 1998). Usually this delay time is measured by a cross-correlation of the direct P and S arrivals for the different events. A particularly robust variation of this idea is based on a measurement of the cross-correlation of the direct P and S waves that is based on an $L_1$ norm and a nearest neighbor approach (Shearer, 1997; Astiz & Shearer, 2000).

In this paper I propose a technique to obtain a measure of the relative source location of two events that is based on coda waves. The main idea is that the energy that constitutes the coda waves is radiated in all directions with a radiation pattern that is determined by the source position. When the source position changes, some wave paths will be longer while other wave paths become shorter. The interference pattern of the scattered waves that contribute to the coda thus changes. Here we use the change in the coda waves to constrain the relative locations of the events. This technique is a
new application of coda wave interferometry (Snieder et al., 2002; Snieder, 2002).

The principles of coda wave interferometry are introduced in section 2. The displacement of an isotropic source in an acoustic medium is treated in section 3. The generalization to an elastic medium that is excited by a point force and of a double couple is presented in sections 4 and 5.1. The source displacement of an explosive source in an elastic medium is discussed in section 5.2.

2 CODA WAVE INTERFEROMETRY AND SOURCE DISPLACEMENT

In this section, I review the elements of coda wave interferometry. A more detailed description is given by Snieder (2002). The idea of coda wave interferometry is based on pat summation (Snieder, 1999). In this approach the wavefield is written as a superposition of the waves that follow all the different paths in the medium:

$$u^{(u)}(t) = \sum_T A_T(t).$$  \hspace{1cm} (1)

The subscript $T$ labels the different trajectories along which the waves have traveled. A trajectory not only specifies the path that a wave has taken through space, it also specifies which of the segments along that path have been traversed as a P-wave or as an S-wave. The summation over trajectories therefore also contains a summation over the different wave modes (P or S) that an elastic wave can take while propagating between the scatterers along each path. The function $A_T(t)$ denotes the contribution of the trajectory $T$ to the waveform recorded at the station under consideration. The superscript $(u)$ in equation (1) denotes that this is the unperturbed waveform, i.e., that associated with the reference source position.

Now suppose that the source location is perturbed. When the source displacement is small, the dominant change to the wave field comes from the travel time perturbation $\tau_T$ for the wave along each trajectory $T$. (In reality the geometrical spreading and radiation of the waves that propagate from the displaced source to a fixed scatterer change as well, but we assume that this contribution is sub-dominant.) Under this assumption the perturbed wave field is given by

$$u^{(p)}(t) = \sum_T A_T(t - \tau_T).$$  \hspace{1cm} (2)

The change in the waveform can be measured by a time-shifted cross-correlation, with shift time $t_s$ computed over a time window of length $2t_w$ and center time $t$:

$$R^{(t-t_w)}(t_s) \equiv \frac{\int_{t-t_w}^{t+t_w} u_i^{(u)}(t') u_i^{(p)}(t' + t_s)dt'}{\left( \int_{t-t_w}^{t+t_w} u_i^{(u)}(t')^2 dt' \int_{t-t_w}^{t+t_w} u_i^{(p)}(t')^2 dt' \right)^{1/2}}.$$  \hspace{1cm} (3)

As shown by Snieder (2002), this cross-correlation attains its maximum value for

$$t_s = \langle \tau \rangle_{(t,t_w)},$$  \hspace{1cm} (4)

where the average in this expression is given by

$$\langle \tau \rangle = \frac{\sum_T A_T^2 \tau_T}{\sum_T A_T^2}.$$  \hspace{1cm} (5)

In this expression the summation is over the trajectories with an arrival time within the interval $(t - t_w, t + t_w)$. The maximum value of the cross-correlation is given by

$$R_{\max}^{(t,t_w)} = 1 - \frac{1}{2} \omega^2 \sigma^2,$$  \hspace{1cm} (6)

where $\sigma$ is the variance of the travel time perturbation defined as

$$\langle \sigma^2 \rangle = \frac{\sum_T A_T^2 (\tau_T - \langle \tau \rangle)^2}{\sum_T A_T^2}.$$  \hspace{1cm} (7)

The frequency $\frac{1}{\omega}$ is given by

$$\frac{1}{\omega} \equiv - \frac{\int_{t-t_w}^{t+t_w} u_i^{(u)}(t') \dot{u}_i^{(u)}(t') dt'}{\int_{t-t_w}^{t+t_w} u_i^{(u)}(t')^2 dt'}. \hspace{1cm} (8)

The cross-correlation (3) can readily be computed given the measured unperturbed and the perturbed waveforms, so both the location of its maximum and the peak value can be measured. With expressions (4) and (6) the mean and variance of the travel time perturbations can therefore be determined from the observations.

When the source location is perturbed, only the length of the wave path to the first scatterer along each path is perturbed, because a perturbation in the source location does not change the relative positions of the scatterers along a path. The travel time change due to a perturbation $\delta$ in the source location leads to a change in the travel time given by

$$\tau_T = - \frac{1}{v} \hat{v} \cdot \delta.$$  \hspace{1cm} (9)

In this expression the unit vector $\hat{v}_T$ points in the direction in which the trajectory $T$ takes off at the source and the velocity $v$ is the velocity of the trajectory as it leaves the source. This can either be the P-velocity or the S-velocity. I assume throughout this paper that the velocity is constant over the region over which the source is displaced and that the velocity is isotropic.

When the scatterers are distributed homogeneously, the summation over all trajectories that leave the source can be replaced by an angular integration over all directions toward which a wave can leave the source. Since the averages (5) and (7) are taken with a weight given by energy of the wave that travels along each the trajectory, the integration over all take-off directions at the source is to be weighted with the radiated energy in those directions. This principle will be used in the following section to compute the mean and variance of the travel time caused by changes in the source location.
3 AN ISOTROPIC SOURCE IN AN ACOUSTIC MEDIUM

In this section we consider the simplest case of a displacement of an isotropic source in an acoustic medium. I assume that the propagation of the waves from the source to the first scatterer along each path can be described by the Green’s function for a homogeneous medium. For an isotropic source at the origin with source spectrum \( S(\omega) \), the waves that propagate to the first scatterer along each path are thus given by

\[
\mathbf{u}(\mathbf{r}) = \frac{e^{ikr}}{4\pi r} S(\omega) .
\]  

(10)

When the scatterers in the medium are distributed homogeneously, the travel time perturbation (5) due to a source displacement \( \delta \) for each path is given by (9) and the mean travel time perturbation (5) is given by

\[
\langle \tau \rangle = - \frac{1}{4\pi v} \int \int \frac{e^{ikr}}{4\pi r} \frac{1}{v} |\hat{\mathbf{r}} \cdot \delta| |S(\omega)|^2 d\Omega d\omega .
\]

(11)

where \( \int \cdots d\Omega \) denotes the angular integration over all outgoing directions, \( \int \cdots d\omega \) denotes an integration over frequency, and \( r \) is the distance to the first scatterer in each direction. When the scatterers are distributed homogeneously, this distance is on average the same for each direction, and expression (11) can be rewritten as

\[
\langle \tau \rangle = - \frac{1}{4\pi v} \int \int (\hat{\mathbf{r}} \cdot \delta) d\Omega .
\]

(12)

The integrand is an odd function of the location \( \hat{\mathbf{r}} \) on the unit sphere. Since we integrate over the full unit sphere this integral vanishes:

\[
\langle \tau \rangle = 0 .
\]

(13)

Physically this reflects the fact that as the source location is moved, some paths become longer while others become shorter. On average the associated imprint on the travel time is zero.

Since the mean travel time perturbation vanishes, \( \sigma^2_\tau = \langle \tau^2 \rangle \). This quantity can be computed using the same reasoning that led to (13), the only difference being that \( \tau = -v^{-1} (\hat{\mathbf{r}} \cdot \delta) \) needs to be replaced by \( \tau^2 = v^{-2} (\hat{\mathbf{r}} \cdot \delta)^2 \). This gives

\[
\sigma^2_\tau = \frac{1}{4\pi v^2} \int (\hat{\mathbf{r}} \cdot \delta)^2 d\Omega .
\]

(15)

This expression is most easily evaluated by using

\[
\hat{\mathbf{r}} = \begin{pmatrix} \cos \varphi \sin \theta \\ \sin \varphi \sin \theta \\ \cos \theta \end{pmatrix} ,
\]

(16)

with \( \theta \) and \( \varphi \) the colatitude and longitude used in a system of spherical coordinates. Aligning the z-axis of the integration variable with the source displacement \( \delta \) reduces the angular integral in (15) to \( \delta^2 \int \cos^2 \theta d\Omega = 4\pi \delta^2/3 \), and hence

\[
\sigma^2_\tau = \frac{1}{3} \frac{\delta^2}{v^2} .
\]

(17)

Using coda wave interferometry, we can infer \( \sigma^2_\tau \) from the changes in the waveform. Then, from (17), we can infer the magnitude \( \delta \) of the source displacement, but not the direction of the source displacement. The change in the arrival time of the first-arriving wave constrains \( (\hat{\mathbf{r}} \cdot \delta) \), with \( \hat{\mathbf{r}} \) the unit vector in the take-off direction of a ray at the source that propagates directly to the receiver. The first-arriving waves and the later-arriving waves thus impose complementary information on the perturbation of the source position by constraining \( (\hat{\mathbf{r}} \cdot \delta) \) and \( \delta \), respectively.

4 A POINT FORCE IN AN ELASTIC MEDIUM

Let us now consider the displacement of a point force in an elastic medium. The far-field displacement due to a point force \( \mathbf{F} \) at the origin in a homogeneous medium given by Aki & Richards (1980):

\[
\mathbf{u} = \frac{e^{ikr}}{4\pi \rho \alpha^2 r} (\hat{\mathbf{r}} \cdot \mathbf{F} + e^{ikr}/4\pi \beta^2 r) (\mathbf{F} - \hat{\mathbf{r}} (\hat{\mathbf{r}} \cdot \mathbf{F})) .
\]

(18)

This expression \( \alpha \) and \( \beta \) are the P and S-velocity respectively, while the wave numbers that appear are given by \( k_\alpha = \omega/\alpha \) and \( k_\beta = \omega/\beta \). For an arbitrary source spectrum \( S(\omega) \), this expression should be multiplied with \( S(\omega) \), but using the same reasoning that led to (13) we conclude that the source spectrum cancels out. For this reason we suppress the presence of the source spectrum \( S(\omega) \) altogether.

When the source is displaced over a distance \( \delta \), the perturbation of the arrival time for the P-waves is given by \(-\alpha^{-1} (\hat{\mathbf{r}} \cdot \delta)\), while the perturbation of the arrival of the S-waves is given by \(-\beta^{-1} (\hat{\mathbf{r}} \cdot \delta)\). In the averages (5) and (7), the averages are taken with the intensities of each path as weight function. Since the P-waves and the S-waves can be considered to be different paths, the mean travel time perturbation is given by expression (T1) of Table 1. This expression can be simplified with the following identities: \( (\hat{\mathbf{r}} (\hat{\mathbf{r}} \cdot \mathbf{F}))^2 = (\hat{\mathbf{r}} \cdot \hat{\mathbf{r}})^2 (\hat{\mathbf{r}} \cdot \mathbf{F})^2 = (\hat{\mathbf{r}} \cdot \mathbf{F})^2 \) and \( |(\mathbf{F} - \hat{\mathbf{r}} (\hat{\mathbf{r}} \cdot \mathbf{F}))|^2 = F^2 - 2 (\hat{\mathbf{r}} \cdot \mathbf{F}) (\hat{\mathbf{r}} \cdot \mathbf{F}) + (\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}) (\mathbf{F} \cdot \mathbf{F}) = F^2 - (\hat{\mathbf{r}} \cdot \mathbf{F})^2 \). Using these identities gives equation (T2) of Table 1. Both integrals in the numerator of that expression vanish because the integrands are odd functions of \( \hat{\mathbf{r}} \) and the integration is carried out over the full unit sphere; therefore \( \langle \tau \rangle = 0 \).

The variance in the travel time can be computed by replacing \(-\alpha^{-1} (\hat{\mathbf{r}} \cdot \delta)\) in the numerator by \(\alpha^{-2} (\hat{\mathbf{r}} \cdot \delta)^2\) and \(-\beta^{-1} (\hat{\mathbf{r}} \cdot \delta)\) by \(\beta^{-2} (\hat{\mathbf{r}} \cdot \delta)^2\). This gives expression
Expressions for the mean and the variance of the travel time.

The integrals over the unit sphere can be carried out using representation (16) for $\alpha$. It is convenient to align the $z$-axis with the point force, so that $\hat{\mathbf{F}} = F^* \cos \theta$. For that coordinate system, the first integral in the numerator is given by

$$\langle \tau \rangle = \frac{\int e^{ik_o r} \mathcal{F}(\mathbf{F})}{4\pi \rho \alpha^2 r} + \frac{1}{\alpha} \int \frac{\mathcal{F}(\mathbf{F})}{4\pi \rho \beta^2 r} d\Omega - \frac{\int e^{ik_o r} \mathcal{F}(\mathbf{F})}{4\pi \rho \beta^2 r} d\Omega.$$

(T1)

$$\langle \tau \rangle = \frac{\int e^{ik_o r} \mathcal{F}(\mathbf{F})}{4\pi \rho \alpha^2 r} + \frac{1}{\alpha} \int \frac{\mathcal{F}(\mathbf{F})}{4\pi \rho \beta^2 r} d\Omega - \frac{\int e^{ik_o r} \mathcal{F}(\mathbf{F})}{4\pi \rho \beta^2 r} d\Omega.$$

(T2)

$$\sigma^2_\tau = \frac{\int e^{ik_o r} \mathcal{F}(\mathbf{F})^2 (\mathbf{F} \cdot \mathbf{\delta})^2 d\Omega + \frac{1}{\beta^2} \int (\mathbf{F} \cdot \mathbf{M})^2 (\mathbf{F} \cdot \mathbf{\delta})^2 d\Omega}{\int \mathcal{F}(\mathbf{F})^2 d\Omega + \frac{1}{\beta^2} \int (\mathbf{F} \cdot \mathbf{M})^2 d\Omega}.$$

(T3)

However, the last line can be rewritten as

$$\frac{4\pi}{15} F^2 (\delta_x^2 + \delta_y^2) + \frac{4\pi}{5} F^2 \delta_z^2.$$

(T4)

$$\sigma^2_\tau = \frac{\int e^{ik_o r} \mathcal{F}(\mathbf{F})^2 (\mathbf{F} \cdot \mathbf{\delta})^2 d\Omega + \frac{1}{\beta^2} \int (\mathbf{F} \cdot \mathbf{M})^2 (\mathbf{F} \cdot \mathbf{\delta})^2 d\Omega}{\int \mathcal{F}(\mathbf{F})^2 d\Omega + \frac{1}{\beta^2} \int (\mathbf{F} \cdot \mathbf{M})^2 d\Omega}.$$

(T5)

Table 1. Expressions for the mean and the variance of the travel time.

(T3) of Table 1. This expression can also be rewritten as

$$\sigma^2_\tau = \left( \frac{1}{\alpha^6} - \frac{1}{\beta^6} \right) \left[ \int (\hat{\mathbf{F}} \cdot \mathbf{F})^2 (\hat{\mathbf{F}} \cdot \mathbf{\delta})^2 d\Omega + \frac{F^2}{\beta^6} \left[ \int \mathcal{F}(\mathbf{F})^2 d\Omega + \frac{1}{\beta^2} \int (\mathbf{F} \cdot \mathbf{M})^2 d\Omega \right] - \int \mathcal{F}(\mathbf{F})^2 d\Omega \right].$$

(19)

The integrations over the unit sphere can be carried out using representation (16) for $\hat{\mathbf{F}}$. It is convenient to align the $z$-axis with the point force, so that $\hat{\mathbf{F}} = F^* \cos \theta$. For that coordinate system, the first integral in the numerator is given by

$$\int (\hat{\mathbf{F}} \cdot \mathbf{F})^2 (\hat{\mathbf{F}} \cdot \mathbf{\delta})^2 d\Omega$$

$$= F^2 \delta_x^2 \int_0^{2\pi} \int_0^\pi \sin^2 \theta \cos^2 \varphi \sin^2 \theta \sin \theta d\theta d\varphi$$

$$+ F^2 \delta_y^2 \int_0^{2\pi} \int_0^\pi \sin^2 \theta \sin^2 \varphi \cos^2 \theta \sin \theta d\theta d\varphi$$

$$+ F^2 \delta_z^2 \int_0^{2\pi} \int_0^\pi \cos \theta \cos^2 \theta \sin \theta d\theta d\varphi$$

$$= \frac{4\pi}{15} F^2 (\delta_x^2 + \delta_y^2) + \frac{4\pi}{5} F^2 \delta_z^2.$$

(20)

The last expression holds for the special case of a coordinate system that is aligned with the point force.

However, the last line can be rewritten as

$$\frac{4\pi}{15} F^2 (\delta_x^2 + \delta_y^2) + \frac{4\pi}{5} F^2 \delta_z^2$$

$$= \frac{4\pi}{15} F^2 (\delta_x^2 + \delta_y^2 + \delta_z^2) + 4\pi \left( \frac{1}{5} - \frac{1}{15} \right) F^2 \delta_z^2,$$

(21)

$$= \frac{4\pi}{15} F^2 \delta^2 + \frac{8\pi^2}{15} (F \cdot \mathbf{\delta})^2.$$

The last identity is invariant under unitary coordinate transformations such as rotations; therefore this expression holds in any coordinate system.

Applying a similar analysis to all terms in (19) gives, after dividing by $F^2$ and some rearrangement,

$$\sigma^2_\tau = \frac{\left( \frac{1}{\alpha^6} + \frac{4}{\beta^6} \right) \delta^2 - 2 \left( \frac{1}{\beta^6} - \frac{1}{\alpha^6} \right) (\hat{\mathbf{F}} \cdot \mathbf{\delta})^2}{5 \left( \frac{1}{\alpha^6} + \frac{2}{\beta^6} \right)}.$$

(22)

where $\hat{\mathbf{F}} \equiv \mathbf{F}/F$ is the unit vector in the direction of the point force.

Note that for the elastic wave generated by a point force, $\langle \sigma^2_\tau \rangle$ depends on not just $\delta^2$ but also on $\langle \hat{\mathbf{F}} \cdot \mathbf{\delta} \rangle$, the projection of the source displacement along the point force. Therefore, the direction of the point force is needed in order to relate the observed value of $\langle \sigma^2_\tau \rangle$ to the source displacement. This contrasts with the acoustic case treated in the previous section where the variance of the travel time was dependent on $\delta$ only.

For a Poisson medium ($\alpha = \sqrt{3}\beta$), expression (22)
results in
\[ \sigma_{v}^{2} \approx \frac{\delta_{v}^{2}}{\beta_{v}^{2}} \left( 0.382 - 0.182 \left( \hat{F} \cdot \hat{\delta} \right)^{2} \right) \] (Poisson medium). \hspace{1cm} (23)

Note that the last term is rewritten in terms of the unit vector \( \hat{\delta} \). The first term gives the contribution that is direction-independent. For the acoustic case the corresponding result (17) is \( \langle \sigma_{v}^{2} \rangle = \delta_{v}^{2}/3v^{3} \approx 0.333\delta_{v}^{2}/v^{2} \), which is close to the coefficient 0.382 in (23) when the shear velocity \( \beta \) is equated to the velocity \( v \) in the acoustic medium. There is a simple reason for this. In expression (22) the velocities \( \alpha \) and \( \beta \) are raised to a fairly high negative power (-4 and -6 respectively). Since \( \beta < \alpha \) this leads to a dominance of the terms that are dependent on \( \beta \). In fact, when the \( \alpha \)-dependent terms in (22) are ignored altogether, expression (22) reduces to
\[ \sigma_{v}^{2} \approx \frac{\delta_{v}^{2}}{\beta_{v}^{2}} \left( 0.4 - 0.2 \left( \hat{F} \cdot \hat{\delta} \right)^{2} \right) \] (\( \alpha \) ignored). \hspace{1cm} (24)

This crude approximation leads to a result that is close to (23) for a Poisson medium. Physically this happens because a point force excites much stronger S-waves than P-waves. Since the travel time averages are weighted with the intensity of both wave types, the contribution of the shear waves dominates. The dominance of the S-wave energy over the P-wave energy has been noted before in different contexts (Aki & Chouet, 1975; Weaver, 1982; Papanicolaou & Ryzhik, 1999; Trégourès & van Tiggelen, 2002; Snieder, 2002).

In some situations one can eliminate the dependence of the variance of the travel time on the direction of the source displacement. As an example, consider a vibrator in a bore hole that vibrates in a direction perpendicular to the bore hole. The relative location of different deployments of the vibrator depends only on the distance along the bore hole. In that case the force \( F \) and the source displacement \( \delta \) are perpendicular, and equation (22) reduces to:
\[ \sigma_{v}^{2} = \frac{1}{5} \left( \frac{1}{\alpha^{2}} + \frac{4}{\beta_{v}^{2}} \right) \delta_{v}^{2} \quad (F \perp \delta). \] (25)

Of course one may question the validity of the employed model for a source in a bore hole since the presence of the bore hole and the properties of its walls modify the radiation of elastic waves.

In this section and the following section I assume that the velocity is isotropic. For an anisotropic elastic medium, the velocity depends on the direction of propagation. In that case the velocity cannot be taken outside the angular integrals. However, the angular integrals can in that case be expanded to take the anisotropy of the velocity into account.

5 A MOMENT TENSOR SOURCE IN AN ELASTIC MEDIUM

In this section we consider the displacement caused by a moment tensor source in an elastic medium. In section 5.1 we consider the case of a double couple and in section 5.2 that of an explosive source. According to Aki and Richards (1980), the displacement in an elastic medium due to a moment tensor source \( M \) is given by
\[ u_{i}(r) = -\frac{i \omega \epsilon^{ik}_{\alpha \beta}}{4\pi \rho \beta^{3} r} \hat{r}_{i} \hat{r}_{j} \hat{r}_{k} M_{jk}. \] (26)

According to expression (5) the travel time perturbation for each trajectory is weighted by the intensity for that trajectory. The intensity corresponding to the different terms in (26) can be computed using the identities \( \langle \hat{r}_{i} \hat{r}_{j} \hat{r}_{k} M_{jk} \rangle = \langle \hat{F} \cdot \hat{F} \rangle (\hat{F} : M)\hat{M}^{2} \approx (\hat{F} : M)^{2} \), and \( \langle \hat{r}_{i} \hat{r}_{j} \hat{r}_{k} M_{jk} \rangle = (\hat{F} : M)\hat{M}^{2} \). By analogy with expression (T2) the mean travel time change due to a perturbation \( \delta \) in the source location is given by equation (T4) in Table 1. Just as is the integral (T2), the integrand in expression (T4) is an odd function of \( \hat{F} \), and the integral vanishes upon integration over the unit sphere so the mean travel time perturbation vanishes: \( \tau = 0 \). Using the same reasoning as that used for (T2), the variance of the travel time is given by equation (T5) in Table 1. The integration over the unit sphere is most easily carried out when a simplified form of the moment tensor is assumed.

5.1 A double couple source

In this section we analyze the variance of the travel time for a double couple source. In the integration we use a coordinate system with the \( z \)-axis perpendicular to the double couple. In that coordinate system the moment tensor is given by
\[ M = \begin{pmatrix} 0 & M & 0 \\ M & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \] (27)

With representation (16) for the unit vector \( \hat{F} \), the contractions that appear in (T5) are given by
\[ (\hat{F} : M)^{2} = M^{2} \sin^{2} \theta, \] (28)
\[ (\hat{F} : M)^{2} = 4M^{2} \sin^{4} \theta \sin^{2} \varphi \cos^{2} \varphi. \] (29)

With these representations the angular integrals that appear in (T5) can be carried out. The resulting integrals are tedious and are given by
\[ \int (\hat{F} : M)^{2} d\Omega = \frac{8\pi}{3} M^{2}, \] (30)
\[ \int (\hat{F} : M)^{2} d\Omega = \frac{16\pi}{15} M^{2}, \] (31)
\[
\int (\hat{r} \cdot \mathbf{M})^2 (\hat{r} \cdot \delta)^2 \, d\Omega = \frac{16\pi}{15} M^2 (\delta_x^2 + \delta_y^2) + \frac{8\pi}{15} M^2 \delta_z^2 .
\] (32)

\[
\int (\hat{r} \cdot \mathbf{M})^2 (\hat{r} \cdot \delta)^2 \, d\Omega = \frac{16\pi}{35} M^2 (\delta_x^2 + \delta_y^2) + \frac{16\pi}{105} M^2 \delta_z^2 .
\] (33)

Inserting these results in expression (28) and using the identity \(\delta_x^2 + \delta_y^2 = \delta^2 - \delta_z^2\) gives after a rearrangement of terms

\[
\sigma_z^2 = \frac{\left(\frac{6}{\alpha^6} + \frac{7}{\beta^6}\right) \delta^2 - \left(\frac{4}{\alpha^6} + \frac{3}{\beta^6}\right) \delta_z^2}{7 \left(\frac{2}{\alpha^6} + \frac{3}{\beta^6}\right)} .
\] (34)

This expression is not quite satisfactory yet because it does not contain the moment tensor in a covariant form. (The only information of the orientation of the double couple is captured in the choice of the \(z\)-direction, which is orthogonal to the double couple.) A covariant formulation can be obtained by defining the following norm of the moment tensor

\[
||\mathbf{M}||^2 \equiv (\mathbf{M} \cdot \mathbf{M}) .
\] (35)

Since this quantity is the double contraction of a tensor of rank two, it is a tensor of rank zero; hence \(||\mathbf{M}||\) is invariant for unitary coordinate transforms. With this norm we can define a normalized moment tensor

\[
\hat{\mathbf{M}} = \frac{\mathbf{M}}{||\mathbf{M}||} .
\] (36)

Since \(\mathbf{M}\) is a tensor of rank two and \(||\mathbf{M}||\) is a scalar, \(\hat{\mathbf{M}}\) is a tensor of rank two. For the moment tensor (27)

\[
\left(\hat{\mathbf{M}} \cdot \delta\right)^2 = \frac{1}{2} (\delta^2 - \delta_z^2) .
\] (37)

This result can be used to eliminate \(\delta_z\) from expression (34) so that

\[
\sigma_z^2 = \frac{\left(\frac{2}{\alpha^6} + \frac{4}{\beta^6}\right) \delta^2 + \left(\frac{4}{\alpha^6} + \frac{3}{\beta^6}\right) \left(\hat{\mathbf{M}} \cdot \delta\right)^2}{7 \left(\frac{2}{\alpha^6} + \frac{3}{\beta^6}\right)} .
\] (38)

This expression is invariant for rotations of the coordinate system and can therefore be applied to a double couple with an arbitrary orientation.

Just as with expression (22) for a point force, the variance of the travel time depends on the magnitude \(\delta\) of the source displacement as well as on the direction of the source displacement. Therefore, it is necessary to know the orientation of the double couple in order to relate the variance in the travel time perturbation as inferred from coda wave interferometry to the source displacement.

For a Poisson medium

\[
\sigma_z^2 \approx \frac{\delta^2}{\beta^2} \left(0.187 + 0.283 \left(\hat{\mathbf{M}} \cdot \delta\right)^2\right) \text{ (Poisson medium)} .
\] (39)

A comparison with expression (23) for a point force shows that the ratio of the isotropic term to the direction-dependent term is, relatively speaking, smaller for the double couple than for the point force. The reason is that for the double couple the angular variation in the radiation pattern is larger than that for a point force.

Equation (38) contains terms that depend on the P-velocity and those that depend on the S-velocity. When the contributions of the P-waves are ignored altogether, the variance of the travel time change is given by

\[
\sigma_z^2 \approx \frac{\delta^2}{\beta^2} \left(0.190 + 0.285 \left(\hat{\mathbf{M}} \cdot \delta\right)^2\right) \text{ (\(\alpha\) ignored)} .
\] (40)

Note that this result is close to the variance of the travel time for a Poisson medium given in (39). For a double couple the radiated energy varies as \(\beta^{-4}\) whereas for a point force it varies as \(\beta^{-5}\). For this reason the dominance of the S-waves in coda wave interferometry is even more pronounced for a double couple than for a point force.

As a special case let us consider the application of this theory to the relative location of aftershocks on a fault. In that case, the relative source location \(\delta\) lies in the fault plane. In the coordinate system used in expression (34), the \(x, y\)-plane is aligned with the fault plane and the component \(\delta_z\) perpendicular to the fault plane is equal to zero so that

\[
\sigma_z^2 = \frac{\left(\frac{6}{\alpha^6} + \frac{7}{\beta^6}\right) \delta^2}{7 \left(\frac{2}{\alpha^6} + \frac{3}{\beta^6}\right)} \text{ (displacement in fault plane)} .
\] (41)

Just as in expression (25) the variance of the travel time perturbation is now related to the absolute value of the source displacement only.

### 5.2 An explosive source

For an explosive source, the moment tensor is given by

\[
\mathbf{M} = \begin{pmatrix} M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & M \end{pmatrix} .
\] (42)

For such a moment tensor \((\hat{\mathbf{r}} \cdot \mathbf{M})^2 = M^2 \hat{r}_i \hat{r}_j \delta_{ij} = M^2 (\hat{r} \cdot \hat{r})^2 = M^2\), and \((\hat{\mathbf{r}} \cdot \mathbf{M})^2 = (\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}) M^2 = M^2\), so that (T5) is given by

\[
\sigma_z^2 = \frac{1}{\alpha^6} (\hat{\mathbf{r}} \cdot \delta)^2 \, d\Omega
\]

\[
\frac{1}{\alpha^6} \int d\Omega
\]

(43)

Note that the terms that depend on the shear velocity \(\beta\) have disappeared; physically this is because an explosive source does not excite shear waves. The angular integration can be carried out in the same way as in
section 3, so that

\[ \sigma_r^2 = \frac{1}{3} \frac{\delta^2}{\alpha^2} . \]  (44)

This result is identical to expression (17) for acoustic waves.

6 DISCUSSION, APPLICATIONS TO SOURCE RELOCATION

As shown in the previous sections, the decorrelation of the coda waves carries information of the source displacement. The cross-correlation (3) can be applied to a number of non-overlapping time windows in the seismic coda. This gives a number of independent measurements on the source displacement that allow for a consistency check of the method and that make it possible to derive an error estimate.

Let us first consider the character of the constraints on the source displacement that follow from coda wave interferometry. For an explosive source in either an acoustic or elastic medium, expressions (17) and (44) state that the source displacement is located on a sphere with radius \( \delta^2/3v^2 \), with \( v \) the velocity of acoustic waves for an acoustic medium and the P-wave velocity for an elastic medium respectively.

For a point force in an elastic medium the constraint on the source location is slightly more complicated. It is possible to decompose the source displacement into a component parallel to the point force and a component perpendicular to the point force:

\[ \delta = \delta_{\parallel F} + \delta_{\perp F} \hat{F} . \]  (45)

With this decomposition, equation (22) can be written as

\[ \sigma_r^2 = \frac{1}{3} \frac{\delta_{\parallel F}^2}{\alpha^6} + \frac{2}{\beta^6} \left( \frac{3}{\alpha^6} + \frac{2}{\beta^6} \right) \delta_{\perp F}^2 . \]  (46)

This expression states the the source displacement is located on an ellipsoid whose symmetry axis is aligned with the point force.

For a double couple the source displacement can be decomposed in a component \( \delta_{\parallel fault} \) parallel to the fault and a component \( \delta_{\perp fault} \) perpendicular to the fault. Following equation (34), the corresponding constraint on the source displacement is given by

\[ \sigma_r^2 = \frac{6}{\alpha^6} + \frac{7}{\beta^6} \delta_{\perp fault}^2 + 2 \left( \frac{1}{\alpha^6} + \frac{2}{\beta^6} \right) \delta_{\parallel fault}^2 . \]  (47)

This expression states that the source displacement is located on an ellipsoid with a symmetry axis perpendicular to the fault plane. In the special case of aftershocks that occur on the same fault, the source displacement is in general in the plane of the fault. In that case the source displacement is constrained to be located on a circle in the fault plane:

\[ \sigma_r^2 = \frac{1}{7} \frac{6}{\alpha^6} + \frac{7}{\beta^6} \delta_{\perp fault}^2 . \]  (48)

In all these situations the source displacement is constrained by coda wave interferometry to be located on a sphere, an ellipsoid, or a circle. This constraint can be used in addition to constraints on the relative source location as inferred from the differential arrival times for the P- and S-waves. Coda wave interferometry thus adds an additional geometrical constraint to the relative source locations of multiple events.

Suppose one knows the location of \( n \) events and that one seeks the relative location of the next event. The location of that event is described by three position variables. Coda wave interferometry gives \( n \) constraints on these variables, which implies that with at least three known events one can locate the subsequent events. In practice one would probably seek an iterative approach that employs the differential arrival times of the P- and S-waves as well. Iterative techniques have been developed to find the relative location of events given constraints on the source displacement between different pairs of events (Menke, 1999; Waldhauser & Ellsworth, 2000). Coda wave interferometry provides additional constraints on the relative location of events.

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R. Snieder


Localizing change with Coda Wave Interferometry: Derivation and validation of the sensitivity kernel

Carlos Pacheco & Roel Snieder

Center for Wave Phenomena, Dept. of Geophysics, Colorado School of Mines

ABSTRACT

Previous formulations of coda wave interferometry make it possible to assess the average change of the medium, but they do not allow for the spatial localization of this change. We present a new theory for localizing changes in the medium using strongly scattered waves, and test it with numerical models for different velocity perturbations. The technique is tested for 2D scalar waves. If the diffusion approximation for the energy transport is valid, we can accurately predict the perturbation in the traveltimes given a perturbation in the velocity field. Using an integral representation for the diffuse wavefield, we derive an expression relating the mean traveltime perturbation due to a small perturbation in the slowness field. These results suggest that we can detect and localize changes in the propagation velocities of the multiple scattering medium. We study the sensitivity of our derived expression for the traveltime change in a multiple scattering medium to different perturbations in the slowness. We validate our theory using synthetic seismograms calculated with a finite-difference algorithm. For the special case of coincident source and receiver, we obtain an analytical solution of the mean traveltime change of the diffuse wavefield. In this case, we are able to relate traveltimes changes to changes in the propagation velocity that are located close (distance \( r < 4l \), where \( l \) is the scattering mean free path) to the source and receiver position. For larger distances, the perturbation is difficult to detect because the changes in the traveltime become too small to be detectable for realistic perturbations in the slowness. For the more general case of a non-coincident source and receiver, we are able to detect the change in the traveltimes for longer distances when the perturbation is located somewhere between the source and the receiver. In general, for localized slowness perturbations, our theory predicts the mean traveltime change of the diffuse wavefield in a multiple-scattering medium. Thus, we need to perform averaging over many realizations of the experiment or over different source-receiver pairs to accurately estimate the mean traveltime change associated with a localized slowness perturbation. The technique presented here can be used in many applications such as medical imaging, non-destructive testing, and reservoir monitoring, to infer temporal changes of the multiple scattering medium.

Key words: random media, multiple scattering, coda wave interferometry, time-lapse measurements

INTRODUCTION

Coda waves have been studied extensively over the past few years. The main goal has been to characterize the transport of energy in media with strong fluctuations in their material properties. Coda waves are thought to be generated by the interaction of waves with either random or periodic structures. Aki and Chouet (1975) proposed that the coda observed in earthquake seismograms is caused by scattering from the small-scale het-
erogentities in the lithosphere that are randomly distributed in space. Following that work, Frankel and Clayton (1984) performed numerical simulations of the wavefield transmitted in 2D random media using a finite-difference algorithm. Multiple scattering has been studied more recently (Tourin & Fink, 2000) in a quasi two-dimensional physical model made of thousands of steel rods randomly distributed and immersed in water. Multiple scattering theory can also be applied to the case of granular porous media in high-frequency regimes (Jia, 1999). Whatever the various scales and range of frequencies involved in different fields of physics, the main problem is to use the multiple scattered waves to estimate the medium, i.e., to solve the inverse problem of estimating medium parameters from observations of multiple scattering.

Progress has been made in the characterization of both the coherent or ballistic energy and the incoherent or diffuse energy in random media. Previous work (Aki & Chouet, 1975; Zhang & Sheng, 1999; Page & Schriemer, 1997; Margerin et al., 1998) has shown that wave transport acquires a diffusive character for strong multiple scattering. As a result, the diffusion approximation has been used with success to characterize a wide range of wave phenomena in strongly scattering media.

Little attention has been paid to the problem of imaging with coda waves, i.e. to localize changes in the medium. Photon density waves have been used to study the formation of images through random media. Jaruwatanadilok and Ishimaru (2002) applied several techniques to improve the quality of such images by reducing multiple scattering or incoherent information. Here we use a different approach, and propose a new theory for localizing the change in acoustic velocity from multiple scattering information. We take advantage of the fact that the wave transport acquires a diffusive character in a strongly scattering medium, and derive an integral representation for the diffusive wavefield in a strongly scattering medium to obtain an expression for the mean or average travelt ime change of the diffuse wavefield caused by a slowness perturbation. Thus, we derive an integral expression that relates the mean travelt ime change with perturbations in the slowness field for any source and receiver configuration. With this representation, we model the mean travelt ime change caused by a slowness perturbation. This constitutes the forward problem; the purpose of this work is to test this new theory with synthetic seismograms computed by finite-differences for different perturbations in the slowness field. The inverse problem, estimating the change in the slowness given the measured travelt ime perturbations, will be addressed in future work.

Recently, coda waves have been used to study the temperature dependence of the seismic velocity in granite (Snieder et al., 2002) using a technique called Coda Wave Interferometry. They found that one can estimate small changes in medium velocity from the multiple scattered waves by measuring the time shift between the unperturbed and perturbed wavefields. The technique takes advantage of the great sensitivity of multiply scattered waves to minute changes in the medium. We extend their work with the modifications to the theory needed to spatially localize the change in the medium.

While Snieder et al. (2002) worked with homogeneous changes in the multiple scattering medium, we introduce localized perturbations in the velocity field and use finite-difference simulations in a 2D velocity medium with strong velocity fluctuations to measure the travelt ime changes. We then assess the validity of our theory by comparing the observed and predicted travelt ime change.

This paper is divided into seven sections. In sections 1 and 2 we review the diffusion approximation for the averaged intensities in a multiple scattering medium and derive an expression for the mean travelt ime change using an integral representation for the diffuse intensities. We also discuss the sensitivity of the derived expression to various parameters, such as distance, time and the diffusion constant. In section 3, we describe the parameters used in the finite-difference calculation of the synthetic seismograms from which we estimated parameters in multiple scattering media. In section 4, we consider the estimation of the propagation velocity in a multiple scattering medium and its relation to the effective velocity of the medium. In section 5, we show estimates of the effective velocity and of the diffusion constant with finite-difference synthetic seismograms. In section 6, we introduce small perturbations in the slowness field and apply the theory developed in section 2 to model the mean travelt ime change and compare it with the actual or measured travelt ime changes from the synthetic seismograms. Finally, in section 7, we discuss the main results and conclusions from our work. For simplicity, the theory and numerical examples are shown for a 2D medium. However, the extension to 3D can be made with some simple substitutions.

1 DIFFUSION APPROXIMATION OF ENERGY TRANSPORT

The transport of energy through a random medium has attracted considerable attention in numerous fields of physics, such as astrophysics, optics, acoustics, solid state physics and heat conduction. In any of these fields, one can generate a pulse of energy that propagates through the medium with a certain intensity $P(r, t)$. In a two-dimensional medium of infinite extent, and in the long time limit (Paaschens, 1997), the intensity can be approximated well by the solution of the diffusion equation,

$$P(r, t) = \frac{1}{4\pi Dt} \exp \left[ -\frac{r^2}{4Dt} - \frac{ct}{l_a} \right], \quad (1)$$

where $P(r, t)$ is the intensity at position $r$ and time $t$, $D$ is the diffusion constant, $c$ is the propagation velocity, $t$ is time, and $l_a$ is an effective acoustic wavelength.
where $r$ is the distance to the source and $D = cl/2$ is the diffusion constant, $l$ is the mean free path (average distance between scatterers), $c$ is the transport or energy velocity and $l_a$ is the absorption length (attributable to intrinsic attenuation). For the purposes of this work, $l_a^{-1}$=0 because we ignore intrinsic attenuation. Eq. (1) represents the temporal evolution of the mean intensity after the waves have scattered multiple times from small-scale heterogeneities. Tourin and Fink (2000) have shown that the diffusion equation describes the propagation of the average intensity in a multiple scattering medium. In their approach, the average intensity is treated as the probability of traveling a distance $r$, varying with time, of a particle undergoing a random walk.

Large deviations from the diffusion approximation can be expected at any $r$, for times less than or close to the arrival time of the coherent or ballistic wave ($t \leq r/c$). Therefore, in order to use the diffusion approximation, we must be in the long-time regime where multiple scattering dominates wave propagation (Paaschens, 1997). For times $t \approx r/c$ more accurate expressions for the probability density of intensity as a function of position have been proposed based on the Boltzmann equation, also known as the Radiative Transfer Equation, of which the diffusion equation is the long-time limit (Chandrasekhar, 1960).

When the diffusion approximation is accurate, we can characterize the scattering with two parameters: the diffusion constant $D$, and the transport or energy velocity $c$. These two parameters are related to the mean free path by

$$D = \frac{cl}{d},$$

where $d$ is the dimension of the problem ($d=2$ in 2D media). In general, these quantities vary with frequency, but we assume that this frequency dependence can be ignored for band-limited signals. In sections 4 and 5 we show independent estimates of both the propagation velocity and the diffusion constant; these estimated values are used to calculate the mean traveltime change of the diffuse wavefield caused by a slowness perturbation in section 6.

2 TRAVELTIME PERTURBATIONS IN THE DIFFUSION REGIME

2.1 Diffusion and Coda Wave Interferometry

In order to localize changes in the scattering medium, we need to derive an expression relating the traveltime change with the localized velocity perturbations in the medium. Snieder et al. (2002) introduced coda wave interferometry whereby multiply scattered waves are used to detect temporal changes in the medium by using the scattering medium as an interferometer. They found that, for a small perturbation in the velocity, estimates of this perturbation can be obtained from multiply scattered waves by a time-windowed cross-correlation. The assumption used is that when the temporal change is weak, the influence of this change on the geometrical spreading and on the scattering coefficient can be ignored, and the dominant effect on the wavefield arises from the change in the traveltime of the wave that travels along each path. If, in contrast, the perturbation is strong, changes in the scattering coefficient and in the trajectories followed by the multiply scattered waves affect both the amplitude and the phase of the scattered wavefield. In this case, the unperturbed and perturbed waveforms become highly uncorrelated and we can not estimate the traveltime change.

For small and homogeneous perturbations in the velocity, the time-windowed cross-correlation of the unperturbed and perturbed wavefields gives us an estimate of the time shift between the two waveforms. With that time shift $\tau$ we can estimate the velocity perturbation with the equation (Snieder et al., 2002),

$$\frac{\delta c}{c} = -\frac{\tau(t,T)}{t},$$

where $\tau$ is the time shift of the time-windowed cross-correlation centered at time $t$ with window size $T$, and $c$ is the propagation velocity in multiple scattering media. Time change $\tau$ is weighted by the energy of the wave along each path. The extreme sensitivity of coda waves to small changes in the medium was used to estimate the nonlinear dependence of the velocity of granite with temperature (Snieder et al., 2002). Here we extend the theory to spatially localize changes in the velocity field.

2.2 An Integral Representation for the Traveltime of the Diffuse Wavefield

In this section we summarize the derivation of the mean or average traveltime change of diffuse waves caused by slowness perturbations. We start from the assumption that the averaged intensities in a multiple-scattering medium can be modeled as a diffusion process. Thus, the intensity at a distance $r$ due to an intensity impulse at the origin is given by $P(r,t)$, as defined in Eq. (1). $P$ represents the temporal evolution of the intensities. In random walk theory, $P$ also represents the joint probability of time and position of a particle arriving at location $r$ at a given time $t$ (Roepstorff, 1994). If, at time $t=0$, a normalized intensity impulse is generated at the source, the total energy within some region $V$ at some later time is given by the integral

$$W(V,t) = \int_V P(r,t)dV(r),$$

Integration over all space gives the total energy of the system, which by the normalization is $W(t) = 1$. The quantity $W(V,t)$ is equal to the probability of a
particle on a random walk of visiting the volume region $V$ in a time $t$.

We now consider the probability that a random walk particle leaves a source at $s$ at time $t = 0$, visits a volume element $dV$ at $r'$ at time $t'$ and arrives at $r$ at time $t$ as is depicted in Figure 1. This probability is equal to the product of two probabilities: the probability of the particle of going from $s$ to $r'$ in a time $t'$, and the probability of going from $r'$ to $r$ in a time $t - t'$, i.e.,

$$P([r', s] \cap [r, r', t - t']) = P(r', s, t') \times P(r, r', t - t'),$$

(5)

where the symbol $\cap$ means intersection. Thus, the probability of going from $s$ to $r'$ and from $r'$ to $r$ is equal to the product of the two probabilities. Using other words, the two events (particle going from $s$ to $r'$ and from $r'$ to $r$) are independent. This is a common property of random motion, i.e., it is a memoryless process. A direct consequence of this property is the Chapman-Kolmogorov equation (Roepstorff, 1994),

$$\int_V P(r, r', t)P(r', r'', t')dV = P(r, r'', t + t').$$

(6)

If we integrate the right side of Eq. (5) over all space, and apply the Chapman-Kolmogorov property, the intensity at the receiver $r$ at time $t$ due to an impulse source at $s$ at time $t = 0$ is equal to

$$P(r, s, t) = \int_V P(r', s, t')P(r, r', t - t')dV(r'),$$

(7)

where $P(r, s, t)$ is the intensity at receiver $r$ at time $t$ due to an intensity impulse at the source $s$ at time $t = 0$. $P(r', s, t')$ is the intensity at time $t'$ at $r'$ due to a source at $s$ activated at time $t = 0$, and $P(r, r', t - t')$ is the intensity at $r$ at time $t$ due to an impulse source at $r'$ on a time $t - t'$. Thus, the probability of a particle undergoing random motion of going from $s$ to $r$ on a time $t$, depends only in the distance between $s$ and $r$ and the traveltime $t$.

We can identify $P(r, r', t - t')$ in Eq. (7) as the Green’s function $G(r, r', t - t')$ which describes the intensity at the receiver $r$ at a time $t - t'$ due to a normalized impulse source located at $r'$ at time $t'$. Thus, we can write Eq. (7) as follows:

$$P(r, s, t) = \int_V P(r', s, t')G(r, r', t - t')dV(r').$$

(8)

This equation holds for all times $0 < t' < t$ (Feynman, 1949), and thus the integral is independent of $t'$. If we integrate both sides of Eq. (8) over time $t'$ we obtain:

$$tP(r, s, t) = \int_0^t \int_V P(r', s, t')G(r, r', t - t')dt'dV(r'),$$

(9)

where we can identify

$$P(r, s, t) = \int_0^t \int_V P(r', s, t')G(r, r', t - t')dt'dV(r'),$$

(10)

We have obtained in Eq. (10) an integral representation for the time $t$ at $r$ of the diffuse intensities due to a source at $s$ at time $t = 0$. As $P(r, s, t)$ represents the mean intensities of the multiple scattering wavefield approximated by the diffusion equation, this time $t$ corresponds to the mean traveltime of the diffuse wavefield in a strongly scattered media. This diffuse wavefield is nothing more than the multiply-scattered wavefield. If we define the kernel $K(r', t) = P(r', s, t')G(r, r', t - t')dt'$ as the temporal convolution of the intensities at $r'$ with the Green’s function $G(r, r', t - t')$. If we divide both sides of Eq. (9) by $P(r, s, t)$, we arrive at the following integral representation for the traveltime of the diffuse wavefield:

$$t = \frac{1}{P(r, s, t)} \int_V P(r', s, t')G(r, r', t - t')dV(r').$$

(11)

In Eq. (11) the kernel $K(r', t)$ represents the traveltime density function. If we integrate $K$ over all volume $V$ we obtain the mean traveltime $t$ of the waves that travel along the multiple-scattering paths, starting at the source at $s$ at $t = 0$ and arriving at the receiver at $r$ in a time $t$. The diffuse wavefield is obtained after summing the contributions from many multiple-scattering paths.

Next, we perturb the slowness in the medium to obtain an expression for the mean traveltime change of the diffuse wavefield caused by the slowness perturbation. We work under the assumption that the perturbation is small so that the scattering coefficient doesn’t change, and the waveform for each scattering path stays approximately the same. Also, the scattering paths remain unchanged so that the only difference between the unperturbed and the perturbed field is the traveltime. The traveltime change for each scattering path depends linearly on the slowness perturbation $\delta s/s$. Inserting a slowness perturbation $\delta s/s$ into Eq. (11) we obtain:
where $\tau(t)$ is the mean traveltime change of the multiple-scattering paths arriving at the receiver at time $t$ of the diffuse wavefield due to the slowness perturbation $\delta s/s$. Thus, the mean traveltime change of the diffuse wavefield is obtained after weighting the traveltime density function $K$ with the slowness perturbation $\delta s/s$ and integrating over all space. Eq. (13) is in the form of a standard linear inverse problem, where $\tau(t)$ is the data and $\delta s/s(r')$ is the model.

To calculate the mean traveltime change for a particular source and receiver configuration we need only to integrate the kernel $K$ weighted by the slowness perturbation $\delta s/s$ over all space. The kernel $K$ is obtained by convolving the intensities at the receiver $P$ with the Green’s function $G$ and dividing the result by the intensities at the receiver $r$ at time $t$ due to a source at $s$ at $t=0$. For source and receiver at different locations, the time convolution $P*G$ does not have an analytical solution so it must be evaluated numerically. For the special case of coincident source and receiver, an analytical solution for this convolution can be obtained. In Appendix A we adapt our theory for the 2D case and calculate $K$ for coincident source and receiver ($r = s = 0$) to obtain:

$$K(r', t) = \frac{1}{2\pi D} \exp \left[ -\frac{r'^2}{2Dt} \right] K_0 \left( \frac{r'^2}{2Dt} \right), \quad (14)$$

where $r'$ is the distance between the source-receiver location and the slowness perturbation, $K_0$ is the modified Bessel function of the second kind, and $D$ is the diffusion constant. If we insert Eq. (14) into Eq. (13) and integrate over area instead of volume we get:

$$\tau(t) = \frac{1}{2\pi D} \int_A e^{-\frac{r'^2}{2Dt}} K_0 \left( \frac{r'^2}{2Dt} \right) \delta s(s) dA(r'). \quad (15)$$

In Eq. (15), we have obtained an expression for the mean traveltime change ($\tau$) of the diffuse wavefield caused by the slowness perturbations ($\delta s/s$), for coincident source and receiver.

In general, for a given perturbation in slowness, we calculate the mean traveltime change $\tau$ for any source and receiver configuration evaluating the integral in Eq. (13). This theory is validated with numerical examples in section 6.
2.3.2 Different Source and Receiver Location

When the source and receiver are not coincident, we can not obtain an analytical solution for the kernel $K$; instead, $K$ must be obtained numerically. We calculate $K(r_s, r_g, t)$ at $t=1$ s for source and receiver separated by a distance of 3000 m, using the diffusion constant value of the previous section. In Figure 5 we plot the value $K$ for different source and receiver location, which shows that $K$ varies slowly in the source-receiver direction (dashed line), while it decreases faster in all other directions. While for coincident source and receiver the kernel $K$ does not depend on the direction between the perturbation and the source-receiver location, for non-coincident source and receiver $K$ varies with direction. This variability with direction can be exploited in future inversion schemes to localize velocity changes using traveltine change information from multiple source and receiver locations.

Another important feature of $K$ for non-coincident source and receiver, is that it is less sensitive to distance, i.e., the area where $K$ is non-vanishing is bigger than in the coincident source and receiver case. As can be seen in Figure 6, where we plot the value of $K$ along the direction depicted by the dashed line in Figure 5, the kernel takes more time for the diffuse waves to visit the more distantly located slowness perturbation and come back to the source. Thus, the change of the slope of $K(r, t)$ with time is an indication of the distance between the perturbation and the source-receiver location.
K does not vanish for perturbations located in the area between the source and receiver. If the perturbation is located somewhere between the source and the receiver, the probability of multiple scattering paths traveling through the perturbed region is large when compared to the probability for coincident source-receiver.

Note that the integration kernel \( K \) decreases rapidly to the left of the source location and to the right of the receiver location (see Figures 5 and 6). Therefore, only changes located within the ellipses of constant \( K \), as shown in Figure 5, are detectable for a given source and receiver configuration. Velocity changes located outside the area defined by these ellipses will have a small contribution to the mean traveltime change of the diffuse wavefield. This property will be useful in the inverse problem formulation.

### 3.1 Multiple-Scattering Medium

To generate synthetic seismograms for our study of multiple scattering we use a fourth-order 2D acoustic finite-difference code that propagates a finite-duration pulse through a specified velocity field. Following Frankel and Clayton (1986), we model the 2D velocity field as a constant-background model with added random velocity fluctuations that scatter the waves. The velocity fluctuations are characterized by a Gaussian autocorrelation function with correlation distance \( a \), and have zero mean and standard deviation \( \sigma \) (see Figure 7 for a representation of the velocity model). The autocorrelation function of the velocity fluctuations \( v_r \) has the form:

\[
< v_r(\mathbf{r}), v_r(\mathbf{r} + \mathbf{r}'> > = \sigma^2 \exp \left(-\frac{r^2}{a^2}\right) .
\]

The total velocity field can be decomposed as

\[
v(\mathbf{r}) = v_0 + v_r(\mathbf{r}),
\]

where \( v_0 \) is the background velocity and \( v_r \) are the random velocity fluctuations with a Gaussian autocorrelation function.

### 3.2 Experiment Setup

The synthetic seismograms were created by transmitting a band-limited pulse as shown in Figure 8 in the 2D velocity medium specified above. To ensure strong scattering, we created a velocity field with a mean velocity of 6000 m/s and a standard deviation of 1500 m/s. The space discretization was chosen to be 20 m in both the horizontal and the vertical direction. Our velocity model has an extent of 1000 \( \times \) 1000 samples; this corresponds to a model of 20 km \( \times \) 20 km. The source is placed in the middle of the model and the receivers are located...
both at the source location and in circular arrays at different distances from the source. The time step used in the finite-difference calculations was 0.8 ms. We calculated the seismograms with traveltime limited to 3 s in order to avoid reflections from the boundaries. Both the space and time intervals were chosen in order to ensure stability and accuracy (Alford & Kelly, 1974) in the finite-difference calculations. For a fourth order finite difference algorithm we have at least 6 grid points per wavelength, which ensures enough accuracy. The value of the correlation length \( a \) in the velocity model was set equal to 40 m, which is small when compared to the dominant wavelength of our signal \( \lambda_d = 240 \) m. Thus, the resultant medium has a component that varies randomly over a length scale that is smaller than the seismic wavelength, and we are between the Rayleigh \( (ka < 1) \) and the Mie scattering regime \( (ka \approx 1) \) (Herraiz, 1987). The employed standard deviation of the velocity fluctuations (1500 m/s) makes the scattering strong.

Figure 9 depicts the typical configuration used for the study, and Figure 10 shows 100 synthetic seismograms computed at a distance of 250 m (left) and 3000 m (right) from the source. Note the strength of the multiple scattered arrivals after the highly attenuated ballistic arrival, especially for a distance of 3000 m from the source. We computed the synthetic seismograms for a receiver located at the source and for arrays of 100 receivers around the source with distances to the source ranging from 250 m to a maximum of 4000 m. In section 5, data from the array of receivers are used to compute the average wavefield and intensity, as well as the transport velocity. As the medium has statistically homogeneous properties, each receiver in the circular array can be treated as a seismogram from a different realization of the random velocity model.

4 EFFECTIVE SLOWNESS IN A MULTIPLE SCATTERING MEDIUM

Page and Sheng (1996) found that when the scattering is strong, the group and phase velocities can become significantly smaller than the material velocities; this is a consequence of the slowing down of the ballistic pulse caused by multiple scattering. In a multiple-scattering medium, energy propagates with an effective velocity that is the energy or transport velocity \( c_{\text{w}} \), which is the same as the velocity that enters Eq. (2). This effective velocity, \( c_{\text{w}} \), is either the velocity of the coherent signal, for non-resonant scattering, or the velocity of the envelope of the trace, i.e., the group velocity for resonant scattering. In the low-frequency or Rayleigh scattering regime (Tregoures & van Tiggelen, 2002), scattering is not resonant because the wavelength is much larger than the scatterer size.
We obtain the effective velocity in a medium with random velocity fluctuations $v_r$ using the effective medium theory approximation (Tatarski, 1963; Sheng, 1995). Let us start with the the expression for the velocity field with fluctuations $v_r$ as defined in Eq. (17),

$$v(r) = v_0 + v_r = v_0(1 + \epsilon(r)), \quad (18)$$

where $v_0$ is the background velocity and $\epsilon(r)$ are the normalized random velocity fluctuations with correlation length $a$ and zero mean. To first order, we can approximate the slowness field resulting from calculating the inverse of the velocity field as

$$s(r) = s_0(1 - \epsilon(r)); \quad (19)$$

with $s_0$ as the mean value of the slowness field. We now consider the Helmholtz equation in 2D:

$$\nabla^2 \Psi(r, w) + k_0^2(1 - \epsilon(r))^2 \Psi(r, w) = \delta(r), \quad (20)$$

where $k_0 = \omega s_0$ is the homogeneous reference medium wave number and $\Psi(r)$ is the wavefield propagating through the random medium. Eq. (20) can be formulated as a random integral equation (Frisch, 1965). By using the first-order smoothing method (Frisch, 1965), we obtain a solution to the mean field $\bar{\Psi}$ radiated by a point source in a random medium, which satisfies the equation :

$$(\nabla^2 + k_0^2)\bar{\Psi}(r', \omega) = k_0^4 \int_A G_0(r', \omega) \Gamma(r') \bar{\Psi}(r', w) \times d^2 r' = \delta(r), \quad (21)$$

where $G_0(r, \omega)$ is the free space Green’s function and $\Gamma(r)$ is the auto-correlation function of the velocity fluctuations. It is essential for the validity of this approximation that $\epsilon k_0^2 a^2$ is small; as $\epsilon < 1$, this is automatically satisfied if $k_0 a << 1$, i.e., if the correlation length is small compared to the wavelength. Eq. (21) reduces to the Helmholtz equation with an “effective” wave number, that from the free-space wave number :

$$(\nabla^2 + k_{eff}^2)\bar{\Psi}(r, w) = \delta(r), \quad (22)$$

$$k_{eff}^2 = k_0^2 - k_0^4 \int_A G_0(r, w) \Gamma(r) d^2 r. \quad (23)$$

Using Eq. (23), we can obtain the effective wavenumber and from it the effective slowness for any random slowness field with known autocorrelation function $\Gamma(r)$. This effective slowness is just the energy or transport slowness, i.e., the slowness with which the coherent pulse propagates in the multiple scattering medium.

5 DIFFUSE INTENSITIES AND EFFECTIVE SLOWNESS FROM FINITE-DIFFERENCE SYNTHETICS

5.1 Diffuse intensities computed from finite-difference synthetics

In this section we obtain an estimate of the diffusion constant $D$ from the averaged intensities in a multiple scattering medium. This diffusion constant will be used to calculate the integration kernel $K$. For propagation distances longer than the mean free path, we study the propagation of the averaged intensity in the medium, which, as we mentioned in section 2.1, obeys the diffusion equation.

The time evolution of the averaged intensity of the synthetic seismograms is determined by squaring the envelope of the calculated waveforms for each distance. After that, we perform averaging over the 100 receivers located around the source at a distance $r$. We assume that because our 2D medium has homogeneous statistical properties, averaging over different receiver locations is equivalent to averaging over different realizations of the random velocity model. This gives us the time evolution of the intensities for each source-receiver offset. We then fit the averaged intensities with the diffusion curve to obtain the value of the diffusion constant for each source-receiver offset. The estimated diffusion constant is obtained after averaging the values for the diffusion constant calculated for all the source-receiver offsets. The resultant estimated value for the diffusion constant is $D = (5.78 \pm 0.462) \times 10^4$ m$^2$/s. Figure 11 shows a good match between the averaged intensity and that predicted with this value of the diffusion constant, for 500, 1500, 2500 and 3500 m source-receiver offset.
5.2 Effective Slowness computed from the synthetic seismograms.

In section 4, we defined the effective velocity as the velocity with which the coherent energy travels in a multiple scattering medium. When multiple scattering is present, we estimate this velocity by measuring the speed with which the coherent pulse propagates through the multiple scattering medium. This coherent or ballistic pulse can be obtained by averaging the wavefield over different realizations of disorder. With our finite-difference synthetics we perform this averaging over different receiver positions in the circular array. Then, calculate the envelope of the average wavefield, and square this envelope to obtain the coherent intensities for each source-receiver offset \( r \).

Figure 12 shows the total wavefield for 10 different receivers located 3000 m away from the source. Note the low amplitude of the ballistic arrival when compared to the multiple scattering events. In order to extract the coherent pulse we average over the waveforms and calculate the envelope of the averaged wavefield. The result of the averaging is shown in Figure 13, where we can easily identify the ballistic pulse of energy as the first arrival. Notice that the incoherent energy has been attenuated by the averaging process, but has not been eliminated, so more averaging is needed to remove the diffuse wavefield completely.

To estimate the speed with which the coherent pulse propagates, we measure the traveltime of the peak of the envelope of the coherent pulse for each distance and perform a linear regression of the moveout of that peak as shown in Figure 14. The slope gives us the group velocity \( v_{\text{group}} = 5415 \pm 110 \) m/s. We can compare this value with the effective velocity calculated using Eq. (23). For our velocity medium with a correlation length \( a = 40 \) m, mean velocity \( v_0 = 6000 \) m/s and standard deviation \( \sigma = 1500 \) m/s the calculated effective velocity for the dominant frequency of 25 Hz is \( v_{\text{eff}} = 5407 \) m/s. Thus, the velocity calculated with effective medium theory agrees well with the transport velocity measured on our synthetic seismograms.

Note that the slowing down of the velocity of propagation is a direct consequence of the multiple scattering. Indeed, the measured group velocity \( (v_{\text{group}} = 5415 \) m/s) is significantly less than the background velocity \( (v_0 = 6000 \) m/s). Table 1 shows a comparison between the calculated effective velocity \( (v_{\text{eff}}) \) and the measured group velocity for different values of the correlation distance \( a \), and \( \epsilon \) (the fractional variation of the slowness fluctuations with respect to the background slowness as
5.3 Perturbations of the Effective Slowness

As mentioned in section 4, we calculate the slowness field as the inverse of our fluctuating velocity field. As a result, we obtain the random slowness field with fluctuations equivalent to those in the velocity field. Let this slowness field be the unperturbed slowness field. We introduce a constant slowness perturbation in our slowness field. The perturbed slowness field can be represented as

\[ s_p = s_{unp} + \delta s_0, \]

where \( s_{unp} \) is the unperturbed slowness field, \( s_p \) is the perturbed field, and \( \delta s \) is the slowness perturbation. The change in the effective slowness is proportional to the change in the mean slowness. The constant of proportionality, which can be obtained from the effective medium theory, depends on the specific parameters of the random slowness fluctuations.

From Eq. (23), we know that the effective wavenumber depends on the mean wavenumber in the following way:

\[ k_{eff}^2 = k_0^2 + F(k_0), \]

where \( F(s_0) \) is the function that describes the dependence of \( k_{eff} \) on \( k_0 \). If we introduce a mean slowness perturbation, it will produce a respective effective wavenumber perturbation. We obtain the perturbed effective wavenumber after adding the effective slowness perturbation to the effective wavenumber, i.e.,

\[ k_{eff}^p = k_{eff} + \delta k_{eff} = k_{eff} + \frac{\delta k_{eff}}{\delta k_0} \delta k_0, \]

where \( k_{eff}^p \) is the perturbed effective wavenumber. If we subtract to Eq. (26) the unperturbed effective wavenumber \( k_{eff} \) and divide by frequency, we obtain the effective slowness perturbation:

\[ \delta s_{eff} = \frac{\delta k_{eff}}{\delta k_0} \delta s_0. \]  

We have just shown in Eq. (27) that the effective slowness perturbation is proportional to the mean slowness perturbation. For a given random slowness field the effective slowness in the multiple-scattering medium is constant and is related to the background or mean slowness in the following way:

\[ s_{eff} = A s_0. \]  

Combining Eqs. (28) and (27) we obtain the relation between the fractional effective slowness perturbation \( \delta s_{eff}/s_{eff} \) and the fractional mean slowness perturbation \( \delta s_0/s_0 \):

\[ \frac{\delta s_{eff}}{s_{eff}} = \frac{\delta k_{eff}/k_0 \delta s_0}{A s_0} = C \frac{\delta s_0}{s_0}. \]

Thus, the fractional perturbation in the effective slowness is directly proportional to the fractional perturbation in the mean slowness. In section 6.1 we calculate the synthetic seismograms after perturbing the slowness field and, with coda wave interferometry, estimate the value of the proportionality constant \( C \) for different homogeneous mean slowness perturbations.

6 ESTIMATING EFFECTIVE SLOWNESS PERTURBATIONS WITH CODA WAVE INTERFEROMETRY

In section 5.2 we estimated the effective velocity in a multiple scattering medium from measurements of the speed of the coherent pulse in the finite-difference synthetics and from Eq. (23). We found that for a medium with homogeneous statistical properties, this effective velocity is constant, and that for small changes in the slowness, the fractional change in the effective slowness \( \delta s_{eff}/s_{eff} \) is proportional to the fractional change in the mean slowness \( \delta s_0/s_0 \). Therefore, we can estimate mean slowness perturbations \( \delta s_0/s_0 \) directly from the effective slowness perturbations \( \delta s_{eff}/s_{eff} \) using Eq. (29).

We now test the theory developed in section 2.2, i.e., we calculate the mean traveltime change with Eq. (15) for different perturbations in the mean slowness \( s_0 \) and compare them with estimated traveltime changes. We have just shown in Eq. (27) that the effective slowness perturbation is proportional to the mean slowness perturbation.

### Table 1

<table>
<thead>
<tr>
<th>( a )</th>
<th>( \epsilon )</th>
<th>( v_{eff}(m/s) )</th>
<th>( v_{group}(m/s) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>.25</td>
<td>5407</td>
<td>5415±110 m/s</td>
</tr>
<tr>
<td>40</td>
<td>.15</td>
<td>5750</td>
<td>5783±40 m/s</td>
</tr>
<tr>
<td>80</td>
<td>.25</td>
<td>5480</td>
<td>5430±180 m/s</td>
</tr>
<tr>
<td>80</td>
<td>.15</td>
<td>5807</td>
<td>5850±55 m/s</td>
</tr>
</tbody>
</table>

### Table 2

<table>
<thead>
<tr>
<th>( \frac{\delta s}{s} )</th>
<th>( \frac{\delta s_{eff}}{s_{eff}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>.01</td>
<td>.0080±.0003</td>
</tr>
<tr>
<td>.02</td>
<td>.0159±.0005</td>
</tr>
<tr>
<td>.03</td>
<td>.0239±.0008</td>
</tr>
<tr>
<td>.04</td>
<td>.0319±.0012</td>
</tr>
<tr>
<td>.05</td>
<td>.0398±.0020</td>
</tr>
<tr>
<td>.07</td>
<td>.0550±.0030</td>
</tr>
<tr>
<td>.10</td>
<td>.0791±.0050</td>
</tr>
</tbody>
</table>

_Coda Wave Interferometry_
the effective slowness for each time window. The estimation process is exemplified in Figure 15. The left panels show both the perturbed and the unperturbed waveforms (top) and their cross-correlation for a specific time window (bottom). The location of the maximum in the cross-correlation corresponds to the time lag between events in the coda of the two seismograms. By repeating this analysis in different time windows of the coda we obtain the time lag as a function of traveltime, from which we obtain estimates for \( \delta s_{\text{eff}}/s_{\text{eff}} \), shown on the right of Figure 15. Note that the slowness perturbation estimated from coda wave interferometry \( \delta s_{\text{eff}}/s_{\text{eff}} \) differs from the mean slowness perturbation \( \delta s_0/s_0 \). They differ by the proportionality constant \( C \) in Eq. (29).

We calculate the slowness perturbation, using coda wave interferometry, for different values of the mean slowness perturbation and the results are shown in Table (2). We estimate the proportionality constant \( C \) from the effective slowness perturbations \( \delta s_{\text{eff}}/s_{\text{eff}} \) and obtain the value of \( C=0.80 \pm 0.03 \). Figure 16 shows as an example, the predicted versus actual traveltime changes for a 2% constant slowness perturbation. The predicted traveltime change includes the proportionality constant \( C=0.80 \) we obtained from coda wave interferometry. Note the good match between observed and predicted traveltime changes.

For an homogeneous perturbation in the slowness field, the traveltimes of all scattering paths are modified by an amount equal to the product of the traveltime of the particular scattering path and the effective slowness perturbation \( \delta s_{\text{eff}}/s_{\text{eff}} \). In other words, each multiple-scattering path has traveled through the slowness perturbation in all of its trajectory. For the case of a constant perturbation, the traveltime change of the scattered waves recorded at one receiver is equal to the mean traveltime change of the diffuse wavefield. Therefore, no averaging is needed in order to estimate the mean traveltime of the diffuse wavefield. We will see in the next section that for a localized perturbation in the slowness, this is no longer true; thus, we need to perform averaging over different realizations of the experiment in order to estimate the mean traveltime change of the diffuse wavefield.

6.1.2 Case 2: Localized Perturbation in the Slowness

In the previous case of a constant change in the slowness for the entire medium, the relation between the time lags and the traveltime is linear as has been measured in the previous work (Snieder et al., 2002). If, instead, we introduce a localized perturbation in the slowness, we evaluate the integral in Eq. (15) numerically.

We consider a localized perturbation within an area in the shape of a ring, centered on the common source and receiver location (see Figure 17). The strength of the slowness perturbation \( \delta s/s \) is constant within the

---

**Figure 16.** Predicted (solid) versus actual (asterisks) traveltime changes for a 2% constant perturbation in the slowness. The source and receiver are coincident.
Figure 15. Estimation of the time lag by cross-correlation of unperturbed and perturbed wavefield (left) and slowness perturbation estimates from coda wave interferometry using Eq. (3) (right). The true slowness perturbation $\delta s/s$ is 0.01.

Figure 17. Localized perturbation in the Slowness in the shape of a ring (dotted area) centered on the source location. Tapers with length of 500 m are applied outside the ring to avoid diffractions from the borders of the perturbation.

Figure 18. Theoretical (solid) versus actual (asterisks) traveltime change for a 2% localized slowness perturbation. The source and receiver are coincident.

ring and, tapers are applied outwards the ring to avoid diffractions generated from the borders. We calculate the perturbed seismograms for this localized slowness perturbation with strength $\delta s/s=0.02$. Next, we measure the traveltime change of the perturbed wavefield with coda wave interferometry, and compare it with the mean traveltime changes calculated numerically after evaluating the integral in Eq. (15).

Figure 18 shows the theoretical and the actual traveltime changes for the localized slowness perturbation. Note that for traveltimes $t$ less than 1.0 s the mean traveltime change is too small to be detectable. After $t=1.0$ s the mean traveltime change is big enough to be detectable, but the measured traveltime changes show large deviations from the theoretical values. This deviations are due to the fact that we have only one realization of the diffuse wavefield and, therefore, the traveltime change from one realization differs from the mean traveltime change of the diffuse wavefield. In order to estimate the slowness perturbation from the measured traveltime change, we need to perform averaging over many realizations of the diffuse wavefield that propagates in the medium with random fluctuations. With
this average we will have an estimate of the mean traveltime change of the diffuse wavefield caused by this localized slowness perturbation.

6.2 Different Source and Receiver Location

6.2.1 Case 1: Constant Perturbation in the Slowness

When the effective slowness perturbation $\delta s_{eff}/s_{eff}$ is constant, the term can be taken out of the integral in Eq. (13), and we obtain as a result that the traveltime change is equal to the product of the slowness, the term can be taken out of the integral in Eq. (13), and we obtain as a result that the traveltime constant, the term can be taken out of the integral in Eq. (13), and we obtain as a result that the traveltime change of the diffuse wavefield in a multiple scattering medium.

6.2.2 Case 2: Localized Perturbation in the Slowness

We now consider the effect of a localized slowness perturbation, as used in section 6.1.2, on the traveltimes of the diffuse wavefield for different source and receiver location. We calculate the finite-difference synthetics after introducing the localized slowness perturbation used in section 6.1.2. The ring is again centered on the source location, and the synthetics seismograms are computed for arrays of 100 receivers around the source for two source-receiver offsets: 2000 and 4000 m.

Figure 19 shows the estimated versus theoretical traveltime changes for one receiver located 2000 m and another located 4000 m away from the source. Note the fluctuations of the estimated traveltime changes, for one receiver, around the theoretical mean traveltime change. Again, this fluctuations arise from the fact that the traveltime change for one realization is different than the mean traveltime change of the diffuse wavefield. In order to estimate the mean traveltime change from the synthetic seismograms, we measure the traveltime change for all 100 receivers in the circular array. Then, we average the resulting 100 traveltime change curves to obtain the mean traveltime change of the diffuse field. Because our random medium is statistically homogeneous and the slowness perturbation is symmetric around the source, the 100 seismograms recoded in the circular array around the source are equivalent to 100 realizations of the diffuse wavefield.

The result of the averaging of the traveltime changes for receivers located 2000 m and 4000 m away from the source is shown in Figure 20. After averaging, the estimated mean traveltime changes show better agreement with the theoretical mean traveltime changes. Thus, for this example, we have shown that our theory accurately predicts the mean traveltime change of the diffuse wavefield in a multiple scattering medium.

7 DISCUSSION AND CONCLUSIONS

For a homogeneous change in the slowness field, the size of the effective slowness perturbation is directly proportional to that of the slowness perturbation. The effective or transport speed is the speed with which the coherent energy travels in the multiple-scattering medium.

Using the results of Tatarski (1963) and Sheng (1993) from effective medium theory, we estimate the approximate effective slowness for a medium with strong fluctuations in the slowness field described by its autocorrelation length $\alpha$ and its standard deviation $\sigma$. The expression obtained for the effective velocity helped support the results of our estimation of velocity perturbations with coda wave interferometry.

We have developed a theory that relates the mean traveltime changes of the diffuse wavefield to localized perturbations of the effective medium for 2D acoustic waves. This theory was developed by means of the representation theory for the diffuse wavefield in a multiple-scattering medium. One of the main results from our theory is that the mean traveltime changes can be obtained by integrating the slowness perturbations weighted by the kernel $K(r,t)$ over the whole area where multiple scattering occurs. The kernel $K(r,t)$ describes the relation of the mean traveltime changes of the diffuse wavefield with the diffusion constant, time, and the distance to the perturbation. The estimated mean traveltime changes from the finite-difference simulations indicate that the theory accurately predicts the time evolution of the mean traveltime perturbations.

The mean traveltime change of the diffuse wavefield depends on various factors. For small slowness perturbations, the traveltime change depends linearly on the perturbation. We found that as long as the waves behave diffusively, the traveltime change is not overly sensitive to the value of the diffusion constant. For a coincident source and receiver, the mean traveltime change is highly sensitive to the distance to the perturbation, i.e., it decays rapidly with distance. For non-coincident source and receiver, small slowness perturbations can be estimated from the mean traveltime changes for larger source-receiver offsets for perturbations located between the source and the receiver location.

In order to detect a localized slowness perturbation we need access to many realizations of the diffuse wavefield propagating through the random velocity model. For the example shown here, we obtain an estimate of the mean traveltime change by averaging over different receiver locations. In general, different realizations of the experiment are not available. In that case, one can average over different source-receiver pairs.

This technique has been developed only for 2D scalar waves. Further work needs to be done to adapt this technique to the more complicated and realistic case of 3D elastic wave propagation. Therefore, application of this technique in geophysics, non-destructive testing, time-lapse reservoir monitoring and other applications
Figure 19. Theoretical (solid) versus actual (asterisks) traveltime change for a receiver located 2000 m (left) and 4000 m (right) away from the source.

Figure 20. Theoretical (solid) versus averaged actual (asterisks) traveltime change for a receiver located 2000 m (left) and 4000 m (right) away from the source.

where elastic waves are used, will require the extension of the present formulation of the integration kernel $K$ that models the mean traveltime change of the diffuse wavefield.

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APPENDIX A: DERIVATION OF $K$ FOR COINCIDENT SOURCE AND RECEIVER IN 2D

We start from the definition of the kernel $K$ in three dimensions:

$$ K(r', t) = \frac{P(r', s, t') \ast G(r, r', t - t')}{P(r, t)} , \quad (A1) $$

where $\ast$ denotes time convolution. $P(r, t)$ is the intensity at the receiver located at $r$ due to a normalized impulse source at the origin at time $t=0$, and is given by (Paaschens, 1997)

$$ P(r, t) = \frac{1}{4\pi D t} \exp \left[ -\frac{r^2}{4Dt} \right] . \quad (A2) $$

$G(r, r', t - t')$ is the Green’s function that describes the intensities at $r$ at time $t$ due to a normalized impulse source at $r'$ at time $t - t'$. The time convolution $P \ast G$ is given by

$$ P(r', s) \ast G(r, r', t) = \int_0^t P(r', t') P(r - r', t - t') dt'. \quad (A3) $$

Substituting Eq. (A2) into Eq. (A3) gives for coincident source and receiver ($r = 0$) the following:

$$ P(r', s) \ast G(r, r', t) = \int_0^t \frac{1}{4\pi Dt} \exp \left[ -\frac{r'^2}{4Dt} \right] \exp \left[ -\frac{r^2}{4D(t-t')} \right] dt'. \quad (A4) $$

After solving the integral in Eq. (A4) we obtain:

$$ P(r', s) \ast G(r, r', t) = \frac{2}{(4\pi D)^2} \exp \left[ -\frac{r'^2}{2Dt} \right] K_0 \left( \frac{r^2}{2Dt} \right). \quad (A5) $$

where $K_0$ is the modified Bessel function of the second kind. Substituting Eqs. (A4) and (A2) into Eq. (A1), we arrive at the expression for the kernel $K(r', t)$ in two dimensions for coincident source and receiver:

$$ K_{2D}(r', t) = \frac{1}{2\pi D} \exp \left[ -\frac{r'^2}{2Dt} \right] K_0 \left( \frac{r^2}{2Dt} \right). \quad (A6) $$