Project Review

Consortium Project on
Seismic Inverse Methods for Complex Structures

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Contents

Acknowledgments iii
Policy on Proprietary Material v
Introduction vii

Velocity Analysis and Parameter Estimation

Numeric implementation of wave-equation migration velocity
analysis operators (CWP-587)
P. Sava & I. Vlad ......................................................... 1

Apparent horizontal displacements in time-lapse seismic images (CWP-588)
D. Hale, B. Cox & P. Hatchell ........................................ 25

Interval anisotropic parameter estimation using velocity-independent
layer stripping (CWP-589)
X. Wang & I. Tsvankin .................................................. 35

Seismic modeling and analysis of the prototype heated nuclear water storage tunnel,
Yucca Mountain, NV (CWP-590)
S. Smith & R. Snieder .................................................... 47

Image Processing

Detection of channels in seismic images using the steerable pyramid (CWP-591)
J. Mathewson & D. Hale .................................................. 55

Defining regions in seismic images by flattening (CWP-592)
D. Parks, W. Harlan & D. Hale ........................................ 67

Compact finite-difference approximations for anisotropic image
smoothing and painting (CWP-593)
D. Hale ................................................................. 75

Anisotropic amplitude and attenuation analysis

Effective reflection coefficients for curved interfaces in TI media (CWP-594)
M. Ayzenberg, I. Tsvankin, A. Aizenberg & B. Ursin .................. 85

Reflection coefficients in attenuative anisotropic media (CWP-595)
J. Behura & I. Tsvankin ................................................... 111

Estimation of interval anisotropic attenuation from reflection data (CWP-596)
J. Behura & I. Tsvankin ................................................... 123
CONTENTS

Imaging

Angle-domain elastic reverse-time migration (CWP-597)
J. Yan & P. Sava ................................................................. 129
Elastic wavefield separation for VTI media (CWP-598)
J. Yan & P. Sava ................................................................. 143
Seismic imaging with Wigner distribution functions (CWP-599)
P. Sava & O. Poliannikov ......................................................... 151
Micro-earthquake monitoring with sparsely-sampled data (CWP-600)
P. Sava .................................................................................. 167
Range and resolution analysis of wide-azimuth angle decomposition (CWP-601)
G. Melo & P. Sava ................................................................. 175
Modeling and imaging with isochron rays (CWP-602)
E.F.F. Silva & P. Sava ............................................................. 193
Accelerating wavefield extrapolation isochron-ray migration (CWP-603)
E.F.F. Silva & P. Sava ............................................................. 205
Asymptotically true-amplitude one-way wave equations in t: modeling, migration
and inversion (CWP-604)
N. Bleistein, Y. Zhang & G. Zhang ........................................... 221

Interferometry

Source distribution in interferometry for wave and diffusion (CWP-605)
Y. Fan & R. Snieder ................................................................. 251
Target-oriented interferometry: Imaging internal multiples from
subsalt VSP data (CWP-578P)
I. Vasconcelos, R. Snieder & B. Hornby .................................... 265
Representation theorems in perturbed acoustic media: Applications to interferometry
and imaging (CWP-584P)
I. Vasconcelos & R. Snieder .................................................... 281
Seismic and electromagnetic controlled-source interferometry in dissipative media
(CWP-606)
K. Wapenaar, E. Slob & R. Snieder ......................................... 297
The extraction of the Green’s function and the generalized optical theorem
(CWP-607)
R. Snieder, F.J. Sánchez-Sesma & K. Wapenaar .......................... 311
The cancellation of spurious arrivals in Green’s function extraction and the
generalized optical theorem (CWP-608)
R. Snieder, K. van Wijk & M. Haney ....................................... 317
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CWP Policy on Proprietary Printed Material

New printed material that is produced at the Center for Wave Phenomena under Consortium support is presented to Sponsors before it is released to the general public. We delay general publication by at least 60 days so that Sponsors may benefit directly from their support of the Center for Wave Phenomena.

During this delay, Sponsors may make whatever use of the material inside their organization that they deem proper. However, we expect that all Sponsors will respect the rights of other Sponsors, and of CWP, by not publishing these results externally and independently, in advance of this 60-day delay (even with attribution to CWP). Please refer to your Consortium Membership Agreement under the paragraph entitled “Sponsor Confidentiality Obligation.”

Those reports in this book that were produced primarily under consortium support and have not been previously distributed or submitted for publication, will be available for general distribution by October 1, 2008.

If you have independently generated results that duplicate or overlap these, and plan to submit them for publication under your own name before this date, please notify us immediately, so that misunderstandings do not arise.
INTRODUCTION

This edition of the report on the Consortium Project at the Center for Wave Phenomena summarizes much of the research conducted within CWP after the 2007 Project Review Meeting. Note that the papers in this report and those presented orally during the Annual Project Review Meeting, May 19–22, 2008, only partially overlap.

Papers in This Report

The papers in this volume are grouped into the following five categories: velocity analysis and parameter estimation, image processing, anisotropic amplitude and attenuation analysis, imaging, and interferometry. These categories show both similarities to and differences from those of the past few years, indicative of both the continuity and expanding breadth of our research program.

The section on velocity analysis and parameter estimation includes four papers. The tutorial by Sava and Vlad is devoted to wave-equation migration velocity analysis operators for zero-offset, survey-sinking and shot-record migration configurations. The paper describes the numeric implementation of these operators and discusses them mainly from an algorithmic point of view. The main focus is on the common elements and the domains of applicability for various velocity estimation frameworks. Hale, Cox, and Hatchell analyze horizontal displacements apparent in time-lapse seismic images. They suggest that all three components of displacement are caused largely by velocity changes above compacting reservoirs, and then test this hypothesis by comparing measured horizontal displacements with those computed via image ray tracing from measured vertical displacements. Wang and Tsvankin develop 2D and 3D algorithms for interval parameter estimation in anisotropic media based on the velocity-independent layer-stripping (VILS) method of Dewangan and Tsvankin. Synthetic tests for typical VTI (transversely isotropic with a vertical symmetry axis) and orthorhombic models demonstrate the superior accuracy and stability of VILS compared to Dix-type techniques. The 3D version of the method is also applied to estimate the interval anellipticity parameters from wide-azimuth data acquired at Rulison field in Colorado. The paper by Smith and Snieder is based on a seismic experiment in Yucca Mountain carried out by the Department of Energy during a heating test. Analysis of the acquired data supported by numeric modeling is used to constrain the temperature dependence of seismic velocities in the proposed nuclear waste repository.

The section on image processing describes algorithms that might typically be applied after seismic imaging or migration to facilitate seismic interpretation, etc. The paper by Mathewson and Hale develops multi-scale steerable filters that can be used to enhance channels in seismic images. These filters are often cited but seldom investigated as an alternative to wavelet or curvelet decompositions. Parks, Harlan, and Hale pose the process of flattening seismic images as an inverse problem, and then show examples of solving this problem with a generic inversion software framework. The paper by Hale illustrates a problem that can arise in using well-known finite-difference approximations in applications such as image smoothing and painting, and then derives new, more appropriate approximations.

Three papers comprise the section on anisotropic amplitude and attenuation analysis. Ayzenberg, Tsvankin, Aizenberg, and Ursin introduce effective reflection coefficients (ERC) for multicomponent data from heterogeneous anisotropic media and implement their formalism for an interface separating isotropic and TI halfspaces. If the reflector is plane, the ERC represents the exact solution that can be substantially different from the corresponding plane-wave reflection coefficient (PWRC), especially at low frequencies and in the post-critical domain. For curved interfaces, ERC provide a significant improvement over PWRC in 3D diffraction modeling of PP and PS reflection data. Two papers by Behura and Tsvankin discuss different aspects of modeling and estimation of anisotropic attenuation. In the first paper, they present linearized PP- and PS-wave reflection coefficients for arbitrarily anisotropic media and simplify them for VTI symmetry using Thomsen-style notation. The analytic solutions and exact numerical modeling show that reflection coefficients for highly attenuative VTI media (such as heavy-oil reservoirs) become sensitive to the attenuation-anisotropy parameters. Also, a nonzero inhomogeneity angle (i.e., the angle between the real and imaginary components of the wave vector) of the incident wave introduces additional terms into the reflection coefficients, which makes conventional
AVO (amplitude-variation-with-offset) analysis inadequate in the presence of strong attenuation. The second paper by Behura and Tsvankin combines the spectral-ratio method with velocity-independent layer-stripping of reflection data to develop an efficient technique for estimation of the interval phase attenuation coefficient. Synthetic examples for layered VTI media confirm the accuracy of this methodology and its potential in the inversion for the interval attenuation-anisotropy parameters. Azimuthal variation of the interval attenuation in fractured formations can provide sensitive attributes for fracture characterization and reconstruction of the stress field.

The section on imaging includes eight papers. The first paper by Yan and Sava presents a method for angle-domain imaging of elastic wavefields in isotropic heterogeneous media, which can be used in elastic reverse-time migration (RTM). The method employs Helmholtz decomposition to separate wave modes prior to the application of an imaging condition. Different wave modes of the source and receiver wavefields can be cross-correlated independently, thus providing information about angle-dependent reflectivity for various combinations of the incident and reflected waves. In their second paper, Yan and Sava discuss a technique for wave-mode separation in anisotropic heterogeneous media applicable to elastic RTM. The mode-separation operators are defined in the space domain and are based on the polarization vectors evaluated at each spatial location by solving the Christoffel equation with the local medium parameters. The derived convolutional operators can separate elastic wavefields for VTI media with arbitrary strength of anisotropy. Sava and Potiannikov introduce a method that combines the conventional seismic imaging condition based on wavefield cross-correlation with seismic interferometry in order to obtain statistically stable images in the presence of rapid velocity variations. The proposed imaging condition consists of applying conventional imaging to Wigner distribution functions of the reconstructed wavefields, which adds minimally to the computational cost of migration and does not change the overall imaging framework. The paper by Sava describes a method for attenuating artifacts in reverse-time migration caused by sparse and irregular data sampling. The sparseness of the receiver array causes artifacts which hamper automatic methods for micro-earthquake location, thus limiting the effectiveness of fluid-flow monitoring near injection wells. The main idea of the proposed technique is to exploit the noise-attenuating properties of pseudo-Wigner distribution functions. Melo and Sava analyze the numeric factors that influence the range and resolution properties of wide-azimuth angle decomposition. Their results indicate that the implementation of conventional extended imaging conditions for wave-equation migration influences the range of reflection and azimuthal angles that can be used for migration velocity analysis or amplitude-variation-with-angle analysis. In their first paper, Silva and Sava present an extension of conventional zero-offset wave-equation modeling and migration to finite-offset data. The method employs the concepts of isochron rays and wavefronts to condition the recorded data prior to migration using conventional wavefield extrapolation techniques. Potential applications of this algorithm include imaging in complex media and fast migration velocity analysis. The second paper by Silva and Sava describes implementation details for isochron-ray wave-equation modeling and migration designed to reduce the computational cost. They propose imaging groups of rays in bundles similar to the ones used for Gaussian beam migration and carrying out redatuming through the simple near-surface model. These techniques can potentially increase the speed of isochron-ray migration by almost an order of magnitude. Bleistein, Zhang, and Zhang address the issue of the amplitude in the solution of one-way wave equations in time. Surprisingly, the current two-term differential equation used in reverse time migration does the “right thing” with some surprises. One of them is that what propagates is the so-called “analytic wave” whose real part looks like a solution to the two-way wave equation and whose imaginary part is the Hilbert transform of the real part (even the Green’s function has this structure). That has ramifications for the inversion of seismic data as opposed to the migration of the same data set.

Six papers in the section on interferometry discuss various aspects of the interferometric method for seismic and electromagnetic fields. The theorems that underpin seismic interferometry are based on a specific distribution of (random) sources. For wave-propagation applications, sources should be located on a closed surface, whereas for diffusion and CSEM (controlled-source electromagnetics) sources should be distributed throughout the volume. Fan and Snieder demonstrate that in practice a limited number of sources is sufficient. They develop sampling criteria for the source distribution and test them with
numerical simulations. Vasconcelos, Snieder, and Hornby apply interferometry of internal multiples to image targets above a borehole receiver array. They employ an interferometric technique, based on representation theorems for perturbed media, which is designed to reconstruct specific primary reflections from multiply reflected waves. The method is tested on a numerical walkaway VSP experiment for the Sigsbee salt model and on field VSP data from the Gulf of Mexico. The paper by Vasconcelos and Snieder develops general reciprocity theorems for perturbed acoustic media that provide explicit integral relations between wavefields and wavefield perturbations. These integral equations are particularly important for reconstructing only field perturbations (e.g., scattered waves) from cross-correlation of measured field quantities. Wapenaar, Slob, and Snieder prove that by inverting a representation integral that relates updoing and downgoing electromagnetic fields, it is possible to remove the imprint of the overburden (including the airwave) from CSEM data. Their method helps to increase the sensitivity of the CSEM response to the electrical conductivity of the reservoir. The paper by Snieder, Sanchez-Sesma, and Wapenaar shows that the autocorrelation of noise yields the imaginary component of the Green’s function at the source, which is related to the energy leaving the source region. While the Green’s function in general is singular at the source location, its imaginary part remains finite. Snieder, van Wijk, and Haney resolve the paradox in seismic interferometry presented by Snieder during the semi-annual CWP meeting at the SEG Convention in San Antonio.

OVERVIEW OF DEVELOPMENTS IN CWP

CWP Faculty

There has been no change in the CWP faculty group and administrative staff since the last Project Review Meeting. The full-time CWP academic faculty includes Dave Hale, Paul Sava, Roel Snieder, and Ilya Tsvankin (director). In accordance with the rotation plan approved by the CWP faculty in 2004, Roel Snieder will assume the position of CWP director in June 2008. Ken Larner and Norm Bleistein remain part of the team in their “retirement,” and are actively involved in many aspects of our research and educational program.

Students and Post-Doctoral Fellows

During the 2007-2008 academic year, 14 graduate students were doing research in CWP. Three new CWP students (Yongxia Liu, Yong Ma, and Tongning Yang; all from China) started their graduate studies in the Fall of 2007. Ran Xuan joined CWP for the Spring 2008 semester.

Eduardo Filpo Ferreira da Silva joined CWP as a post-doctoral fellow in March 2007 for a 16-month period. Eduardo is a geophysicist with Petrobras, Brazil, and his company provides full support for his post-doctoral position with CWP.

Center Support

Currently the Consortium is supported by 26 companies including three new sponsors that have joined after the last Project Review Meeting: TGS-Nopec, SINOPEC, and Woodside Energy, Ltd. We thank the representatives of our sponsors for their continued support. A full list of sponsor companies over the term of the past year appears on the acknowledgment page at the beginning of this volume.

We have received about $840K of additional support since June 2007 from the U.S. Department of Energy, National Science Foundation, Petroleum Research Fund of the American Chemical Society, U.S. Geological Survey, ExxonMobil, Shell, Statoil, and ADNOC (for details, see below). Also, in 2005 Landmark Graphics committed to support a research fellowship for a Ph.D. student (currently Derek Parks) in computer science and/or geophysics for four years, toward the goal of developing new methods for modeling the earth’s subsurface. Our industrial and government support for research and education complement one another; each gains from, and strengthens, the other. As a net result, for the annual 2007-08 fee of $49.8K, a company participates in a research project whose total funding level is close to $1.84M, which gives the leverage factor close to 37.
Joint Projects with Industry and Non-Profit Corporations

Roel Snieder and his students continue their work with Shell within the framework of the company’s Gamechanger program. Shell has provided funding for the three-year project “Stripping the overburden from the seismic and electromagnetic earth response” started by Roel in collaboration with Kees Wapenaar and Evert Slob of Delft University.

Paul Sava has initiated two new industry-funded projects. The first project, supported by a grant from ExxonMobil, addresses the problem of microearthquake location for 4D fluid monitoring. This two-year project is executed in collaboration with Roel Snieder and graduate student Ran Xuan. The second project is supported by a grant from ENI and dedicated to the development of migration velocity analysis using reverse-time migration techniques. This four-year project will provide full support for one graduate student. In addition, Paul continues a three-year project supported by Statoil on wave-equation velocity analysis and imaging for wide-azimuth data.

Ilya Tsvankin and his students have begun working on the project “Analysis of multicomponent seismic data from carbonate reservoirs” in cooperation with Karl Berteussen of the Petroleum Institute (PI), Abu Dhabi. This three-year project is supported by PI and ADNOC as part of the Cooperative Research Program with CSM. Ilya has also received two-year funding from the Research Partnership to Secure Energy for America (RPSEA) for his project “Azimuthal AVO and attenuation analysis for fracture characterization.” RPSEA is a non-profit corporation formed by a consortium of premier U.S. energy research universities, industry and independent research organizations.

CWP encourages directly sponsored research with companies and non-profit corporations that could lead to sharing of results with the Consortium.

13th International Workshop on Seismic Anisotropy

Ilya Tsvankin and Ken Larner, along with James Gaiser and Edward Jenner of ION Geophysical/GXT Imaging Solutions, are the organizers of the 13th International Workshop on Seismic Anisotropy planned for August 10-15, 2008, in Winter Park, Colorado. The previous workshops, which proved instrumental in moving seismic anisotropy to the forefront of exploration seismology, have been held every other year since the 1980’s at various locations all over the world. If your company is interested in sponsoring 13IWSA (e.g., by providing student travel grants), please contact one of the organizers. 13IWSA registration information may be found at http://www.cwp.mines.edu/13IWSA/13iwsa.htm.

Interaction with Other Research Projects at CSM and Elsewhere

During this past year, as in previous years, faculty and students of CWP have interacted closely with other industry-funded research projects in the CSM Department of Geophysics. These include the Reservoir Characterization Project (RCP), led by Tom Davis; the Center for Rock Abuse, led by Mike Batzle; and the Gravity/Magnetics Project, led by Yaoguo Li.

In addition, the CWP faculty have engaged in collaborative efforts with researchers elsewhere. Roel Snieder currently is on sabbatical leave and works on outreach and education on energy with support from the Global Climate and Energy Project (GCEP) at Stanford University. Roel has made several trips to Stanford University as part of this collaboration. His public lecture “The Global Energy Challenge” can be downloaded from Roel’s webpage. Roel has also given several seminars at Stanford University on a variety of topics.

Other collaborations of the CWP faculty include:

- Norm Bleistein
  - Sam Gray and Yu Zhang (CGGVeritas)
  - Guanquan Zhang (Chinese Academy of Sciences)
• Dave Hale
  – Barbara Cox and Paul Hatchell (Shell)
  – Craig Artley, Bill Harlan and Andreas Rüger (Landmark Graphics)

• Paul Sava
  – Clara Andreoletti (ENI)
  – Joe Dellinger (BP)
  – Tong Fei (Saudi Aramco)
  – Sergey Fomel (UT Austin)
  – Paul Fowler (WesternGeco)
  – James Gaiser and Ivan Vasconcelos (ION Geophysical)
  – Scott Morton (Hess)
  – Michael Payne, Jie Zhang, Anupama Venkataraman, and Rongrong Lu (ExxonMobil)
  – Ioan Vlad (StatoilHydro)
  – Stewart Wright (Dawson Geophysical)
  – Yu Zhang (CGVeritas)

• Roel Snieder
  – Joan Gomberg, Matt Haney, Stephanie Prejean, and Bill Stephenson (US Geological Survey)
  – Peter Malin (Duke University)
  – Malcolm Sambridge (Australian National University)
  – Jon Sheiman (Shell International E&P)
  – Ivan Vasconcelos (ION Geophysical)
  – Anupama Venkataraman and Mike Payne (ExxonMobil)
  – Kees Wapenaar and Evert Slob (Delft Institute of Technology)

• Ilya Tsvankin
  – Andrey Bakulin and Vladimir Grechka (Shell International E&P)
  – Karl Bertussen (Petroleum Institute, Abu Dhabi)
  – James Gaiser (ION Geophysical)
  – Martin Landrø (Norwegian University of Science and Technology)
  – Peter Leary (Consultant, OYO Geospace)
  – Ivan Pšenčík (Czech Academy of Sciences)
  – Sergey Shapiro (Free University of Berlin)

**Travels and Activities of CWP People**

Interactions and collaborations that have taken place away from Golden include the following:

• Norm Bleistein
  – Presented a full-day course on Gaussian beams at the SEG Annual Meeting in San Antonio (September 2007).
  – Paper on Gaussian beam modeling, migration, and inversion presented at the SEG Annual Meeting was recognized as one of the top 30 papers at the meeting.
  – Presented a one-week course on Gaussian beams and other topics in modeling, migration, and inversion at Total in Pau, France (November 2007).
  – Presented a one-week course on Gaussian beams and other topics in modeling, migration, and inversion at the University of Campinas (May 2008).

• Dave Hale
  – Presented image processing research to Landmark Graphics in Denver (June 2007).
  – Traveled to Houston to participate in the review of joint (CoRE) research between Chevron and CSM, and presented research on algorithms to enhance channels in seismic images (October 2007).
– Traveled to Houston to meet with Chevron geoscientists to discuss algorithms for measuring and analyzing small vector displacements in time-lapse seismic images, and to assist with their implementation (December 2007).
– Met in Boulder with Craig Artley, Bill Harlan, and Andreas Rüger from Landmark Graphics to discuss image processing research (January 2008).
– Collaborated with Barbara Cox and Paul Hatchell of Shell on analysis of apparent displacements in time-lapse seismic images (February 2008).

• Ken Larner

– Spent the Fall 2007 semester with the Stanford Exploration Project working with students and filling in for Biondo Biondi during his sabbatical leave.
– In cooperation with Roel Snieder wrote the book “The Art of Being a Scientist” submitted for publication to Cambridge University Press.
– Served as Chair of the Board of Directors of the SEG Advanced Modeling Project (SEAM).
– Served on the Organizing Committee of the 13th International Workshop on Seismic Anisotropy (13IWSA).

• Paul Sava

– Received the Reginald Fessenden Award from SEG at the Society’s 2007 Annual Meeting for development of angle-domain common-image gathers.
– Presented a paper at the EAGE Annual Meeting in London (June 2007).
– Traveled to Saudi Arabia to collaborate with colleagues from Saudi Aramco (June 2007).
– Traveled to Houston to collaborate with colleagues from BP, TGS, and GXT (July 2007), ExxonMobil and CGGVeritas (December 2007).
– Traveled to Rio de Janeiro to present the course “Wavefield seismic imaging” prior to the SBGF convention. Gave an introduction to reproducible scientific computing during the Open Source Software Workshop (November 2007).
– Worked on the Madagascar software package with Sergey Fomel (ongoing).

• Roel Snieder

– Spent two weeks at the Alaska Volcano Observatory in Anchorage, working with Matt Haney and Stephanie Prejean (August 2007).
– Wrote the book “The Art of Being a Scientist” with Ken Larner (see above). Taught a short course on this topic at Utrecht University (January 2008).
– Gave invited presentation “Monitoring with seismic waves and tilt measurements: Application to Yucca Mountain and to fluid transport in the near surface” at the DOE Symposium “Basic Research Relevant to CO2 Sequestration” in Washington, DC (March 2008).
– Visited Delft University of Technology and presented the lecture “Changing the boundary conditions in seismic interferometry” (February 2008).
– Visited Ivan Vasconcelos (ION Geophysical) in Barcelona and presented two seminars at the University of Barcelona (April 2008).
– Served on the selection panel of the Spinoza Award of the Netherlands Organisation for Scientific Research. Made two trips to the Netherlands for meetings of the panel.
– With co-conveners Kasper van Wijk, Matt Haney, and Albena Mateeva, is organizing the session “Innovations in Geophysics: A Tribute to Rodney Calvert” at the upcoming SEG Annual Meeting in Las Vegas.

• Ilya Tsvankin

– Presented keynote lecture “Seismic inversion for azimuthally anisotropic models of fractured formations” at the EAGE/SEG Summer Research Workshop on Fractured Reservoirs in Pergia, Italy (September 2007).
Taught two-day SEG Continuing Education Course “Seismic anisotropy: Basic theory and applications in exploration and reservoir characterization” at the SEG Annual Meeting in San Antonio (September 2007).


Taught a short (1.5 days) course on seismic anisotropy and gave an invited seminar on anisotropic velocity analysis at ConocoPhillips in Houston (March 2008).

Chaired the Organizing Committee of the 13th International Workshop on Seismic Anisotropy (13IWSA).

Our students traveled considerably as well. Most of them gave presentations at the SEG Annual Meeting in San Antonio (September 2007).

- Ivan Vasconcelos gave a paper at the EAGE Annual Meeting in London (June 2007).
- Jyoti Behura attended the Workshop on Heavy Oils in Edmonton (July 2007) and COMSOL software courses in Albuquerque (August 2007). He also traveled to Houston to collaborate with geophysicists from Chevron on a modeling project (December 2007).
- Yuanzhong Fan made a trip to Houston to collaborate with geophysicists from the Shell Bellaire Research Center (November 2007). He spent three months (February-April 2008) at Shell Research in the Netherlands working on a joint project.
- Ran Xuan traveled to Houston to collaborate with geophysicists from ExxonMobil (February 2008).
- Steve Smith presented a paper at a meeting of the Seismological Society of America in Santa Fe (April 2008).

Visitors to CWP

CWP has benefited again this year from visits by a number of scientists and friends from other universities and industry. We strongly encourage visits from our sponsor representatives, whether it be for a single day, or for an extended period.

- James Gaiser (ION Geophysical) and Paul Fowler (WesternGeco) have regularly participated in the A(nisotropy)-Team seminar and collaborated with CWP faculty and students.
- Walter Sollner of PGS in Norway is spending his six-month leave in CWP (March-October 2008).
- Alexandre Araman, from Total S.A. in Pau (France), came for a one-year visit (January 2007 – January 2008) to collaborate with CWP faculty and students and take courses in the Professional M.S. Program.
- As mentioned above, Eduardo Filpo Ferreira da Silva from Petrobras (Brazil) is spending 16 months at CWP as a post-doctoral fellow.
- Yong Zheng (visiting scientist) and Weitao Wang (visiting student) came to CSM from China for a six-month visit (September 2007 – February 2008) to do research with Roel Snieder.
- Milana Ayzenberg, Ph.D. student from NTNU, Norway, visited CWP to work on a project with Ilya Tsvankin (October 2007 – February 2008).
- Lennart de Groot, M.S. student from Utrecht University, came to collaborate with Roel Snieder (February – June 2008).

We also had a number of short-term visitors:

- John Anderson (ExxonMobil, Houston)
- Vladimir Grechka (Shell, Houston)
- Boris Gurevich (Curtin University, Australia)
• Panos Kelamis (Aramco, Saudi Arabia)
• Evgeny Landa (OPERA, France)
• Malcolm MacNeill (Woodside Energy, Australia)
• Gaspar Monsalve (CU Boulder)
• Javier A. Perez (PDVSA, Venezuela)
• Malcolm Sambridge (Australian National University)
• Andrey Shabelansky (Paradigm Geophysical, Israel)
• Mamoru Takanashi (Japan Oil, Gas and Metals National Corporation)
• Albert Tarantola (Institut de Physique du Globe de Paris)
• Yu Zhang (CGGVeritas, Canada)

Papers at SEG

Once again, CWP students and faculty presented a large number of papers at the SEG Annual Meeting. During the 2007 Annual Meeting in San Antonio, they had a total of 22 oral presentations, poster papers, and workshop contributions. A number of these presentations result from collaborations with sponsor companies and academic groups.

For other meetings where CWP personnel presented papers, see the section “Travels and Activities of CWP People” above.

Publications

As in past years, a significant number of papers authored or co-authored by CWP faculty and students have been published in leading journals. The complete list of CWP papers from 1984 onward is on our web site at http://www.cwp.mines.edu/bookshelf.html. Most papers are available there for downloading as PDF files.

The Ph.D. theses of of Kurang Mehta and Ivan Vasconcelos were distributed to sponsors and others following the 2007 Annual Project Review meeting. If you would like to receive a copy of these, or of any other CWP publications, contact Barbara McLenon at “barbara@dix.mines.edu”.

Computing Environment

The CWP research computing environment includes a 32 processor Linux cluster system. Each of the 16 nodes consists of a dual processor Pentium Xeon 2.4 GHz PC system with 2 GB of RAM available per processor, and about 160 GB total of hard-disk storage for each node. We have used this unit for nearly four years.

Each student and faculty member has a desktop system of 3 GHz or faster, running Linux, with 250-1000 GB of storage space per desktop. We have purchased five 64-bit Dell systems, each with 8 GB of RAM and one Terrabyte of hard-disk space. In addition, we have doubled our home directory space. The total amount of disk space available on the CWP Net exceeds 10 Terabytes, roughly double of that available last year.

For data transport, our preferred medium consists of USB hard drives, formatted with the ext3 filesystem. CWP faculty and students make regular use of the following commercial packages: Mathematica, Matlab, the Intel C and Fortran compilers, as well as the NAG95 (Fortran 90/95 compiler).

In addition to the CWP internal computing facilities, the CSM campus facilities include a “petascale” cluster, which is scheduled to become available later this year.
Software Releases

CWP releases open-source software as well as software that is confidential to the Consortium. Most confidential codes depend heavily on the free software environment, so both are relevant to the Consortium. The period of confidentiality is three years. Some of the codes developed at CWP are part of government-funded research projects, and have to be released as open source. Software developed using in-house resources of sponsor companies generally is not available to us for release.

Of interest to Consortium members are SU-style codes for nonhyperbolic moveout inversion of wide-azimuth P-wave data from anisotropic media by Jia Yan and Ivan Vasconselos, previously used by Xiaoxia Xu. These items have been discussed in previous reports, but have required additional testing before release. Another new collection of codes is a Matlab package by Yuanzhong Fan for the investigation of source arrays for virtual source experiments. Finally, Matlab codes written by Steve Smith for the analysis of seismic waves scattered by a tunnel are available.

A widely used vehicle of open software distribution is the CWP/SU:Seismic Un*x (SU) package. This package has been installed at more than 3700 sites at locations defined by 68 internet country codes, as determined by voluntary direct emails. Another measure of the user base is the active membership in the “seisunix” listserver group (more than 800 as of April 3, 2008), and general interest via downloads of more than 10 per day, though these may be more reflecting of internet bots, rather than real users.

Release 41 of SU was issued on April 10, 2008, and contained many updates and new software. For details, please download the release notes from http://www.cwp.mines.edu/cwpcodes.

The open-source Mines Java Toolkit is available online from Dave Hale’s home page at http://www.mines.edu/~dhale/jtk/. This software is the foundation for most of Dave’s teaching and research, and is also being used by commercial software companies. Anyone with a web browser can view and download the always up-to-date source code repository.

Paul Sava and his students continue to work with and develop software for Madagascar, an open-source software package for geophysical data processing and reproducible numerical experiments. Its mission is to provide a convenient and powerful environment and a technology transfer tool for researchers working with digital image and data processing. The technology developed using the Madagascar project management system is transferred in the form of recorded processing histories, which become “computational recipes” to be verified, exchanged, and modified by users of the system. This open-source package is available from http://rsf.sourceforge.net/.

Annual Project Review Meeting

This year’s Annual Project Review Meeting will be held on May 19–22, 2008, on the CSM campus in Golden, Colorado. A tradition of recent years is that, prior to the Meeting, we hold a tutorial for sponsors on a topic of particular interest within CWP. This year, in the afternoon of May 19, Malcolm Sambridge of the Australian National University will give a tutorial entitled “Making inferences from data: From Newton to Bayes.” During the following three days, students and faculty will present more than 20 research papers. In addition, the program will include two guest speakers: Bill Harlan from Landmark Graphics and John Sherwood of Applied Geophysical Services.

WELCOME

With great pleasure, we welcome representatives of our sponsor companies to the 24th Annual Project Review Meeting, and look forward to the opportunity to exchange with you ideas and thoughts about this past year’s projects and plans for the future.

Ilya Tsvankin, Director
Center for Wave Phenomena
May 2008
Representation theorems in perturbed acoustic media: Applications to interferometry and imaging

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ABSTRACT

General reciprocity theorems for perturbed acoustic media are provided in the form of convolution- and correlation-type theorems. The results presented here differ from previous derivations because they provide explicit integral relations between wavefields and wavefield perturbations which alone do not satisfy the acoustic wave equations. Using Green’s functions to describe perturbed and unperturbed waves in two distinct wave states, representation theorems are derived from the reciprocity relations in perturbed media. While the convolution-type theorems can be manipulated to obtain the Lippmann-Schwinger integral, the correlation-type theorems can be used to retrieve the perturbation response of the medium by cross-correlations. By cross-correlating wavefield perturbations recorded at a given receiver with unperturbed waves at another, one obtains a pseudo-experiment where only wavefield perturbations propagate from one receiver to the other as if one of them were a source. This can be done when the pressure fluctuations arise from random volume sources; or from coherent excitation by surface sources when the receivers lie away from the perturbed portion of the medium. The perturbation-based integral equations presented here are particularly important for the description of scattering problems, and for reconstructing only field perturbations (e.g., scattered waves) from cross-correlating measured field quantities.

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Key words: 3D, 9C, wide-azimuth seismic data, fracture characterization, multiple fracture sets, orthorhombic anisotropy

1 INTRODUCTION

Reciprocity theorems have long been used to describe important properties of wave propagation phenomena. Lord Rayleigh (Rayleigh, 1878) used a local form of an acoustic reciprocity theorem to demonstrate source-receiver reciprocity. Time-domain reciprocity theorems were later generalized to relate two wave states with different field, material and source properties in absorbing, heterogeneous media (de Hoop, 1988).

Fokkema and van den Berg (1993) show that acoustic reciprocity theorems can be used for modeling wave propagation, for boundary and domain imaging, and for estimation of the medium properties. In the field of exploration seismology, an important application of convolution-type reciprocity theorems lies in removing multiple reflections, also called multiples, caused by the Earth’s free-surface (Fokkema & van den Berg, 1993; Berkhout & Verschuur, 1997). These approaches rely on the convolution of single-scattered waves to create multiples, which are then adaptively subtracted from the recorded data. Other approaches for the elimination of multiples from seismic data rely on inverse scattering methods (Weglein et al., 2003; Malcolm et al., 2004). The inverse-scattering based methodologies are typically used separately from the representation theorem approaches (Fokkema & van den Berg, 1993; Berkhout & Verschuur, 1997) in predicting multiples. Note: Rep-
presentation theorems are derived from reciprocity theorems using Green’s functions; e.g., see Section 3 of this paper.

Recent forms of reciprocity theorems have been derived for the extraction of Green’s functions (Wapenaar et al., 2002; Wapenaar & Fokkema, 2006; Wapenaar et al., 2006), showing that the cross-correlations of waves recorded by two receivers can be used to obtain the waves that propagate between these receivers as if one of them behaves as a source. These results coincide with other studies based on cross-correlations of diffuse waves in a medium with an irregular boundary (Lobkis & Weaver, 2001), caused by randomly distributed uncorrelated sources (Weaver & Lobkis, 2001; Wapenaar et al., 2002; Shapiro et al., 2005), or present in the coda of the recorded signals (26). (Snieder, 2004). An early analysis by Claerbout (1968) shows that the reflection response in a 1D medium can be reconstructed from the autocorrelation of recorded transmission responses. This result was later extended for cross-correlations in heterogeneous 3D media by Wapenaar et al. (2004), who used one-way reciprocity theorems in their derivations. Green’s function retrieval by cross-correlations has found applications in the fields of global (Shapiro et al., 2005; Saba et al., 2005) and exploration seismology (Schuster et al., 2004; Willis et al., 2006; Bakulin & Calvert, 2006), ultrasonics (Malcolm et al., 2004; wan Wijk 2006), helioseismology (Rickett & Claerbout, 1999), structural engineering (Snieder & Şafak, 2006; Thompson & Snieder, 2007) and ocean acoustics (Roux et al., 2004; Saba et al., 2005).

Although the correlation-based Green’s function retrieval has proven for special cases by methods other than reciprocity theorems (e.g., [Lobkis et al., 2001; Bakulin & Calvert, 2006]), the derivations based on reciprocity theorems have provided for generalizations beyond lossless acoustic wave propagation to elastic wave propagation and diffusion (40; 27). (Wapenaar, 2004; Snieder, 2006). More general forms of reciprocity theorems have been derived (Wapenaar et al., 2006; Snieder et al., 2007; Weaver, 2008) which include a wide range of differential equations such as the acoustic, elastodynamic, and electromagnetic wave equations, as well as the diffusion, advection and Schrödinger equations, among others.

In this paper, we derive reciprocity theorems for acoustic perturbed media. The perturbations of the wavefield due to the perturbation of the medium can be used for imaging or for monitoring. For imaging, the unperturbed medium is assumed to be so smooth that is does not generate reflected waves, while the perturbation accounts for the reflections. In monitoring application, the perturbation consists of the time-lapse changes in the medium. Although previous derivations of reciprocity theorems account for arbitrary medium parameters that are different between two wave states (de Hoop, 1988; Fokkema & van den Berg, 1993), they do not explicitly consider the special case of perturbed media. In perturbed media, there are special relations between the unperturbed and perturbed wave states (e.g., in terms of the physical excitation) that make the reciprocity theorems in such media differ in form with respect to their more general counterparts (Fokkema & van den Berg, 1993; Wapenaar et al., 2006). We discuss some of these differences. Another important aspect of studying representation theorems in perturbed media lies in retrieving wavefield perturbations from cross-correlations, in a manner analogous to that presented earlier (Wapenaar et al., 2006; Snieder, 2007). As we show here, wavefield perturbations by themselves do not satisfy the wave equations and thus their retrieval does not follow directly from these earlier derivation. In parallel with the work we present here, Douma (2007) derived convolution- and correlation-type reciprocity theorems for acoustic perturbed media that are related to the Lippmann-Schwinger series, with the purpose of accomplishing time-lapse monitoring by non-linear inversion.

We first derive general forms of convolution- and correlation-type reciprocity theorems by manipulating the perturbed and unperturbed wave equations for two wave states. In the Section that follows, we write representation theorems using the Green’s functions for unperturbed and perturbed waves in the two states. We show that the convolution-type theorems result in scattering-like integrals that can be used to describe the field perturbations between two observation points. Next we analyze how the correlation-type theorems can be used to extract the field perturbations from cross-correlations of observed fields, for different case of medium and experiment configurations. Finally, we discuss the applications of these representation theorems in recovering the perturbation response between two sensors from random medium fluctuations and from coherent surface sources; and also their applications in tomography.

2 RECIROCITY THEOREMS IN CONVOLUTION AND CORRELATION FORM

We define acoustic wave states in a domain $\mathcal{V}$, bounded by $\partial\mathcal{V}$ (Figure 1). The outward pointing vector normal to $\partial\mathcal{V}$ is represented by $\mathbf{n}$. We consider two wave states, which we denote by the superscripts $A$ and $B$, respectively. Each wave state is defined in an unperturbed medium with compressibility $\kappa_0(\mathbf{r})$ and density $\rho_0(\mathbf{r})$; as well as in a perturbed medium described by the material properties $\kappa(\mathbf{r})$ and $\rho(\mathbf{r})$. Using the Fourier convention $u(t) = \int u(\omega) \exp(-i\omega t) d\omega$, the acoustic wavefield equations for state $A$ in an unperturbed medium are, in the
frequency-domain,
\[ \nabla p_0^A(r, \omega) - i\omega \rho_0(r) v_0^A(r, \omega) = 0 \]  
(1)
\[ \nabla \cdot v_0^A(r, \omega) - i\omega \rho_0(r) p_0^A(r, \omega) = q^A(r, \omega), \]
where \( p^A(r, \omega) \) and \( v^A(r, \omega) \) represent pressure and particle velocity, respectively. The quantity \( q^A(r, \omega) \) describes the source distribution as a volume injection rate density. Similar equations describe waves in state A in a perturbed medium:
\[ \nabla p^A(r, \omega) - i\omega \rho(r) v^A(r, \omega) = 0 \]  
(2)
\[ \nabla \cdot v^A(r, \omega) - i\omega \rho(r) p^A(r, \omega) = q^A(r, \omega), \]
with perturbed pressure and particle velocity given by \( p^A(r, \omega) \) and \( v^A(r, \omega) \), respectively. The acoustic field equations that describe wave propagation in state B follow by replacing the subscript \( A \) by \( B \) in equations 2 and 1. Note that the source distribution \( q^A(r, \omega) \) is the same for both the unperturbed (equation 1) and perturbed (equation 2) cases. We assume that no external volume forces are present by setting the right-side of the vector equations in equations 1 and 2 equal to zero. The perturbed pressure for either wave state is given by \( p = p_0 + p_B \), where the subscript \( S \) indicates the wavefield perturbation caused by medium changes.

To derive Rayleigh's reciprocity theorem (Rayleigh, 1878; de Hoop, 1988; Fokkema & van den Berg, 1993) we insert the equations of motion and stress-strain relations for states \( A \) and \( B \) (equations 1 and 2) in
\[ v_0^B \cdot E_0^A + p_0^A E_0^B - v_0^A \cdot E_0^B - p_0^B E_0^A; \]
(3)
where \( E \) and \( p \) represent the equation of motion and the stress-strain relation in equation 1, respectively. The subscripts in equation 3 indicate that equations and field parameters (\( p \) and \( v \)) are considered in the unperturbed case. The superscripts indicate whether equations and field parameters pertain to state \( A \) or \( B \). For brevity, we omit the parameter dependence on \( r \) and \( \omega \). From equation 3 we isolate the interaction quantity \( \nabla \cdot (p_0^B v_0^B - p_0^B v_0^A) \) (de Hoop, 1988). Next, we integrate the result of equation 3 over the domain \( V \) and apply Gauss’ divergence theorem. This results in
\[ \int_{r \in V} [p_0^B v_0^B - p_0^B v_0^A] \, ds = \int_{r \in V} [p_0^B q_0^B - p_0^B q_0^A] \, dV; \]
(4)
which is referred to as a reciprocity theorem of the convolution type (de Hoop, 1988; Fokkema & van den Berg, 1993), because the frequency-domain products of field parameters represent convolutions in the time domain. A correlation-type reciprocity theorem (de Hoop, 1988; Fokkema & van den Berg, 1993) can be derived from isolating the interaction quantity \( \nabla \cdot (p_0^B v_0^B - p_0^B v_0^A) \) from
\[ v_0^B \cdot E_0 + p_0^A E_0^B + v_0^A \cdot E_0^B + p_0^B E_0 \]
(5)
where \( * \) denotes complex conjugation. Subsequent volume integration and application of the divergence theorem yields
\[ \int_{r \in V} [p_0^B v_0^B + p_0^B v_0^A] \, ds = \int_{r \in V} [p_0^B q_0^B + p_0^B q_0^A] \, dV; \]
(6)
where complex conjugates represents time-domain cross-correlations of field parameters. For this reason, equation 6 is a reciprocity theorem of the correlation type (de Hoop, 1988; Fokkema & van den Berg, 1993).

The theorems in equations 4 and 6 also hold when the material properties in states \( A \) and \( B \) are the same. General reciprocity theorems that account for arbitrarily different source and material properties between two wave states have been derived (de Hoop, 1988; Fokkema & van den Berg, 1993).

Here, we further develop these reciprocity theorems...
for the special case of perturbed acoustic media. First, we consider
\[
\mathbf{v}_0^B \cdot \mathbf{E}^A + p^A E_0^B - \mathbf{v}^A \cdot \mathbf{E}_0^B - p_0^B E^A,  \tag{7}
\]
which involves the equations and field parameters for state \(B\) in an unperturbed medium (e.g., equation 1), along with equations and field parameters for state \(A\) in a perturbed medium (equation 2). From equation 7 we isolate the interaction quantity \(\nabla \cdot (p^A \mathbf{v}_0^B - p_0^B \mathbf{v}^A)\). After separating this quantity, we integrate equation 7 over \(\mathcal{V}\) and apply Gauss' theorem. Next, given that \(p = p_0 + p_S\) and \(\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_S\), we use the result from equation 4, which gives
\[
\iint_{\mathcal{V}} \left[ p_0^B \mathbf{v}_0^B - p_0^B \mathbf{v}_0^A \right] \cdot dS = \iint_{\mathcal{V}} i(\kappa_0 - \kappa) p^A p_0^B dV,  \tag{8}
\]
which is a convolution-type reciprocity theorem for perturbed media. This expression is a new form of the reciprocity theorem because it relates the wavefields \(p_0^B\) and \(\mathbf{v}_0^B\) with the wavefield perturbations \(p_S^B\) and \(\mathbf{v}_S^B\). Previously derived reciprocity theorems (de Hoop, 1988; Fokkema & van den Berg, 1993; Wapenaar & Fokkema, 2006) provide, for two arbitrary wave states, relations between field parameters that satisfy wave equations (e.g., equations 1 and 2). Wavefield perturbations such as \(p_S^B\) and \(\mathbf{v}_S^B\), by themselves, do not satisfy the wave equations for perturbed media (e.g., equation 2). Although equation 8 accounts for compressibility changes only, it can be modified to include density perturbations. Such modification involves adding, to the right-hand side of the equation, a third volume integral whose integrand is proportional to \((p_0 - p)\) and the wavefields \(\mathbf{v}^A\) and \(\mathbf{v}_0^B\) (Fokkema & van den Berg, 1993).

The correlation-type counterpart of equation 8 can be derived from the interaction quantity \(\nabla \cdot (p^A \mathbf{v}_0^B + p_0^B \mathbf{v}^A)\), which can be isolated from
\[
\mathbf{v}_0^{B*} \cdot \mathbf{E}^A + p^A E_0^{B*} + \mathbf{v}^A \cdot \mathbf{E}_0^{B*} + p_0^{B*} E^A.  \tag{9}
\]
After performing the same steps as in the derivation of equation 8 we obtain
\[
\iint_{\mathcal{V}} \left[ p_S^A \mathbf{v}^B_S + p_0^{B*} \mathbf{v}^A_S \right] \cdot dS = \iint_{\mathcal{V}} i(\kappa_0 - \kappa) p^A p_0^{B*} dV,  \tag{10}
\]
which is a correlation-type reciprocity theorem for perturbed acoustic media. Again, we assume that both \(\kappa\) and \(\kappa_0\) are real (i.e., no attenuation). As with its convolution counterpart (equation 8), equation 10 is novel because it provides a relation between the wavefield perturbations in state \(A\) and the unperturbed waves in state \(B\). Density perturbations can be included in equation 10 in a manner analogous to that discussed for equation 8 (de Hoop, 1988; Fokkema & van den Berg, 1993). By interchanging the superscripts in equations 7 and 9 we derive convolution- and correlation-type reciprocity theorems that relate the perturbations \(p_S^B\) and \(\mathbf{v}_S^B\) to \(p_0^A\) and \(\mathbf{v}_0^A\). These theorems have the same form as the ones in equations 8 and 10, except \(A\) is interchanged with \(B\) in equation 8, and with \(B^*\) in equation 10.

Since the source term \(q^A(r, \omega)\) is the same both in the unperturbed and perturbed cases, it follows from equations 1 and 2 that
\[
\nabla p_0^A(r, \omega) - i\omega p_0^A(r, \omega) = \nabla p^A(r, \omega) - i\omega p(r) \mathbf{v}^A(r, \omega)
\]
\[
= \nabla \cdot \mathbf{v}_0^A(r, \omega) - i\omega \kappa_0^A p_0^A(r, \omega)
\]
\[
= \nabla \cdot \mathbf{v}^A(r, \omega) - i\omega \kappa(r) p^A(r, \omega).  \tag{11}
\]
These relations can be used to derive others reciprocity theorems in perturbed media. To derive a convolution-type theorem, we first consider the combination
\[
\mathbf{v}_0^B \cdot \mathbf{R}^A + p_0^B \mathbf{R}_0^B - \mathbf{v}_0^A \cdot \mathbf{R}_0^A - p_0^A \mathbf{R}^A + p^A \mathbf{R}^B - \mathbf{v}^A \cdot \mathbf{R}^B - p^A \mathbf{R}^B \tag{12}
\]
where \(\mathbf{R}\) and \(R\) are the vector and scalar relations in equation 11; the superscripts indicate whether they pertain to wave state \(A\) or \(B\). Equation 12 is then integrated over volume. With Gauss' theorem, the resulting equation can be simplified by using the identities \(p = p_0 + p_S\) and \(\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_S\). In simplifying equation 12, we also use the convolution-type reciprocity theorem in equation 8 as shown, and with interchanged \(A\) and \(B\) superscripts. These steps result in
\[
\iint_{\mathcal{V}} \left[ p_S^A \mathbf{v}^B_S - p_0^{B*} \mathbf{v}^A_S \right] \cdot dS = \iint_{\mathcal{V}} i\omega (\kappa - \kappa_0) \left[ p_S^A p_0^B - p_0^{A*} p_S^B \right] dV.  \tag{13}
\]
Because the frequency-domain products in the integrands correspond to convolutions in the time domain, this integral theorem is of the convolution type. The convolution-type reciprocity theorem in equation 13 relates wavefield perturbations in both wave states \(A\) and \(B\) over the surface \(\partial V\) with wavefield perturbations, unperturbed waves, and the medium perturbation \((\kappa - \kappa_0)\) within the volume \(V\). Equation 13 can be extended to include density perturbations by adding an extra volume integral with the integrand proportional to \((p_0 - p)\) \(\left[ v_0^B v^B_0 - v_0^A v^A_0 \right] \) (Fokkema & van den Berg, 1993).

A correlation-type theorem of the same form as equation 13 can be derived from modifying the relations in equation 12 to account for the time-reversed wave state \(B^*\) (e.g., equations 3 and 5). Again, we integrate the result over \(V\), use Gauss' theorem, and simplify it by representing perturbed wavefields in terms of unperturbed waves and wavefield perturbations. Using the theorem in equation 10 as is, and with \(A\) interchanged
with \( B^* \), we obtain
\[
\int_{V \in \partial V} \left[ p^A_S v^B_S + p^B_S v^A_S \right] \cdot dS = \int_{V \in V} i \omega (\kappa - \kappa_0) \left[ p^A_S p^B_S - p^B_S p^A_S \right] dV ,
\]
which is the correlation-type counterpart of the theorem in equation 13. This theorem can be extended to include small density perturbations in a manner analogous to that described for equation 13. Like the reciprocity theorems in equations 8 and 10, equations 13 and 14 provide relations between wavefield perturbations and wavefields that satisfy the unperturbed acoustic wave equation.

3 REPRESENTATION THEOREMS IN PERTURBED MEDIA

In this Section, we use Green’s functions to derive representation theorems from the reciprocity theorems in the previous Section. We focus the discussion on the role of Green’s functions in the reciprocity theorems in equations 10 and 14 because of the applicability of these theorems to remote sensing experiments (treated in the next Section). The Green’s function forms of the theorems in equations 4 and 6 are treated by others (Wapenaar et al., 2002; Wapenaar & Fokkema, 2006; Draganov et al., 2006). The convolution theorem in equation 4 leads to the well-known acoustic source-receiver reciprocity relation (Rayleigh, 1878; Fokkema & van den Berg, 1993; Wapenaar & Fokkema, 2006). From the correlation-type theorem in equation 6, it follows that the surface integration of the cross-correlated Green’s function results in the causal and anticausal Green’s functions that propagate between two receivers (Wapenaar, 2004; Wapenaar & Fokkema, 2006). We discuss this in in more detail below.

We introduce the Green’s functions, in the frequency domain, by setting
\[
q^{A,B} = \delta (r - r_{A,B}) ,
\]
where the positions \( r_{A,B} \) denote the wave states \( A \) and \( B \), respectively. This choice for \( q \) allows for expressing the field quantity \( p \) in terms of the Green’s functions \( G \), i.e.,
\[
p^{A,B}(r, \omega) = G(r, r_{A,B}, \omega) = G_0(r, r_{A,B}, \omega) + G_S(r, r_{A,B}, \omega) ,
\]
where the subscripts 0 and \( S \) stand for unperturbed waves and wavefield perturbations, respectively. Note that these are the Green’s functions for sources of the volume injection rate type. The derivation below can also be reproduced using volume injection sources (Wapenaar & Fokkema, 2006). It follows from equations 16 and 2 that \( v^{A,B}(r, \omega) = (i \omega \rho)^{-1} \nabla G(r, r_{A,B}, \omega) \).

Let us first analyze the Green’s form of the convolution-type theorem in equation 8, by using the definitions in equations 15 and 16. For the moment we assume homogeneous boundary conditions at \( \partial V \) that make the surface integral vanish. These conditions are satisfied when (i) a radiation boundary condition holds when the surface extends to infinity, (ii) or the pressures \( G_{A,B} \) are zero at \( \partial V \) and/or when (iii) \( \nabla G_{A,B} \cdot \mathbf{n} = 0 \). Under these conditions we obtain
\[
G_S(r, r_A) = \int_{V \in V} \frac{1}{i \omega \rho} G(r, r_A) V(r) G_0(r, r_B) dV ;
\]
where \( V(r) = \omega^2 \rho (\kappa(r) - \kappa_0(r)) \) is the perturbation operator or scattering potential (Rodberg & Thaler, 1967). For brevity we omit the dependence on the frequency \( \omega \). Equation 17 is the integral equation known as the Lippmann-Schwinger equation (Rodberg & Thaler, 1967), commonly used in scattering problems. When none of the surface boundary conditions listed above apply, the surface integral of equation 8 is present on the right-hand side of equation 17.

Next, we turn our attention to the correlation-type reciprocity theorem in equation 10. Substituting the Green’s functions (equation 16) for the wavefields \( p \) and \( v \) in equation 10 gives
\[
\int_{V \in V} G_S(r, r_A) \delta(r - r_B) dV = \int_{V \in V} \frac{1}{i \omega \rho} \left[ G_S(r, r_A) \nabla G_0^0(r, r_B) + G_0^0(r, r_A) \nabla G_S(r, r_A) \right] dS
\]
\[
+ \int_{V \in V} i \omega (\kappa_0 - \kappa) G(r, r_A) G_0^0(r, r_B) dV ;
\]
or simply,
\[
G_S(r_B, r_A) = \int_{V \in V} \frac{1}{i \omega \rho} \left[ G_S(r, r_A) \nabla G_0^0(r, r_B) + G_0^0(r, r_A) \nabla G_S(r, r_A) \right] dS
\]
\[
+ \int_{V \in V} i \omega (\kappa_0 - \kappa) G(r, r_A) G_0^0(r, r_B) dV .
\]
The left-hand side describes causal wavefield perturbations that propagate from \( r_A \) to \( r_B \) as if the observation point at \( r_B \) acts as a source. This equation shows that the causal wavefield perturbation \( G_S(r_B, r_A) \) is obtained from the surface integral of the cross-correlation of wavefield perturbations at \( r_A \) with unperturbed waves at \( r_B \). Along with this surface integral, the volume integrals in equation 19 are, in general, necessary to recover \( G_S(r_B, r_A) \). By taking equation 19 with interchanged superscripts \( A \) by \( B^* \), and using reciprocity
I. Vasconcelos & R. Snieder

Figure 2. Schematic illustrations of the role of the volume integrals in reconstructing $G_S(r, r_A)$ for Case I. Medium perturbations are restricted to the volume $P$, placed away from the observation points. The solid arrows represent the paths of unperturbed waves in $G_0(r, r_{A,B})$, while the dotted arrows illustrate the paths of the field perturbations $G_S(r, r_A)$. For some sources on $\partial V$ the medium perturbation in $P$ does not affect the path of the measured unperturbed waves, as in the example of the source position $r_1$ in panel (a). In contrast, the path of the observed unperturbed waves is affected by $P$ for source positions such as $r_2$ in panel (b). Here, the source positions $r_1$ in (a) and $r_2$ in (b) are stationary points for the integrands in equation 18.

$$(G_{0,S}(r_B, r_A) = G_{0,S}(r_A, r_B)), \text{we obtain}$$

$$G_S^*(r_B, r_A) = \int_{r \in \partial V} \frac{1}{i\omega \rho} [G_S^*(r_B, r_A) \nabla G_0(r, r_A)$$

$$+ G_0(r, r_A) \nabla G_S^*(r, r_B)] \cdot dS$$

$$+ \int_{r \in V} i\omega (\kappa_0 - \kappa) G_0^*(r, r_B) G_0(r, r_A) dV$$

$$+ \int_{r \in V} i\omega (\kappa_0 - \kappa) G_S^*(r, r_B) G_0(r, r_A) dV. \quad (20)$$

Equations 19 and 20 are similar in form to the expressions derived by Wapenaar et al. (2006) and Snieder (2007) and Snieder et al. (2007). In those studies, the cross-correlation of recorded waves leads to a superposition of causal and anticausal wavefields $G$ or $G_0$. For attenuative media, we derive an integral expression analogous to equation 19 in Appendix B. In attenuative media, apart from the integrals in equation 19, it is necessary to evaluate two other volume integrals to retrieve $G_S(r_B, r_A)$ (see equation B1 in Appendix B).

There are two important differences between equations 19 and 20 and previous expressions for Green’s function retrieval (Wapenaar et al., 2006; Snieder et al., 2007). The first difference is that here we obtain the wavefield perturbations $G_S$, which by themselves do not satisfy the acoustic wave equations (e.g., equation 2), from cross-correlations of $G_S$ with $G_0$. Second, the proper manipulation of unperturbed waves $G_0$ and perturbations $G_S$ in the integrands of equations 19 and 20 allows for the separate retrieval of causal and anticausal wavefield perturbations $G_S(r_B, r_A)$ in the frequency-domain rather than their superposition. Since the correlation-type representation theorems for $G$ or $G_0$ (Wapenaar et al., 2006; Snieder et al., 2007) result in the superposition of causal and anticausal responses in the frequency-domain, their time-domain counterparts retrieve two-sides of the signal, i.e., they retrieve the wavefield at both positive and negative times. Because of this, we refer to the theorems in refs. (Wapenaar et al., 2006; Snieder et al., 2007; Snieder, 2007) as two-sided theorems. The theorems in equations 19 and 20 recover the time-domain field perturbation response for either positive (equation 19) or negative (equation 20) times only. Therefore, we call the theorems in equations 19 and 20 one-sided theorems.

To address the physical meaning of the different terms in equation 19, we break our discussion apart into the analysis of different scenarios for medium properties, acquisition geometries and boundary conditions. In the first scenario, which we refer to as Case I (Figure 2), the medium perturbations (including scatterers) are restricted to a volume $P$ which is located somewhere away from the two observation points. In this case, equation 19 becomes

$$G_S(r_B, r_A) = \int_{r \in \partial V} \frac{1}{i\omega \rho} [G_S(r, r_A) \nabla G_0^*(r, r_B)$$

$$+ G_0^*(r, r_B) \nabla G_S(r, r_A)] \cdot dS$$

$$+ \int_{r \in P} i\omega (\kappa_0 - \kappa) G_0^*(r, r_A) G_0^*(r, r_B) dV$$

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and field perturbations, respectively. Figure 2, solid and dashed arrows denote unperturbed waves the medium perturbation occupies all of the volume \( P \) illustrated by the arrows in Figure 2. For some sources fields in the integrands of equation 21. These paths are the stationary points along the paths of the different and can be evaluated asymptotically by the stationary-in equation 21 are Fourier-type oscillatory integrals, in equation 21 are not affected by the perturbations because the paths of the recorded unperturbed waves categorized by solid lines). This is not true for the remaining sources \( r \in \partial V \) (e.g., \( r_2 \) in Figure 2b) for which the observed unperturbed waves pass through \( P \), where the contributions of both volume integrals in equation 21 are important. Let us now consider only the sources \( r \in \partial V_1 \), where \( \partial V_1 \) is a segment of \( \partial V \) that contains all source positions for which the paths of \( G_0(r, r_A, B) \) (equation 21) are not affected by \( P \). As discussed above we can then neglect the first volume integral in equation 21. The last volume integral is of higher order in the medium perturbation than the surface integral. Thus, in Case I, for \( r \in \partial V_1 \) and to leading order in \( V \), equation 21 can be reduced to

\[
G_S(r_B, r_A) \approx \int_{r \in \partial V_1} \frac{1}{i\omega\rho} \left[ G_S(r, r_A) \nabla G_0(r, r_B) + G_0(r, r_B) \nabla G_S(r, r_A) \right] \cdot dS. \tag{22}
\]

For this special case, equation 22 is convenient because it allows for the reconstruction of \( G_S(r_B, r_A) \) by cross-correlating observed quantities due to sources over \( \partial V_1 \) without prior knowledge of the perturbations in \( P \). This has applications in the retrieval of the medium’s perturbation response from coherent surface sources, as we discuss in detail in the next Section. It is important to point out that although the truncation of the surface integration domain in equation 21 results in equation 22, this type of truncation can introduce artifacts in \( G_S(r_B, r_A) \) if the medium perturbations introduce multiple scattering (Snieder et al., 2006; Wapenaar, 2006). Below (along with Appendix A) we provide a stationary-phase derivation to a simple model that illustrates the role of the surface and volume integrals in Case I.

In general, the volume integrals in equation 19 cannot be ignored. Let us consider another example, Case II, illustrated by Figure 3. Now the medium configuration is the same as in Case I, but one of the receivers lies inside the perturbation volume \( P \). So for Case II, it is impossible to find source positions on \( \partial V \) for which the direct waves observed at \( r_B \) are not affected by the medium perturbation. Therefore, all integrals in equation 21 must always be evaluated. Another such example is Case III in Figure 4, where the perturbations occur over the entire volume \( V \).

Another interesting case (Case IV) arises when we take the medium configuration to be the same as in Case I (Figure 2), except the perturbation is now characterized by a smooth wavespeed change, i.e., there are no scattered or reflected waves. In this case \( G_S(r_B, r_A) = 0 \), and equation 21 becomes

\[
\int_{r \in P} i\omega(\kappa_0 - \kappa)G_S(r, r_A)G_0^*(r, r_B)dV
\]

\[
= \int_{r \in \partial V} \frac{1}{i\omega\rho} \left[ G_S(r, r_A) \nabla G_0^*(r, r_B) \right] \cdot dS
\]

\[
+ \int_{r \in \partial V} \frac{1}{i\omega\rho} \left[ G_0^*(r, r_B) \nabla G_S(r, r_A) \right] \cdot dS. \tag{23}
\]

In Case IV, \( G_S(r_B, r_A) = 0 \), and equation 23 shows that waves observed at \( r_A \) and \( r_B \) due to sources over the
surface $\partial V$ can be used to reconstruct the perturbations inside the volume.

The last case we consider, Case V, is when homogeneous boundary conditions apply on $\partial V$ so that contribution of the surface integral vanishes. These are the same conditions listed in the derivation of equation 17. Without the surface integral, and for arbitrary experimental and medium configurations, equation 19 is

$$G_S(\mathbf{r}_B, \mathbf{r}_A) = \int_{V} \frac{1}{i \omega \rho} G(\mathbf{r}, \mathbf{r}_A) V(\mathbf{r}) G_0(\mathbf{r}, \mathbf{r}_B) dV .$$

(24)

This equation, as we discuss on the next Section, plays an important role in the retrieving $G_S(\mathbf{r}_B, \mathbf{r}_A)$ from random sources within the volume $V$. Note that equation 24 is similar to the Born approximation, which follows by replacing $G$ is the right hand side of expression 17 by the unperturbed wave:

$$G_S(\mathbf{r}_B, \mathbf{r}_A) = \int_{V} \frac{1}{i \omega \rho} G_0(\mathbf{r}, \mathbf{r}_A) V(\mathbf{r}) G_0(\mathbf{r}, \mathbf{r}_B) dV .$$

(25)

The only difference between these expressions is the complex-conjugation of the reference field $G_0^*(\mathbf{r}, \mathbf{r}_B)$ in the integrand. Just as in the classical form of the Lippmann-Schwinger integral in equation 17, equation 24 describes $G_S(\mathbf{r}_B, \mathbf{r}_A)$ describes scattering interactions between the scattering potential and the unperturbed fields in the integrand. The scattered field $G_S(\mathbf{r}_B, \mathbf{r}_A)$ can be equally described by causal and anti-causal reference fields $G_0(\mathbf{r}, \mathbf{r}_B)$ (equation 25) and $G_0^*(\mathbf{r}, \mathbf{r}_B)$ (equation 24). It thus follows that in the time domain $G_S(\mathbf{r}_B, \mathbf{r}_A)$ is equally described by either time-advanced (i.e., causal) or time-retarded (i.e., anti-causal) reference fields $G_0$. This property in fact holds for any kind of wave propagation, and is restricted to systems that are invariant under time-reversal (i.e., in the absence of attenuation) as discussed in (Vasconcelos, 2007).

Finally, we turn our attention to the reciprocity theorem in equation 14. Using the Green’s functions as defined in equation 16, we express equation 14 as

$$\int_{\partial V} [G_S(\mathbf{r}, \mathbf{r}_A) \nabla G_S^*(\mathbf{r}, \mathbf{r}_B)] \cdot d\mathbf{S} +$$

$$\int_{\partial V} [G_S^*(\mathbf{r}, \mathbf{r}_B) \nabla G_S(\mathbf{r}, \mathbf{r}_A)] \cdot d\mathbf{S}$$

$$= i \omega \int_{V} (\kappa - \kappa_0) G_0(\mathbf{r}, \mathbf{r}_A) G_0^*(\mathbf{r}, \mathbf{r}_B) dV$$

$$- i \omega \int_{V} (\kappa - \kappa_0) G_S(\mathbf{r}, \mathbf{r}_A) G_0^*(\mathbf{r}, \mathbf{r}_B) dV .$$

(26)

This equation relates the surface integral of cross-correlated wavefield perturbations to a volume integral whose integrand is proportional to the medium perturbation and cross-correlations between unperturbed waves and wavefield perturbations. Note that equation 26 does not depend on the source term $q$ (equation 15); the same holds for the reciprocity theorem in equation 14. Although not immediately applicable for the reconstruction of the Green’s function as equations 19 or 20, equation 26 is suitable for other purposes in remote sensing experiments. These purposes are discussed in the next Section.

**Example of Case I.**

We present here a stationary-phase derivation where we evaluate all terms in equation 21 for the model in Figure 5. In this model, the background medium is a homogeneous space and the perturbation volume $P$ is characterized by a thin-sheet where $\kappa - \kappa_0 \neq 0$ and is
constant. This sheet extends infinitely in the horizontal \([x, y]\)-plane and is placed far away from the observation points. The medium perturbation is such that it generates scattering. Since the perturbation sheet is thin (i.e., its thickness is orders of magnitude smaller than the dominant wavelength), waves reflected off the sheet can be thought of as single-scattered waves. Likewise, the waves transmitted across the sheet differ from unperturbed direct-waves only by a transmission amplitude factor.

To perform our stationary-phase evaluation, we begin with equation 21, which is of the form

\[
G_S(r_B, r_A) = \int_{\Omega} G_S(r_B, r_A) \cdot dS + \int_{\Omega_1} G_S(r_B, r_A) \cdot dS ;
\]

where for brevity we omit the integrands of equation 21. Although in principle it is possible to evaluate both the surface and the volume integrals on the right-hand side of equation 27 using the stationary-phase method (Bleistein & Handelsman, 1975), instead we evaluate the volume integral according to

\[
\int_{r \in \Omega_1} G_S(r_B, r_A) \cdot dS + \int_{r \in \Omega_2} G_S(r_B, r_A) \cdot dS ;
\]

The total surface \(\partial V\) in Figure 5 consists of two surfaces \(\partial V_t\) and \(\partial V_b\) parallel to the \(x, y\)-plane on the top and bottom of the perturbation (Figure 6). The total contribution of the surface integral is given by

\[
\int_{r \in \partial V} G_S(r_B, r_A) \cdot dS = \int_{r \in \partial V_t} G_S(r_B, r_A) \cdot dS + \int_{r \in \partial V_b} G_S(r_B, r_A) \cdot dS .
\]

For simplicity, in this example we consider waves only in the far-field regime (see Appendix A and Born & Wolf (1959). Moreover, we simplify the surface integral in equation 21 by imposing the far-field radiation boundary condition introduced by Wapenaar et al. (2005), where \(\nabla G \cdot dS = (\omega/c) G \cdot dS \). Under these conditions, the integration over the top surface (Appendix A) is

\[
\int_{r \in \partial V_t} \frac{2}{\rho_c} G_S(r_B, r_A)G^*_S(r_B, r_B) \, dS = \frac{1}{\omega \cos \theta} G_S(r_B, r_A); \tag{30}
\]

where \(\theta\) is the scattered-wave incidence angle at the stationary point (Figure 7b). The equation above shows that with the top sources alone we can retrieve the desired response \(G_S(r_B, r_A)\). This is an example of the case described by equation 22 and illustrated more generally by Figure 2a. For all arbitrary sources on the top surface (Figure 7a), the dominant contribution to equation 30 comes from the stationary source position represented in Figure 7b, where the path of the unperturbed wave observed at \(r_B\) is aligned with the portion of the down-going lag of the reflected wave recorded at \(r_A\) that passes through \(r_B\). Note that to obtain \(G_S(r_B, r_A)\) from equation 30, we must multiply it by \(-\omega \cos \theta\). The multiplication by \(\omega \cos \theta\) translates into a time-domain differentiation (Snieder et al., 2006; Snieder, 2007). Multiplying equation 30 by \(\cos \theta\) is possible for this theoretical example, but unpractical in an experiment where the depth of the sheet is unknown. If the sheet is far from the observation points such that \(\cos \theta \approx 1\), equation 30 (along with the time-domain differentiation) yields \(G_S(r_B, r_A)\) exactly. When the sheet is close to the observation points and \(\cos \theta \approx 0\), the amplitude of

---

**Figure 7.** Cartoons representing the stationary-phase evaluation of the integral over the top plane \(\partial V_t\). We choose the depth of the integration plane to be \(z = 0\); while the perturbation sheet lies at \(z = D\). In panel (a) we depict an arbitrary source position \(r\), for which the corresponding unperturbed and scattered waves are denoted by the solid and dotted arrows, respectively. The point \(r\) in panel (b) is the stationary source point on \(\partial V_t\) that yields the causal scattered wave that propagates from \(r_B\) to \(r_A\). Panel (b) illustrates the stationary condition which requires \(\psi_B = \theta_A = \theta\).
$G_S(r_B, r_A)$ is severely distorted (equation 30). As we discuss below, the proper compensation for this effect comes from the volume integral in equations 27 or 24.

The asymptotic evaluation of the integral over the bottom surface gives

$$\int_{r \in \partial V_b} \frac{2}{\rho c} G_S(r, r_A) G_0^*(r, r_B) \, dS = -\frac{t}{i \omega \cos \psi'} G_0^*(r_B, r_A); \quad (31)$$

where $t$ is a transmission coefficient (Appendix A), and the angle $\psi'$ is the stationary-phase angle defined in Figure 8b. The bottom integration results in an arrival corresponding to the unperturbed wave that propagates between $r_B$ and $r_A$. Because of the complex conjugation in equation 31, the resulting arrival is anticausal in the time-domain. The bottom surface integral in equation 31 results in $G_0$ instead of the desired $G_S(r_B, r_A)$, so its contribution must be canceled by the volume integral (e.g., equation 27). This shows that for sources on the bottom surface, the volume integrals cannot be ignored in reconstructing $G_S$. This case is an example of Case II (i.e., for source positions like $r_2$) in Figure 2b.

Once the surface integrals in equations 30 and 31 are evaluated, we can derive the contribution of the volume integrals using equation 28. Moreover, because we separated the contributions from the top and bottom surfaces, we can also separate the volume integral contributions that correspond to the stationary points on each surface. For sources on the top surface (Figures 6 and 7), only one of the two volume integrals in equation 21 is nonzero, following the discussion regarding $G_0(r_B, r_A)$.

Case I. So, using equations 28, 30 and 21 we obtain

$$\int_{r \in V} i \omega (\kappa_0 - \kappa) G_S(r, r_A) G_0^*(r, r_B) \, dV = \left( 1 + \frac{1}{i \omega \cos \theta} \right) G_S(r_B, r_A). \quad (32)$$

For sources on the bottom surface both volume integrals in equation 21 must be taken into account. Therefore, from equations 28, 31 and 21,

$$\int_{r \in V} i \omega (\kappa_0 - \kappa) G(r, r_A) G_0^*(r, r_B) \, dV = \frac{t}{i \omega \cos \psi'} G_0^*(r_B, r_A). \quad (33)$$

The volume integral in equation 32 provides the amplitude correction necessary for the result of the surface integral in equation 30 to be equal to $G_S(r_B, r_A)$. While the integral in equation 33 cancels the undesired arrival generated by the surface integral in equation 31.

4 EXTRACTING THE PERTURBED GREEN’S FUNCTION

Retrieving $G_S$ from random sources in $V$

The first application we propose for the representation theorems presented in the previous Section arises when homogeneous boundary conditions apply at the surface, canceling the surface integral in equation 19. This yields

$$G_S(r_B, r_A) = \int_{r \in V} i \omega (\kappa_0 - \kappa) G(r, r_A) G_0^*(r, r_B) \, dV. \quad (34)$$
When these homogeneous boundary conditions apply (see derivation of equation 17), the pressure observed at any given observation point \( r_0 \) is given by

\[
p_0(r_0) = \int G_0(r, r_0)q(r)dV
\]

\[
p(r_0) = \int G(r, r_0)q(r)dV;
\]

(35)

for the unperturbed and perturbed states, respectively. \( q \) is the source term in equations 1 and 2. Next we consider random uncorrelated sources throughout the volume \( V \), such that

\[
q(r_1, \omega)q(r_2, \omega) = \Delta \kappa(r_1, \omega) \delta(r_1 - r_2) \vert R(\omega) \vert^2;
\]

(36)

where \( \Delta \kappa = \kappa_0 - \kappa \) and \( \vert R(\omega) \vert^2 \) is the power spectrum of a random excitation function. Note from equation 36 that the source intensity is proportional to the local perturbation \( \Delta \kappa \) at every source position. We then multiply equation 34 by \( \vert R(\omega) \vert^2 \) to obtain

\[
G_S(r_B, r_A)\vert R(\omega) \vert^2
\]

\[
= i\omega \int \Delta \kappa(r_1, \omega) \delta(r_1 - r_2) \vert R(\omega) \vert^2
\]

\[
G(r_1, r_A)G_0(r_2, r_B)dV_1dV_2
\]

\[
= i\omega \int G(r_1, r_A)q(r_1)dV_1
\]

\[
\left( \int G_0(r_2, r_B)q(r_2)dV_2 \right)^*.
\]

(37)

Using the definitions in equation 36, equation 37 yields

\[
G_S(r_B, r_A) = \frac{i\omega}{\vert R(\omega) \vert^2} p(r_A)p_0^* (r_B).
\]

(38)

This equation shows that the perturbation response between \( r_B \) and \( r_A \) can be extracted simply by cross-correlating the perturbed pressure field observed at \( r_A \) with the unperturbed pressure measured at \( r_B \). This cross-correlation must be compensated for the spectrum \( \vert R(\omega) \vert^2 \) and multiplied by \( i\omega \) (i.e., differentiated with respect to time). The result in equation 38 also holds for perturbed attenuative media, as we show with equation B4 in Appendix B. Unlike the volume sources in the lossless case in equation 36, the random volume sources in attenuative media (equation B3 in Appendix B) are locally proportional to both the real and imaginary parts of the compressibility perturbation, and to the attenuation in the unperturbed medium.

Equation 38 is not practical because volume sources like those described by equation 36 are not easily reproducible in either lab or field experiments. The real usefulness of equation 38 lies in understanding the energy partitioning requirements for the reconstruction of the desired scattered-wave response. Let us consider, for example, a medium configuration like the one described by Figure 2. In that case, according to equation 36, the volume sources would only exist inside the perturbation volume \( P \). This results in a nonzero net energy flux that is outgoing at the boundary of \( P \). As a consequence, there is also a preferred direction of energy flux at the observation points. This situation is completely different than the condition of equipartitioning required for the reconstruction of \( G_0 \) or \( G \) (Wapenaar et al., 2006; Snieder et al., 2007), which requires that the total energy flux within any direction at the receivers be equal to zero. Vassconcelos (2007) provides a detailed discussion on the energy considerations of retrieving the medium’s perturbation responses for general physical systems.

**Retrieving \( G_S \) from coherent sources on \( \partial V \)**

Equations 22 yields \( G_S(r_B, r_A) \): the perturbed waves that propagate between \( r_B \) and \( r_A \), as if the observation point at \( r_B \) acts as a source. This is accomplished by the cross-correlation of unperturbed fields observed at \( r_B \) with field perturbations at \( r_A \). Let us consider coherent sources over \( \partial V_1 \) such that the observed fields are given by

\[
p(r_A, r_B) = W(r, \omega)G(r_A, r_B)
\]

and

\[
v(r_A, r_B) = (i\omega\rho)^{-1}W(r, \omega)\nabla G(r_A, r_B),
\]

where \( W(r, \omega) \) is an excitation function. The observed unperturbed fields are defined analogously to the perturbed ones. In this case, the desired response \( G_S(r_B, r_A) \) can be obtained by

\[
G_S(r_B, r_A) \approx \int_{\partial V_1} \mathcal{F}(\omega)p_B(r_A)v_S(r_A) - dS;
\]

(39)

where \( \mathcal{F}(\omega) = \mathcal{F}(r, \omega) \) is a waveform shaping filter designed to suppress the effect of \( |W| |\omega| \), and is ideally given by

\[
\mathcal{F}(r, \omega) = [i\omega|\omega|W(r, \omega)]^{-1}.
\]

There are experiments where the measurement of the particle velocity field is not available (e.g., in most marine seismic exploration surveys). In that case we can use a far-field radiation condition to set \( \nabla G \cdot dS = (i\omega/c)G dS \) (Wapenaar et al., 2005), which simplifies equation 22 to

\[
G_S(r_B, r_A) \approx \int_{\partial V_1} \frac{2}{pc} G_S(r_A)G_0(r_B) dS.
\]

(40)

Then using the same observed fields present in equation 39, we can extract \( G_S(r_B, r_A) \) by

\[
G_S(r_B, r_A) \approx \int_{\partial V_1} \mathcal{F}'(\omega)p_B(r_A)p_0^*(r_B) dS;
\]

(41)

where now \( \mathcal{F}'(\omega) = \mathcal{F}'(r, \omega) \) is a shaping filter with the same objective of the filter \( \mathcal{F}(\omega) \) in equation 39, given by

\[
\mathcal{F}'(r, \omega) = 2|\omega|c|W(r, \omega)|^{-1}.
\]

**5 DISCUSSION AND CONCLUSION**

By manipulating the acoustic wave equations for unperturbed and perturbed media we derive convolution- and correlation-type reciprocity theorems for perturbed acoustic media. These theorems differ from previous forms of reciprocity theorems (de Hoop, 1988; Fokkema & van den Berg, 1993) because they provide explicit
relations between the wavefields and wavefield perturbations. Note that the wavefield perturbations by themselves do not satisfy the perturbed wave equations. We present a suite of integral equations based on the perturbed acoustic wave equations that may be useful both for theoretical considerations and for applications in imaging acoustic scattered waves and in inferring medium perturbations.

When the fields are described by Green’s functions observed at two distinct points, one of the convolution-type theorems presented here retrieves the field perturbations that propagate between these two points through a volume integral that is similar in form to the Lippmann-Schwinger integral (Rodberg & Thaler, 1967). This convolution-type theorem result in the Lippmann-Schwinger integral when the two observation points coincide. With the Green’s function form of the correlation-type representation theorems, we show that by cross-correlating wavefield perturbations measured at one receiver with unperturbed waves recorded by another we obtain the wavefield perturbations that propagate between the receivers as if one of the receivers were a source. This concept relates to previous work in the field of Green’s function retrieval from diffuse-wave correlation (Weaver & Lobkis, 2001; Malcolm et al., 2004; Larose et al., 2006) and from correlation of deterministic wavefields (Wapenaar, 2004; Snieder, 2004; Wapenaar & Fokkema, 2006). These studies show that cross-correlations can be used to recover a superposition of the causal and anticausal parts of the wavefields $G$ or $G_0$ (i.e., unperturbed or perturbed). Our expressions recover the wavefield perturbations $G_S$ separately from its anticausal counterpart $G_S^\ast$. For systems that are invariant under time reversal, Green’s function retrieval by wavefield cross-correlations require only a surface integration (Larose et al., 2006; Wapenaar & Fokkema, 2006; Snieder et al., 2007), whereas the retrieval of the perturbations $G_S$ from correlations of wavefield perturbations with unperturbed wavefields requires an additional volume integral.

The extraction of wavefield perturbations $G_S$ that propagate between receivers as if one of them acts as a source can be a useful tool for remote sensing experiments such as those in medical imaging, ocean acoustics and seismology, for example. When medium perturbations are localized within a volume $\mathbb{P}$ that does not contain the observation points, the wavefield perturbations propagating between receivers can be obtained only from the surface integral of cross-correlated perturbations and unperturbed wavefields (i.e., the volume integral can be neglected). Neglecting the volume integrals can yield distortions in the reconstructed response. The extracted perturbed Green’s function can be used for the purpose of detecting, locating or monitoring medium perturbations.

When the boundary conditions are such that the surface integral vanishes, the field perturbations propagating between two receivers can be extracted from the cross-correlation of unperturbed fluctuations measured at one receivers with the perturbed response measured at the other. This only holds if the measured responses are due to sources distributed throughout the entire volume that are locally proportional to the medium perturbation. Although this result is of limited practical applications, it gives an important insight into the energy considerations for reconstructing only the field perturbations. While energy equipartitioning is required for the accurate reconstruction of the perturbed or unperturbed wavefields, the reconstruction of the wavefield perturbations (e.g., scattered waves only) requires a non-zero net energy flux at the observation points. This is discussed in detail in Vasconcelos (2007).

We have proposed direct applications of the interferometric retrieval of wavefield perturbations as proposed in this paper (Vasconcelos & Snieder, 2007; Vasconcelos et al., 2007). Vasconcelos et al. (2007) use the retrieval of wavefield perturbations according to the formulation in this paper to image salt structures in the Earth’s subsurface from the interference of multiply scattered waves measured inside a deep borehole in off-shore Gulf of Mexico. Vasconcelos and Snieder (2007) use the concepts developed here to interpret the physical meaning of their deconvolution-based interferometry approach. Their deconvolution interferometry method (based partly on equations 21 and 40) has been validated with numerical experiments (Vasconcelos & Snieder, 2008; Vasconcelos et al., 2007a, Vasconcelos et al., 2007b) and has been applied to imaging with drill-bit noise.

The interferometric retrieval of wavefield perturbations we describe here can be used for targeting the imaging of particular portions of the medium (Vasconcelos et al., 2007a) where this technique is called target-oriented interferometry. Such an application has also been implicitly proposed by Bakulin and Calvert (2006) and by Mehta et al. (2007), in the so-called Virtual Source method. With this method, transmission and reflection responses can be regarded as unperturbed waves and wavefield perturbations, respectively. Although most of the examples cited come from the field of geophysics, our results are immediately applicable to other fields in acoustics such as physical oceanography, laboratory and medical ultrasonics, and non-destructive testing.

Another important potential use for the exact form of the correlation-type representation theorems that express the wavefield perturbations that propagate between two receivers lies in the calculation of Fréchet derivatives (Tarantola, 1987; Hettlich, 1998; Sava & Biondi, 2004), which consist of the partial derivatives of the wavefield perturbations with respect to the medium perturbations. These derivatives which can be derived from the theorems we provide here. For example, introducing an infinitesimal medium parameter change in
equation 25 yields the derivative $\partial G_S/\partial \kappa(r)$. These derivatives are important for the computation of sensitivity kernels used in full-waveform inversion (Tarantola, 1987), in imaging (Hettlich, 1998) or in linearized formulations of wave-equation based tomography (Sava & Biondi, 2004).

From correlation- and convolution-type representation theorems in perturbed media, we suggest the application of estimating the medium perturbations by combining theorems presented here with inverse scattering theory (Weglein, et al., 2003; Malcolm et al., 2004). This type of approach can potentially be used for imaging of the wavefield perturbations (Tarantola, 1987; Weglein, et al., 2003), as well as for the targeting the extraction of a particular desired subset of the wavefield perturbations (Malcolm et al., 2004). In this context, the use of our expressions can be simplified by linearizing the wavefield perturbations in the medium changes (one such approximation is given by equation 25). This would yield, for example, a Born approximation (Born & Wolf, 1959; Snieder, 1990; Weglein et al., 2003) of the theorems presented here.

Apart from imaging applications, our results (both in terms of retrieving wavefield perturbations and for estimating medium perturbations) can be used for monitoring temporal changes in the medium. In geoscience, this could be applied to remotely monitoring the depletion of aquifers or hydrocarbon reservoirs; or monitoring the injection of CO$_2$ for carbon sequestration. In material science, our results can be used to monitor material integrity with respect, for example, to temporal changes in temperature or changes due to crack formation. The detection of earthquake damage is a potential application in the field of structural engineering. Within medical imaging applications, our expressions can be tailored, for instance, to observe tumor evolution from a series of time-lapse ultrasonic measurements.

6 ACKNOWLEDGMENTS

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APPENDIX A: STATIONARY-PHASE EVALUATION OF THE THIN-LAYER CASE.

Integration over \(\partial V_i\).

The wavefields shown in Figure 7 can be described by the ray-geometric impulse responses

\[
G_0(r, r_B) = -\frac{\rho}{4\pi|r_B - r|} e^{ik|r_B - r|}, \text{ and } \quad G_S(r, r_A) = -r^2 \rho e^{ik(|r_A' - r_A| + |r - r_A'|)}.
\]

where \(r_A'\) are the specular reflection points for the receiver-source pairs \((r, r_A)\). \(G_0\) and \(G_S\) are the far-field acoustic Green’s functions we use to describe direct- and specularly-reflected waves, respectively. \(r\) is the reflection coefficient, which for simplicity we assume to be constant with respect to incident angle. In our model, \(r = (x, y, z = 0)\) for \(r \in \partial V_i\) and \(r_{A,B} = (x_{A,B}, y_{A,B} = 0, z_{A,B})\). The distances in the phases and denominators in equation A1 can be expressed in terms of the corresponding ray-lengths in Figure 7. Using the Green’s functions in equation A1 to express

\[
I_{\partial V_i} = \int_{r \in \partial V_i} \frac{2}{\rho c} G_S(r, r_A) G_0(r, r_B) dS,
\]

we get

\[
I_{\partial V_i} = \frac{2\rho r}{c(4\pi)^2} \int \frac{e^{ik(L_1 + L_2 - L_B)}}{(L_1 + L_2) L_B} dx dy,
\]

where \(\varphi = ik(L_1 + L_2 - L_B)\) is the phase of the integrand, and \(k = \omega/c\). \(L_B = |r_B - r|\) is the direct-wave path in Figure 7; and \(L_1 = |r_A' - r|\) and \(L_2 = |r_A - r_A'|\) are the down- and up-going lags of the reflected-wave path in Figure 7, respectively. The source position that gives a stationary contribution to the integral in equation A3 satisfies (Snieder et al., 2006)

\[
\theta_A = \psi_B
\]

and

\[
y = 0,
\]

where \(\psi_B\) is the angle defined between the direct wave and the vertical at \(r_B\), and \(\theta_A = \theta\) is the incidence angle, at the interface, of the reflected wave recorded at \(r_A\) (Figure 7).

Because \(y_{A,B} = 0\) and the model is a flat reflector in a homogeneous and isotropic medium, the stationary contribution for all terms comes from sources at \(y = 0\) The condition \(\theta_A = \psi_B\) (equation A4) states that the stationary source is the one that sends a direct wave which is passes straight through \(r_B\), gets reflected and is then recorded at \(r_A\). This is the same stationary condition as for the \(C_{AB}^*\) term analyzed by Snieder et al. (2006).

To approximate the integral in equation A3 with the stationary-phase method we must evaluate, at the stationary point, the second derivatives

\[
\frac{\partial^2 L_1}{\partial x^2} = \frac{L_1}{(L_1 + L_2)^2} \cos^2 \theta, \quad (A6)
\]

\[
\frac{\partial^2 L_2}{\partial x^2} = \frac{L_2}{(L_1 + L_2)^2} \cos^2 \theta, \quad (A7)
\]

and

\[
\frac{\partial^2 L_1}{\partial y^2} = \frac{L_1}{(L_1 + L_2)^2}; \quad (A9)
\]

\[
\frac{\partial^2 L_2}{\partial y^2} = \frac{L_2}{(L_1 + L_2)^2} \quad (A10),
\]

Based on these second derivatives, the zero-order stationary-phase approximation (Bleistein and Handelsman, 1975) to equation A3 is

\[
I_{\partial V_i} = \frac{2\rho r}{c(4\pi)^2} \frac{e^{ik(L_1 + L_2 - L_B)}}{(L_1 + L_2) L_B} \left( e^{-i\pi/4} \right)^2 \frac{2\pi}{k \cos \theta} \times \frac{1}{L_B - \frac{1}{L_1 + L_2}}^{-1},
\]

which yields

\[
I_{\partial V_i} = \frac{2\rho r}{c(4\pi)^2} \frac{e^{ik(L_1 + L_2 - L_B)}}{\left( i\omega \cos \theta \right)} (L_1 + L_2 - L_B).
\]

At the stationary source point, where \(\theta_A = \psi_B\), the distance \(L_1 + L_2 - L_B\) is equivalent to the ray-length of the scattered wave that propagates between \(r_A\) and \(r_B\).
(see Figure 7b). Thus, in the stationary-phase approximation, \( I_{\theta V_l} \) is given by

\[
I_{\theta V_l} = \frac{1}{-i\omega \cos \theta} G_s(r_B, r_A) ; \tag{A13}
\]

showing how the \( I_{\theta V_l} \) term results in a causal scattered wave propagating from \( r_B \) to \( r_A \).

**Integration over \( \partial V_b \).**

The stationary-phase evaluation of the bottom surface integral is analogous to that of the top surface integral, but with the impulse responses given by

\[
G_0(r, r_B) = -t \rho e^{ik|r_B - r|}/4\pi |r_B - r| ,
\]

and

\[
G_s(r, r_A) = -t \rho e^{i\kappa |r_A - r|}; \tag{A14}
\]

where \( t \) is a transmission coefficient, such that the total transmission coefficient in \( G(r, r_A) \) is given by \((1 + t)\). The bottom surface integral can be then written as

\[
I_{\theta V_b} = \int_{r \in \partial V_b} \frac{2 \rho e^{ik|r_B - r|}}{4\pi |r_B - r|} G_s(r, r_A) G_0^*(r, r_B) \, dS
\]

\[
= \frac{2pt}{c(4\pi)^2} \int e^{ik(L_A - L_B)} \, dx dy , \tag{A15}
\]

with \( L_B \) and \( L_A \) being the ray-lengths corresponding to the solid and dashed arrows in Figure 8, respectively. These lengths are defined by

\[
L_{A,B} = \sqrt{(x - x_{A,B})^2 + y^2 + (2D - z_{A,B})^2} . \tag{A16}
\]

Given that the phase of the integrand on the right-hand side of equation A15 is controlled by \( \phi' = (L_A - L_B) \), the stationary source point can be found by setting

\[
0 = \frac{\partial \phi'}{\partial y} = \frac{y}{L_A} - \frac{y}{L_B} , \tag{A17}
\]

and

\[
0 = \frac{\partial \phi'}{\partial x} = \frac{x - x_A}{L_A} - \frac{x - x_B}{L_B} = \sin \psi_{A}' - \sin \psi_{B}' ; \tag{A18}
\]

the angles \( \psi_{A}' \) and \( \psi_{B}' \) are defined in Figure 8. It follows from equations A17 and A18 that the phase of the integrand of equation A15 is stationary when

\[
\psi_{A}' = \psi_{B}' = \psi' \quad \text{and} \quad y = 0 . \tag{A19}
\]

Since the sources are restricted to the bottom surface, the condition in equation A19 is satisfied only when the ray-paths in Figure 8a are aligned as in Figure 8b.

Next we evaluate, at the stationary point (equation A19), the second derivatives

\[
\frac{\partial^2 \phi'}{\partial x^2} = \frac{(2D - z_{A,B})^2}{L_A^2} - \frac{(2D - z_{B})^2}{L_B^2} = \cos^2 \psi' \left( \frac{1}{L_A} - \frac{1}{L_B} \right) , \tag{A20}
\]

and

\[
\frac{\partial^2 \phi'}{\partial y^2} = \frac{L_A^2}{L_A} - \frac{L_B^2}{L_B} = \frac{1}{L_A} - \frac{1}{L_B} . \tag{A21}
\]

The second derivatives are used to approximate the integral in equation A15, according to

\[
I_{\theta V_b} = \frac{2pt}{c(4\pi)^2} \exp \left( \frac{ik(L_A - L_B)}{L_{A,B}} \right) \times e^{-i\pi/4} \sqrt{\frac{2\pi}{k}} \sqrt{\frac{1}{\cos^2 \psi' \left( \frac{1}{L_A} - \frac{1}{L_B} \right)}} \times e^{-i\pi/4} \sqrt{\frac{2\pi}{k}} \sqrt{\frac{1}{L_A} - \frac{1}{L_B}} , \tag{A22}
\]

which gives

\[
I_{\theta V_b} = \frac{pt}{-i\omega \cos \psi'} \exp \left( \frac{ik(L_A - L_B)}{L_{A,B}} \right) . \tag{A23}
\]

Note that at the stationary point shown in Figure 8b, \( L_B - L_A \) is the ray-length of an unperturbed wave propagating between \( r_B \) and \( r_A \). Therefore, equation A23 can be represented by

\[
I_{\theta V_b} = \frac{t}{-i\omega \cos \psi'} G_0^*(r_B, r_A) . \tag{A24}
\]

**APPENDIX B: EXTENSION TO ATTENUATIVE MEDIA**

While in the main text we only treat the case of acoustic waves in loss-less media, incorporating attenuation into our expression can be done in a straightforward manner by making \( \kappa_0 \) and \( \kappa \) a complex quantity (Snieder, 2007; Douma, 2008), and by replacing \((\kappa_0 - \kappa)\) by \((\kappa_0^* - \kappa)\) where appropriate (with the exception of equation 20, where it should be replaced by \((\kappa_0 - \kappa^*)\)). Then, for example, equation 19 in attenuative media reads

\[
G_s(r, r_A) = \int_{r \in \partial V} \frac{1}{i\omega \rho} [G_s(r, r_A) \nabla G_0^*(r, r_B) + G_0^*(r, r_B) \nabla G_s(r, r_A)] \, dS
\]

\[
- \int_{r \in V} i\omega \Re \{\Delta \kappa'\} G(r, r_A) G_0^*(r, r_B) \, dV
\]

\[
+ \int_{r \in V} 2i\omega \Im \{\kappa_0\} G(r, r_A) G_0^*(r, r_B) \, dV
\]

\[
- \int_{r \in V} \omega \Im \{\Delta \kappa'\} G(r, r_A) G_0^*(r, r_B) \, dV ; \tag{B1}
\]

where \( \Re \) and \( \Im \) stand for the real and imaginary parts of a complex quantity, respectively, and \( \Delta \kappa' = \kappa_0^* - \kappa \).

The first volume integral in equation B1 yields the integrals in equation 19, while the other volume integral account for attenuation. Note that in attenuative media, even if there’s no perturbation (i.e., \( \Delta \kappa' = 0 \)), the second volume integral in equation B1 is nonzero. This case is analyzed by Snieder (2007).

If homogeneous boundary conditions apply on \( \partial V \)
that cancel the contribution of the volume integral (equation 34), equation B1 simplifies to

\[
G_S(r_B, r_A) = \int_{V} \omega \Re \{(\kappa_0 - \kappa)\} G(r, r_A)G^*_0(r, r_B) dV
+ \int_{V} \omega \Im \{(\kappa_0 - \kappa)\} G(r, r_A)G^*_0(r, r_B) dV. \tag{B2}
\]

We now consider random volume sources similar to equation 36 but now described by

\[
q(r_1, \omega)q(r_2, \omega) = \Delta \kappa'(r_1, \omega) \delta(r_1 - r_2) |R(\omega)|^2. \tag{B3}
\]

Note that at every point in the volume, equation B3 describes a superposition of three sources which are locally proportional to \(i \Re \{\Delta \kappa'\}\), \(\Im \{\Delta \kappa'\}\) and \(\Im \{\kappa_0\}\), respectively. Through a derivation analogous to equation 37, equation B2 gives

\[
G_S(r_B, r_A) = \frac{i \omega}{|R(\omega)|^2} p(r_A)p^*_0(r_B), \tag{B4}
\]
same as in equation 38.

When the surface integral is nonzero, the volume integrals cannot be ignored. In particular, for attenuative media the discussion that leads equation 22 is not valid because the second volume integral in equation B1 above in general cannot be neglected. Nevertheless, results from field exploration seismology surveys (Bakulin & Calvert, 2006; Mehta et al., 2007; Vasconcelos et al., 2007) suggest that only evaluating the surface integral retrieves the desired with correct kinematics and geometrical spreading. It possible that in some cases (like Case I, Figure 2) the volume integrals provide an amplitude correction to what is extracted from the surface integral alone, but this issue is yet to be resolved (Snieder, 2007).
Numeric implementation of wave-equation migration velocity analysis operators

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ABSTRACT

Wave-equation migration velocity analysis (MVA) is a technique similar to wave-equation tomography because it is designed to update velocity models using information derived from full seismic wavefields. On the other hand, wave-equation MVA is similar to conventional, traveltime-based MVA because it derives the information used for model updates from properties of migrated images, e.g. focusing and move-out. The main motivation for using wave-equation MVA is derived from its consistency with the corresponding wave-equation migration, which makes this technique robust and capable of handling multipathing characterizing media with large and sharp velocity contrasts. The wave-equation MVA operators are constructed using linearizations of conventional wavefield extrapolation operators, assuming small perturbations relative to the background velocity model. Similarly to typical wavefield extrapolation operators, the wave-equation MVA operators can be implemented in the mixed space-wavenumber domain using approximations of different orders of accuracy. As for wave-equation migration, wave-equation MVA can be formulated in different imaging frameworks, depending on the type of data used and image optimization criteria. Examples of imaging frameworks correspond to zero-offset migration (designed for imaging based on focusing properties of the image), survey-sinking migration (designed for imaging based on moveout analysis using narrow-azimuth data) and shot-record migration (also designed for imaging based on moveout analysis, but using wide-azimuth data). The wave-equation MVA operators formulated for the various imaging frameworks are similar because they share common elements derived from linearizations of the single square-root equation. Such operators represent the core of iterative velocity estimation based on diffraction focusing or semblance analysis, and their applicability in practice requires efficient and accurate implementation. This tutorial concentrates strictly on the numeric implementation of those operators and not on their use for iterative migration velocity analysis.

Key words: wave-equation, migration, velocity analysis

1 INTRODUCTION

Accurate wave-equation depth imaging requires accurate knowledge of a velocity model. The velocity model is used in the process of wavefield reconstruction, irrespective of the method used to solve the acoustic wave-equation, e.g. by integral methods (Kirchhoff migration), or by differential/spectral methods (migration by wavefield extrapolation and reverse-time migration).

Generally-speaking, there are two possible strategies for velocity estimation from surface seismic data in the context of wavefield depth migration. The two strategies differ by the domain in which the information used to update the velocity model is collected. The first strategy is formulated in the data space, prior to migration, and it involves matching of recorded
and simulated data using an approximate (background) velocity model. Techniques in this category are known by the name of tomography (or inversion). The second strategy is formulated in the image space, after migration, and it involves measuring and correcting image features that indicate model inaccuracies. Techniques in this category are known as migration velocity analysis (MVA), since they involve migrated images and not the recorded data directly.

Tomography and migration velocity analysis can be implemented in various ways that can be classified based on the carrier of information from the data or image to the velocity model. Thus, we can distinguish between ray-based methods and wave-based methods. This terminology is applicable to both tomography and migration velocity analysis. For ray-based methods, the carrier of information are wide-band rays traced using a background velocity model from picked events in the data (or image). Methods in this category are known as traveltime tomography (Stork, 1992; Al-Yahya, 1987; Fowler, 1988; Etgen, 1990; Chavent & Jacewitz, 1995; Clement et al., 2001; Chauris et al., 2002a,b; Billette et al., 2003; Lambare et al., 2004; Clapp et al., 2004). For wave-based methods, the carrier of information are band-limited wavefields constructed using a background velocity model. Methods in this category are known as wave-equation tomography (Gauthier et al., 1986; Tarantola, 1987; Mora, 1989; Woodward, 1992; Pratt, 1999, 2004), and wave-equation MVA (Biondi & Sava, 1999; Sava & Biondi, 2004a,b; Shen et al., 2005; Albertin et al., 2006b; Maharramov & Albertin, 2007). This paper concentrates on methods from the latter category.

The volume of information used for model updates using wave-based methods is at least one order of magnitude larger than the volume of information used for ray-based methods. Thus, a fundamental question we should ask is what is gained by using wave-based methods over ray-based methods. First, modern imaging applications using wave-based methods (downward continuation or reverse-time extrapolation) require consistent velocity estimation methods which interact with model in the same frequency band as the migration methods. Second, wave-based methods are robust (i.e. stable) for models with large and sharp velocity variations (e.g. salt). Third, wave-based methods describe naturally all propagation paths, thus they can easily handle multi-pathing characterizing wave propagation in media with large velocity variations.

Wave-equation tomography and wave-equation MVA have both similarities and differences. Wave-equation tomography uses the advantage that the residual used for velocity updating is obtained by a direct comparison between recorded and measured data. In contrast, wave-equation MVA uses the property that the residual used for velocity updating is obtained by a comparison between a reference image and an improved version of it. On the other hand, wave-equation tomography has the disadvantage that the kinematics of events in the data domain are more complex than in the image domain. In addition, the dimensionality of the space in which to evaluate the misfit between recorded and simulated data is higher too, potentially making a comparison more complex. Also, wave-equation MVA optimizes directly the desired end product, i.e. the migrated image, which potentially makes this technique more “interpretive” and less of a computational “black-box”.

One important component of MVA methods is what type of measurement on migrated images is used to evaluate its deficiencies. Although strictly related to one-another, we can describe two kinds of information available to quantify image quality. First is focusing analysis, which evaluates whether point-like events, e.g. fault truncations, are focused in migrated images at their correct position. Image enhancement for wave-equation MVA can be formulated purely based on this type of information, which makes the techniques fast since it only operates with zero-offset data (Sava et al., 2005). Second is moveout analysis, which is the case for all conventional MVA techniques, whether using rays or waves. In this case, we can formulate wave-equation MVA based on analysis of move-out of common-image gathers using velocity scans (Biondi & Sava, 1999; Sava & Biondi, 2004a,b) or based on analysis of differential semblance of nearby traces in similar common-image gathers (Shen et al., 2005).

Moveout analysis requires construction of common-image gathers (CIGs) characterizing the dependence of reflectivity function of various parameters used to parametrize multiple experiments used for imaging. There are two main alternatives for common-image gather construction. First, we can construct offset-domain CIGs (Yilmaz & Chambers, 1984), when reflectivity depends on source-receiver offset on the acquisition surface, which is a data parameter. Second, we can construct angle-domain CIGs (de Bruin et al., 1990; Prucha et al., 1999; Mosher & Foster, 2000; Rickett & Sava, 2002; Xie & Wu, 2002; Sava & Fomel, 2005; Soubaras, 2003; Fomel, 2004; Biondi & Symes, 2004), when reflectivity depends on the angles of incidence at the reflection point, which is an image parameter. For wave-equation migration, offset-domain CIGs are not a practical solution, since the information from multiple offsets (or multiple seismic experiments) mixes in the process of migration. Furthermore, angle-domain CIGs suffer from fewer artifacts than offset-domain CIGs, due to the fact that reflectivity parametrization for angle-gathers occurs after wavefield reconstruction in the subsurface, as oppose to offset-gathers when reflectivity parametrization is related to data parameters (Stolk & Symes, 2004).

Both wave-equation tomography and wave-equation MVA methods are based on a fundamental “small perturbation” assumption, which requires a reasonably-good starting model. This requirement represents a drawback which is responsible for the main difficulty of methods in both categories. For wave-equation tomography or inversion, we can update models based on differences between recorded and simulated data. If the starting background model is not accurate enough, we run the risk of subtracting wavefields corresponding to different events. Likewise, for wave-equation MVA, we update the model based on differences between two images, one simulated in the background model and an enhanced version of this image. If the enhanced version of the image goes too far from the reference image, we run the risk of subtracting events
corresponding to different reflectors. This phenomenon is usually referred to in the literature as "cycle skipping" and various strategies have been designed to ameliorate this problem, e.g. by optimal selection of frequencies used for velocity analysis (Sirgue & Pratt, 2004; Albertin, 2006). However, alternative methods used to evaluate image accuracy, e.g. differential semblance (Symes & Carazzone, 1991; Shen et al., 2003), have the best potential to ameliorate this situation. In this case wave-equation MVA analyzes difference between image traces within common-image gathers which are likely to be similar enough from one-another such as to avoid the cycle-skipping problem. Even in this case, the assumption made is that the nearby traces in a gather are sampled well-enough, i.e. the seismic events differ by only a fraction of the wavelet, which is a function of image sampling and frequency content. In practice, there is no absolute guarantee that nearby events are closely related to one another, although this is more likely to be true for DSO than it is for direct image differencing.

In this paper, we concentrate on the implementation of the wave-equation migration velocity analysis operators for various wave-equation migration configurations. The main objective of the paper is to derive the linearized operators linking perturbations of the slowness model to the corresponding perturbations of the seismic wavefield and image. All our theoretical development is formulated under the single scattering (Born) approximation applied to acoustic waves. We begin by describing the MVA operators corresponding to zero-offset, survey-sinking and shot-record wave-equation migration frameworks. We describe the theoretical background for each operator and emphasize the similarities and differences between the different operators. Throughout the paper, we use pseudo-code to illustrate implementation strategies and data flow for the various wave-equation MVA operators. Finally, we illustrate the wave-equation MVA operators with impulse responses corresponding to simple and complex models. We leave outside the scope of this paper the procedure in which the discussed operators are used for migration velocity analysis.

2 WAVE-EQUATION MIGRATION AND VELOCITY ANALYSIS OPERATORS

The conceptual framework of wave-equation MVA is similar to that of conventional (ray-based) MVA in that the source of information for velocity updating is extracted from features of migrated images. This is in contrast with wave-equation tomography (or inversion), where the source of information is represented by the mismatch between recorded and simulated data. The main difference between wave-equation MVA and ray-based MVA is that the carrier of information from the migrated images to the velocity model is represented by the entire extrapolated wavelet and not by a rayfield constructed from selected points in the image based on an approximate velocity model.

The key element for the wave-equation MVA technique is a definition of an image perturbation corresponding to the difference between the image obtained with a known background velocity model and an improved image. Such image perturbations can be constructed using straight differences between images (Biondi & Sava, 1999; Albertin et al., 2006a), or by examining moveout parameters in migrated images (Sava & Biondi, 2004a,b; Shen et al., 2005; Albertin et al., 2006b; Maharramov & Albertin, 2007). Then, using wave-equation MVA operators, the image perturbations can be translated into slowness perturbations which update the model. The direct analogy between wave-based MVA and ray-based MVA is the following: wave-based methods use image perturbations and back-propagation using waves, while ray-based methods use traveltime perturbations and back-propagation using rays. Thus, wave-equation MVA benefits from all the characteristics of wave-based imaging techniques, e.g. stability in areas of large velocity variation, while remaining conceptually similar to conventional traveltime-based MVA.

We can formulate wave-equation migration and velocity analysis for different configurations in which we process the recorded data. There are two main classes of wave-equation migration, survey-sinking migration and shot-record migration (Claerbout, 1985), which differ in the way in which recorded data are processed. Both wave-equation imaging techniques use similar algorithms for downward continuation and, in theory, produce identical images for identical implementation of extrapolation operators and if all data are used for imaging (Berkhout, 1982; Biondi, 2003). The main difference is that shot-record migration is used to process separate seismic experiments (shots) sequentially, while survey-sinking migration is used to process all seismic experiments (shots) simultaneously. The shot-record operators are more computationally expensive but less memory intensive than the survey-sinking operators. A special case of survey-sinking migration assumes the sources and receivers are coincident on the acquisition surface, a technique usually described as the exploding reflector model (Loewenthal et al., 1976) applicable to zero-offset data. All operators described here can be used in models characterized by complex wave propagation (multipathing).

In all situations, wave-equation migration can be formulated as consisting of two main steps. The first step is wavefield reconstruction (abbreviated W.R. for the rest of this paper) at all locations in space and using all frequencies from the recorded data as boundary conditions. This step requires numeric solutions to a form of wave equation, typically the one-way acoustic wave equation. The second step is the imaging condition (abbreviated I.C. for the rest of this paper), which is used to extract from the reconstructed wavefield(s) the locations where reflectors occur. This step requires numeric implementation of image processing techniques, e.g. cross-correlation, which evaluate properties of the wavefield indicating the presence of reflectors. Needless to say, the two steps are not implemented sequentially in practice, since the size of the wavefield is usually large and cannot be handled efficiently on conventional computers. Instead, wavefield reconstruction and imaging condition are implemented on-the-fly, avoiding expensive data storage and retrieval. Wave-equation MVA requires implementation of an additional procedure which links
image and slowness perturbations. This link is given by a wavefield scattering operation (abbreviated w.s. for the rest of this paper) which is derived by linearization from conventional wavefield extrapolation operators.

In the following sections, we describe the migration and velocity analysis operators for the various imaging configurations. We begin with zero-offset imaging under the exploding reflector model, because this is the simplest wave-equation imaging framework and can aid our understanding of both survey-sinking and shot-record migration and velocity analysis frameworks. We then continue with a description of the wave-equation migration velocity analysis operator for multi-offset data using the survey-sinking and shot-record migration configuration. For each configuration, we describe the implementation of the forward operator (used to translate model perturbations into image perturbations) and of the adjoint operator (used to transform image perturbations into model perturbations). Both forward and adjoint operators are necessary for the implementation of efficient numeric conjugate gradient optimization (Claerbout, 1985). Throughout this paper, we are using the following notations and naming conventions:

- \( \omega \): angular frequency
- \( z \): depth coordinate
- \( m = \{ m_x, m_y \} \): midpoint coordinates
- \( h = \{ h_x, h_y \} \): half-offset coordinates
- \( k_{m} = \{ k_{m_x}, k_{m_y} \} \): midpoint wavenumbers
- \( k_{h} = \{ k_{h_x}, k_{h_y} \} \): half-offset wavenumbers
- \( s (m) \): medium slowness
- \( s_{0} (m) \): background slowness
- \( u (m) \) or \( u (m, h) \): wavefield at frequency \( \omega \) and depth \( z \) for zero-offset and multiple-offset data, respectively
- \( \Delta u (m) \) or \( \Delta u (m, h) \): scattered wavefield at frequency \( \omega \) and depth \( z \) for zero-offset and multiple-offset data, respectively
- \( r (m) \) or \( r (m, h) \): image at depth \( z \) for zero-offset and multiple-offset data, respectively
- \( \hat{F}^{\pm} [] \): forward scattering operator (causal for +, anticausal for -)
- \( \hat{A}^{\pm} [] \): adjoint scattering operator (causal for +, anticausal for -)

2.1 Zero-offset migration and velocity analysis

Wavefield reconstruction for zero-offset migration based on the one-way wave-equation is performed by recursive phase-shift in depth starting with data recorded on the surface as boundary conditions. In this configuration, the imaging condition extracts the image as time \( t = 0 \) from the reconstructed wavefield at every location in space. Thus, the surface data need to be extrapolated backward in time which is achieved by selecting the appropriate sign of the phase-shift operation (which depends on the sign convention for temporal Fourier transforms):

\[
u_{z + \Delta z} (m) = e^{-ik_{z} \Delta z} u_{z} (m) .
\]  

In equation 1, \( u_{z} (m) \) represents the acoustic wavefield at depth \( z \) for a given frequency \( \omega \) at all positions in space \( m \), and \( u_{z + \Delta z} (m) \) represents the same wavefield extrapolated to depth \( z + \Delta z \). The phase shift operation uses the depth wavenumber \( k_{z} \) which is defined by the single square-root (SSR) equation

\[
k_{z} = \sqrt{2\omega s (m)^{2} - |k_{m}|^{2}} ,
\]

where \( s (m) \) represents the spatially-variable slowness at depth level \( z \). Equations 1-2 describe wavefield extrapolation using a pseudo-differential operator, which justifies our use of laterally-varying slowness \( s (m) \). As indicated earlier, the image is obtained from this extrapolated wavefield by selection of time \( t = 0 \), which is typically implemented as summation of the extrapolated wavefield over frequencies:

\[
r_{z} (m) = \sum_{\omega} u_{z} (m, \omega) .
\]

Phase-shift extrapolation using wavenumbers computed using equations 1 and 2 is not feasible in media with lateral variation. Instead, implementation of such operators is done using approximations implemented in a mixed space-wavenumber domain (Stoffa et al., 1990; Ristow & Ruhl, 1994; Huang et al., 1999). A brief summary the mixed-domain implementation of the split-step Fourier (SSF) operator is presented in Appendix A.

For velocity analysis, we assume that we can separate the total slowness \( s (m) \) into a known background component \( s_{0} (m) \) and an unknown component \( \Delta s (m) \). With this convention, we can linearize the depth wavenumber \( k_{z} \) relative to the background slowness \( s_{0} \) using a truncated Taylor series expansion

\[
k_{z} \approx k_{z0} + \left. \frac{dk_{z}}{ds} \right|_{s_{0}} \Delta s (m) ,
\]

where the depth wavenumber in the background medium characterized by slowness \( s_{0} (m) \) is

\[
k_{z0} = \sqrt{2\omega s_{0} (m)^{2} - |k_{m}|^{2}} .
\]

Here, \( s_{0} (m) \) represents the spatially-variable background slowness at depth level \( z \). Using the wavenumber linearization from equation 4, we can reconstruct the acoustic wavefields in the background model using a phase-shift operation

\[
u_{z + \Delta z} (m) = e^{-ik_{z0} \Delta z} u_{z} (m) .
\]

We can represent wavefield extrapolation using a generic solution to the one-way wave-equation using the notation \( u_{z + \Delta z} (m) = \hat{FZOM} [s_{0} (m), u_{z} (m)] \). This notation indicates that the wavefield \( u_{z + \Delta z} (m) \) is reconstructed from the wavefield \( u_{z} (m) \) using the background slowness \( s_{0} (m) \). This operation is repeated independently for all frequencies \( \omega \).
A typical implementation of zero-offset wave-equation migration uses the following algorithm:

**ZERO-OFFSET MIGRATION ALGORITHM**

\[
\omega = \omega_{\text{min}} \ldots \omega_{\text{max}} \left\{ \begin{array}{l}
\text{read } u(\mathbf{m}) \\
\text{read } r(\mathbf{m}) \\
\text{write } u(\mathbf{m}) \\
\text{write } r(\mathbf{m}) \\
u(\mathbf{m}) = E_{\text{ZOM}}[2s_0(\mathbf{m}), u(\mathbf{m})]
\end{array} \right.
\]

A seismic image is produced using migration by wavefield extrapolation as follows: for each frequency, read data at all spatial locations \( \mathbf{m} \); then, for each depth, sum the wavefield into the image at that depth level (i.e. apply the imaging condition) and then reconstruct the wavefield to the next depth level (i.e. perform wavefield extrapolation). The “\( \omega \)” sign in this algorithm indicates that extrapolation is anti-causal (backward in time), and the factor “2” indicates that we are imaging data recorded in two-way traveltime with an algorithm designed under the exploding reflector model. Wavefield extrapolation is usually implemented in a mixed domain (spacewavenumber), as briefly summarized in Appendix A.

The wavefield perturbation \( \Delta u(\mathbf{m}) \) caused at depth \( z + \Delta z \) by a slowness perturbation \( \Delta s(\mathbf{m}) \) at depth \( z \) is obtained by subtraction of the wavefields extrapolated from \( z \) to \( z + \Delta z \) in the true and background models:

\[
\Delta u_{z+\Delta z}(\mathbf{m}) = e^{-ik_s \Delta z} u_z(\mathbf{m}) - e^{-ik_{s0} \Delta z} u_{z0}(\mathbf{m}) = e^{-ik_{s0} \Delta z} \left[ e^{-i \frac{ds}{ds_0} |\Delta s(\mathbf{m})| \Delta z} - 1 \right] u_z(\mathbf{m}), (8)
\]

Here, \( \Delta u(\mathbf{m}) \) and \( u(\mathbf{m}) \) correspond to a given depth level \( z \) and frequency \( \omega \). A similar relation can be applied at all depths and all frequencies.

Equation 8 establishes a non-linear relation between the wavefield perturbation \( \Delta u(\mathbf{m}) \) and the slowness perturbation \( \Delta s(\mathbf{m}) \). Given the complexity and cost of numeric optimization based on non-linear relations between model and wavefield parameters, it is desirable to simplify this relation to a linear relation between model and data parameters. Assuming small slowness perturbations, i.e. small phase perturbations, the exponential function \( e^{i \Delta s(\mathbf{m}) \Delta z} \) can be linearized using the approximation \( e^{i \phi} \approx 1 + i \phi \) which is valid for small values of the phase \( \phi \). Therefore the wavefield perturbation \( \Delta u(\mathbf{m}) \) at depth \( z \) can be written as

\[
\Delta u(\mathbf{m}) \approx -i \Delta z \left[ \omega u(\mathbf{m}) \right] \Delta s(\mathbf{m})), (9)
\]

Equation 9 defines the zero-offset forward scattering operator \( F_{\text{ZOM}}[u(\mathbf{m}), 2s_0(\mathbf{m}), \Delta s(\mathbf{m})] \), producing the scattered wavefield \( \Delta u(\mathbf{m}) \) from the slowness perturbation \( \Delta s(\mathbf{m}) \), based on the background slowness \( s_0(\mathbf{m}) \) and background wavefield \( u(\mathbf{m}) \) at a given frequency \( \omega \). The image perturbation at depth \( z \) is obtained from the scattered wavefield using the time \( t = 0 \) imaging condition, similar to the procedure used for imaging in the background medium:

\[
\Delta r(\mathbf{m}) = \sum_{\omega} \Delta u(\mathbf{m}, \omega). (10)
\]

Given an image perturbation \( \Delta r(\mathbf{m}) \), we can construct the scattered wavefield by the adjoint of the imaging condition

\[
\Delta u(\mathbf{m}, \omega) = \Delta r(\mathbf{m}), (11)
\]

for every frequency \( \omega \). Then, the slowness perturbation at depth \( z \) and frequency \( \omega \) caused by a wavefield perturbation at depth \( z \) under the influence of the background wavefield at the same depth \( z \) can be written as

\[
\Delta s(\mathbf{m}) \approx +i \omega |s_0(\mathbf{m})| \Delta r(\mathbf{m}) (\mathbf{m}) \approx +i \Delta z \frac{2\omega u(\mathbf{m}) \Delta s(\mathbf{m})}{\sqrt{1 - \left( \frac{|k_m|}{2\omega s_0(\mathbf{m})} \right)^2}}. (12)
\]

Equation 12 defines the adjoint scattering operator \( A_{\text{ZOM}}^*[u(\mathbf{m}), 2s_0(\mathbf{m}), \Delta u(\mathbf{m})] \), producing the slowness perturbation \( \Delta s(\mathbf{m}) \) from the scattered wavefield \( \Delta u(\mathbf{m}) \), based on the background slowness \( s_0(\mathbf{m}) \) and background wavefield \( u(\mathbf{m}) \). A typical implementation of zero-offset forward and adjoint scattering uses the following algorithms:

**ZERO-OFFSET FORWARD SCATTERING ALGORITHM**

\[
\omega = \omega_{\text{min}} \ldots \omega_{\text{max}} \left\{ \begin{array}{l}
\text{initialize } \Delta u(\mathbf{m}) = 0 \\
\text{read } u(\mathbf{m}) \\
\text{read } \Delta s(\mathbf{m}) \\
\text{read } \Delta u(\mathbf{m}) + = F_{\text{ZOM}}[u(\mathbf{m}), 2s_0(\mathbf{m}), \Delta s(\mathbf{m})] \\
\text{write } \Delta r(\mathbf{m}) \\
\text{write } \Delta u(\mathbf{m}) = E_{\text{ZOM}}[2s_0(\mathbf{m}), \Delta u(\mathbf{m})]
\end{array} \right.
\]

**ZERO-OFFSET ADJOINT SCATTERING ALGORITHM**

\[
\omega = \omega_{\text{min}} \ldots \omega_{\text{max}} \left\{ \begin{array}{l}
\text{initialize } \Delta u(\mathbf{m}) = 0 \\
\text{read } u(\mathbf{m}) \\
\text{read } \Delta s(\mathbf{m}) \\
\text{read } \Delta u(\mathbf{m}) = E_{\text{ZOM}}^{*}[2s_0(\mathbf{m}), \Delta u(\mathbf{m})]
\end{array} \right.
\]
The forward zero-offset wave-equation MVA operator follows a similar pattern to the implementation of the downward continuation operator: for each frequency and for each depth, read the slowness perturbation \( \Delta s(m) \) at all spatial locations \( m \), then apply the scattering operator \( W(s,m) \) given equation 12 to the slowness perturbation and accumulate the scattered wavefield for downward continuation; then, apply the imaging condition \( (1.3) \) producing the image perturbation \( \Delta r \) at depth \( z \) and then reconstruct the scattered wavefield backward in time using the wavefield extrapolation operator \( W(r,s,m) \) to the next depth level. The adjoint zero-offset wave-equation MVA operator also follows a similar pattern to the implementation of the downward continuation operator: for each frequency and for each depth, reconstruct the scattered wavefield forward in time using the wavefield extrapolation operator \( W(r,s,m) \) to the next depth level, then apply the adjoint of the imaging condition \( (1.3) \) by adding the image to the scattered wavefield and then apply the adjoint wavefield scattering operator \( W(s,m) \) to obtain the slowness perturbation \( \Delta s(m) \). Both forward and adjoint scattering algorithms require the background wavefield, \( u \), to be precomputed at all depth levels, although more efficient implementations using optimal checkpointing are possible (7). Scattering and wavefield extrapolation are implemented in the mixed space-wavenumber domain, as briefly explained in Appendix A.

### 2.2 Survey-sinking migration and velocity analysis

Wavefield reconstruction for multi-offset migration based on the one-way wave-equation under the survey-sinking framework (Claerbout, 1985) is implemented similarly to the zero-offset case by recursive phase-shift of prestack wavefields

\[
u_{z+\Delta z}(m, h) = e^{-ikz_0\Delta z} u_z(m, h),
\]

followed by extraction of the image at time \( t = 0 \). Here, \( m \) and \( h \) stand for midpoint and half-offsets coordinates, respectively, defined according to the relations

\[
m = \frac{r+s}{2},
\]

\[
h = \frac{r-s}{2},
\]

where \( s \) and \( r \) are coordinates of sources and receivers on the acquisition surface. In equation 13, \( u_z(m, h) \) represents the acoustic wavefield for a given frequency \( \omega \) at all midpoint positions \( m \) and half-offsets \( h \) at depth \( z \), and \( u_{z+\Delta z}(m, h) \) represents the same wavefield extrapolated to depth \( z + \Delta z \). The phase shift operation uses the depth wavenumber \( k_z \) which is defined by the double square-root (DSR) equation

\[
k_z = \sqrt{\omega^2 s(m-h)^2 - \frac{k_m-k_h^2}{2}} + \sqrt{\omega^2 s(m+h)^2 - \frac{k_m+k_h^2}{2}}.
\]

The image is obtained from this extrapolated wavefield by selection of time \( t = 0 \), which is usually implemented as summation over frequencies:

\[
r_z(m, h) = \sum_{\omega} u_z(m, h, \omega).
\]

Similarly to the derivation done in the zero-offset case, we can assume the separation of the extrapolation slowness \( s(m) \) into a background component \( s_0(m) \) and an unknown perturbation component \( \Delta s(m) \). Then we can construct a wavefield perturbation \( \Delta s(m, h) \) at depth \( z \) and frequency \( \omega \) related linearly to the slowness perturbation \( \Delta s(m) \). Linearizing the depth wavenumber given by the DSR equation 16 relative to the background slowness \( s_0(m) \), we obtain

\[
k_z \approx k_{z0} + \frac{dk_z}{ds_0} \Delta s(m-h) + \frac{dk_z}{ds_0} \Delta s(m+h),
\]

where the depth wavenumber in the background medium is

\[
k_{z0} = \sqrt{\omega^2 s_0(m-h)^2 - \frac{k_m-k_h^2}{2}} + \sqrt{\omega^2 s_0(m+h)^2 - \frac{k_m+k_h^2}{2}}.
\]

Here, \( s_0(m) \) represents the spatially-variable background slowness at depth level \( z \). Using the wavenumber linearization given by equation 18, we can reconstruct the acoustic wavefields in the background medium using a phase-shift operation

\[
u_{z+\Delta z}(m, h) = e^{-ik_{z0}\Delta z} u_z(m, h).
\]

We can represent wavefield extrapolation using a generic solution to the one-way wave-equation using the notation \( u_{z+\Delta z}(m, h) = E_{SSM}[s_0(m), u_z(m, h)] \). This notation indicates that the wavefield \( u_{z+\Delta z}(m, h) \) is reconstructed from the wavefield \( u_z(m, h) \) using the background slowness \( s_0(m) \). This operation is repeated independently for all frequencies \( \omega \). A typical implementation of survey-sinking wave-equation migration uses the following algorithm:

**SURVEY-SINKING MIGRATION ALGORITHM**

\[
\omega = \omega_{min} \ldots \omega_{max}\{
\text{read } u(m, h)\}
\]

\[
z = z_{min} \ldots z_{max}\{
\text{write } u(m, h)\}
\]

\[
r(m, h) + = u(m, h)\}
\]

\[
\text{write } r(m, h)\}
\]

\[
u(m, h) = E_{SSM}[s_0(m), u(m, h)]\}
\]
This algorithm is similar to the one described in the preceding section for zero-offset migration, except that the wavefield and image are parametrized by midpoint and half-offset coordinates and that the depth wavenumber used in the extrapolation operator is given by the DSR equation using the background slowness $s_0(\mathbf{m})$. Wavefield extrapolation is usually implemented in a mixed domain (space-wavenumber), as briefly summarized in Appendix A.

Similarly to the derivation of the wavefield perturbation in the zero-offset migration case, we can write the linearized wavefield perturbation for survey-sinking migration as

$$
\Delta u(\mathbf{m}, \mathbf{h}) \approx -i \frac{dk_z}{ds} \Delta s(\mathbf{m} - \mathbf{h}) \Delta z u(\mathbf{m}, \mathbf{h})
$$

Equation 21 defines the forward scattering operator $\mathcal{F}_{SSM}^{\pm} [u(\mathbf{m}, \mathbf{h}), s_0(\mathbf{m}), \Delta s(\mathbf{m}, \mathbf{h})]$, producing the scattered wavefield $\Delta u(\mathbf{m}, \mathbf{h})$ from the slowness perturbation $\Delta s(\mathbf{m})$, based on the background slowness $s_0(\mathbf{m})$ and background wavefield $u(\mathbf{m}, \mathbf{h})$. The image perturbation at depth $z$ is obtained from the scattered wavefield using the time $t = 0$ imaging condition, similar to the procedure used for imaging in the background medium:

$$
\Delta r(\mathbf{m}, \mathbf{h}) = \sum_{\omega} \Delta u(\mathbf{m}, \mathbf{h}, \omega). \tag{22}
$$

Equation 25 defines the adjoint scattering operator $\mathcal{A}_{SSM}^{\pm} [u(\mathbf{m}, \mathbf{h}), s_0(\mathbf{m}), \Delta u(\mathbf{m}, \mathbf{h})]$, producing the slowness perturbation $\Delta s(\mathbf{m})$ from the scattered wavefield $\Delta u(\mathbf{m}, \mathbf{h})$, based on the background slowness $s_0(\mathbf{m})$ and background wavefield $u(\mathbf{m}, \mathbf{h})$. A typical implementation of survey-sinking forward and adjoint scattering follows the algorithms:

\begin{verbatim}
SURVEY-SINKING FORWARD SCATTERING ALGORITHM
\omega = \omega_{\text{min}} \ldots \omega_{\text{max}}
\Delta u(\mathbf{m}, \mathbf{h}) = 0
z = z_{\text{min}} \ldots z_{\text{max}}
read u(\mathbf{m}, \mathbf{h})
read \Delta s(\mathbf{m})
\Delta u(\mathbf{m}, \mathbf{h}) = \mathcal{F}_{SSM}^{\pm} [u(\mathbf{m}, \mathbf{h}), s_0(\mathbf{m} - \mathbf{h}), \Delta s(\mathbf{m} - \mathbf{h})]
\Delta r(\mathbf{m}, \mathbf{h}) = \Delta u(\mathbf{m}, \mathbf{h})
write \Delta r(\mathbf{m}, \mathbf{h})
\Delta u(\mathbf{m}, \mathbf{h}) = \mathcal{F}_{SSM}^{\pm} [s_0(\mathbf{m}), \Delta u(\mathbf{m}, \mathbf{h})]
\end{verbatim}

\begin{verbatim}
SURVEY-SINKING ADJOUT SCATTERING ALGORITHM
\omega = \omega_{\text{min}} \ldots \omega_{\text{max}}
\Delta u(\mathbf{m}, \mathbf{h}) = 0
z = z_{\text{min}} \ldots z_{\text{max}}
read u(\mathbf{m}, \mathbf{h})
\Delta s(\mathbf{m} - \mathbf{h}) = \mathcal{F}_{SSM}^{\pm} [s_0(\mathbf{m}), \Delta u(\mathbf{m}, \mathbf{h})]
\Delta r(\mathbf{m}, \mathbf{h}) = \Delta s(\mathbf{m} - \mathbf{h})
write \Delta s(\mathbf{m} - \mathbf{h})
\Delta s(\mathbf{m} + \mathbf{h}) = \mathcal{A}_{SSM}^{\pm} [u(\mathbf{m}, \mathbf{h}), s_0(\mathbf{m} + \mathbf{h}), \Delta u(\mathbf{m}, \mathbf{h})]
\end{verbatim}

These algorithms are similar to the ones described in the preceding section for zero-offset migration, except that the wavefield and image are parametrized by midpoint and half-offset coordinates. Furthermore, the two square-roots corresponding to the DSR equation update the slowness model separately, thus characterizing the source and receiver propagation paths to the image positions. Both forward and adjoint scattering algorithms require the background wavefield, $u(\mathbf{m}, \mathbf{h})$, to be precomputed at all depth levels. Scattering and
wavefield extrapolation are implemented in the mixed space-wavenumber domain, as briefly explained in Appendix A.

2.3 Shot-record migration and velocity analysis

Wavefield reconstruction for multi-offset migration based on the one-way wave-equation under the shot-record framework is performed by separate recursive extrapolation of the source and receiver wavefields, \( u_s \) and \( u_r \). The wavefield extrapolation progresses forward in time (causal) for the source wavefield and backward in time (anti-causal) for the receiver wavefield:

\[
\begin{align*}
\mathbf{u}_{s,z+\Delta z} &= e^{i k_s \Delta z} \mathbf{u}_{s,z} \\
\mathbf{u}_{r,z+\Delta z} &= e^{-i k_r \Delta z} \mathbf{u}_{r,z}
\end{align*}
\]

In equations 26-27, \( u_{s,z}(m) \) and \( u_{r,z}(m) \) represent the source and receiver acoustic wavefield for a given frequency \( \omega \) at all positions in space \( \mathbf{m} \) at depth \( z \), and \( u_{s,z+\Delta z}(m) \) and \( u_{r,z+\Delta z}(m) \) represent the same wavefields extrapolated to depth \( z + \Delta z \). The phase shift operation uses the depth wavenumber \( k_z \) which is defined by the single square-root (SSR) equation

\[ k_z = \sqrt{[\omega s(m)]^2 - |k_m|^2} \]  

The image is obtained from the extrapolated wavefields by selection of the zero cross-correlation lags in space of time between the source and receiver wavefields, an operation which is usually implemented as summation over frequencies:

\[ r_z(m) = \sum_{\omega} u_{s,z}(m,\omega) u_{r,z}(m,\omega) \]  

An alternative imaging condition (Sava & Fomel, 2006) preserves the space and time cross-correlation lags in the image.

Linearizing the depth wavenumber given by the equation 28 relative to the background slowness \( s_0 (m) \) similarly to the case of zero-offset migration, we can reconstruct the acoustic wavefields in the background model using a phase-shift operation

\[ u_{s,z+\Delta z} = e^{i k_s \Delta z} u_{s,z} \]  

which define the causal \( \mathcal{E}_s^{SRM}[s_0(m),u_s(m)] \) and the anti-causal \( \mathcal{E}_s^{SRM}[s_0(m),u_s(m)] \) wavefield extrapolation operators for shot-record migration constructed using the background slowness \( s_0(m) \) and producing the wavefields \( u_{s,z+\Delta z} \) and \( u_{r,z+\Delta z} \) at depth \( z + \Delta z \) from the wavefields \( u_{s,z} \) and \( u_{r,z} \) at depth \( z \), respectively. A typical implementation of shot-record wave-equation migration follows the algorithm:

---

**SHOT-RECORD MIGRATION ALGORITHM**

\[ \omega = \omega_{min} \ldots \omega_{max} \{
\]

read \( u_s(m) \) and \( u_r(m) \)
\[ z = z_{min} \ldots z_{max} \{
\]

write \( u_s(z) \) and \( u_r(z) \)
\[ r(m) = r_s(m) u_r(m) \]

write \( r(m) \)

\[ u_s(m) = \mathcal{E}_s^{SRM}[s_0(m),u_s(m)] \]

\[ u_r(m) = \mathcal{E}_r^{SRM}[s_0(m),u_r(m)] \]

\[ \}
\]

---

This algorithm is similar to the one used for zero-offset or survey sinking migration, except that the source and receiver wavefields are reconstructed separately using wavefield extrapolation. Unlike the zero-offset extrapolation operator, the shot-record extrapolation operator uses the background slowness \( s_0 \) since the operation involves sinking of the source and receiver wavefields from the surface toward the image positions. Wavefield extrapolation is usually implemented in a mixed domain (space-wavenumber), as briefly summarized in Appendix A.

Similarly to the derivation of the wavefield perturbation in the zero-offset migration case, we can write the linearized wavefield perturbation for shot-record migration as

\[ \Delta u_s(m) \approx +i \frac{dk_s}{ds} z u_s(m) \Delta s(m) \]

\[ \approx +i \Delta z \omega u_s(m) \Delta s(m) \]

and

\[ \Delta u_r(m) \approx -i \frac{dk_r}{ds} z u_r(m) \Delta s(m) \]

\[ \approx -i \Delta z \omega u_r(m) \Delta s(m) \]

Equations 32-33 define the forward scattering operators \( \mathcal{F}_s^{SRM}[u(m),s_0(m),\Delta s(m)] \) producing the scattered wavefields \( \Delta u(m) \) from the slowness perturbation \( \Delta s(m) \), based on the background slowness \( s_0(m) \) and background wavefield \( u(m) \). In this case, the symbol \( u \) stands for either \( u_s \) or \( u_r \), given the appropriate choice of sign in the forward scattering operator. The image perturbation at depth \( z \) is obtained from the source and receiver scattered wavefields using the relation

\[ \Delta r(m) = \sum_{\omega} \left( u_s(m,\omega) \Delta u_r(m,\omega) + \Delta u_s(m,\omega) u_r(m,\omega) \right) \]

which corresponds to the frequency-domain zero-lag cross-correlation of the source and receiver wavefields required by the imaging condition.

Given an image perturbation \( \Delta r \), we can construct the scattered source and receiver wavefields by the adjoint of the imaging condition

\[ \Delta u_s(m) = u_r(m) \Delta r(m) \]

\[ \Delta u_r(m) = u_s(m) \Delta r(m) \]

for every frequency \( \omega \). Then, the slowness perturbations due to the source and receiver wavefields at depth \( z \) under the influ-
ence of the background source and receiver wavefields at the same depth \( z \) can be written as

\[
\Delta s_\nu(m) \approx -i \frac{dk_\nu}{ds} |_{s_0} \Delta z u_\nu(m) \Delta u_\nu(m)
\]

\[
\approx -i \Delta z \frac{\omega u_\nu(m) \Delta u_\nu(m)}{\sqrt{1 - \left[ \frac{|k_m|}{\omega s_0(m)} \right]^2}}
\]  \tag{37}

and

\[
\Delta s_r(m) \approx -i \frac{dk_r}{ds} |_{s_0} \Delta z u_r(m) \Delta u_r(m)
\]

\[
\approx -i \Delta z \frac{\omega u_r(m) \Delta u_r(m)}{\sqrt{1 - \left[ \frac{|k_m|}{\omega s_0(m)} \right]^2}}
\]  \tag{38}

Equations 37-38 define the adjoint scattering operators \( A_{SRM}^\pm [u(m), s_0(m), \Delta u(m)] \), producing the slowness perturbation \( \Delta s(m) \) from the scattered wavefield \( \Delta u(m) \), based on the background slowness \( s_0(m) \) and background wavefield \( u(m) \). In this case, \( u \) stands for either \( u_\nu \) or \( u_r \), given the appropriate choice of sign in the adjoint scattering operator. A typical implementation of shot-record forward and adjoint scattering follows the algorithms:

### SHOT-RECORD FORWARD SCATTERING ALGORITHM

\[
\begin{align*}
\omega = \omega_{\text{min}} \ldots \omega_{\text{max}} \{ \\
\text{initialize } \Delta u_\nu(m) = 0 \text{ and } \Delta u_r(m) = 0 \\
\text{read } u_\nu(m) \text{ and } u_r(m) \\
\text{read } \Delta s(m) \\
\Delta s(m) + = F_{SRM}^+ [u_\nu(m), s_0(m), \Delta s(m)] \\
\Delta u_\nu(m) + = F_{SRM}^- [u_\nu(m), s_0(m), \Delta u_\nu(m)] \\
\text{read } \Delta r(m) \\
\Delta r(m) + = u_\nu(m) \Delta u_r(m) \\
\Delta r(m) + = \Delta u_\nu(m) u_r(m) \\
\text{write } \Delta r(m) \\
\Delta u_\nu(m) = E_{SRM}^+ [s_0(m), \Delta u_\nu(m)] \\
\Delta u_r(m) = E_{SRM}^- [s_0(m), \Delta u_r(m)]
\end{align*}
\]

### SHOT-RECORD ADJOINT SCATTERING ALGORITHM

\[
\begin{align*}
\omega = \omega_{\text{min}} \ldots \omega_{\text{max}} \{ \\
\text{initialize } \Delta u_\nu(m) = 0 \text{ and } \Delta u_r(m) = 0 \\
\text{read } u_\nu(m) \text{ and } u_r(m) \\
\Delta u_\nu(m) = E_{SRM}^+ [s_0(m), \Delta u_\nu(m)] \\
\Delta u_r(m) = E_{SRM}^- [s_0(m), \Delta u_r(m)] \\
\text{read } \Delta r(m) \\
\Delta r(m) + = u_\nu(m) \Delta u_r(m) \\
\Delta u_\nu(m) + = u_r(m) \Delta r(m) \\
\text{read } \Delta s(m) \\
\Delta s(m) + = A_{SRM}^+ [u_r(m), s_0(m), \Delta u_r(m)]
\end{align*}
\]

These algorithms are similar to the one used for zero-offset or survey sinking migration, except that the source and receiver wavefields are reconstructed separately using wavefield extrapolation. Unlike the zero-offset scattering operators, the shot-record scattering operators use the background slowness \( s_0 \) since the operation involves sinking of the source and receiver wavefields from the surface toward the image positions. Both forward and adjoint scattering algorithms require the background wavefields, \( u_\nu(m) \) and \( u_r(m) \), to be precomputed at all depth levels. Scattering and wavefield extrapolation are implemented in the mixed space-wavenumber domain, as briefly explained in Appendix A.

### 3 EXAMPLES

We illustrate the wave-equation migration velocity analysis operators using impulse responses corresponding to different imaging configurations. We concentrate on imaging in the zero-offset and shot-record frameworks, since they also implicitly characterize the essential elements of the survey-sinking framework. In all cases, we use wavefield reconstruction based on one-way wavefield extrapolation with the multi-reference split-step Fourier method (Stoffa et al., 1990; Popovici, 1996).

A fundamental question concerning the wavefield scattering operator (W.S.) is what is its sensitivity for a given perturbation of the image or of the slowness model. This sensitivity is usually characterized using the so-called “sensitivity kernels” which are often discussed in the literature in the context of tomography problems. For wave-equation MVA, this topic was discussed in the context of zero-offset imaging by Sava & Biondi (2004a,b). The important topic of sensitivity and model resolution falls outside the scope of this paper, so we do not discuss it here in any detail. We merely concern ourselves with describing the behavior of the wave-equation MVA operators described earlier.

We can analyze the sensitivity of the wavefield scattering operator in two ways. The first option is to assume a localized slowness perturbation, compute image perturbations using the forward scattering operator and then return to the slowness perturbation using the adjoint scattering operator. The second option is to assume a localized image perturbation, compute the slowness perturbation using the adjoint scattering operator and then return to the image perturbation using the forward scattering operator.

As discussed in the preceding sections, the main difference between ray-based and wave-based MVA techniques is that the connection between measurements on the image and updates to the model is done with rays and waves, respectively. The impact of this fundamental difference is best seen if we analyze impulse responses of the wave-equation MVA...
Figure 1. Schematic representation of the forward and adjoint operators for ray-based MVA and wave-based MVA. The forward operator $F$ applied to a slowness anomaly $\Delta s$ generates a traveltime perturbation (a) or an image perturbation (b). The ray-based adjoint MVA operator $A$ applied to the traveltime perturbation generates a slowness perturbation uniformly distributed along a ray normal to the reflector (c). The wave-based adjoint MVA operator $A$ applied to the image perturbation generates a slowness perturbation with a wider space distribution but with a relative focus at the location of the original slowness anomaly (d).

and compare them with those of conventional traveltime tomography. Figure 1 shows a one-to-one comparison between the forward and adjoint operators for ray-based MVA (traveltime tomography) on the left and wave-based MVA on the right in the context of zero-offset imaging. Assuming a small slowness perturbation $\Delta s$, we can construct using the forward MVA operators a traveltime perturbation and an image perturbation corresponding to ray-based MVA (a) and wave-based MVA (b), respectively. For this zero-offset configuration, the ray-based MVA produces a traveltime anomaly strictly located on the reflector under the slowness anomaly, while the wave-based MVA produces an image anomaly distributed in space in the vicinity of the reflector. Then, we can construct respective slowness updates if we apply the ray-based and wave-based adjoint MVA operators to the traveltime perturbation and image perturbation, respectively. For the ray-based MVA, the slowness update spreads uniformly along a ray orthogonal to the reflector (c), while for wave-based MVA, the slowness update is distributed in space from the image perturbation to the surface, but with a concentration at the location of the true anomaly (d). Similar behavior characterizes wave-equation MVA under shot-record or survey-sinking frameworks.

The first set of examples corresponds to a simple model consisting of a linear $v(z)$ velocity model and a horizontal reflector, Figures 2(a)-2(b). The velocity is linearly increasing from 1.5 km/s to 2.75 km/s. We simulate zero-offset data, Figure 4(a), and one shot corresponding to horizontal position $x = 6$ km, Figure 4(b).

Assuming a localized slowness perturbation, Figure 3(a), we can compute image perturbations using the forward scattering operators, as defined in the preceding sections. Figure 5(a) shows the image perturbation for the zero-offset case and Figure 6(a) shows the similar image perturbation for the shot-record case. As illustrated in Figure 1, the image perturbations are distributed in the vicinity of the reflector. Two interfering
events are seen for the shot-record case, corresponding to the source and receiver wavefields, respectively.

Similarly, we can compute slowness perturbations using the adjoint scattering operators. Figure 5(b) shows the slowness perturbation for the zero-offset case computed from the image perturbation in Figure 5(a) and Figure 6(b) shows the similar slowness perturbation for the shot-record case computed from the image perturbation in Figure 6(a). As illustrated in Figure 1, the slowness perturbations are distributed in an area connecting the reflector to the surface, but with a relative focus at the location of the original anomaly. For the shot-record case, the back-projection splits toward the source and receivers, corresponding to the upward continuation of the source and receiver wavefields.

We can also analyze the wave-equation MVA operator sensitivity in another way. Assuming a localized image perturbation, Figure 3(b), we can compute slowness perturbations using the adjoint scattering operators, as defined in the preceding sections. Figure 7(a) shows the slowness perturbation for the zero-offset case and Figure 8(a) shows the similar slowness perturbation for the shot-record case. Here, too, we see slowness perturbations distributed in an area connecting the reflector to the surface, but in this case, there is no relative focus of the anomaly because the image perturbation is strictly localized on the reflector. For the shot-record case, the back-projection splits toward the source and receivers, corresponding to the upward continuation of the source and receiver wavefields. This case corresponds to the case of practical MVA where measurements of defocusing features are made on the image itself.

As we have done in the preceding experiment, we can also compute image perturbations using the forward scattering operators based on the back-projections created using the adjoint scattering operators. Figure 7(b) shows the image perturbation for the zero-offset case computed from the slowness perturbation in Figure 7(a) and Figure 8(b) shows the similar image perturbation for the shot-record case computed from the slowness perturbation in Figure 8(a). We can observe that the resulting image perturbations spread beyond the original location, indicating wider sensitivity of the wave-based MVA kernels to image perturbations than that of the corresponding ray-based MVA kernels.

Similar sensitivity can be observed for the more complex Sigsbee 2A model (Paffenholz et al., 2002), Figures 9(a)-9(b). Similarly to the preceding example, we simulate zero-offset data, Figure 11(a), and one shot corresponding to horizontal position \( x = 14.6 \) km, Figure 11(b).

Figures 12(a) and 13(a) correspond to the image perturbations for the slowness anomaly shown in Figure 10(a). We can observe image perturbations that spread in the vicinity of the reflector, similarly to the simpler example described earlier. The multi-pathing from the source to the reflector generates the multiple events characterizing the image perturbations. Figures 12(b) and 13(b) correspond to the slowness perturbations constructed by applying the zero-offset and shot-record adjoint scattering operators to the image perturbations from Figures 12(a) and 13(a). We see similar back-projection patterns to the ones observed in the preceding example, except that the propagation patterns are more complicated due to the presence of the salt body in the background model.

Figures 14(a) and 15(a) correspond to the slowness perturbations for the image anomaly shown in Figure 10(b).
can observe slowness perturbations that spread in the vicinity of the reflector, similarly to the simpler example described earlier. Finally, Figures 14(b) and 15(b) correspond to the image perturbations for the slowness perturbations constructed by the adjoint MVA operators shown in Figures 14(a) and 15(a) for the zero-offset and shot-record cases, respectively.

4 CONCLUSIONS

The wave-equation MVA operator discussed in this paper, can be implemented in various imaging frameworks, e.g. zero-offset (exploding reflector), survey-sinking or shot-record. In all cases, the forward and adjoint operators follow similar patterns involving combinations of scattering, imaging and extrapolation. The forward and adjoint operators share common elements and can be implemented in the mixed space-wavenumber domain, similarly to the implementation of the wavefield extrapolation operators.

The real challenges in using wave-based MVA are two-fold. First, the image perturbations need to be generated by techniques that do not compare image features that are too far from one-another, which is a property partially addressed by techniques based on differential semblance. Second, the cost of the wave-equation MVA operator is large, therefore a feasible implementation requires clever numeric implementation, e.g. by frequency decimation similarly to the approach taken in waveform inversion.

The examples shown in this paper illustrate the main characteristics of the various wave-equation MVA operators, i.e. stability during back-projection in background models with sharp velocity variation (e.g. salt), natural ability to characterize multi-pathing and wide area of sensitivity which is commensurate with the frequency band of the recorded data.

5 ACKNOWLEDGMENTS

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Figure 4. (a) Simulated zero-offset data and (b) simulated shot-record data for the model depicted in Figures 2(a)-2(b) with a source located at coordinates $x = 0$ km and $z = 0$ km.


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Figure 5. (a) Zero-offset image perturbation obtained by the application of the forward scattering operator to the slowness perturbation from Figure 3(a) and (b) zero-offset slowness perturbation obtained by the application of the adjoint scattering operator to the image perturbation from panel (a).

Figure 6. (a) Shot-record image perturbation obtained by the application of the forward scattering operator to the slowness perturbation from Figure 3(a) and (b) shot-record slowness perturbation obtained by the application of the adjoint scattering operator to the image perturbation from panel (a).
Figure 7. (a) Zero-offset slowness perturbation obtained by the application of the adjoint scattering operator to the image perturbation from Figure 3(b) and (b) zero-offset image perturbation obtained by the application of the forward scattering operator to the slowness perturbation from panel (a).

Figure 8. (a) Shot-record slowness perturbation obtained by the application of the adjoint scattering operator to the image perturbation from Figure 3(a) and (b) shot-record image perturbation obtained by the application of the forward scattering operator to the slowness perturbation from panel (a).
Figure 9. (a) Sigsbee 2A synthetic model and (b) a sub-salt horizontal reflector.

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Figure 10. (a) Slowness perturbations used to demonstrate the WEMVA operators in Figures 12(a)-13(b), and (b) image perturbation used to demonstrate the WEMVA operators in Figures 14(a)-15(b).

Figure 11. (a) Simulated zero-offset data and (b) simulated shot-record data for the model depicted in Figures 9(a)-9(b) with a source located at coordinates $x = 14.6$ km and $z = 1.52$ km.

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Figure 12. (a) Zero-offset image perturbation obtained by the application of the forward scattering operator to the slowness perturbation from Figure 10(a) and (b) zero-offset slowness perturbation obtained by the application of the adjoint scattering operator to the image perturbation from panel (a).

Past, present and future.: EAGE, 66th Conference and Exhibition of the EAGE, Paris, France.


Figure 13. (a) Shot-record image perturbation obtained by the application of the forward scattering operator to the slowness perturbation from Figure 10(a) and (b) shot-record slowness perturbation obtained by the application of the adjoint scattering operator to the image perturbation from panel (a).

Figure 14. (a) Zero-offset slowness perturbation obtained by the application of the adjoint scattering operator to the image perturbation from Figure 10(b) and (b) zero-offset image perturbation obtained by the application of the forward scattering operator to the slowness perturbation from panel (a).

6 APPENDIX A: MIXED-DOMAIN OPERATORS

For the case of the phase-shift operation in media with lateral slowness variation, the mixed-domain solution involves forward and inverse Fourier transforms (denoted fFT and iFT in our algorithms) which can be implemented efficiently using standard Fast Fourier Transform algorithms. The numeric implementation is summarized in the following table:

\[
\text{MIXED-DOMAIN IMPLEMENTATION OF THE EXTRAPOLATION OPERATOR } \mathcal{E}_{ZOM} \]

\[
\begin{align*}
(u(m)) & \xrightarrow{2D} u(k_m) \\
(u(k_m)) & \ast = e^{\pm ik_z k \Delta z}
\end{align*}
\]

**References:**


In this chart, \( \hat{k}_z \) denotes the \( \omega - k \) component of the depth wavenumber and \( \hat{k}_{z_x} \) denotes the \( \omega - x \) component of the depth wavenumber. An example of mixed-domain implementation is the Split-Step Fourier (SSF) method, where \( \hat{k}_z \) represents the SSR equation computed with a constant reference slowness \( \tilde{s} \), and \( \hat{k}_{z_x} = \omega (s - \tilde{s}) \) represents a space-domain correction (Stoffa et al., 1990).

Based on the equation 28, the derivative of the depth wavenumber relative to slowness is

\[
\left. \frac{d k_z}{d s} \right|_{s_0} = \frac{\omega}{\sqrt{1 - \frac{k_m}{\omega s(s)}}}.
\]  

The numeric implementation of the pseudo-differential equation A-1 is as complicated in media with lateral slowness variation as its phase-shift counterpart (equation 2). However, we can construct efficient and robust numeric implementations using similar approximations as the ones employed for the phase-shift relation, e.g. mixed-domain numeric implementation.

The linearized scattering operator can also be implemented in a mixed-domain by expanding the square-root from
relation A-1 using a Taylor series expansion
\[
\frac{dk}{ds} \bigg|_{s_0} \approx \omega \left( 1 + \sum_{j=1}^{N} c_j \left( \frac{\left| k_m \right|}{\omega s_0(m)} \right)^2 \right), \quad (A-2)
\]
where \(c_j\) are binomial coefficients of the Taylor series.

Therefore, the wavefield perturbation at depth \(z\) caused by a slowness perturbation at depth \(z\) under the influence of the background wavefield at the same depth \(z\) (adjoint scattering operator 9) can be written as
\[
\Delta u \approx \pm i\omega \Delta z \left( 1 + \sum_{j=1}^{N} c_j \left( \frac{\left| k_m \right|}{\omega s_0(m)} \right)^2 \right) u \Delta s. \quad (A-3)
\]

Similarly, the slowness perturbation at depth \(z\) caused by a wavefield perturbation at depth \(z\) under the influence of the background wavefield at the same depth \(z\) (adjoint scattering operator 12) can be written as
\[
\Delta s \approx \mp i\omega \Delta z \left( 1 + \sum_{j=1}^{N} c_j \left( \frac{\left| k_m \right|}{\omega s_0(m)} \right)^2 \right) \mp i\omega \Delta u. \quad (A-4)
\]

The mixed-domain implementation of the forward and adjoint scattering operators A-3 and A-4, is summarized on the following tables:

### 7 APPENDIX B: SUMMARY OF OPERATORS

All wave-equation migration velocity analysis operators described in the preceding sections are similar in that they relate perturbations of the image with perturbations of the (slowness) model. In all cases, this velocity estimation procedure takes advantage of features of migrated images which indicate incorrect imaging. The imaging inaccuracies can have several causes, i.e. incorrect downward continuation, irregular illumination, limited acquisition aperture etc., but the wave-equation MVA operators translate all inaccuracies in model updates. This feature, however, is a fundamental limitation of all migration velocity analysis techniques and we do not expand on this topic further.

Since the migration velocity analysis operators link image perturbations with slowness perturbations, they are all composed of several common parts, but with implementations that are specific for each imaging configuration. Thus, a wave-equation MVA operator is composed of an extrapolation operator (for wavefield reconstruction from recorded data), an imaging operator (for image construction from reconstructed wavefields) and a scattering operator (for relating wavefield perturbations to slowness perturbations). The following table summarizes the wave-equation MVA operator components in different imaging configurations, as described in detail in the preceding sections.
<table>
<thead>
<tr>
<th></th>
<th>extrapolation operator</th>
<th>imaging operator</th>
<th>scattering operator</th>
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<td></td>
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<td>velocity</td>
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<td>DSR</td>
<td>full</td>
<td>backward</td>
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<tr>
<td>shot-record</td>
<td>SSR</td>
<td>full</td>
<td>forward (source)</td>
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Apparent horizontal displacements in time-lapse seismic images

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ABSTRACT
In addition to vertical time shifts commonly observed in time-lapse seismic images, horizontal displacements are apparent as well. These apparent horizontal displacements may be small relatively to seismic wavelengths, perhaps only 5 m at depths of 5 km, but they consistently suggest an outward lateral expansion of images away from a compacting reservoir. It is well known that apparent vertical displacements are caused mostly by a decrease in seismic wave velocities above compacting reservoirs. Those same velocity changes contribute to horizontal displacements. This contribution can be computed from the velocity changes that, in turn, can be estimated from measured vertical displacements. Horizontal displacements computed in this way are similar to those measured, and this similarity suggests that horizontal as well as vertical displacements may be largely due to velocity changes.

Key words: time-lapse seismic imaging

1 INTRODUCTION
Physical displacements of rocks during reservoir compaction cause apparent displacements in time-lapse seismic images. Figures 1 and 2 illustrate an example of apparent vertical displacements (time shifts) measured from two seismic images of a high-pressure high-temperature reservoir in the North Sea.

Figure 1 shows three orthogonal slices from a 3-D seismic image acquired in 2002. Lines in each of the three slices indicate where they intersect in 3-D. The one point where all three slices intersect lies just beneath the target reservoir. To monitor changes in that reservoir, a second 3-D seismic image (not shown) was acquired in 2004. By cross-correlating these two 3-D images for many overlapping windows, we obtain estimates of apparent vertical displacements $\delta t$ like those shown in Figure 2.

These measured apparent vertical displacements $\delta t$ are mostly positive; that is, vertical two-way reflection times generally increase from 2002 to 2004. Hatchell and Bourne (2005a, 2005b) show that such increases are caused mostly by a decrease in seismic wave velocity in rocks above the reservoir as those rocks are stretched by reservoir compaction.

In this sense, Figure 2 is an image of a low-velocity lens. And if not accounted for in seismic migration (as it was not here), this lens above the reservoir will cause apparent horizontal displacements that may obscure any concurrent physical horizontal displacements of subsurface rocks.

Displacements in 3-D are vectors with three components - vertical, inline and crossline - and with careful processing we can measure all three. Figures 3 and 4 show measured inline ($\delta x$) and crossline ($\delta y$) components of apparent displacements corresponding to the same two images acquired in 2002 and 2004. Like the vertical displacements in Figure 2, these apparent lateral displacements are not constant. Both $\delta x$ and $\delta y$ vary as functions of vertical time $t$, inline distance $x$ and crossline distance $y$. Together the three components $\delta t$, $\delta x$ and $\delta y$ comprise a 3-D apparent displacement vector field.

Figures 3 and 4 suggest that apparent displacements in the inline and crossline directions are less consistent than those in the vertical direction. Most measured lateral displacements are less than 5 m, which is
Figure 1. Three orthogonal slices of a 3-D seismic image recorded in 2002. A second image (not shown) was recorded in 2004. Lines in each slice show the intersections of the other two slices.

Figure 2. Vertical components $\delta t$ of apparent displacement vectors, measured in ms.
Apparent horizontal displacements

**Figure 3.** Inline components $\delta x$ of apparent displacement vectors, measured in m.

**Figure 4.** Crossline components $\delta y$ of apparent displacement vectors, measured in m.
relatively small compared with horizontal sampling intervals (inline and crossline trace spacings) of 25 m.

But the largest of these measured lateral displacements appear to be spatially correlated with the reservoir location. Specifically, they imply that, near the reservoir, the seismic image is expanding horizontally.

In this paper we show that much of this apparent horizontal expansion can be explained by the low-velocity lens above the reservoir in 2004. Using the concept of image rays, we show that horizontal displacements like those in Figures 3 and 4 are consistent with an expected expansion outward, away from the center of the reservoir where vertical compaction is largest.

Moreover, from the measured vertical displacements displayed in Figure 2, we can estimate the location, size and shape of the low-velocity lens and then compute expected magnitudes of corresponding horizontal displacements. We show below that these computed magnitudes are approximately 5 m near the reservoir, with a spatial pattern that is consistent with measured displacements.

2 WHICH WAY?

The processing used to measure the apparent displacements shown in Figures 2–4 began with local 3-D prediction-error filtering of the two 3-D seismic images (Hale, 2007). This processing was local in that for each image sample we computed a different 3-D prediction-error filter from a 3-D autocorrelation of only nearby samples. When applied to the 3-D seismic image of Figure 1, local 3-D prediction-error filtering yields the less coherent image displayed in Figure 5.

As expected, local prediction-error filtering has attenuated laterally coherent features, while preserving those features in our images that best enable us to resolve all three components of displacement. Without this filtering, displacements are well defined in only those directions perpendicular to features that (in seismic images) tend to be planar or linear within small cross-correlation windows. Local 3-D prediction-error filtering attenuates such locally predictable features, thereby leaving only a less predictable, more random texture from which we can measure three components of displacement.

Furthermore, by highlighting point-like features that tend to scatter seismic waves in all directions, this filtering simplifies our analysis of lateral displacements. We need not be concerned with specular reflections from subsurface interfaces dipping at various angles. Instead, we need only consider diffractions from scattering points.

Figure 6 illustrates diffractions in 2002 and 2004.
for a single stationary point in the subsurface. In this example, we assume that the point is located on the left side of a compacting reservoir. Compaction causes a dilation of rocks above the reservoir and a corresponding decrease in velocity, so that velocity in 2004 tends to decrease from left to right above this point.

This left-to-right decrease in velocity will cause a right-to-left displacement of diffractions. Velocities directly above the diffracting point are lower in 2004, so that seismic waves propagating there will arrive at the surface later than they did in 2002. The shortest two-way traveltime in 2004 corresponds to an image ray (Hubral, 1975, 1977; Larner et al., 1981) that emerges to the right of their true locations.

Therefore, where velocity decreases from left to right above this point, the peaks of diffractions will shift from right to left.

Now imagine the effect of migration on these two diffractions, where the same migration velocities (the 2002 velocities) are used for both. We would typically use the same velocities, in part because we might not yet know how velocities have changed in 2004. Migration will collapse the diffractions to construct point-like features in migrated images. In migrated time sections, each imaged point will appear at the apex of its corresponding diffraction curve. The point in 2004 will be imaged to the left of the point in 2002.

Assuming that the point in Figure 6 was correctly imaged in 2002, then that same point will be incorrectly imaged in 2004 to the left of its true location.

On the opposite side of the reservoir, where velocity in 2004 decreases from right to left, point-like features in 2004 will be imaged to the right of their true locations. Therefore, the effect of the low-velocity lens above the reservoir in 2004 is an apparent expansion of the image beneath that lens.

Of course, the horizontal displacement of diffractions in Figure 6 is greatly exaggerated. Recall that measured displacements are roughly 5 m at depths of 5 km. This illustration explains only the direction, not the magnitude, of the displacements of diffractions and imaged points that we may expect as velocity above a compacting reservoir decreases.

3 HOW FAR?

To quantify apparent horizontal displacements, we must first quantify the change in velocity from 2002 to 2004, and then compute image rays for 2004 like the one shown in Figure 6. We can estimate the change in velocity from the apparent vertical displacements in time $\delta t$ shown in Figure 2.

To do this, we begin with the distance $\Delta z$ that a seismic wave in 2002 travels vertically downward in time $\Delta t$:

$$\Delta z = v \Delta t,$$

where $v$ denotes the seismic wave velocity in 2002. Assuming that changes from 2002 to 2004 are small, we have

$$\Delta(z) = \delta v \Delta t + v \Delta(\delta t).$$

Dividing equation 2 by equation 1, we obtain

$$\frac{\Delta(z)}{\Delta z} = \frac{\delta v}{v} + \frac{\Delta(\delta t)}{\Delta t},$$

which in the limit of infinitesimal $\Delta z$ and $\Delta t$ becomes

$$\epsilon_{zz} \equiv \frac{d(\delta z)}{dz} = \frac{\delta v}{v} + \frac{d(\delta t)}{dt}.$$  \hspace{1cm} (4)

Here $\epsilon_{zz}$ denotes vertical strain. The quantity $d(\delta t)/dt$ is sometimes called time strain (e.g., Rick-ett, 2006). An alternative phrase that extends well to three components of apparent displacement is apparent strain.

Hatchell and Bourne (2005a, 2005b) show that the fractional change in velocity $\delta v/v$ is approximately proportional to physical vertical strain:

$$\frac{\delta v}{v} = -Re_{zz},$$ \hspace{1cm} (5)

where $R \approx 5$ for rocks above many different compacting reservoirs around the world. Combining equations 4 and 5, we see that this same fractional change in velocity is also proportional to apparent vertical strain:

$$\frac{\delta v}{v} = -\frac{R}{1 + R} \frac{d(\delta t)}{dt}.$$ \hspace{1cm} (6)

In other words, given measured apparent displacements $\delta t(t, x, y)$ like those in Figure 2 and interval velocities $v(t)$ measured in 2002, we can estimate changes in velocity $\delta v(t, x, y)$.

With $\delta v(t, x, y)$, we may then use image ray tracing to estimate apparent horizontal displacements $\delta x(t, x, y)$ or $\delta y(t, x, y)$. The trajectory of a ray can be found by integrating a system of ordinary differential equations.
Quantities in equations 7 are illustrated in Figure 7. At any point \((x, y, z)\) along the raypath, the vector \(\mathbf{p} \equiv (p_x, p_y, p_z)\) with length \(1/v\) points in the direction of propagation.

For general velocity functions \(v = v(x, y, z)\), the ray equations are coupled. We can solve any one of them only by solving all of them simultaneously. For some initial conditions at time \(t = 0\), we typically trace a ray by numerical integration of these equations for finite time steps \(\Delta t\). To compute small apparent displacements in time-lapse seismic imaging, such explicit ray tracing is unnecessary because the relevant image rays are nearly vertical, and the ray equations become essentially uncoupled.

To understand why, consider the two image rays in Figure 6. The defining characteristic of image rays is that their raypaths are vertical at the surface \(z = 0\), where \(p_x = p_y = 0\) and \(p_z = \pm 1/v\). The image raypath in 2002 is entirely vertical because velocity has not yet significantly with time, and so these ray parameters remain constant. For the image raypath in 2004, the ray parameters \(p_x\) and \(p_y\) may increase or decrease slightly as the ray is perturbed by small lateral variations in velocity. However, the raypath in 2004 will remain almost vertical with small \(p_x\) and \(p_y\).

Why almost vertical? Recall that measured lateral displacements in Figures 3 and 4 are about 5 m near the reservoir at a depth of about 5 km. This implies that \(x\) and \(y\) coordinates in equations 7 are almost constant along each raypath, so that each pair of equations 7 is almost entirely decoupled from the other two pairs. And if velocity \(v\) is available as a function \(v(t)\) of vertical time \(t\) (as it usually is in the course of seismic data processing), then we may integrate each pair of ray-tracing equations independently.

To extract apparent lateral displacements from the ray-tracing equations, we need only the first two pairs of differential equations 7. We first rewrite them as

\[
\begin{align*}
\frac{dx}{dt} &= v^2 p_x, \\
\frac{dp_x}{dt} &= -\frac{1}{v} \frac{\partial v}{\partial x}, \\
\frac{dy}{dt} &= v^2 p_y, \\
\frac{dp_y}{dt} &= -\frac{1}{v} \frac{\partial v}{\partial y}, \\
\frac{dz}{dt} &= v^2 p_z, \\
\frac{dp_z}{dt} &= -\frac{1}{v} \frac{\partial v}{\partial z}.
\end{align*}
\]

Equations 9 represent differential equations for apparent lateral displacements \(\delta x\) and \(\delta y\) caused by velocity changes \(\delta v\). In deriving equations 9 from equations 8, we may ignore any terms with \(p_x, p_y, \partial v/\partial x\) and \(\partial v/\partial y\) because, as illustrated in Figure 6, those quantities are all zero for the vertical image rays in 2002.

Dividing both sides of equations 9 by \(dt\) and using equation 6,

\[
\begin{align*}
\frac{d(\delta x)}{dt} &= v^2 \delta p_x, \\
\frac{d(\delta p_x)}{dt} &= \frac{R}{1 + R} \frac{\partial (\delta t)}{\partial x}, \\
\frac{d(\delta y)}{dt} &= v^2 \delta p_y, \\
\frac{d(\delta p_y)}{dt} &= \frac{R}{1 + R} \frac{\partial (\delta t)}{\partial y}.
\end{align*}
\]

Now assume that apparent vertical displacements \(\delta t\) (and, hence, \(\delta p_x\) and \(\delta p_y\)) are zero at the surface \(z = t = 0\), and that the ratio \(R/(1+R)\) does not vary significantly with time, and integrate both \(d(\delta p_x)/dt\) and \(d(\delta p_y)/dt\):

\[
\begin{align*}
\frac{d(\delta x)}{dt} &= v^2 \delta p_x, \\
\delta p_x &= \frac{R}{1 + R} \frac{\partial (\delta t)}{\partial x}, \\
\frac{d(\delta y)}{dt} &= v^2 \delta p_y, \\
\delta p_y &= \frac{R}{1 + R} \frac{\partial (\delta t)}{\partial y}.
\end{align*}
\]

Eliminating \(\delta p_x\) and \(\delta p_y\) from these equations,

\[
\begin{align*}
\frac{d(\delta x)}{dt} &= \frac{R}{1 + R} \int_0^t v^2 \frac{\partial (\delta t)}{\partial x} \, dt, \\
\frac{d(\delta y)}{dt} &= \frac{R}{1 + R} \int_0^t v^2 \frac{\partial (\delta t)}{\partial y} \, dt.
\end{align*}
\]

Integrating once more,

\[
\begin{align*}
\delta x &= \frac{R}{1 + R} \int_0^t v^2 \frac{\partial (\delta t)}{\partial x} \, dt, \\
\delta y &= \frac{R}{1 + R} \int_0^t v^2 \frac{\partial (\delta t)}{\partial y} \, dt.
\end{align*}
\]

As written, equations 13 are not quite consistent with measured horizontal displacements displayed in Figures 3 and 4. First, times \(t\) and time displacements \(\delta t\) in equations 13 above are one-way times corresponding to only the downgoing halves of image rays. To obtain the equations for two-way times, we may simply divide velocities \(v\) by 2. Second, the displacement \(\delta x\) at the surface in Figure 6 has a sign opposite that of the tip of a
downgoing image ray, so we must negate the horizontal displacements in equations 13. With these corrections,

\[
\begin{align*}
\delta x &= -\frac{R}{1 + R} \int_0^t \frac{v^2}{4} \frac{\partial(\delta t)}{\partial x} \, dt, \\
\delta y &= -\frac{R}{1 + R} \int_0^t \frac{v^2}{4} \frac{\partial(\delta t)}{\partial y} \, dt.
\end{align*}
\]

(14)

As suggested by Cox and Hatchell (2008), we may express both lateral and vertical displacements in terms of a single “time-shift potential” function \( \phi = \phi(t, x, y) \) defined by

\[
\phi \equiv -\int_0^t \frac{v^2}{4} \delta t \, dt,
\]

so that

\[
\begin{align*}
\delta x &= \frac{R}{1 + R} \frac{\partial \phi}{\partial x}, \\
\delta y &= \frac{R}{1 + R} \frac{\partial \phi}{\partial y}, \\
\delta t &= -4 \frac{\partial \phi}{v^2} \delta t.
\end{align*}
\]

(15)

Equations 15 and 16 provide rather simple relationships between apparent horizontal displacements \( \delta x(t, x, y) \) and \( \delta y(t, x, y) \) and apparent vertical displacements \( \delta t(t, x, y) \). Because we have measured all three components of displacement (displayed in Figures 2–4), we can test these relationships.

4 TESTING

Equations 15 and 16 enable us to test the hypothesis that apparent horizontal displacements, like apparent vertical displacements, are caused mostly by decreases in seismic velocities above a compacting reservoir. If this hypothesis is valid, then \( \delta x(t, x, y) \) and \( \delta y(t, x, y) \) that we compute via equations 15 and 16 from measured \( \delta t(t, x, y) \) should be comparable to our measured \( \delta x(t, x, y) \) and \( \delta y(t, x, y) \). Discrepancies might suggest other explanations not considered here, such as physical horizontal displacements or changes in seismic anisotropy, as well as errors in our measurements.

Any \( \delta x(t, x, y) \) and \( \delta y(t, x, y) \) that we compute from measured \( \delta t(t, x, y) \) will of course have errors. In this respect, the integration over time \( t \) in equation 15 has both advantages and disadvantages. On the one hand, this integration performs a smoothing that will tend to attenuate errors in \( \delta t(t, x, y) \) that are amplified by lateral derivatives \( \partial \phi/\partial x \) and \( \partial \phi/\partial y \). On the other hand, integration enables errors in \( \delta t(t, x, y) \) for small times to alter the \( \delta x(t, x, y) \) or \( \delta y(t, x, y) \) computed for all times.

Unfortunately, errors in \( \delta t(t, x, y) \) can be large for small times \( t \), where image quality is often degraded as seismograms recorded with large source-receiver offsets are muted. In time-lapse imaging, any variations in seismic acquisition or muting patterns have a more significant effect on earlier reflections than on later ones. Therefore, we should generally avoid integrating from time \( t = 0 \) in equation 15.

Although not displayed in Figure 2, measured vertical displacements \( \delta t \) for times \( t = 0 \) to 2.4 s are negligible in this example, except for a brief range (less than 100 ms) of small times \( t \) where the errors cited above are largest. Therefore, when computing \( \phi \) via equation 15, we replaced the lower limits of integration \( t = 0 \) with \( t = 2.4 \) s, the beginning of the time window displayed in Figure 2.

In addition to measured apparent vertical displacements \( \delta t \), application of equations 15 and 16 requires estimates for interval velocities \( v(t) \) and \( R \). We estimated the function \( v(t) \) using depths and times measured in a checkshot survey. The parameter \( R \) is more difficult to estimate, but for large values of \( R \approx 5 \) the ratio \( R/(1 + R) \) is fairly insensitive to uncertainties in \( R \). We assumed a constant \( R = 5 \).

For the sampled images in this example, we used a simple sum to approximate the integral over time \( t \) in equation 15, and a two-sample centered finite-difference approximation to the partial derivatives \( \partial \phi/\partial x \) and \( \partial \phi/\partial y \) in equations 16. The apparent horizontal displacements computed in this way are displayed in Figures 8 and 9.

Computed horizontal displacements displayed in Figures 8 and 9 are clearly not the same as the measured horizontal displacements in Figures 3 and 4. In particular, computed horizontal displacements are noticeably smoother in time than measured horizontal displacements.

Still, the spatial patterns of the larger displacements are comparable. Both computed and measured displacements imply an apparent lateral expansion. And near the reservoir magnitudes of computed horizontal displacements are approximately 5 m, consistent with measured horizontal displacements.

Moreover, although our derivation and computations here are somewhat different, the computed horizontal displacements shown in Figures 8 and 9 are generally consistent with those shown by Cox and Hatchell (2008). They also show that displacements \( \delta x \) and \( \delta y \) computed from measured \( \delta t \) are comparable to those computed by ray-tracing for a velocity function \( v + \delta v \) estimated from a geomechanical model.

5 CONCLUSIONS

This research began with the unexpected observation that time-lapse seismic images appear to expand laterally away from a compacting reservoir. The simplest explanation for these apparent horizontal displacements is that they are caused primarily by a decrease in seismic wave velocities above the reservoir.

That decrease in velocities is well understood to be largely responsible for the apparent vertical displacements that we today measure routinely in time-lapse imaging. The concept is simple. Waves that travel
Figure 8. Computed inline components $\delta x(t, x, y)$ of apparent displacement vectors, in m. Compare with Figure 3.

Figure 9. Computed crossline components $\delta y(t, x, y)$ of apparent displacement vectors, in m. Compare with Figure 4.
through thick layers of rocks with velocities that have decreased slightly will arrive slightly later in time.

The extension of this concept to horizontal displacements is only a bit more complex, requiring only an understanding of how waves are focused by a low-velocity lens, and how that focusing alters seismic images that have not been processed to account for it.

To account for changes in velocity, we must first quantify them. The method presented in this paper uses lateral derivatives of measured apparent vertical displacements to estimate relevant lateral changes in velocity. Image ray approximations then provide a simple method for computing apparent horizontal displacements in time-lapse seismic images.

Our ability to compute apparent horizontal displacements caused by velocity changes leads to an interesting question. If we subtract any horizontal displacements that we compute from those we measure, are we left with physical horizontal displacements?

Before we can answer this question, we must better understand the accuracy with which we can measure displacements from seismic images, as well as the spatial resolution of those measurements. The processing used here was tuned to enable measurements of all three components of displacements, and care was taken to maximize the fidelity of each step in this processing. But the displacements we measure may be small fractions of seismic wavelengths, and a tradeoff between accuracy and resolution is unavoidable.

The results shown in this paper suggest that future work to improve our understanding of accuracy and resolution in time-lapse seismic imaging is worthwhile. We should ideally measure displacements from unstacked seismic images, because the effects of a low-velocity lens will vary for different source-receiver offsets. We might also consider additional contributions to apparent lateral displacements, such as those resulting from changes in seismic anisotropy. Similarities and differences like those shown here between computed and measured horizontal displacements are interesting. It remains to be seen whether they can be made meaningful.

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Interval anisotropic parameter estimation using velocity-independent layer stripping

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ABSTRACT
Moveout analysis of long-spread P-wave data is widely used to estimate the key time-processing parameter $\eta$ in layered VTI media. Inversion for interval $\eta$ values, however, suffers from instability caused by the tradeoff between the effective moveout parameters and by the subsequent error amplification during Dix-type layer stripping. Here, we propose an alternative approach to interval parameter estimation based on the velocity-independent layer-stripping method of Dewangan & Tsvankin (2006). This method makes it possible to compute the interval moveout in a layer of interest without knowledge of the medium parameters in the overburden. Then the interval traveltimes are inverted for the normal-moveout velocity ($V_{nmn}$) and $\eta$ using the single-layer nonhyperbolic moveout equation. Thus, the layer stripping of $\eta$ in the conventional method is replaced by the much more stable stripping of reflection traveltimes. We also develop the 3D version of the layer-stripping method and apply it to interval parameter estimation for orthorhombic media using wide-azimuth P-wave data. The superior accuracy and stability of our algorithm is illustrated on ray-traced synthetic data for typical layered VTI and orthorhombic models. Even small correlated noise in reflection traveltimes produces substantial distortions in the effective $\eta$, and for some models, even in the effective $V_{nmn}$. As a result, the interval $\eta$ values estimated by the conventional Dix-type method may be highly inaccurate. In contrast, the output of our layer-stripping algorithm proves to be insensitive to mild correlated traveltine errors. The algorithm is also tested on wide-azimuth P-wave reflection data recorded above a fractured reservoir at Rulison field in Colorado. The interval moveout parameters estimated by the velocity-independent layer stripping in the shale layer above the reservoir are more plausible and stable than those obtained by the Dix-type method.

Key words: Anisotropy, interval parameters, layer stripping, nonhyperbolic moveout, wide-azimuth data, layer-cake models

1 INTRODUCTION
Traveltime analysis of surface reflection data yields effective moveout parameters for the whole section above the reflector. However, for purposes of migration velocity analysis, AVO (amplitude-variation-with-offset) inversion, and seismic fracture characterization, it is necessary to estimate interval parameters, which is done using layer-stripping (e.g., Dix, 1955; Grechka & Tsvankin, 1998; Grechka et al., 1999) or tomographic (e.g., Stork, 1992; Grechka et al., 2002) methods.

The conventional Dix (1955) equation, derived for horizontally layered isotropic media, helps to obtain the interval NMO (normal-moveout) velocity using the NMO velocities for the reflections from the top and bottom of a layer. The Dix equation remains valid for NMO velocities of all pure (non-converted) modes in layer-cake VTI (transversely isotropic with a vertical symmetry axis) media. For 3D wide-azimuth data from layered azimuthally anisotropic media, the effective NMO velocity can be obtained by Dix-type averaging of the interval NMO ellipses (Grechka et al., 1999). Then, as
long as the model is laterally homogeneous, the interval NMO velocity or ellipse can be found using Dix-type layer stripping.

NMO velocity and conventional-spread reflection moveout, however, are often insufficient to build the velocity field for anisotropic media, even in the time domain. This explains the importance of using nonhyperbolic (long-spread) reflection moveout in anisotropic parameter estimation. In VTI media, all P-wave time-domain signatures depend on just two parameters, the NMO velocity from a horizontal reflector \( V_{nmo} \) and the anellipticity coefficient \( \eta \) (Alkhalifah & Tsvankin, 1995; Tsvankin, 2005). While \( V_{nmo} \) controls the conventional-spread reflection moveout of horizontal P-wave events, \( \eta \) is responsible for the deviation from hyperbolic moveout at long offsets. Most implementations of nonhyperbolic moveout inversion for VTI media (e.g., Alkhalifah, 1997; Toldi et al., 1999; Grechka & Tsvankin, 1998) are based on the moveout equation of Alkhalifah & Tsvankin (1995) which represents an adaptation of the more general Tsvankin–Thomsen (1994) equation for P-waves.

An alternative algorithm for \( \eta \) estimation operates with the dip dependence of P-wave NMO velocity (Alkhalifah & Tsvankin, 1995). Although the dip-moveout inversion is relatively stable, it requires the presence of dipping reflectors under the formation of interest.

The parameters \( V_{nmo} \) and \( \eta \) are usually estimated from a 2D semblance scan on long-spread data (the maximum offset should reach two reflector depths) from a horizontal reflector. Despite its relative simplicity, nonhyperbolic moveout inversion suffers from the instability caused by the tradeoff between \( V_{nmo} \) and \( \eta \) on any finite spread. Grechka & Tsvankin (1998) found that even small traveltime errors, which could be considered as insignificant in data processing, may cause large errors in the estimated \( \eta \). For layered VTI media, the error is amplified in the layer-stripping process, which may cause unacceptable distortions in the interval \( \eta \) values.

The Alkhalifah–Tsvankin (1995) equation was extended to azimuthally anisotropic models by employing the azimuthally dependent NMO velocity (i.e., the NMO ellipse) and parameter \( \eta \) (Grechka & Tsvankin, 1999; Xu & Tsvankin, 2006). Here, we consider azimuthally anisotropic media with orthorhombic symmetry typical for fractured reservoirs (Schoenberg & Helbig, 1997; Bakulin et al., 2000; Grechka & Kachanov, 2006). Nonhyperbolic moveout of P-waves in an orthorhombic layer with a horizontal symmetry plane is governed by the azimuths of the vertical symmetry planes, the symmetry-plane NMO velocities \( V_{nmo}^{(1)} \) and \( V_{nmo}^{(2)} \) responsible for the NMO ellipse, and three anellipticity coefficients \( \eta^{(1,2,3)} \) (Grechka & Tsvankin, 1999).

For layered orthorhombic media, P-wave moveout is described by the Alkhalifah–Tsvankin equation with the effective moveout parameters (Xu & Tsvankin, 2006; Vasconcelos & Tsvankin, 2006). Since the symmetry-plane NMO velocities and parameters \( \eta^{(1,2,3)} \) depend on the fracture compliances and orientations (Bakulin et al., 2000), nonhyperbolic moveout inversion can help in building physical models for reservoir characterization. Also, the parameters \( V_{nmo}^{(1,2)} \) and \( \eta^{(1,2,3)} \) are sufficient to perform all time-processing steps in orthorhombic models, such as NMO and DMO (dip moveout) correction, and prestack and poststack time migration (Grechka & Tsvankin, 1999). Still, estimation of the interval parameters \( \eta^{(1,2,3)} \) for layered orthorhombic media still suffers from the instability caused by the tradeoff between the effective parameters \( V_{nmo} \) and \( \eta \) and by the error amplification in the layer stripping.

Here, we propose an alternative approach to interval moveout parameter estimation based on the velocity-independent layer-stripping method of Dewangan & Tsvankin (2006). This layer-stripping algorithm, which operates with reflection traveltimes, produces accurate interval long-spread reflection moveout, which can then be inverted for the interval parameters. We discuss both the 2D version of the method designed for layered VTI media and the 3D implementation for wide-azimuth data from layered orthorhombic media. Using ray-traced synthetic data for typical layered anisotropic media, we demonstrate that in contrast to Dix-type inversion, our method remains robust in the presence of typical correlated noise in reflection traveltimes. Finally, we apply the layer-stripping algorithm to wide-azimuth P-wave data acquired over a fractured reservoir at Rulison field in Colorado.
can be computed as

\[
\tilde{t}_{\text{int}}(T, R) = t_{\text{eff}}(x^{(1)}, x^{(2)}) - \frac{1}{2} \left[ t_{\text{ovr}}(x^{(1)}, x^{(3)}) + t_{\text{ovr}}(x^{(2)}, x^{(4)}) \right].
\]

Here, the superscripts eff and ovr refer to the reflection traveltimes from the target reflector and the bottom of the overburden, respectively. The corresponding source-receiver pair (T,R) has the following horizontal coordinates:

\[
x_T = \frac{x^{(1)} + x^{(3)}}{2}, \quad x_R = \frac{x^{(2)} + x^{(4)}}{2}.
\]

Equation 1 and 2 yield the interval reflection moveout function in the target zone without any information about the velocity model.

### 2.2 Layer stripping in 3D

The 3D version of the layer-stripping algorithm does not impose any restriction on the target zone, but each layer in the overburden still has to be laterally homogeneous with a horizontal symmetry plane. For wide-azimuth data (Figure 2), identifying the target and overburden reflections with the same ray segments requires estimating two orthogonal horizontal slowness components from time slopes on wide-azimuth data. In Figure 2, the horizontal slownesses of the target (eff) and overburden (ovr) reflections at location \(x^{(i)} = [x_1^{(i)}, x_2^{(i)}]\) can be obtained from

\[
p_i^{\text{eff}}(x^{(1)}, x^{(2)}) = \frac{\partial t_{\text{eff}}(x, x^{(2)})}{\partial x_i} \bigg|_{x=x^{(1)}}, \quad (i = 1, 2) \quad (3)
\]

and

\[
p_i^{\text{ovr}}(x^{(1)}, x^{(3)}) = \frac{\partial t_{\text{ovr}}(x, x^{(3)})}{\partial x_i} \bigg|_{x=x^{(1)}}, \quad (i = 1, 2). \quad (4)
\]

By equalizing time slopes, we find the location \(x^{(3)}\), for which the horizontal slownesses of the two events are identical,

\[
p_i^{\text{eff}}(x^{(1)}, x^{(2)}) = p_i^{\text{ovr}}(x^{(1)}, x^{(3)}).
\]

Therefore, the reflections \(x^{(1)}TQRx^{(2)}\) and \(x^{(1)}Tx^{(3)}\) share the same downgoing leg \(x^{(1)}T\). Similarly, equalizing the time slopes at point \(x^{(2)}\) helps to find the overburden reflection \(x^{(2)}Rx^{(4)}\) that shares the same upgoing leg \(Rx^{(2)}\) with the target event \(x^{(1)}TQRx^{(2)}\). The interval reflection traveltimes can then be obtained from equation 1 discussed above. Since T and R represent the midpoints of the corresponding source-receiver pairs, their horizontal coordinates can be easily found from the coordinates of points \(x^{(1)}, x^{(2)}, x^{(3)},\) and \(x^{(4)}\).

Thus, the velocity-independent layer-stripping algorithm makes it possible to construct the interval moveout functions both in 2D and 3D, which can then be used for interval parameter estimation.
3 TESTS ON SYNTHETIC DATA

Here, we test our layer-stripping algorithm on 2D and 3D long-spread P-wave data for layered VTI and orthorhombic models generated by anisotropic ray tracing (Gajewski & Psenčík, 1987). The interval moveout parameters in the target layer are estimated from both our method and Dix-type equations. To evaluate the stability of the two techniques, we add several types of correlated noise to the input traveltimes.

3.1 2D inversion for VTI media

Nonhyperbolic moveout of P-waves in a single VTI layer can be accurately described by the following equation (Alkhalifah & Tsvankin, 1995; Tsvankin, 2005):

\[ t^2 = t_0^2 + \frac{x^2}{V_{nmo}^2} - \frac{2\eta x^4}{V_{nmo}^2[t_0^2 V_{nmo}^2 + (1 + 2\eta)x^2]}. \]

where \( x \) is the offset, \( t_0 \) is the two-way zero-offset reflection traveltime, \( V_{nmo} \) is the normal-moveout velocity and \( \eta \) is the anellipticity coefficient. For layer-cake VTI media, all moveout parameters become effective quantities (Alkhalifah & Tsvankin, 1995; Tsvankin, 2005):

\[ t^2(N) = t_0^2(N) + \frac{x^2}{V_{nmo}^2(N)} - \frac{2\eta(N) x^4}{V_{nmo}^2(N)[t_0^2(N) V_{nmo}^2(N) + (1 + 2\eta(N))x^2]}. \]

The effective NMO velocity is obtained from the Dix equation,

\[ V_{nmo}^2(N) = \frac{1}{t_0(N)} \sum_{i=1}^{N} (V_{nmo}^{(i)})^2 t_0^{(i)}, \]

Table 1. Interval parameters of a three-layer VTI model (model 1)

<table>
<thead>
<tr>
<th>Layer 1</th>
<th>Layer 2</th>
<th>Layer 3 (target)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thickness (km)</td>
<td>0.7</td>
<td>0.3</td>
</tr>
<tr>
<td>( t_0 ) (s)</td>
<td>0.70</td>
<td>0.25</td>
</tr>
<tr>
<td>( V_{nmo} ) (km/s)</td>
<td>2.10</td>
<td>2.52</td>
</tr>
<tr>
<td>( \eta )</td>
<td>0</td>
<td>0.10</td>
</tr>
</tbody>
</table>

Figure 3. (a) Synthetic long-spread reflections from the top and bottom of layer 3 (target) in model 1. (b) The \( t^2(x^2) \) function (solid lines) for both events in plot (a). The dashed lines mark the hyperbolic moveout function, \( t^2 = t_0^2 + \frac{x^2}{V_{nmo}^2} \). The model parameters are listed in Table 1.

Figure 4. Random traveltime error with the maximum magnitude about 10 ms.
where $t_0^{(i)}$ and $V_{nmo}^{(i)}$ are the inteval values. The effective parameter $\eta$ is approximated by

$$
\eta(N) = \frac{1}{8} \left\{ \frac{1}{V_{nmo}^{(N)} t_0(N)} \right\} \left[ \sum_{i=1}^{N} (V_{nmo}^{(i)})^4 (1 + 8\eta^{(i)}) t_0^{(i)} \right] - 1 \right\}.
$$

a: Command not found.

The best-fit effective parameters $V_{nmo}$ and $\eta$ for the top and bottom of a layer of interest are obtained by applying semblance-based nonhyperbolic moveout inversion to long-spread P-wave data. Then the interval $V_{nmo}$ in layer $i$ can be obtained from the Dix equation 8,

$$
(V_{nmo}^{(i)})^2 = \frac{V_{nmo}^2(t_0^{(i)}) - V_{nmo}^2(t_0^{(i-1)})}{t_0^{(i)} - t_0^{(i-1)}},
$$

and the interval $\eta$ can be found from equation 9:

$$
\eta^{(i)} = \frac{1}{8(V_{nmo}^{(i)})^4} \left[ g(i)t_0^{(i)} - g(i-1)t_0^{(i-1)} \right] - (V_{nmo}^{(i)})^4; 
$$

where $g(N) = V_{nmo}^4(N) \left[ 1 + 8\eta(N) \right]$.

Although equations 6 and 7 provide a good approximation for nonhyperbolic moveout in VTI media, the estimated $\eta$ is sensitive to small errors in $V_{nmo}$ even if the maximum offset-to-depth ratio ($x_{max}/h$) is between two and three. The tradeoff between the effective $V_{nmo}$ and $\eta$ (along with the slight bias of the nonhyperbolic moveout equation) causes the instability in the $\eta$ estimation, which is amplified in the Dix-type layer stripping (Grechka & Tsvankin, 1998).

### 3.1.1 Model 1

The first numerical test was performed for the three-layer VTI model with the parameters listed in Table 1. We applied the velocity-independent layer stripping (VLS) and the Dix-type method to the synthetic long-spread ($x_{max}/h = 2$ for the bottom of the model) data from the top and bottom of the target (third) layer (Figure 3). Note that, in Figure 3, the $t^2(x^2)$ curves for both events deviate obviously from the hyperbolic moveout approximation at large offsets. The absolute error of the interval $\eta$ estimated by the VLS is 0.02, while it reaches 0.06 when the Dix-type method is used. The error of the VLS is mostly caused by the small bias of the single-layer moveout equation 6.

### 3.1.2 Error analysis

To study the influence of realistic noise on the interval parameter estimation, we added random, linear and sinusoidal time error to the reflection moveout from the bottom of the target layer. The traveltimes from the top of the target were left unchanged.

First, we used random errors with the magnitude of up to 10 ms (Figure 4). The errors in the interval $\eta$ estimated by VLS do not exceed 0.02, while the Dix-type method produces errors in the range of 0.05-0.08. Since both methods are based on semblance analysis, they remain stable in the presence of random noise.

The second type of noise used in our tests is linear, which can simulate long-period static errors. For a relatively large error that changes from 6 ms at zero offset to -6 ms at the maximum offset, VLS estimates the interval $V_{nmo}$ and $\eta$ with errors of 4% and 0.07, respectively. The distortions in $V_{nmo}$ and $\eta$ after the Dix-type layer stripping are much larger (15% and 0.34, respectively), which makes the inversion results practically useless.

Next, we added the sinusoidal time errors, which can represent short-period static errors: $t = A \sin(n \pi x / x_{max})$. The errors in the interval $V_{nmo}$ and $\eta$ estimated by both methods for different values of $A$ and $n$ are listed in Table 2. The error in the interval $\eta$ produced by VLS reaches only 0.08 even for $A = 8$ ms, while the Dix-type method breaks down for $A > 3$ ms.

### 3.2 3D inversion for orthorhombic media

The azimuthally-dependent P-wave reflection moveout in a single orthorhombic layer can be well-approximated with azimuthally varying parameters $V_{nmo}$ and $\eta$ in equation 6 (Xu & Tsvankin, 2006; Vasconcelos & Tsvankin, 2006):

$$
\tilde{t}^2(x, \alpha) = t_0^2 + \frac{x^2}{V_{nmo}^{(1)}(\alpha)} - \frac{2\eta(\alpha) x^4}{V_{nmo}(\alpha) t_0 \frac{V_{nmo}^{(1)}(\alpha)}{V_{nmo}(\alpha) + (1 + 2\eta(\alpha)) x^2}},
$$

where $\alpha$ is the source-to-receiver azimuth. The azimuthally-dependent $V_{nmo}$ is obtained from the equation of the NMO ellipse:

$$
V_{nmo}^{(2)}(\alpha) = \frac{\sin^2(\alpha - \varphi) + \cos^2(\alpha - \varphi)}{\left[ V_{nmo}^{(1)}(\alpha) \right]^2} + \frac{\cos^2(\alpha - \varphi)}{\left[ V_{nmo}^{(2)} \right]^2},
$$

where $\varphi$ is the azimuth of the $[x_1, x_3]$ symmetry plane, and $V_{nmo}^{(1)}$ and $V_{nmo}^{(2)}$ are the NMO velocities in the vertical symmetry planes $[x_2, x_3]$ and $[x_1, x_3]$, respectively. The parameter $\eta$ is approximated by Pech & Tsvankin (2004):

$$
\eta(\alpha) = \eta^{(1)} \sin^2(\alpha - \varphi) + \eta^{(2)} \cos^2(\alpha - \varphi)
$$

$$
- \eta^{(3)} \sin^2(\alpha - \varphi) \cos^2(\alpha - \varphi),
$$

where $\eta^{(1)}$, $\eta^{(2)}$, and $\eta^{(3)}$ are the anellipticity coefficients defined in the $[x_2, x_3]$, $[x_1, x_3]$, and $[x_1, x_2]$ symmetry planes, respectively.

For layered orthorhombic media, all moveout parameters become effective quantities. The effective
Error parameters $A = 3\text{ ms}, n = 3$ $A = 3\text{ ms}, n = 2$ $A = 8\text{ ms}, n = 3$

<table>
<thead>
<tr>
<th>Inversion error</th>
<th>$V_{\text{nmo}}$ (%)</th>
<th>$\eta$</th>
<th>$V_{\text{nmo}}$ (%)</th>
<th>$\eta$</th>
<th>$V_{\text{nmo}}$ (%)</th>
<th>$\eta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>VILS</td>
<td>0.6</td>
<td>0.01</td>
<td>0.0</td>
<td>0.00</td>
<td>2.1</td>
<td>0.08</td>
</tr>
<tr>
<td>Dix</td>
<td>11</td>
<td>0.19</td>
<td>8.2</td>
<td>0.13</td>
<td>21</td>
<td>0.41</td>
</tr>
</tbody>
</table>

Table 2. Percentage errors of the interval $V_{\text{nmo}}$ and absolute errors of the interval $\eta$ estimated by the velocity-independent layer stripping (VILS) and the Dix-type method (Dix) in the presence of a sinusoidal time error for model 1.

\[ V_{\text{nmo}}^{(1)}, V_{\text{nmo}}^{(2)} \text{ and } \varphi \text{ can be obtained from the generalized Dix equation for averaging of the interval NMO ellipses (Grechka et al., 1999). If the vertical symmetry planes in different layers are misaligned, the principal directions for the effective $\eta$ are described by a separate azimuth, } \varphi_1 \text{ (Xu & Tsvankin, 2006):} \]

\[ \eta(\alpha) = \eta^{(1)} \sin^2(\alpha - \varphi_1) + \eta^{(2)} \cos^2(\alpha - \varphi_1) - \eta^{(3)} \cos^2(\alpha - \varphi_1) \sin^2(\alpha - \varphi_1). \]

(15)

The effective $\eta$ value for each azimuth can be computed from equation 9, since kinematic signatures in each vertical plane of layered orthorhombic media can be approximately described by the corresponding VTI equation (Tsvankin, 1997, 2005). Then the effective $\eta^{(1)}$, $\eta^{(2)}$, $\eta^{(3)}$ and $\varphi_1$ are found by fitting the effective $\eta$ values to equation 15.

We use the semblance-based 3D nonhyperbolic moveout inversion algorithm of Vasconcelos & Tsvankin (2006) to estimate the best-fit effective moveout parameters $V_{\text{nmo}}^{(1,2)}$, $\eta^{(1,2,3)}$, $\varphi$ and $\varphi_1$ for the top and bottom of the target layer. Then the interval NMO ellipse is obtained from the generalized Dix equation (Grechka et al., 1999), and the interval $\eta$ value for each azimuth is computed from the VTI equation 11. Finally, the interval parameters $\eta^{(1,2,3)}$ are obtained by fitting the azimuthally varying $\eta$ values to equation 14.

To apply VILS to 3D wide-azimuth data, we have to estimate both horizontal slowness components. In principle, the two orthogonal horizontal components of the slowness vector (equations 3 and 4) can be computed directly from reflection traveltimes on common-shot or common-receiver gathers. A more numerically stable option, however, is to use equation 12 with the best-fit parameters to estimate the horizontal slownesses at each surface location. Despite the tradeoffs between the effective moveout parameters, equation 12 provides sufficient accuracy for long-spread P-wave moveout and, therefore, for the horizontal slowness components.
Interval parameter estimation using VILS

<table>
<thead>
<tr>
<th>Error parameters</th>
<th>A = 3 ms, n = 3, m = 0</th>
<th>A = 3 ms, n = 3, m = 2</th>
<th>A = 10 ms, n = 3, m = 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inversion error</td>
<td>V_{nmo} (%)</td>
<td>η</td>
<td>V_{nmo} (%)</td>
</tr>
<tr>
<td>VILS</td>
<td>1.2</td>
<td>0.04</td>
<td>1.0</td>
</tr>
<tr>
<td>Dix</td>
<td>4.4</td>
<td>0.09</td>
<td>2.3</td>
</tr>
</tbody>
</table>

Table 3. Percentage maximum errors of the interval $V_{nmo}^{(1,2)}$ and the absolute maximum errors of the interval $\eta_{nmo}^{(1,2,3)}$ in the presence of a sinusoidal time error in model 2. The errors in the estimated azimuth $\varphi$ do not exceed 0.5° for both methods.

<table>
<thead>
<tr>
<th>Layer 1</th>
<th>Layer 2</th>
<th>Layer 3 (target)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Symmetry type</td>
<td>ISO</td>
<td>VTI</td>
</tr>
<tr>
<td>Thickness (km)</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>$t_0$ (s)</td>
<td>0.50</td>
<td>0.41</td>
</tr>
<tr>
<td>$V_{nmo}^{(1)}$ (km/s)</td>
<td>2</td>
<td>2.49</td>
</tr>
<tr>
<td>$V_{nmo}^{(2)}$ (km/s)</td>
<td>2</td>
<td>2.49</td>
</tr>
<tr>
<td>$\eta^{(1)}$</td>
<td>0</td>
<td>0.05</td>
</tr>
<tr>
<td>$\eta^{(2)}$</td>
<td>0</td>
<td>0.05</td>
</tr>
<tr>
<td>$\eta^{(3)}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\varphi$ (°)</td>
<td>30</td>
<td></td>
</tr>
</tbody>
</table>

Table 4. Interval parameters of a three-layer model that includes an orthorhombic layer (model 2).

### 3.2.1 Model 2

The second model includes a target orthorhombic layer beneath the overburden composed of isotropic and VTI layers (Table 3). We applied both layer-stripping methods to the synthetic long-spread ($x_{max}/h = 2$ for the bottom of the model), wide-azimuth data from the top and bottom of the target layer (Figure 5). Note that the traveltimes from the bottom of the target layer in Figure 5 vary with azimuth. Without traveltime noise, both methods give similar accuracy in the interval moveout parameters.

### 3.2.2 Error analysis

As before, we added linear and sinusoidal time error to the reflection moveout from the bottom of the target layer in model 2. For the linear error that changes from 6 ms at zero offset to -6 ms at the maximum offset for each azimuth, the interval $V_{nmo}^{(1,2)}$ and $\eta_{nmo}^{(1,2,3)}$ estimated by VILS are distorted by no more than 3% and 0.06, respectively. In contrast, the Dix-type method produces the maximum errors of 9% and 0.16 in the interval $V_{nmo}^{(1,2)}$ and $\eta_{nmo}^{(1,2,3)}$, respectively. The errors in the azimuth $\varphi$ for both methods are negligible.

We also contaminated the traveltimes with several sinusoidal functions of the form $A \sin(n\pi x/x_{max}) \sin m\alpha$. The coefficients $n$ and $m$ control the period of the error function in the radial and azimuthal directions, respectively. When $m = 0$ (i.e., no azimuthal variation in the error), both methods get more accurate results when $n$ is an even number (Table 4), which agrees with the conclusions of Xu & Tsvankin (2006). If the noise varies with azimuth, the errors in the inversion results are higher for the even values of $m$. VILS produces the errors not exceeding 0.09 in the interval $\eta$ even for $A = 10$ ms, while Dix-type method gives unacceptably large errors of up to 0.22.

Next, we studied the influence of the thickness of the target layer on the inversion results. Any layer-stripping method becomes less accurate as the layer gets thinner. We added the sinusoidal error with $A = 3$ ms, $n = 3$ and $m = 0$ to the traveltimes from the bottom of the target layer in model 2 and reduced the layer thickness from 0.5 km to 0.15 km (Table 5). Under this noise, VILS gives accurate values of both $V_{nmo}$ and $\eta$ even when the thickness is 0.2 km, while the error in the interval $V_{nmo}$ estimated by the Dix-type method exceeds 8%. When the thickness decreases to 0.15 km, even the errors from VILS become unacceptably large.

### 3.2.3 Model 3

The third model includes the target orthorhombic layer beneath the overburden composed of isotropic and orthorhombic layers (Table 6). Note that the vertical symmetry planes in the orthorhombic layers are misaligned, so the parameter $\eta$ is described by equation 15. As was the case for model 2, the accuracy of both methods for noise-free data is similar.

The sinusoidal error $t = A \sin(n\pi x/x_{max})$ applied to the traveltimes from the bottom of the target layer
produces much more significant distortions in the output of the Dix-type method compared to VILS. For instance, when $A = 6$ ms and $n = 3$, the maximum errors in the interval $V_{nmo}$ and $\eta^{1,2,3}$ estimated by VILS are 4% and 0.09, respectively, while the Dix-type method produces the errors of 13% and 0.25, respectively.

### 4 FIELD-DATA EXAMPLE

We applied our 3D VILS algorithm to wide-azimuth P-wave data acquired at Rulison field, a basin-centered gas accumulation located in South Piceance Basin, Colorado. The reservoir (Williams Fork formation) is capped by the UMV shale, which served as the target layer in our study (Figure 6).

Xu & Tsvankin (2007) applied a comprehensive processing sequence designed for layered azimuthally anisotropic media to the data and analyzed the azimuthal AVO response and the effective and interval NMO ellipses. We used the same data set as Xu & Tsvankin (2007), which was preprocessed for purposes of azimuthal moveout and AVO analysis. Since the subsurface structure is close to layer-cake (Figure 7), the moveout equations discussed above should give an accurate description of reflection traveltimes. To increase azimuth and offset coverage, we combined CMP gatherings into 5x5 superbins, as suggested by Xu & Tsvankin (2007). We carried out the moveout inversion in the center of the RCP survey area (Figure 8), where the azimuthal coverage is sufficient for eliminating the acquisition footprint (Xu & Tsvankin, 2007).

Because the average offset-to-depth ratio at the bottom of the reservoir is close to unity, nonhyperbolic moveout inversion cannot be applied to the reservoir formation. Therefore, the target layer is chosen to be the UMV shale (cap rock), which overlies the reservoir. In the center of the study area, the offset-to-depth ratio at the bottom of the shale is between 1.9 and 2.2.
Table 7. Interval parameters $\eta_{1,2,3}$ estimated for two superbin gathers near the center of the study area.

<table>
<thead>
<tr>
<th>Superbin</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\eta^{(1)}$</td>
<td>$\eta^{(2)}$</td>
</tr>
<tr>
<td>VILS</td>
<td>0.38</td>
<td>0.47</td>
</tr>
<tr>
<td>Dix</td>
<td>0.74</td>
<td>1.24</td>
</tr>
</tbody>
</table>

To estimate the interval moveout parameters, we used the VILS and Dix-type algorithms for the layered orthorhombic model discussed above.

Our tests show that the NMO ellipticity is small for both the top and bottom of the target layer over most of the area. Therefore, the principal directions of the effective and interval NMO ellipses are poorly constrained by the data. However, as long as the offset-to-depth ratio is close to two, the interval $\eta_{1,2,3}$ can be estimated in a reliable fashion. The interval parameters $\eta_{1,2,3}$ for two superbin gathers near the center of the area are listed in Table 7. The interval $\eta$ values obtained by the Dix-type method are implausibly large for shale formations (Wang, 2002; Tsvankin, 2005).

To test the inversion stability of both methods, we also added a linear time error (from 4 ms at zero offset to -4 ms at the maximum offset for each azimuth) to the reflection moveout from the bottom of the shale layer in the second superbin. The interval $\eta_{1,2,3}$ estimated by VILS change only by -0.06, -0.07, 0.01, respectively, and the changes produced by the Dix-type method are much larger: -0.12, -0.21, 0.13, respectively. Hence, VILS is much more stable than the Dix-type method in the presence of correlated time errors, as was established above for the synthetic data.

The NMO ellipticity is pronounced only near the east boundary of the study area (Xu & Tsvankin, 2007). Due to the small offset-to-depth ratio (between 1 and 1.3) for the bottom of the target layer near the edges of the area, the effective parameters $\eta_{1,2,3}$ may contain large errors. The estimated interval $V_{\text{nmo}}^{(1,2,3)}$ and $\varphi$ for two adjacent superbin gathers near the east boundary are listed in Table 8. As before, we added a linear time error (from 2 ms at zero offset to -2 ms at the maximum offset for each azimuth) to the traveltimes from the bottom of the shale in the second superbin, which causes the deviation of 3% in the effective $V_{\text{nmo}}^{(1,2)}$. As a result, the interval $V_{\text{nmo}}^{(1,2,3)}$ and $\varphi$ estimated by VILS change by 7%, 8% and 1°, respectively. In contrast, the changes from the Dix-type method are 13%, 16% and 4°, respectively. Due to the thin layer and relatively low semblance values (about 0.4) in 3D moveout analysis, the noise causes some significant deviations even in VILS. However, it is still beneficial to apply VILS for interval NMO velocity estimation from conventional-spread data.

5 DISCUSSION AND CONCLUSIONS

We applied the velocity-independent layer-stripping method (VILS) of Dewangan & Tsvankin (2006) to non-hyperbolic moveout inversion for layered VTI and orthorhombic media. While Dix-type differentiation algorithms operate with effective moveout parameters, VILS is based on stripping of reflection traveltimes. If the overburden is laterally homogeneous and has a horizontal symmetry plane, VILS produces the exact interval traveltimes without any information about the velocity field. Then the interval traveltimes are inverted for the relevant parameters of the target layer using moveout equations for a homogeneous medium.

Because effective traveltimes are much better constrained by reflection data than effective moveout parameters, VILS gives more stable interval parameter estimates than Dix-type techniques. In particular, our synthetic tests on noise-contaminated data confirm that VILS can substantially increase the accuracy of non-hyperbolic moveout inversion for the interval time-processing parameter $\eta$ in VTI media. The addition of small linear or sinusoidal time errors causes pronounced distortions in the effective $\eta$ values, which get further enhanced by Dix-type layer stripping. In contrast, the interval moveout function produced by VILS is weakly sensitive to moderate levels of noise in the input traveltimes, which ensures a higher stability of the interval $\eta$ estimates.

We also discussed the extension of VILS to 3D wide-azimuth P-wave data from azimuthally anisotropic models that include orthorhombic and TI layers. To identify the target and overburden reflections that share the same ray segments, we compute the horizontal slowness components from the best-fit effective moveout parameters, which helps to avoid direct differentiation of traveltimes and to reduce the computational cost. Then the interval moveout produced by VILS is inverted for the azimuths of the vertical symmetry planes, symmetry-direction NMO velocities, and the anellipticity parameters $\eta_{1,2,3}$. Wide azimuthal coverage helps to increase the stability of $\eta$ estimation using 3D Dix-type layer stripping. Still, our tests clearly demonstrate the superior performance of VILS for typical orthorhombic models, including those with the depth-varying azimuths of the symmetry planes.

The 3D version of the method was successfully tested on wide-azimuth P-wave reflections from an anisotropic shale layer at Rulison field in Colorado. For long-spread superbin gathers in the center of the study area, VILS yields more plausible and stable values of the interval parameters $\eta_{1,2,3}$ than the Dix-type method. Near the east boundary of the study area, where the offset-to-depth ratio is smaller and the $\eta$-parameters are poorly constrained, application of VILS helps to obtain a more stable estimate of the interval NMO ellipse.

While our results clearly demonstrate the advantages of our method over Dix-type techniques, the su-
Table 8. Interval NMO ellipses estimated for two superbin gathers near the east boundary of the study area.

<table>
<thead>
<tr>
<th>Superbin</th>
<th>V_{nmo}^{(1)} (km/s)</th>
<th>V_{nmo}^{(2)} (km/s)</th>
<th>φ (°)</th>
<th>V_{nmo}^{(1)} (km/s)</th>
<th>V_{nmo}^{(2)} (km/s)</th>
<th>φ (°)</th>
</tr>
</thead>
<tbody>
<tr>
<td>VILS</td>
<td>4.22</td>
<td>3.99</td>
<td>115</td>
<td>4.33</td>
<td>3.84</td>
<td>122</td>
</tr>
<tr>
<td>Dix</td>
<td>4.26</td>
<td>3.82</td>
<td>104</td>
<td>4.41</td>
<td>3.91</td>
<td>139</td>
</tr>
</tbody>
</table>

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Seismic modeling and analysis of the prototype heated nuclear waste storage tunnel, Yucca Mountain, NV

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ABSTRACT
The 1997-2002 Yucca Mt. heated drift scale test (DST) used a specialized system of heaters to simulate thermal effects from stored nuclear waste on a tunnel, surrounding volcanic tuff, and groundwater flow. Seismic calibration shots were recorded on a receiver array installed around the tunnel while temperatures inside were elevated to approximately 200°C. Receiver gathers show classic wave propagation behavior as given in the literature for pulses incident upon cylinders/tunnels. However, a combination thermal/groundwater process causes direct and reflected P-S arrival changes in the surrounding tuff as temperatures increase. Data also show these waveforms effectively do not change once the target temperature is obtained at the tunnel wall. Laboratory data for granite was used as a physical analog to develop velocity versus temperature models for Yucca Mt. tuff. Barring 3-D effects/out-of-plane reflections, 2-D spectral element waveform simulation coupled with well-constrained thermal models has consistently replicated receiver data in the plane of the calibration source. Field/model waveform agreement is a function of velocity gradient for \( V_{p,s}(x, z, t, T) \) models derived by fitting and adjusting said granite data. Waveforms using velocity model sets with a \( V^{-1}dV/dT \) gradient of \( -0.5\% \) per 100°C show improved agreement with field data over those with a \( -2.5\% \) per 100°C gradient. This optimal velocity change is lower than values stated in literature. Because the velocity gradient is small, velocity changes may be controlled by fracturing due to thermal expansion and fragment compression, not large bulk modulus changes associated with groundwater phase transitions. In addition, our numerical modeling shows a diffracted/“Franz” wave propagating perpendicular to the tunnel wall in the thermal transition region. This wave is present in field and model data propagating at speeds that vary with conditions near the tunnel. P-S separation and Franz wave velocity are therefore potential tools for seismic measurement and monitoring of conditions near the tunnel.

Key words: tunnels, boreholes, seismic modeling, seismic inversion, monitoring, rock physics, diffracted waves, time-lapse

1 INTRODUCTION
This project has focused on the processing and analysis of seismic data collected during the second through fourth years of the 1998-2002 heated drift scale test (DST) at the Yucca Mountain facility, 90 miles NW of Las Vegas, NV (Tsang et al., 1999; Rutqvist, 2004; TRW Environmental Safety Systems, Inc., 1998). The facility is located within a large block of tuff (volcanic ash) half-
way between the surface and water table, and hundreds of meters from bounding fault planes (Day et al., 1998). Over four years the tunnel was heated to approximately 200°C by two sets of heating elements to simulate thermal emission from decaying nuclear waste. One set of heaters is a group of canisters located along the axis of the tunnel, with the second set of “wing” heaters extending 15 meters into the rock on either side (Figure 1(a)). The entire system was monitored for hydrological, chemical, seismic and other indications of environmental change. Check/calibration shots were collected to verify the functionality of a receiver array installed to monitor seismic activity (Lehtonen, 2005; Rutledge et al., 2003). Seismic data show an increasing separation between P and S wave arrivals with increasing temperature. Velocity models were based on numerical fits of wavewave measurements of granite, and adjusted to velocities of tuff using differential arrival times between receivers. Elastic waveform modeling using data constrains in-situ velocity changes in the rock to approximately −0.5% of ambient velocity per 100°C. These velocity changes, when related to recorded waveforms by numerical modeling, may be used for monitoring changes in the surrounding rock caused by heating.

2 FIELD DATA PROCESSING AND TRENDS

A joint effort by the U.S. national labs collected seismic data about the 47.5 meter long, 5 meter diameter tunnel using receivers locations shown by the triangles in Figure 1(b). Each receiver is a single-component instrument assumed to be aligned along the axis of the radially oriented borehole. The seismic source was a sledgehammer located in the adjoining instrumentation/observation drift 28 meters away, and six meters above the axis of the experimental drift (Figure 1(a)) (TRW Environmental Safety Systems, Inc., 1998; Rutledge, 2006). Calibration shots comprising the data set were used to check the functionality of the array after it failed to record during the first year of heating. We processed the data to remove of coherent, bandlimited noise and spurious samples (degliching). Waveforms recorded on the same day were then stacked. Because the raw data lack a trigger signal, waveforms were aligned by first breaks for correlation analysis. This analysis revealed the P-S separation trends evident in Figure 2. Increasing P-S separation occurs up to approximately two years into the experiment. Around that date temperature stabilizes to within 20°C of the 200°C target temperature and the P-S separation remains roughly constant. This trend is observed to varying extent in all of the processed receiver gathers, and we hypothesize that the trend is caused by changing wavewaves in the neighborhood of the tunnel resulting from thermal effects and changing groundwater saturation.

3 VELOCITY MODELING

2D Elastic waveform modeling was conducted to constrain wavewaves changes in the surrounding rock. Ground-truth data consisting of temperature versus radius curves at 12 month intervals above and below the tunnel (Rutqvist, 2004) permits the definition of the temperature at the tunnel wall (Figure 3(a)) and minor axis of an elliptical thermal zone caused by the core and “wing” heaters (Figure 3(b)). Ellipticity of velocity models is based on the ellipticity of a published thermodynamic model of the transition zone at 12 months (Rutqvist, 2004), and is assumed to remain constant for all temperatures/events. Wavewave speeds as a function of temperature from two separate sets of measurements on granite were used as physical analog for the Yucca Mountain tuff (Figures 3(c) and 3(d)). Only the section of the hysteresis cycle of Figure 3(d) indicated by the arrow was used in the second model. Granite is used to model tuff because both are igneous rocks with 65–75% quartz content (Carmichael, 1989; Grét et al., 2006). Grain size is not considered a primary velocity control as fracturing occurs both around and through grains at sufficiently high temperatures and pressures (Batzie et al., 1980). Because the data lacks a trigger, measurements of differential path-lengths vs. arrival times were used to determine an ambient compressional velocity of approximately 3600 m/s. This value agrees well with ambient velocities of Yucca Mountain Topopha tuff given by New England Research (New England Research, 2007) and Indian Springs tuff from the Mines Commonground Database (Batzie & Scales, 2001-present). Polynomial coefficients of the granite data were adjusted to match the ambient velocity of tuff form the differential travel times, generating velocity vs. temperature curves for tuff (Figures 3(e) and 3(f)). These models have −0.5% and −2.5% slopes with respect to ambient velocities over a 100°C temperature domain. Red squares in Figure 3(a) and circles on lines in Figures 3(e) and 3(f) indicate days/temperaturest where calibration shot data is available. An example compressional velocity model for the initial event (≈ 135°C) is shown in Figure 4. Note that the seismic calibration shots do not begin at ambient temperature/velocity conditions or continue into the cooling phase when the heaters were shut down.

4 MODEL/DATA COMPARISON

Model waveforms were generated using SEM2DPAK spectral element software (Ampuero, 2007), a 2D package that limits analysis to the receivers in the plane of the source. Spectral element modeling (Komatitsch & Tromp, 1999) uses a variational integral formulation and interpolating polynomials across computational mesh elements to solve for displacement. It does not suffer from much of the dispersion effects and numerical instabilities seen in finite element and finite difference al-
Figure 1. (a) 3-D Visualization of Yucca Mountain DST tunnel from Rutqvist (2004) showing experimental tunnel with temperature monitoring boreholes and “wing” heaters. Seismic source (sledgehammer) location is in nearby observation tunnel. (b) Seismic array receiver locations (triangles), distributed along the length of DST tunnel (end-on view). Labeled receivers used on this study are in the plane of the source (CH1 to CH6). Shaded regions around waste package (center/red) indicate hypothetical groundwater dryout (near, red), and phase transition (medium, blue) after Spycher (2003).

Figure 2. Receiver gather for instrument opposite tunnel from source, chronologically from bottom to top (years on right). Note increasing separation of primary (compressional) and secondary (shear) arrivals during heating of the tunnel (indicated by red bar) that remain fixed through period of maximum operating temperature of 200°C (blue bar), indicating changes to the rock occurring during tunnel heating.
Figure 3. Velocity modeling data as described in section 3 of the text. (a) Tunnel wall temperature. (b) Minor axis of thermal transition zone. (c) and (d) Granite wavespeed data (see text). (e) and (f) Extrapolated tuff wavespeeds having $-0.5\%$ and $-2.5\%$ slopes per $100^\circ$C with respect to ambient velocity, respectively.
Figure 4. Example tuff compressional velocity at ≈ 135°C model derived from granite/tuff data in Figures 3(c) and 3(e). Geometry is similar to thermal models shown by Rutqvist et al. (2004). Source and receivers in the source-plane perpendicular to the tunnel are shown (see figure 1(b)).

Good agreement between modeled data (heavy/blue) and field data (thin/black) is shown in Figure 5. The data shown include all calibration events for a receiver with a source/receiver path crossing most of the thermal transition zone (CH4). Figures 6(a) and 6(b) show alignment of shear wave arrivals for the −0.5% and −2.5% per 100°C tuff models respectively. Field data waveforms are the output of the preprocessing stage and have been aligned to the first break of the model data for each event. The vertical line across the shear wave arrivals serves as a reference for the central peak of both model and field waveforms (heavy and thin lines respectively). Acceptable agreement between modeled and field waveforms was achieved for models with −1% change for the total temperature range of 200°C, as shown in Figure 6(a) (0.5%/100°C). This alignment can be contrasted with the −2.5%/100°C velocity change shown in Figure 6(b) where the arrival time of the shear wave clearly increases. The −0.5%/100°C temperature change is less than numbers stated in literature (Guéguen & Palciauskas, 1994), and clearly constrain the in-situ changes of velocity to be small. This suggests that the controlling factor of wavespeed change is fracturing caused by thermal expansion or compression of fragments trapped in existing fractures (Batzle et al., 1980). Occurrence of fracturing is supported by the localization of microseisms by Rutledge et al. (Rutledge et al., 2003). Were water present in the fractures surrounding the tunnel, the total bulk modulus could vary by several orders of magnitude, resulting in a much higher wavespeed in the transition zone (Wang, 2001). Unfortunately, as seismic data was not collected from ambient temperature and velocity conditions, we can’t study potential effects of groundwater entering/re-entering the surroundings. Also, without cooling cycle data, we cannot determine if microfracture damage occurs as demonstrated by the change in slope of the top branch of the hysteresis data used for velocity modeling (Figure 3(d)) (Grêt et al., 2006).

In addition to constraints on the velocity, numerical modeling shows the presence of a diffracted wave (“Franz” wave) propagating perpendicular to the surface of the tunnel in the shadow zone. This wavefront has been described analytically (Glibert & Knopoff, 1959) and observed experimentally (Neubauer & Dragone, 1970), and is clearly seen in modeled data (Figure 7(a)). Subtle indications of the Franz wave in the Yucca Mt. seismic can be observed by comparing arrivals in modeled shot gathers with wavelet variations in receiver gathers. However, poor signal-to-noise ratios (SNR) at later times in the field data preclude it from
analysis. Low Franz wave SNR occurs because the receiver array did not include elements located sufficiently close to the thermal transition zone. Franz wave sensitivity to velocity changes near the tunnel suggests that future monitoring of rock conditions may be achieved by observing transit times around the surface of the tunnel or differential arrival times of the curved wavefront by a linear array extending radially from the tunnel wall (Figure 7(b)).

5 DISCUSSION/CONCLUSIONS

Processing of seismic data from the heated drift scale test (DST) of 1998-2002 shows clear changes in P-S wavelet separation vs time and temperature. Multiple velocity models based on granite and adjusted to meet ambient velocity values were used to model waveforms at receivers in a plane with seismic source. Good agreement with wavespeed models having $-0.5\%/100^\circ C$ velocity change from ambient values constrains velocity changes due to heating to be quite weak. The velocity change is likely due to additional fracturing caused by compression of fragments trapped in existing fractures during thermal expansion. Sufficient amounts of groundwater would cause a large bulk modulus/velocity changes. Localizations of induced seismicity support thermal expansion/fracturing as the primary velocity control (Rutledge et al., 2003). Diffracted phases propagating in the shadow zone of the tunnel may be useful monitoring tools if the proper receiver array is installed.

ACKNOWLEDGEMENTS

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Figure 6. Shear wave arrival shot gathers for extrapolated tuff data of Figures 3(e) and 3(f) having $-0.5\%/100^\circ C$ (a) and $-2.5\%/100^\circ C$ (b) slopes, respectively. Model waveforms are darker, heavier traces. Data is from Channel 4, located below tunnel opposite from source. Both figures show increasing time/temperature as a function of event number, chronologically from to top bottom. Shear wave arrivals show good agreement for models based on $-0.5\%$ per $100^\circ C$ slope. Note: all field and model waveforms in (b), right, arrive at later times because all field waveforms are aligned with model waveforms computed using a higher $dV/dT$ than (a) left ($-2.5\%/100^\circ C$ vs. $-0.5\%/100^\circ C$).

Figure 7. (a) Numerical modeling shows diffracted/Franz wave propagating over tunnel crown normal to the tunnel surface. (b) Because this wave circles the tunnel in within the zone most effected by thermal change, its arrival time around the tunnel and differential arrival times vs. radius are potential tools for monitoring using the suggested array configurations (diamonds).
ledge (LANL), Jonny Rutqvist (LBNL), Erik Saenger (Fachrichtung Geophysik, Freie Universität Berlin), John Stockwell (CSM/CWP), Jeroen Tromp (Caltech), and Kenneth Hurst-Williams (LBNL), Shuki Ronen (Chevron, formally VeritasDGC), and Sergio Grion (CGGVeritas) for information, discussions, and suggestions.

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Detection of channels in seismic images using the steerable pyramid

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ABSTRACT
Channels have always been important geologic features in the exploration for oil and gas. With 3-D seismic data they can often be mapped easily on time or depth slices. In other situations they can be difficult to detect, due to structural complexity or other factors. There are a number of image-processing algorithms that can be used to enhance linear features such as channels in 3-D seismic volumes.

One way involves the use of steerable pyramid filters to partition a seismic image in terms of scale and orientation. Features can then be characterized according to dimensionality and direction using the partitioned image. Here, we explain our implementation of the steerable pyramid in 2-D and 3-D, and show how it can be used to enhance image features. Examples of channel enhancement on synthetic seismic images demonstrate the efficacy of this processing.

Key words: seismic image processing, interpretation

1 INTRODUCTION
Sands associated with channels often make good reservoirs for hydrocarbons. For this reason, the detection of channels is an important part of seismic interpretation for oil and gas exploration. From 3-D seismic data it is possible, for horizontal layering, to perform detailed mapping of channel systems directly on time slices. For example, Figure 1 is a view of a synthetic 3-D seismic dataset with two channel systems, which are clearly visible on the displayed time slice. However, in the majority of cases layers are not horizontal, and horizon flattening needs to be done first.

In theory, it should be possible to automatically detect channels whether or not they are horizontal. They have unique characteristics that differentiate them from other types of features that we encounter in 3-D seismic data. Channels are long, sinuous objects with arbitrary orientation. They are locally linear in that their extent is much greater in one direction than in any other direction, at a given point. Other features in a 3-D seismic volume are either locally planar (e.g. bedding planes, faults) or locally isotropic (e.g. pinnacle reefs, salt bodies). This difference in dimensionality between different features in the input image can be used to detect and enhance (or attenuate) particular types of features.

Previous study of the dimensionality and orientation of image features has mostly involved eigenvalue decomposition of the structure tensor. This analysis can

Figure 1. Synthetic 3-D seismic volume with channels
also be made using the steerable pyramid, which is a method of image decomposition into scale and orientation subbands first introduced in the early 1990’s (Freeman and Adelson 1991, Simoncelli et al 1992).

The steerable pyramid has some nice features that may be important in the detection and enhancement of channels. For example, it is inherently a multi-scale process. Given the diversity of shapes and sizes of channels this is potentially an advantage over other approaches. Also, the steerable pyramid may have an advantage in interactive work, as the precomputed filters can be used for rapid analysis and enhancement of image features. Finally, little work has been done on dimensionality estimation using the steerable pyramid, and we believe that it has potential for this type of analysis.

2 THE 2-D STEERABLE PYRAMID

The steerable pyramid is a transform in which steerable filters are used in a multi-scale recursive scheme, resulting in a decomposition in terms of scale and orientation. Figure 2 shows an illustration of this decomposition in the wavenumber domain, for a 2-D steerable pyramid with three scale levels and three basis orientations.

2.1 Review of 2-D steerable filters

To illustrate the concept of a steerable filter we can consider the directional derivative of a Gaussian. A two-dimensional symmetric filter \( g \) with Gaussian impulse response is

\[
g(x, y) = e^{-(x^2 + y^2)/2}.
\]

Differentiation of this function with respect to \( x \) and \( y \) gives us 1st-derivative-of-Gaussian filters with impulse responses

\[
h(x, y; \theta = 0) = \frac{\partial}{\partial x} g(x, y) = -x g(x, y)
\]

and

\[
h(x, y; \theta = \pi/2) = \frac{\partial}{\partial y} g(x, y) = -y g(x, y)
\]

A directional 1st-derivative-of-Gaussian filter with arbitrary orientation \( \theta \) can be made by performing a weighted sum of these filters,

\[
h(x, y; \theta) = \cos(\theta) h(x, y; \theta = 0) + \sin(\theta) h(x, y; \theta = \pi/2). \tag{4}
\]

which is the concept of steering in its simplest form. If we then convolve the filter \( h(x, y; \theta) \) with an input image \( p(x, y) \), we get a directionally filtered output image \( q(x, y; \theta) \):

\[
q(x, y; \theta) = \int dr \int ds \ h(r, s; \theta) p(x - r, y - s), \tag{5}
\]

where \( r \) and \( s \) are dummy integration variables.

The problem is that steering of the filters is not useful when \( \theta \) varies spatially. For this case we must steer the filtered images. Let us define two basis filters \( b_j(x, y) \) and their Fourier transforms \( B_j(k_x, k_y) \):

\[
b_0(x, y) \equiv h(x, y; \theta = 0) \iff B_0(k_x, k_y)
\]

\[
b_1(x, y) \equiv h(x, y; \theta = \pi/2) \iff B_1(k_x, k_y).
\]

When convolved with input image \( p(x, y) \), these produce output images

\[
q_j(x, y) = \int dr \int ds \ b_j(r, s; \theta) p(x - r, y - s),
\]

such that

\[
q(x, y; \theta) = \cos(\theta) q_0(x, y) + \sin(\theta) q_1(x, y). \tag{9}
\]

For the case where \( \theta \) varies spatially, i.e. \( \theta = \theta(x, y) \),
\[ q(x, y) = \cos(\theta x, y) q_0(x, y) + \sin(\theta x, y) q_1(x, y). \]  

Using this result, the output \( q(x, y) \) can be steered for any \( \theta(x, y) \) from the precomputed \( q_j(x, y) \).

The concept of steering extends to any order of directional derivative and to other types of functions as well. For the 1st-derivative-of-Gaussian case there were only two basis filters \( b_0 \) and \( b_1 \), scaled by weighting functions \( \cos \theta \) and \( \sin \theta \). A general expression for a steerable filter in terms of basis filters \( b_j \) and weighting functions \( w_j \) is

\[ h(x, y; \theta) = \sum_{j=0}^{M-1} w_j(\theta) b_j(x, y), \]

where \( M \) is the number of basis filters. We can also write the general form of equation 10:

\[ q(x, y) = \sum_{j=0}^{M-1} w_j(x, y) q_j(x, y). \]

In order to extend steering to functions other than the directional first-derivative-of-Gaussian filter, we need answers to the following questions:

- For what sorts of basis filters \( b_j \) does steering work?
- How many basis filters do we need? (What is \( M \)?)
- What directions \( \theta_j \) should we use?
- What are the weights \( w_j(\theta) \)?

These questions are answered in steering theorems developed by Freeman and Adelson (1991), which can be found in Appendix A of this paper.

Let's consider the extension of steerable filters to the directional 2nd-derivative-of-Gaussian filter. First, we wish to determine the required number of basis directions \( M \). For \( \theta = 0 \) this filter is

\[ b_0(x, y) = h(x, y; \theta = 0) = (x^2 - 1)e^{-(x^2 + y^2)}. \]

This can be written in polar coordinates as

\[ b_0(r, \phi) = (r^2 \cos^2 \phi - 1)e^{-r^2}, \]

where \( r = \sqrt{x^2 + y^2} \) and \( \phi = \arg(x, y) \), or

\[ b_0(r, \phi) = (r^2 e^{-2\phi} + 2(r^2 - 1)e^0 + r^2 e^{i2\phi}) e^{-r^2}. \]

Therefore, the number of non-zero Fourier coefficients is three. By Theorem 1 at least three basis filters are required, so \( M = 3 \). In fact it turns out that the number of basis directions needed to steer a directional derivative in 2-D is in general one greater than the order of the derivative. In this case, equation 11 then becomes

\[ h(x, y; \theta) = \sum_{j=0}^{2} w_j(\theta) b_j(x, y). \]

The directions of the \( M \) basis filters should be chosen such that

\[ \theta_j = j\pi/M, \ j = 0, 1, \cdots, M - 1, \]

which gives uniformly spaced basis directions. This is not the only set of \( \theta_j \) that can be used, but the steering is more accurate than for other choices, and works for any angle of rotation. In this case, we choose \( \theta_j = j\pi/3 \), \( j = 0, 1, 2 \), consistent with equation (17), and the basis filters are

\[ b_j(x, y) = h(x, y; \theta = j\pi/3). \]

Using Theorem 1, we can now solve for the weighting functions \( w_j \). After removing all but the non-zero terms, equation A3 becomes

\[ \begin{bmatrix} 1 \\ e^{i2\theta} \\ e^{i2\theta_1} e^{i2\theta_2} e^{i2\theta_3} \end{bmatrix} \begin{bmatrix} w_1(\theta) \\ w_2(\theta) \\ w_3(\theta) \end{bmatrix}. \]

When we solve this system of equations, with \( \theta_0 = 0 \), \( \theta_1 = \pi/3 \), and \( \theta_2 = 2\pi/3 \), we find that

\[ w_j(\theta) = \frac{1}{3}[1 + 2 \cos(2(\theta - j\pi/3))]. \]

The basis filters \( b_j \) need not be directional derivatives. Other filters can be steered (Freeman and Adelson 1991). In particular, we may choose \( b_j(x, y) \), with Fourier transforms \( B_j(k_x, k_y) \), such that

\[ \sum_{j=0}^{M-1} B_j(k_x, k_y) = C \]

for all wavenumbers \( (k_x, k_y) \), where \( C \) is a constant independent of \( (k_x, k_y) \). For example, for \( M = 3 \) we might choose (Castleman, Schulze, and Wu, 1998)

\[ B_j(k_x, k_y) = \cos^2(\theta - \theta_j), \]

where \( \theta_j = j\pi/3 \), \( j = 0, 1, 2 \). For this filter, the constant \( C = 3/2 \). The equation for the filter may also be written using direction cosines \( \alpha_j \) and \( \beta_j \):

\[ B_j(k_x, k_y) = \frac{(\alpha_j k_x + \beta_j k_y)^2}{k_x^2 + k_y^2}. \]
domain. The part of the energy attenuated by \( L \) wavenumber energy in the corners of the wavenumber zone of the filter, and
\[
|k| = \sqrt{k_x^2 + k_y^2}.
\]
The purpose of filter \( L_0 \) is to attenuate high-wavenumber energy in the corners of the wavenumber domain. The part of the energy attenuated by \( L_0 \) can be passed to the output, but it doesn’t contribute anything useful to the output image and we generally discard it. Low-pass filter \( L_1 \) removes any data that will be aliased in downsampling. For \( L_0 \) we have been using \( k_a = 0.75\pi \) and \( k_b = \pi \). Filter cutoff parameters for \( L_1 \) are half those used for \( L_0 \), so \( k_a = 0.375\pi \) and \( k_b = 0.5\pi \).

The results of filtering with \( L_0 \) and \( L_1 \) are subtracted to produce a bandpass-filtered image to which the basis directional filters \( B_j \) are applied, also by multiplication in the wavenumber domain, producing bandpass-filtered images \( q_{00}, q_{01} \) and \( q_{02} \). This makes up one level of the pyramid.

The output of filter \( L_1 \) is then downsampled by selecting every other sample, giving us an image one quarter the size of the input image. The \( L_1 \) filter is then applied, the output subtracted from the downsampled input, and the basis filters \( B_j \) are applied exactly as described above, producing bandpass-filtered images \( q_{10}, q_{11} \) and \( q_{12} \). This makes up the next level of the pyramid.

The process can be applied recursively as many times as we like, until we run out of samples. At each level the size of the image is one quarter that of the previous one. The final pyramid level, \( l \) contains only very low wavenumbers. The basis filters \( B_j \) are not applied to this final level. Figure 7 is a display of a time slice from the 3-D synthetic dataset from Figure 1 which has been processed with a three level, three directions steerable pyramid transform.

Reconstruction of the image is straightforward. First, the weighting functions \( w_{10}, w_{11} \) and \( w_{12} \) are applied to the directionally filtered images \( q_{10}, q_{11} \) and \( q_{12} \), and the low-wavenumber image, \( l \), is upsampled. In the next section we will describe how we determine \( w_j \). For the moment let’s just say it is a spatially-variant scaling function designed to optimally steer the directionally filtered images.

The four resulting intermediate images are summed. Filtered images \( q_{00}, q_{01} \) and \( q_{02} \) are multiplied by their respective weighting functions \( w_{00}, w_{01} \) and \( w_{02} \). Note that different levels of the pyramid have...
different weighting functions, so they can be steered in different directions. These outputs are added to the summed result from the previous level of the pyramid. The output is the reconstructed image.

If the steering weights $w_j$ are all equal to 1, the output image is identical to the input, less the high-wavenumber energy that is removed by filter $L_0$. However, if weighting functions have been applied to enhance features in the image, the output may look quite different than the input. For example, figure 9 shows the time slice seen in figure 7 before and after steerable pyramid processing with steering weights designed to enhance coherent features in the image. The two channel systems are enhanced while noise is generally reduced. The question is, how do we determine the optimum weights?

### 2.3 2-D Estimation of feature orientation

There are a number of effective ways to determine the directionality of features in an image. Previous work in this area has been done with the structure tensor (van...
Figure 10. Steerable filter result for a simple image of tilted lines. Input (a) and steered versions with weights calculated for the correct orientation (b) and perpendicular to the correct orientation (c).

Figure 11. Plot of absolute amplitude vs. $\theta$ for a single steered output sample from the input image shown in figure 10.

Vliet and Verbeek, 1995), plane-wave destruction filters (Fomel, 2002), and other techniques (van Spaendonck, 2000).

The steerable pyramid can give very accurate estimates of direction. To see how this is done, consider the simple image of tilted lines in figure 10. Three basis filters as given in equation 22 were applied to this image for directions $\theta = 0$, $\pi/3$, and $2\pi/3$. From the orientation of the planes in the input image and of the basis filters, we can calculate the proper weighting functions $w_j$ by using equation 20. If we use weights calculated for some different direction to steer the basis images, the amplitude of the reconstructed image is reduced. For an error of $\pi/2$ the output amplitude is near zero, as in figure 10(c). If the weights are correct, the amplitude of the reconstructed image is the same as the input, as in figure 10(b).

Figure 11 shows the amplitude of a single reconstructed output sample given the input image shown in figure 10(a). For this image, $w_j$ were calculated for $0 \leq \theta \leq 180^\circ$. The amplitude maximum occurs at $138^\circ$, which is consistent with the direction of the lines in the input image.

The direction that gives us the largest output amplitude can be found analytically. From equation 12, for the case where the number of basis filters $M = 3$,

$$q(x, y) = \sum_{j=0}^{2} w_j(x, y)q_j(x, y). \quad (25)$$

We also have the equation for $w_j$ as a function of $\theta$ (equation 20). Combining these gives us

$$q(x, y; \theta) = \sum_{j=0}^{2} \frac{1}{3} [1 + 2 \cos(2(\theta(x, y) - \theta_j))]q_j(x, y). \quad (26)$$

We find an extremum of this function by setting $\partial q/\partial \theta = 0$, and solve for $\theta$ in terms of the basis image amplitudes, giving us

$$\theta(x, y) = \frac{1}{2} \arctan \left( \frac{\sqrt{3}(q_1(x, y) - q_2(x, y))}{2q_0(x, y) - q_1(x, y) - q_2(x, y)} \right). \quad (27)$$

As we can see in figure 11, this function has two extrema separated by $90^\circ$, and we select the one with the larger absolute amplitude. Using this method we are able to calculate the local dominant direction at every sample location in the image. This estimate is somewhat unstable for sample locations where the input amplitude is small and in the presence of noise, so we apply a smoothing filter for stability.

To illustrate the estimation of direction, we have applied a steerable pyramid to the time slice from figure 1, and calculated orientation estimates at every sample location in the second pyramid level image. This result is shown in figure 12. The line segments superimposed on the seismic image depict the dominant direction calculated in the middle of the line segment. Lines are displayed for every fifth sample, though they are calculated for all samples.
3 THE 3-D STEERABLE PYRAMID

3.1 3-D steerable filters

When we extend the concept of steerable filters to three dimensions, we need to take another look at the questions that we answered for the 2-D case. For example, how many directional filters will we need in three dimensions? What directions should these filters take?

We will continue with an approximation to the second derivative, as we did for the two-dimensional case. According to Theorem 3 (Freeman and Adelson 1991), we will need six basis filters.

The determination of which directions to use for our filters is more complicated for three dimensions than it was for two. For the 2-D case we stated that filters which equally divide the angle range are most accurate, and can be steered to any orientation. This holds for the 3-D case also, though we note again that there is more than one set of angles that will work.

To find directions that will equally partition the full range of angles, a good approach is to consider the regular polyhedra (Delle Luche et al, 2004). The regular icosahedron has twenty equilateral triangles for faces, and twelve vertices. Lines between opposite vertices of the regular icosahedron define six directions that evenly partition the range of 3-D angles. The angle difference between any two of these directions is $63.43495^\circ$. Also, these directions define three perpendicular rectangles for which the ratio of the length of the sides is the golden ratio $1 : (1 + \sqrt{5})/2$. The fact that our filter directions can lie on these perpendicular planes simplifies the math a good deal, if we set the directions of the planes parallel to our coordinate axes.

We have found an alternative set of directions that is every bit as good as the scheme described above, based on the vertices of the cuboctahedron. The cuboctahedron can be formed by shrinking the three perpendicular rectangles that define the vertices of the regular icosahedron, so that they become three squares. For this set of filter directions, the angle difference between one direction and the four others that are not coplanar is $60^\circ$. The other direction in the same plane is perpendicular. Figure 13 shows how the filter directions are constructed using the two polyhedra.

We have found that the two geometries give essentially identical results. The system based on the icosahedron has the disadvantage that the mathematical expressions are more complex. For this reason we have used filter directions based on the vertices of the cuboctahedron in our research.

The three-dimensional directional filters are applied by multiplication in the wavenumber domain, and they are just a 3-D extension of the filters that we applied in 2-D. In our discussion of 2-D steerable filters, the directional filters are defined in equation 22. In 3-D we can define the six directional filters in the same way.

In figure 14, the vector $\hat{OC}$ is parallel to the axis of symmetry of one of the six basis filters $B_j$. Its direction can be described using angles $\theta_j$ and $\phi_j$. The direction of vector $OM$ is from the origin ($k_x = k_y = k_z = 0$) to some point $M$, and its direction can be described by angles $\Theta$ and $\Phi$. Let $\Omega$ be the angle between vectors $\hat{OC}$ and $OM$.

$$B_j(\theta, \phi) = \cos^2 \Omega(\theta, \phi; \theta_j, \phi_j), \ j = 0, 1, \cdots, 5,$$ (28)

The calculation of $\cos \Omega$ is straightforward using the dot product $\hat{OM} \cdot \hat{OC}$ (Delle Luche et al, 2004):
Table 1. Direction cosines for the axes of symmetry of six basis filters $B_j$ with geometry based on the vertices of the cuboctahedron.

<table>
<thead>
<tr>
<th>$j$</th>
<th>$\alpha_j$</th>
<th>$\beta_j$</th>
<th>$\gamma_j$</th>
</tr>
</thead>
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<td>$\frac{1}{\sqrt{2}}$</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
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<td>$-\frac{1}{\sqrt{2}}$</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{1}{\sqrt{2}}$</td>
<td>0</td>
<td>$\frac{1}{\sqrt{2}}$</td>
</tr>
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<tr>
<td>5</td>
<td>0</td>
<td>$\frac{1}{\sqrt{2}}$</td>
<td>$-\frac{1}{\sqrt{2}}$</td>
</tr>
</tbody>
</table>

A simpler way to define the directions for the 3-D case is using the direction cosines:

$$\alpha = \sin \theta \cos \phi \quad \beta = \sin \theta \sin \phi \quad \gamma = \cos \theta$$

The directions of the axes of symmetry of the six basis filters $B_j$ are given in terms of $\alpha_j$, $\beta_j$, and $\gamma_j$ in Table 1. Similarly to the 2-D case, the equation for the basis filters may also be written using direction cosines:

$$B_j(k_x, k_y, k_z) = \left(\frac{\alpha_j k_x + \beta_j k_y + \gamma_j k_z}{\sqrt{k_x^2 + k_y^2 + k_z^2}}\right)^2.$$ 

So far we have discussed cosine-squared filters, but we can also use $1 - \cos^2 = \sin^2$ filters. In 2-D the only difference between these is a 90° rotation, but they are very different in 3-D. The $\cos^2$ filters are conic in shape and the $\sin^2$ filters have an annular shape. These are applied by multiplication in the wavenumber domain, followed by inverse 3-D Fourier transform to produce the image in the space domain. Figure 16a shows a single $\cos^2$ filter in yellow, and the equivalent $\sin^2$ filter in blue, in the wavenumber domain. Figure 16b shows the identical filters after inverse Fourier transform, in the space domain. The conic $\cos^2$ filter in wavenumber domain becomes an annulus in space domain, and the annular $\sin^2$ filter becomes conic. Amplitudes in these displays have been clipped so that we can see the shapes of the filters.

The effect of the conic $\cos^2$ filters applied in the wavenumber domain is to enhance planar features in the space domain, whose normal direction is parallel to the axis of symmetry of the filter. The $\sin^2$ filters enhance linear features aligned with the filter axis of symmetry.

Along with directional filters $B_j$, radial low-pass filters $L_0$ and $L_1$ are also applied as for the 2-D steerable pyramid. These are identical to the filters in 2-D, applied using equation 24 by multiplication in the wavenumber domain. The only difference is that for 3D, $|k| = \sqrt{k_x^2 + k_y^2 + k_z^2}$.

With the basis filters $B_j$ and radial filters defined, the steerable pyramid transform and subsequent im-

Figure 15. Image in 3-D wavenumber domain of six directional filters based on the cuboctahedron. Amplitudes less than 92% have been zeroed so that we can see the conic shape of the filters. Three views (a-c) have been displayed, allowing us to appreciate the symmetry of the directional filters.

Figure 16. Conic $\cos^2$ filter (yellow) transforms to annular filter and annular $\sin^2$ filter (blue) transforms to conic filter, going from wavenumber domain (a) to space domain (b).
agram reconstruction proceed according to the 2-D process flows in figures 5 and 8, except that there are now six basis images at every pyramid level.

To find the weighting functions for the 3-D steerable filters, we use equation A5:

\[
\begin{bmatrix}
\alpha^2 \\
\alpha \beta \\
\alpha \gamma \\
\beta^2 \\
\beta \gamma \\
\gamma^2
\end{bmatrix}
= \begin{bmatrix}
\alpha_0^2 & \alpha_1^2 & \alpha_2^2 & \alpha_3^2 & \alpha_4^2 & \alpha_5^2 \\
\alpha_0 \beta_0 & \alpha_1 \beta_1 & \alpha_2 \beta_2 & \alpha_3 \beta_3 & \alpha_4 \beta_4 & \alpha_5 \beta_5 \\
\alpha_0 \gamma_0 & \alpha_1 \gamma_1 & \alpha_2 \gamma_2 & \alpha_3 \gamma_3 & \alpha_4 \gamma_4 & \alpha_5 \gamma_5 \\
\beta_0^2 & \beta_1^2 & \beta_2^2 & \beta_3^2 & \beta_4^2 & \beta_5^2 \\
\beta_0 \gamma_0 & \beta_1 \gamma_1 & \beta_2 \gamma_2 & \beta_3 \gamma_3 & \beta_4 \gamma_4 & \beta_5 \gamma_5 \\
\gamma_0^2 & \gamma_1^2 & \gamma_2^2 & \gamma_3^2 & \gamma_4^2 & \gamma_5^2
\end{bmatrix}
\begin{bmatrix}
w_0 \\
w_1 \\
w_2 \\
w_3 \\
w_4 \\
w_5
\end{bmatrix}
\] (32)

The direction cosines \(\alpha_j\), \(\beta_j\), and \(\gamma_j\) for our filters based on the cuboctahedron are listed in Table 1. These values are inserted into equation 32 and we can then solve for \(w_j(\alpha, \beta, \gamma)\), giving the following set of equations:

\[
\begin{align*}
w_0 &= \frac{1}{2}(\alpha + \beta)^2 - \gamma^2 \\
w_1 &= \frac{1}{2}(\alpha - \beta)^2 - \gamma^2 \\
w_2 &= \frac{1}{2}(\gamma + \alpha)^2 - \beta^2 \\
w_3 &= \frac{1}{2}(\gamma - \alpha)^2 - \beta^2 \\
w_4 &= \frac{1}{2}(\beta + \gamma)^2 - \alpha^2 \\
w_5 &= \frac{1}{2}(\beta - \gamma)^2 - \alpha^2
\end{align*}
\] (33)

As in the 2-D case, we multiply the filtered images \(q_j\) by the weights \(w_j\) to produce a steered image:

\[
q(x, y, z) = \sum_{j=0}^{5} w_j(x, y, z)q_j(x, y, z).
\] (34)

### 3.2 3-D Estimation of Orientation

The 3-D steerable pyramid can be used to estimate the orientation of image features exactly as we did in two dimensions, but it is more complicated. We are now calculating a three-dimensional angle. Also, the type of steerable filter, \(\cos^2\) or \(\sin^2\), will depend on whether we are looking for planar or linear features.

For an example of orientation analysis in three dimensions, let’s consider a synthetic 3-D data volume comprising parallel planes of known orientation, shown in figure 17. The orientation of the planes is given by the direction of the normal vector, with \(\theta = 72^\circ\) and \(\phi = 33^\circ\), as we defined them in figure 14. We calculated the absolute amplitude of a single steered output sample for all values of \(\theta\) and \(\phi\). This was done by calculating and applying the appropriate \(w_j\), then summing the basis images, iterating through all angles. The result is shown in figure 18.

In the display, we note that there is a maximum at \(\theta = 72^\circ, \phi = 33^\circ\), which is consistent with the known orientation of the planes. We also see maxima at 180° from this, and the minimum amplitude is essentially
zero, at 90° from the correct direction. Obviously, we can’t do this kind of search for every sample in a 3-D image. In two dimensions we were able to calculate the optimum steering angle analytically from the amplitudes of the basis images, and this should be possible in three dimensions as well.

4 DETECTION AND ENHANCEMENT OF CHANNELS

We noted earlier in this paper that steerable \( \cos^2 \) filters enhance planar features, while \( \sin^2 \) filters should enhance linear features. This gives us an idea of how to use these filters in channel detection.

In most seismic datasets the dominant features are planar reflectors. In many situations this can make detection of channels difficult, as they can be obscured by the reflections. The steerable pyramid can be made to enhance planar events, using \( \cos^2 \) filters. The enhanced reflections can then be subtracted from the seismic data, revealing the channels. Figure 19 shows an example of this approach. Strong planar events are greatly attenuated by the process, but amplitudes of channels are similar to the input.

Another approach that we have been studying is to use the 3-D steerable pyramid with \( \sin^2 \) filters to enhance the channels. Early trials using the 3-D \( \sin^2 \) filters were unsuccessful, and we have been testing variations of this method on a time slice from the 3-D synthetic dataset using 2-D filters. Figure 20 shows one result of this testing. Four cascaded iterations of 2-D steerable pyramid adaptive smoothing were applied to the image. The amount of smoothing applied is varied over the image using a linearity attribute that is calculated for every sample. The result of this variable smoothing is that the continuity of channels is improved while surrounding areas are essentially unchanged.

5 CONCLUSION

The steerable pyramid transform is potentially a useful tool for analysis of orientation and dimensionality, though it has seen little use in geophysical applications until now. During our research we have established that it can give us accurate and detailed estimates of the orientation of image features. We have also found linearity attributes in 2-D using steerable pyramid filters.

One thing that we noted regarding the steerable pyramid was the lack of research into 3-D implementation. The 3-D steerable pyramid concept was introduced by Freeman and Adelson in 1991, but there are only a couple of technical papers in this area. We have done quite a lot of research in 3-D and have discovered a number of new developments. We have found a set of filter directions based on the vertices of the cuboctahedron that simplify the math while giving accurate results. We have also found analytic expressions for feature orientation in terms of image amplitudes.

Quite a lot of work remains to be done. We are currently working to improve our orientation and dimensionality estimates for 3-D data, which is absolutely necessary for detection of channels. We are also working to improve the program efficiency so that we can work with larger datasets. We also plan to test the applicabil-

![Figure 19. Example of use of 3-D steerable pyramid for attenuation of planar events. Strong horizontal reflections on 3-D synthetic input (a) are enhanced using the 3-D steerable pyramid, then subtracted (b).](image-url)
APPENDIX A: STEERING THEOREMS

Below are steering theorems from Freeman and Adelson (1991) that were used in this research:

Given a function \( f^\theta \) rotated by an angle \( \theta \), and \( f^{0} \), which is the same function rotated by a number of angles \( \theta_j, j = 0, 1, ..., M \). The general steering constraint is

\[
f^\theta(x, y, z) = \sum_{j=0}^{M-1} w_j(\theta) f^{\theta_j}(x, y, z), \tag{A1}
\]

where \( M \) is the number of basis functions required for steering and \( w_j(\theta) \) are the weighting functions.

Let \( f \) be any function which can be expanded in a Fourier series in polar angle, \( \phi \), with \( r = \sqrt{x^2 + y^2} \) and \( \phi = \text{arg}(x, y) \),

\[
f(r, \phi) = \sum_{n=-N}^{N} a_n(r)e^{in\phi}. \tag{A2}
\]

**Theorem 1:** The steering condition, equation A1, holds for functions expandable in the form of equation A2 if and only if the weighting functions \( w_j(\theta) \) are solutions of:

\[
\begin{bmatrix}
1 \\
e^{i\theta} \\
e^{i2\theta} \\
\vdots \\
e^{iN\theta}
\end{bmatrix}
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
e^{i\theta_1} & e^{i\theta_2} & \cdots & e^{i\theta_M} \\
\vdots & \vdots & \ddots & \vdots \\
e^{iN\theta_1} & e^{iN\theta_2} & \cdots & e^{iN\theta_M}
\end{bmatrix}
\begin{bmatrix}
w_1(\theta) \\
w_2(\theta) \\
\vdots \\
w_M(\theta)
\end{bmatrix} = 0,
\tag{A3}
\]

If, for any \( n, a_n(r) = 0 \), then the corresponding \( n \)th row of the left side and of the matrix of the right side of equation (A3) should be removed.

**Theorem 2:** Let \( T \) be the number of positive or negative frequencies \( -N \leq n \leq N \) for which \( f(r, \phi) \) has non-zero coefficients \( a_n(r) \). Then the minimum number of basis functions which are sufficient to steer \( f(r, \phi) \) by Eq. (A1) is \( T \) (i.e., \( M \geq T \)).

**Theorem 3:** Given a three dimensional axially symmetric function \( f(x, y, z) = A(r)P_n(x) \), where \( P_n(x) \) is an even or odd symmetry \( N \)th order polynomial in \( x \). Let \( \alpha \), \( \beta \) and \( \gamma \) be the direction cosines.
of the axis of symmetry of \( f^R(x, y, z) \) and \( \alpha_j, \beta_j \) and 
\( \gamma_j \) be the direction cosines of the axis of symmetry of 
\( f^{R_j}(x, y, z) \). Then the steering equation,

\[
f^R(x, y, z) = \sum_{j=0}^{M-1} w_j(\alpha, \beta, \gamma) f^{R_j}(x, y, z), \tag{A4}
\]

holds if and only if \( M \geq (N+1)(N+2)/2 \) and 
\( \text{(b) the } w_j(\alpha, \beta, \gamma) \text{ satisfy} \)

\[
\begin{align*}
\begin{bmatrix}
\alpha^N \\
\alpha^{N-1}\beta \\
\alpha^{N-1}\gamma \\
\alpha^{N-2}\beta^2 \\
\gamma^N
\end{bmatrix}
&= 
\begin{bmatrix}
\alpha_1^N & \alpha_2^N & \cdots & \alpha_M^N \\
\alpha_1^{N-1}\beta_1 & \alpha_2^{N-1}\beta_2 & \cdots & \alpha_M^{N-1}\beta_M \\
\alpha_1^{N-1}\gamma_1 & \alpha_2^{N-1}\gamma_2 & \cdots & \alpha_M^{N-1}\gamma_M \\
\vdots & \vdots & \vdots & \vdots \\
\gamma_1^N & \gamma_2^N & \cdots & \gamma_M^N
\end{bmatrix}
\begin{bmatrix}
w_1(\alpha, \beta, \gamma) \\
w_2(\alpha, \beta, \gamma) \\
w_3(\alpha, \beta, \gamma) \\
\vdots \\
w_M(\alpha, \beta, \gamma)
\end{bmatrix}. \tag{A5}
\end{align*}
\]
Defining regions in seismic images by flattening

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ABSTRACT
Flattening a seismic image removes the effects of geologic processes and transforms the image into layers as they were deposited in geologic time. In a flattened image, we can easily define regions that correspond to geologic layers.

Estimates of local dip are required to compute the shifts in depth (or time) needed to flatten every event in a seismic image. Our flattening process uses structure tensors to estimate the local dip of every sample in a seismic image and a regularized least squares inversion to solve for shifts that are consistent with these local dips.

Key words: flattening, structure tensors, inversion

1 INTRODUCTION
Our goal is to create an algorithm to allow for the easy definition of regions within a 3D seismic image. Often these regions are geologic layers, and to define such layers in seismic images, we might track top and bottom isochron surfaces. Figure 1b displays an example of a complete isochron surface with no holes. For surfaces like these, traditional horizon tracking methods may require more user interaction than is necessary.

Traditional horizon tracking methods may require more user interaction than is necessary.

The algorithm presented by Lomask et al. (2006) computes such surfaces by flattening a seismic image. This flattening algorithm reverses the effects of geologic processes and transforms an image back into layers as they were laid down in geologic time by solving for a shift field (Figure 2b). This shift field provides a mapping between the original image and the flattened image.

After the image has been flattened (Figure 2d), tracking isochron surfaces of constant geologic time is only a matter of selecting a plane of constant time in the flattened image and reversing the flattening process. Thus, by calculating the shifts needed to flatten the volume, all of the isochron surfaces in the image are tracked in 3D at once.

Our process for flattening a seismic image only varies in the details from Lomask et al. (2006). Both algorithms calculate the shift field from dips of seismic reflections. Therefore, the first step of the flattening process is the estimation of a local dip at every sample of a seismic image. Our method uses structure tensors (van Vliet and Verbeek, 1995) to compute the unit normal vector of the best fitting plane in a local window around every sample in a seismic image. After the dip is estimated, both algorithms solve for the flattened image by applying a vertical shift to every sample in the input image. Our method computes the shift field by using a regularized least squares inversion. Posing the shift field calculation as an inversion problem allows us to take advantage of preexisting inversion frameworks, such as the one described by Harlan (2004).

2 THE STRUCTURE TENSOR
Our implementation of the flattening process uses the structure tensor (Fehmers and Höcker, 2003) or the gradient-squared tensor (van Vliet and Verbeek, 1995) to efficiently calculate the normal vector of the best fitting plane in a local window around every sample of a seismic image.

The gradient of an image can be used to obtain information about the orientation of features in an image. Unfortunately, we cannot use the simple gradient to estimate the orientation of features in an image because gradients with similar orientation but opposite signs will cancel each other. Computing the outer product of gradients resolves this problem and causes gradients with opposite directions to reinforce each other. After the
outer product of gradient has been computed, one can obtain the orientation of the structure in an image at different scales by averaging over a local window.

Smoothing of the gradient outer products may be implemented using recursive Gaussian filters (RGF) (Hale, 2006). To perform this local averaging, we convolve the image with a Gaussian function. The recursive nature of the RGF allows for a more efficient computation than a simple convolution. RGFs are also separable filters. This means that applying a 1D version of the filter in all three coordinate directions is equivalent to applying a 3D version of the filter. Therefore, we only need to implement the RGF in 1D, and apply it independently in each coordinate direction.

When computing the local average of the gradient outer products, the local window is defined in terms of the half-width $\sigma$ of the Gaussian function that we are convolving with the image. Since, we apply smoothing independently in each coordinate direction, we can ad-

**Figure 1.** (a) A screenshot of a 3D seismic image. (b) The same seismic image with an isochron surface tracked in 3D. (c) A close up of the isochron surface shown in (b).

**Figure 2.** (a) A vertical slice of the seismic image $f(x, y_0, z)$ shown in Figure 1. (b) A vertical slice of the calculated shift field $s(x, y_0, z)$ needed to flatten the input image $f(x, y_0, z)$. Blue indicates the section will be shifted up, and conversely, red sections will be shifted down to make the output image flat. (c) The seismic image and the shift field superimposed. (d) The flattened image $g(x, y_0, z)$. 
just the half-width of the Gaussian function in each direction independently. A half-width of four samples, in each coordinate direction, was used in the computation of images shown in this paper.

We now have all of the components of the structure tensor $G$ which is defined as

$$G(x, y, z) = \begin{pmatrix} <g_x g_z> & <g_x g_y> & <g_x g_z> \\ <g_y g_z> & <g_y g_y> & <g_y g_z> \\ <g_z g_z> & <g_z g_y> & <g_z g_z> \end{pmatrix},$$

where $< \cdot >$ is a Gaussian smoothing operator with a half-width of $\sigma$. The $x$, $y$, and $z$ components of the gradient are represented by $g_x$, $g_y$, and $g_z$, respectively.

The eigenvector corresponding to the largest eigenvalue of $G$ is the normal to the best fitting plane in the local window. Note that $G$ is symmetric positive-definite. Therefore, we can efficiently solve for its eigenvalues and eigenvectors. This is important because this decomposition is performed for every sample in an image.

We are able to use the two largest eigenvalues of $G$ to estimate the planarity $p(x, y, z)$ of events:

$$p(x, y, z) = \frac{\lambda_1(x, y, z) - \lambda_2(x, y, z)}{\lambda_1(x, y, z)} \in [0, 1],$$

where $\lambda_1$ is the largest eigenvalue and $\lambda_2$ is the second largest eigenvalue. The planarity is close to one when there is a well defined planar event in the local window and close to zero when the event is noisy or non-planar. We use the planarity value $p(x, y, z)$ to estimate the quality of our normals when solving for the shift field $s(x, y, z)$.

### 3D Shift Field Calculation

To easily introduce our process of solving for the shift field, we first consider the problem in 2D and later extend the process to 3D. We also assume that the vertical axes of our images is depth. However, the flattening process works equally well for either depth or time.

Once the local dips have been estimated, the next step in the flattening process is to find the distance every sample in the seismic image must be shifted to flatten the image (Figure 4a). These shifts will only move a sample up or down in depth. They cannot move a sample into a different trace. All samples in a trace are not statically shifted together. A trace is allowed to be stretched in some parts and squeezed in others to make the final image flat.

The required shift at every sample comprises our shift field $s(x, z)$. A positive value in the shift field indicates the corresponding sample needs to be pushed down to greater depth to create a flat image. Conversely, a negative value of the shift field indicates the sample needs to be moved up. The shift field $s(x, z)$ is then used to map an input image $f(x, z)$ to a flattened image $g(x, z)$.

Figure 4c illustrates how we solve for this shift at every sample. The partial derivative of the shift field with respect to $x$ is equal to the $x$ component of the normal vector divided by the $z$ component of the normal vector. This ratio also equals the tangent of the dip angle $\theta$:

$$\frac{\partial s(x, z)}{\partial x} = \frac{n_x(x, z)}{n_z(x, z)} = \tan \theta.$$  \hfill (3)

Therefore, the shift field $s(x, z)$ is calculated by integrating all the dip tangents $n_x(x, z)/n_z(x, z)$ along an event:

$$s(x, z) = \int_0^x \frac{n_x(\eta, z)}{n_z(\eta, z)} d\eta.$$  \hfill (4)

Thus, the shift at some location $(x_0, z_0)$ is a function of all of the dip tangents $n_x(x, z)/n_z(x, z)$ along the event.

In Figures 4a and 4b, we sketch a hypothetical seismic event before and after flattening, respectively. To flatten the event, the sample at $(x_0, z_0)$ in the input image $f$ will be moved to $(x_0, z_1)$ in the output image $g$:

$$g(x_0, z_1) = f(x_0, z_0).$$  \hfill (5)

In our current algorithm, the output image $g$ is computed by interpolating samples from the input image $f$. Consequently, when computing the output at $(x_0, z_1)$, we must find the corresponding sample at $(x_0, z_0)$ in the input image. The $x$ coordinate of this location is known but the $z$ coordinate is not known. Fortunately, we can use the shift field to find the $z$ coordinate at which to interpolate the input image. Figure 4a shows that the unknown $z$ coordinate $z_0$ is equal to the output location $z_1$ minus the shift at the event.
We can generalize this to all samples in the output image:

\[ s(x_0, z_0) = z_1 - s(x_0, z_0). \]  

(6)

Combining equations 5 and 6 yields

\[ g(x_0, z_1) = f(x_0, z_1 - s(x_0, z_0)). \]  

(7)

Unfortunately, the right-hand-side of this relation still has an unknown \( z_0 \). The location \((x_0, z_0)\) of the event in the input image is a function of the shift \( s(x_0, z_0) \) at the location of the event. To overcome this circular dependency, we assume that the shift at the location of the event in the input is approximately equal to the shift at the output location:

\[ s(x_0, z_0) \approx s(x_0, z_1). \]  

(8)

With this approximation we can calculate one sample \( g(x_0, z_1) \) via

\[ g(x_0, z_1) \approx f(x_0, z_1 - s(x_0, z_1)). \]  

(9)

We can generalize this to all samples in the output image:

\[ g(x, z) \approx f(x, z - s(x, z)). \]  

(10)

From equation 4, we know that this approximation is best if the integral of dip tangents at \((x_0, z_0)\) is equal to the integral of dip tangents at \((x_0, z_1)\):

\[ \int_0^x n_z(\eta, z_0) \, d\eta \approx \int_0^x n_z(\eta, z_1) \, d\eta. \]  

(11)

This is equivalent to assuming that all of the events are close to parallel or that the depths \( z_0 \) and \( z_1 \) are almost equal. Thus, as \(|z_0 - z_1|\) approaches zero, the error in our approximation approaches zero.

Since, in practice, the error is never zero, the output image \( g(x, z) \) may not be completely flattened, even after the flattening process has been applied. We compensate for this error by repeatedly applying the flattening process (dip estimation, shift field calculation, and flattening) until \(|z_0 - z_1|\) becomes negligible. Figure 5 shows a graph of the average dip angle \( \theta \) from equation 3) per flattening iteration for the 3D image in Figure 1.

### 4 3D SHIFT FIELD CALCULATION

In 3D, we use the same process and assumptions as in 2D. Our mapping from an input image \( f(x, y, z) \) to a flattened image \( g(x, y, z) \) becomes

\[ g(x, y, z) \approx f(x, y, z - s(x, y, z)). \]  

(12)

The major difference, in 3D, is that the normal vector \( \mathbf{n}(x, y, z) \) of the best fitting plane at every sample now includes a \( y \) component. Thus, we have two dip tangents for every sample in the input image. Whereas, in 2D, there was only one dip tangent for every sample

\[ \frac{n_x(x, y, z)}{n_z(x, y, z)}. \]  

(13)

and another for the \( y \) component of the normal

\[ \frac{n_y(x, y, z)}{n_z(x, y, z)}. \]  

(14)

Accordingly, there are now two relevant partial derivatives that relate to the shift field \( s(x, y, z) \)

\[ \frac{\partial s(x, y, z)}{\partial x} = \frac{n_x(x, y, z)}{n_z(x, y, z)} \]  

(15)

and

\[ \frac{\partial s(x, y, z)}{\partial y} = \frac{n_y(x, y, z)}{n_z(x, y, z)}. \]  

(16)
With two partial derivatives, we can no longer simply integrate the dip tangents to solve for the shift field \( s(x, y, z) \). Instead, we use a least squares approach to honor both \( x \) and \( y \) dip tangents in the computation of the shift field. This approach has the advantage that we never need to explicitly define the operator that is applied to the dip tangents to solve for the shift field \( s(x, y, z) \). We only need to formulate the shift field \( s(x, y, z) \) in terms of its partial derivatives, as in equations 15 and 16.

This least squares approach has another advantage in that it takes the form of an inversion problem. In a typical inversion problem we attempt to find model \( m \) that when transformed by the function \( t(\cdot) \) approximates some data \( d \):

\[
d = t(m).
\]

Formulating the shift field calculation as an inversion problem allows us to use the geophysical inversion framework of Harlan (2004).

We simplify notation by packing the samples of the shift field \( s(x, y, z) \) into a vector \( s \) such that

\[
s[i + N_z(j + N_y k)] = s(x_i, y_j, z_k),
\]

where \( N_x \), \( N_y \), and \( N_z \) are the number of samples in the \( x \), \( y \), and \( z \) directions, respectively, and \( x_i, y_j, \) and \( z_k \) are the \( x \), \( y \), and \( z \) locations of our sampled values for every sample in the shift field. The index of the shift field \( s \) runs from 1 to \( N = N_x \times N_y \times N_z \) for every sample in the original shift field \( s(x, y, z) \), packed one after another:

\[
s \equiv \begin{bmatrix} s(1) \\ s(2) \\ \vdots \\ s(N) \end{bmatrix}.
\]

We perform a similar packing to create a vector \( d \) that contains both \( x \) and \( y \) dip tangents:

\[
d \equiv \begin{bmatrix} n_x(1)/n_z(1) \\ \vdots \\ n_x(N)/n_z(N) \\
 n_y(1)/n_z(1) \\ \vdots \\ n_y(N)/n_z(N) \end{bmatrix}.
\]

Note that the vector \( d \) contains \( 2N \) elements.

Next, we create a vector that contains both horizontal partial derivatives of the shift field vector \( s \), we obtain a vector of length \( 2N \) that contains both sets of horizontal partial derivatives:

\[
\nabla_{xy} s = \begin{bmatrix} s_{,x}(1) \\ \vdots \\ s_{,x}(N) \\
 s_{,y}(1) \\ \vdots \\ s_{,y}(N) \end{bmatrix}.
\]

Using our newly defined vectors, we can combine equations 15 and 16 into one expression:

\[
\nabla_{xy} s = d.
\]

This is an overdetermined system. There are \( N \) unknown shifts \( s \) and \( 2N \) equations. Equation 23 takes the form of a typical inversion problem where we wish to find the model that when transformed is equal to a dataset (equation 17). In our case, the shift field vector \( s \) is the model, the horizontal gradient \( \nabla_{xy} \) is the transform, and the dip tangents vector \( d \) is the dataset that we wish to fit. Using this information we create an error function that the inversion framework will minimize to find the shift field \( s \):

\[
E = [d - \nabla_{xy} s]^T [d - \nabla_{xy} s].
\]

Note that for steeply dipping reflectors, \( n_x/n_z \) and \( n_y/n_z \) become very large, even infinite. To avoid this,
we redefine \( \mathbf{d} \) to be
\[
\mathbf{d} \equiv \begin{bmatrix}
n_x(1) \\
\vdots \\
n_x(N)
n_y(1) \\
\vdots \\
n_y(N)
\end{bmatrix}.
\] (25)

Then, we define a new vector \( \mathbf{n}_a \) that contains the \( z \) components of the estimated normal vectors packed in a similar way to \( \mathbf{d} \):
\[
\mathbf{n}_a \equiv \begin{bmatrix}
n_z(1) \\
\vdots \\
n_z(N)
n_s(1) \\
\vdots \\
n_s(N)
\end{bmatrix}.
\] (26)

Note that the vector \( \mathbf{n}_a \) contains \( 2N \) samples. Instead of dividing the \( x \) and \( y \) components of the estimated normal vectors by the \( z \) component, we multiply the horizontal gradient of the shift field by \( \mathbf{n}_a \). Therefore, equation 23 becomes
\[
\mathbf{n}_a \times \nabla_{xy} \mathbf{s} = \mathbf{d}.
\] (27)

and our new error function is
\[
E = [\mathbf{d} - \mathbf{n}_a \nabla_{xy} \mathbf{s}]^T [\mathbf{d} - \mathbf{n}_a \nabla_{xy} \mathbf{s}].
\] (28)

Next, we weight the data misfit error differently. We would like to allow for more misfit in areas where we know that the data is of poor quality, and conversely, we would like to force the error function to have less misfit in areas where we have good quality data. We use the planarity value computed from the structure tensor as an indicator of data quality. The planarity, defined by equation 2, varies from zero to one. It will be close to zero when there is a large planar event and close to one elsewhere. We force the error function to put more emphasis on the parts of the image that have a stronger planarity value by packing the planarity value for every sample into a diagonal matrix \( \mathbf{P} \):
\[
\mathbf{P} = \begin{bmatrix}
p(1) \\
\vdots \\
p(N)
p(1) \\
\vdots \\
p(N)
\end{bmatrix}.
\] (29)

\( \mathbf{P} \) is a \( 2N \times 2N \) matrix. We incorporate the planarity value into the error function by scaling each sample by the corresponding planarity value:
\[
E = [\mathbf{d} - \mathbf{n}_a \nabla_{xy} \mathbf{s}]^T \mathbf{P} [\mathbf{d} - \mathbf{n}_a \nabla_{xy} \mathbf{s}].
\] (30)

An iterative search for solutions to equation 30 will introduce high-frequency variations in early iterations. We would prefer that the framework solve for an over simplified shift field, if it has not fully converged. To accomplish this, we add a regularization term to our error function. The regularization term will discourage complexity in the shift field by increasing the error \( E \) when the shift field contains large variations. We apply a derivative-like roughening operator \( \mathbf{R} \) to the shift field in the regularization term. This operator will enhance the high frequencies of the shift field, but unlike a true derivative, this operator will not remove a constant value from the shift field. This has the added benefit of forcing the magnitude of the shift field \( \mathbf{s} \) to be small.

Forcing the shifts to be small reduces the error in the shift field calculation caused by our earlier approximation (equation 8) that shifts are small. Therefore, our solution to the shift field solves for the smallest possible amount of shift that can account for the dip tangents.

This means that we do not need to shift all traces in reference to a particular trace. All samples in the image will be shifted by the minimum amount needed to flatten the image.

We account for differences in the magnitudes of the data misfit term and the regularization term by introducing a scaling \( \epsilon \) to the regularization term. This scaling \( \epsilon \) may be used to adjust for differences in the units of the two terms. The scaling \( \epsilon \) also allows us to adjust the importance of keeping the shifts smooth and small versus the importance of accurately fitting the dip tangents. We use the heuristic from Harlan (2004) that the shifts “should be allowed to attain magnitudes 10 to 100 times their most reasonable values before the two terms are equal”. Adding the regularization term to our error function gives
\[
E = [\mathbf{d} - \mathbf{n}_a \nabla_{xy} \mathbf{s}]^T \mathbf{P} [\mathbf{d} - \mathbf{n}_a \nabla_{xy} \mathbf{s}] + \epsilon [\mathbf{R} \mathbf{s}]^T [\mathbf{R} \mathbf{s}].
\] (31)

Instead of solving for the shift field \( \mathbf{s} \) directly, we solve for a rough version of the shift field \( \tilde{\mathbf{s}} \). We solve for this rough version of the shift field by performing a change of variables (Harlan, 1995):
\[
\mathbf{s} = \mathbf{S} \tilde{\mathbf{s}},
\] (32)

where \( \mathbf{S} \) is a Gaussian smoothing filter (Hale, 2006) in all three dimensions. We have used a Gaussian smoothing filter with half-widths of four, four, and ten samples in the \( x, y, \) and \( z \) directions, respectively, to compute the images in this paper. Our error function, after the change of variables, becomes
\[
E = [\mathbf{d} - \mathbf{n}_a \nabla_{xy} \mathbf{s}]^T \mathbf{P} [\mathbf{d} - \mathbf{n}_a \nabla_{xy} \mathbf{s}] + \epsilon [\mathbf{R} \tilde{\mathbf{s}}]^T [\mathbf{R} \tilde{\mathbf{s}}].
\] (33)
Next, we explicitly defined the roughening operator $R$ as the inverse of the smoothing operator $S$. Therefore,

$$RS = I.$$  \hspace{1cm} (34)

Now, we see the roughening operator $R$ has all of the properties that we described earlier. It will enhance high-frequency variations in the shift field because we know that the Gaussian smoothing filter $S$ will attenuate high-frequency variations. It will also not remove constant shifts from the shift field $s$ because applying a smoothing filter to a constant shift field will have no effect.

Using equation 34, to simplify the error function, we obtain

$$E = [d - n_a \nabla_y S] \mathbf{P} [d - n_a \nabla_y S] + \epsilon \tilde{s}^T \tilde{s}. \hspace{1cm} (35)$$

This error function was minimized in all of the flattening examples shown in this paper.

5 FAULTS

Our current algorithm has limited success flattening seismic images with faults. In Figure 6a, we show a 3D synthetic seismic image with a large vertical fault running down the middle of the image. The calculated shift field (Figure 6d) smoothly transitions from negative shift to positive shift across the fault. These shifts do not fully flatten the image. To accurately flatten the image, the shifts should be very sharp at the fault plane. This leads to the inaccurate isochron surface in Figure 6c. This isochron surface should be vertical at the fault, but it linearly transitions from one side of the fault to the other.

This artifact may be a result of our regularization term which forces the shift field to be smooth. This smoothing can obscure discontinuities in structures at faults. We would instead prefer to have a spatially varying smoother that does not smooth across faults (where the planarity is low), while smoothing elsewhere. We hope to add this improvement to future versions of our algorithm.

6 CONCLUSIONS

We have presented an algorithm that allows for the computation and display of isochron surfaces. Our flattening process is an extension of that by Lomask et al. (2006). This process uses structure tensors to efficiently calculate estimates of the local dip in the seismic image and employs a least squares inversion to compute the vertical shifts needed to flatten the image. Isochron surfaces, calculated from a flattened image, can then be used to define regions within a seismic image.

Figure 6. (a) A 3D synthetic seismic image with a fault running vertically through the middle of the volume. (b) A 3D isochron surface for one of the layers in the synthetic seismic volume shown in (a). (c) A close-up of the isochron surface shown in (b) that highlights the undesirable smoothing of the isochron surface across the fault. (d) A 2D slice through the center of the shift field that is used to flatten the image in (a).

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Compact finite-difference approximations for anisotropic image smoothing and painting

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ABSTRACT
Finite-difference approximations are succinctly represented by their stencils, a set of weights that when applied to adjacent samples of a function approximate some differential operator. In image processing the samples are pixels or voxels, and the differential operators must be inverted for smoothing or painting applications. For efficiency in such applications requiring inverses, the finite-difference stencils should be compact, using only a small $3 \times 3$ set of nine pixels or a $3 \times 3 \times 3$ set of twenty-seven voxels.

From $2 \times 2$ and $2 \times 2 \times 2$ approximations to gradient operators I obtain $3 \times 3$ (9-point) and $3 \times 3 \times 3$ (27-point) stencils that approximate Laplacian operators. The latter may include tensor coefficients that make them anisotropic. By deriving finite-difference stencils for anisotropic Laplacians in this way, that is, from approximations to gradients, we guarantee that our approximations to anisotropic Laplacians are symmetric and positive-semidefinite. And by choosing the gradient approximations carefully, discretization errors can be made isotropic to leading order.

Key words: finite-difference numerical methods

1 INTRODUCTION
A Laplacian is a symmetric differential operator that roughens. When applied to a function, it removes any constant bias or linear trend while enhancing higher frequencies.

This behavior is apparent in the Fourier transform of the Laplacian operator. For if $F(k)$ denotes the Fourier transform of a function $f(x)$

$$f(x) \iff F(k),$$

then

$$-\nabla \cdot \nabla f(x) \iff k^2 F(k).$$

The minus sign in $-\nabla \cdot \nabla$ makes this symmetric differential operator positive-semidefinite (SPS); its Fourier transform $k^2 = k^T k$ is non-negative and implies amplification of higher wavenumbers.

An isotropic Laplacian roughens a function equally in all directions; its Fourier transform $k^2$ depends on only the magnitude $k$ of the wavenumber vector $k$.

To roughen a function in one direction more than others, we may include a symmetric positive-semidefinite (SPS) tensor $D$ in our Laplacian operator:

$$-\nabla \cdot D \nabla f(x) \iff k^T D k F(k).$$

For this anisotropic Laplacian operator, the direction of maximum roughening is aligned with that eigenvector of $D$ having largest eigenvalue.

In practical applications involving images, the function $f(x)$ is sampled and partial derivatives must be approximated with finite differences. If we arrange all image samples into a single column vector $f$, then an anisotropic Laplacian may be approximated by

$$-\nabla \cdot D \nabla f \approx G^T D G f,$$

where $G$ is some sparse matrix that represents a finite-difference approximation to a gradient. In particular, for a 2-D image with $N = N_1 \times N_2$ samples, $G$ is a $2N \times N$ matrix

$$G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix},$$

with $N \times N$ sparse matrix components $G_1$ and $G_2$ that...
approximate partial derivatives along the first and second image dimensions, respectively.

In the finite-difference approximation of equation 4, the tensor field $D$ is represented by a block-diagonal matrix

$$D = \begin{bmatrix} D_{11} & D_{12} \\ D_{12} & D_{22} \end{bmatrix},$$

composed of $N \times N$ diagonal matrices $D_{11}, D_{12}$ and $D_{22}$. The $3N$ possibly non-zero elements of those matrices determine the directions and extents of roughening, which may vary from sample to sample.

1.1 Image smoothing

If an anisotropic Laplacian roughens in certain directions, then the inverse of an anisotropic Laplacian smooths in those same directions, and one application of such an inverse operator is structure-oriented image smoothing (e.g., Fehmers and Höcker, 2003). Given an input image $f$, an anisotropically smoothed output image $g$ may be obtained as the solution to:

$$(I + G^T DG)g = f. \tag{7}$$

The addition of the identity matrix $I$ to the anisotropic Laplacian $G^T DG$ has two consequences. First, it makes the symmetric matrix $I + G^T DG$ positive-definite (SPD) so that a solution $g$ can be found efficiently using conjugate-gradient iterations. Second, it implies that no smoothing occurs in eigenvector directions for which corresponding eigenvalues of the tensors in $D$ are zero.

The directions and extents of smoothing via equation 7 are determined entirely by the elements of the block-diagonal matrix $D$. Those $3N$ values may be computed from the structure of the input image $f$ (van Vliet et al., 1995), which may vary from sample to sample. Indeed, our desire to perform spatially-varying smoothing is the primary reason that we cannot simply use fast Fourier transforms to solve equation 7.

1.2 Image painting

Another application of anisotropic Laplacians is image painting, in which missing samples of a painting $p$ are computed to be consistent with the tensors in $D$ and known samples of $p$. Following Claerbout (1992), let $M$ denote a diagonal masking matrix that describes the locations of the missing samples:

$$M[i, i] = \begin{cases} 1 & : p[i] \text{ missing} \\ 0 & : p[i] \text{ known}, \end{cases} \tag{8}$$

and let $K$ be the diagonal complement of $M$ such that $K + M = I$. Then to compute the missing parts $Mp$ of $p$ we may solve

$$(K + MG^T DGM)p = (K - MG^T DGK)p. \tag{9}$$

To facilitate the solution of equation 9, I constructed the matrix on the left-hand-side to be symmetric and positive-definite (SPD). Any solution $p$ to equation 9 is also a solution to an alternative symmetric positive-semidefinite (SPS) system analogous to that proposed by Claerbout (1992, p. 178):

$$MG^T DGMp = -MG^T DGKp. \tag{10}$$

Using $K + M = I$, this system is equivalent to

$$MG^T DGp = 0. \tag{11}$$

which simply states that the missing parts of $p$ should vary linearly in spatially-varying directions determined by eigen-decomposition of the tensors in $D$.

In other words, solution of equation 9 yields a structure-oriented multi-dimensional linear interpolation of the known samples $Kp$ of the painting $p$. In practice, those known samples could be specified interactively or determined from other information. For seismic images of the earth’s subsurface, the known samples could correspond to locations where well logs are available.

Figures 1 and 2 show an example of image painting via equation 9. In this example, I painted one pixel red near the center of the image $p$ shown in Figure 1. The solution to equation 9 is shown in Figure 2. Although this painting is an extreme example in which only a single pixel is known, it demonstrates that paint flows to other samples according to the image structure represented by the tensors in $D$.

The painted image in Figure 2 may vary depending on the finite-difference approximations $G_1$ and $G_2$ in equation 9. In fact, my first attempts at image painting yielded results more like the one shown in Figure 3, in which artifacts roughly orthogonal to image features are
Finite-difference approximations

Figure 2. The image of Figure 1 painted with a finite-difference approximation to an anisotropic Laplacian. A single pixel inside the white circle was constrained to be red. All other pixels were painted using structure estimated from the image.

Figure 3. The image of Figure 1 painted with a poor finite-difference approximation to an anisotropic Laplacian. A single pixel inside the white circle was constrained to be red. All other pixels were painted using structure estimated from the image.

Artifacts such as those highlighted in Figure 4 are caused by poor finite-difference approximations to anisotropic Laplacians. The purpose of this paper is to explain these artifacts and to derive the improved approximations that were used to obtain the painting in Figure 2.

I begin by designing a family of 2-D finite-difference approximations to isotropic Laplacians. The design method used facilitates the derivation of approximations to anisotropic Laplacians. I then use this same method to obtain new 3-D finite-difference approximations for both isotropic and anisotropic Laplacians.

Both 2-D and 3-D approximations are compact in that they approximate derivatives for any image sample using only nearest neighbor samples: 9 samples for 2-D and 27 samples for 3-D Laplacians. Both 2-D and 3-D approximations maintain the SPS property of $G^T DG$, which ensures SPD systems of equations 7 and 9. Compact SPD finite-difference approximations lead to the most efficient methods for solving such equations.

2 ISOTROPIC LAPLACIANS

Let us first consider approximations to isotropic Laplacians in two dimensions:

$$-\nabla \cdot \nabla \approx G^T G = G_1^T G_1 + G_2^T G_2.$$  \hspace{1cm} (12)

Finite-difference approximations $G_1$ and $G_2$ to partial derivatives are most succinctly described by stencils that specify the weights applied to adjacent image samples. To obtain compact stencils for $G^T G$, I consider...
approximations \( G_1 \) and \( G_2 \) with the following stencils:

\[
G_1 = \begin{pmatrix}
-s & -s \\
 s & r \\
 r & s \\
\end{pmatrix}, \quad G_2 = \begin{pmatrix}
-s & s \\
 r & -r \\
 r & -r \\
\end{pmatrix},
\]

for \( r \geq s \geq 0 \), \( r + s = 1 \). (13)

The condition \( r + s = 1 \) ensures that \( G_1^T \) and \( G_2^T \) are 2nd-order approximations, assuming unit image sampling intervals.

Gathering image samples with weights provided by the stencils in equations 13 is equivalent to multiplication by the matrices \( G_1 \) and \( G_2 \). Scattering with the same weights is equivalent to multiplication by their transposes \( G_1^T \) and \( G_2^T \). Here is a small fragment of a computer program that both gathers and scatters in this way to compute \( g = G^T Gf \):

```pseudo
for (int i2=1; i2<n2; ++i2) {
    for (int i1=1; i1<n1; ++i1) {
        float f1r = f[i2][i1]; // gather begins
        float f1s = f[i2][i1];
        float f2r = f[i2][i1];
        float f2s = f[i2][i1];
        float g1 = r*f1r+s*f1s;
        float g2 = r*f2r+s*f2s; // scatter ends
        g[i2][i1] = g1*r+g2*r;
        g[i2][i1] = g1*r+g2*s;
        g[i2][i1-1] = g1*s+g2*r;
        g[i2-1][i1] = g1*s+g2*s;
        g[i2-1][i1-1] = g1*s+g2*s;
    }
}
```

Although this program fragment ignores boundaries, it can be easily modified to implement zero-value or zero-slope boundary conditions.

Gathering and scattering in this way using the \( 2 \times 2 \) stencils of equations 13 is equivalent to simply gathering with the following \( 3 \times 3 \) stencil:

\[
G^T G = \begin{pmatrix}
0 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 0 \\
\end{pmatrix} + 2rs \begin{pmatrix}
-1 & 2 & -1 \\
2 & -4 & 2 \\
-1 & 2 & -1 \\
\end{pmatrix}. \quad (14)
\]

Compact finite-difference approximations to Laplacians are often specified in this way, as a \( 3 \times 3 \) stencil for \( G_f G_f \) (e.g., Patra and Karttunen, 2005). I focus instead on the \( 2 \times 2 \) stencils for \( G_1 \) and \( G_2 \), the components of \( G \), because those will facilitate compact SPS finite-difference approximations \( G^T D G \) to anisotropic Laplacians with spatially-varying tensors in \( \mathbf{D} \).

The conditions \( r \geq s \geq 0 \) and \( r + s = 1 \) imply a family of finite-difference approximations for which \( 0 \leq rs \leq 1/4 \). Figure 5 shows three different members of this family, corresponding to \( rs = 0, 1/12, \) and \( 1/4 \), with their Fourier transforms.

The three approximations displayed in Figure 5 are comparable near the origin, for low wavenumbers \( k_1 \) and \( k_2 \), where contours are nearly circular with amplitudes that well approximate the ideal Fourier transform \( k^2 = k_1^2 + k_2^2 \). This isotropy can be important in applications such as image smoothing and painting, where results should not depend on sampling directions.

For higher wavenumbers, errors are visible in all three approximations, but contours for \( rs = 1/12 \) appear most nearly circular. In fact, errors for this middle approximation are well-known to be isotropic to 2nd order, and among all compact 9-point stencils this approximation of the isotropic Laplacian operator is optimal in that sense (e.g., Patra and Karttunen, 2005).

Let us define discretization error as a function \( E(k) \) of wavenumber \( k \) by the Fourier transform pair

\[
G^T G \iff k^2 [1-E(k)]. \quad (15)
\]
The Fourier transform of our finite-difference stencil \( G^T G \) (equation 14) is the product of the ideal factor \( k^2 \) and a second factor \( [1 - E(k)] \). The second factor includes the error \( E(k) \) that we would like to be zero or, failing that, at least isotropic to \( O(k^2) \). From the series expansion of the Fourier transform of \( G^T G \) we find that for \( rs = 1/12 \) the discretization error

\[
E(k) = \frac{k^2}{12} + O(k^4) \tag{16}
\]

is indeed isotropic to \( O(k^2) \). For \( rs \neq 1/12 \), the first 2nd-order term of this error is anisotropic; it depends on the wavenumber vector \( k \) and not just its magnitude \( k \).

The coefficients \( r \) and \( s \) of \( G_1 \) and \( G_2 \) for the optimal approximation are easily found from the conditions \( r \geq s \geq 0 \), \( r + s = 1 \) and \( rs = 1/12 \). They are

\[
r = 1/2 + 1/\sqrt{6}, \quad s = 1/2 - 1/\sqrt{6}.
\]

The coefficients for \( rs = 1/4 \) are simply \( r = s = 1/2 \). These coefficients imply a simple averaging of finite differences in \( G_1 \) and \( G_2 \) defined by equations 13. Observe however that the Fourier transform displayed in Figure 5c implies that this approximation attenuates high wavenumbers near Nyquist, in addition to low wavenumbers near the origin.

This observation can help us understand the image painting artifacts in Figure 4. When the finite-difference operator of Figure 5c is applied to an input constant or checkerboard image \( f \), the output image \( g = G^T Gf \) is zero. The inverse of such an operator will tend to amplify the same constant or checkerboard features.

### 3 ANISOTROPIC LAPLACIANS

From finite-difference approximations \( G_1 \) and \( G_2 \) we may also construct approximations

\[
-\nabla \cdot D \nabla \approx G^T D G = \begin{bmatrix} G_1^T & G_2^T \end{bmatrix} \begin{bmatrix} D_{11} & D_{12} \\ D_{12} & D_{22} \end{bmatrix} \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \tag{17}
\]

to anisotropic Laplacians. We simply gather with \( G \), multiply by \( D \), and scatter with \( G^T \). The corresponding computer program might look like this:

```c
for (int i2=1; i2<n2; ++i2) {
    for (int i1=1; i1<n1; ++i1) {
        float f1r = f[i2 ][i1 ]-f[i1 ][i1 ];
        float f1s = f[i1-1][i1 ]-f[i1-1][i1-1];
        float f2r = f[i2 ][i1 ]-f[i2-1][i1 ];
        float f2s = f[i2 ][i1-1]-f[i2-1][i1-1];
        float f1 = r*f1r+s*f1s;
        float f2 = r*f2r+s*f2s;
        float g1 = f1*d11[i2][i1]+f2*d12[i2][i1];
        float g2 = f1*d12[i2][i1]+f2*d22[i2][i1];
        float g1r = g1*r;
        float g1s = g1*s;
        float g2r = g2*r;
        float g2s = g2*s;
        g[i2 ][i1 ] = g1r+g2s;
    }
}
```

The gather-scatter symmetry in this program ensures an SPS implementation of \( G^T D G \). As above, this implementation is equivalent to one that gathers with a 9-point stencil. However, the latter implementation would require a more complex evaluation of stencil coefficients that vary from sample to sample inside the innermost loop.

### 3.1 The problem

Although the approximation implemented by this computer program is SPS, it is a rather poor approximation except for the special case \( rs = 1/4 \). To see the problem, let us first consider the approximation with \( rs = 0 \), for three different sets of constant coefficients \( d_{11}, d_{12} \) and \( d_{22} \) of the tensor \( D \).

Figure 6a shows the 9-point stencil for the case \( d_{11} = 1, d_{12} = d_{22} = 0 \). As expected, this stencil approximates a second derivative in the direction of the vertical axis \( x_1 \).

The problem is apparent in the other two stencils shown in Figures 6b and 6c; these stencils approximate second derivatives in the directions \( x_1 = x_2 \) and \( x_1 = -x_2 \), respectively. Both stencils approximate the appropriate derivatives; but they do so with very different discretization errors.

The stencil in Figure 6c should ideally be a rotated version of the stencil in Figure 6b, but this clearly is not the case. Image smoothing or painting with such stencils will yield surprising differences for features oriented at angles of \( -45 \) degrees and \( +45 \) degrees with respect to the sampling grid.

Stencils for \( rs = 1/12 \) show the same problem, as indicated in Figures 7. Although \( rs = 1/12 \) is optimal for the isotropic case \( d_{11} = d_{22} = 1, d_{12} = 0 \) shown in Figures 5, this value is clearly not optimal for the anisotropic cases shown in Figures 7.

Stencils for \( rs = 1/4 \), shown in Figures 8, do show the expected \( (\pm 45\text{-degree}) \) symmetry and in this sense are better approximations to anisotropic Laplacians. However, they unfortunately also exhibit the same attenuation at high wavenumbers near Nyquist that we have already seen in Figures 5.

This attenuation of high wavenumbers in approximations to anisotropic Laplacians will cause amplification of those same wavenumbers in image smoothing and painting applications. These wavenumbers near the spatial Nyquist frequencies are apparent in Figure 3, where they appear as a checkerboard pattern of painting artifacts. These high wavenumbers are enhanced when we invert approximations like those shown in Figures 8.
3.2 A solution

Recall that the approximation for \( rs = 1/12 \) is in one sense optimal; it is the most isotropic approximation of the isotropic Laplacian. For anisotropic Laplacians we can improve this approximation by considering two additional approximations \( G_1 \) and \( G_2 \) to the gradient operators.

\[
G_{1a} = \begin{bmatrix} -s & -r \\ s & r \end{bmatrix}, \quad G_{2a} = \begin{bmatrix} -s & s \\ -r & r \end{bmatrix}, \quad G_{1b} = \begin{bmatrix} -r & -s \\ r & s \end{bmatrix}, \quad G_{2b} = \begin{bmatrix} -r & r \\ -s & s \end{bmatrix},
\]

In hindsight, we had no reason above to prefer one or the other of the approximations \( G_{1a} \) or \( G_{1b} \); likewise for \( G_{2a} \) and \( G_{2b} \). In equations 13 above, I chose the stencils \( G_{1a} \) and \( G_{2a} \), but I might just as well have chosen \( G_{1b} \) or \( G_{2b} \).

With two choices for each of the finite-difference approximations \( G_1 \) and \( G_2 \), we can form a total of four different \( G^T DG \). Let us average all four to obtain

\[
-\nabla \cdot D \nabla \approx \frac{1}{4} \left( \begin{bmatrix} G_{1a}^T G_{2a} & G_{1a}^T G_{2b} \\ G_{1b}^T G_{2a} & G_{1b}^T G_{2b} \end{bmatrix} D_{11} D_{12} D_{12} D_{22} \right) \begin{bmatrix} G_{1a} \\ G_{2a} \\ G_{1b} \\ G_{2b} \end{bmatrix}.
\]

This average of four SPS matrices is clearly SPS. For \( rs = 1/12 \) and constant tensor coefficients, this average yields the 9-point stencils shown in Figures 9. These stencils exhibit desired symmetries — the stencil in Figure 9c is a rotated version of that in Figure 9b — and they do not attenuate wavenumbers near the spatial Nyquist frequencies. The finite-difference approx-
Finite-difference approximations

The implementation in equation 19 is the one that I used in the image painting example of Figure 2.

The same average of four approximations for \( r_s = 0 \) yields a slightly different finite-difference approximation that was derived from a mixed finite-element method by Arbogast et al. (1997). I favor the approximation shown here with \( r_s = 1/12 \) because as discussed above its errors are more isotropic.

Implementation of the approximation in equation 19 need not require a factor of four increase in computational cost, because the averaging can be performed analytically. The derivation is tedious but the result is simple, even for the case of spatially varying tensor coefficients. For that case, define symmetric tensors \( D \) at the four corners of an image sample with indices \((i_1, i_2)\) as follows:

\[
D(i_1 - \frac{1}{2}, i_2 - \frac{1}{2}) \equiv \begin{bmatrix} a_{00} & b_{00} \\ b_{00} & c_{00} \end{bmatrix}
\]

\[
D(i_1 - \frac{1}{2}, i_2 + \frac{1}{2}) \equiv \begin{bmatrix} a_{10} & b_{10} \\ b_{10} & c_{10} \end{bmatrix}
\]

\[
D(i_1 + \frac{1}{2}, i_2 - \frac{1}{2}) \equiv \begin{bmatrix} a_{01} & b_{01} \\ b_{01} & c_{01} \end{bmatrix}
\]

\[
D(i_1 + \frac{1}{2}, i_2 + \frac{1}{2}) \equiv \begin{bmatrix} a_{11} & b_{11} \\ b_{11} & c_{11} \end{bmatrix}
\]

Then, using the finite-difference approximation of equation 19, the 9-point stencil centered on the image sample
with indices \((i_1, i_2)\) is

\[
\begin{array}{ccc}
-b_{00} & -a_{00} - a_{10} & b_{10} \\
-a_{00} - c_{00} & a_{00} + a_{10} + a_{11} + b_{00} - b_{10} + b_{11} + c_{00} + c_{10} + c_{11} & -c_{10} - c_{11} \\
b_{01} & -a_{01} - a_{11} & -b_{11} \\
-a_{00} - c_{00} & a_{00} + a_{10} + a_{11} + b_{00} - b_{10} + b_{11} + c_{00} + c_{10} + c_{11} & -a_{10} - c_{10} \\
-a_{01} - c_{01} & a_{01} + a_{11} + b_{01} - b_{11} + c_{01} + c_{11} & -a_{11} - c_{11}
\end{array}
\]  

For \(rs = 0\) only the first part of this stencil is significant, and this part is the finite-difference approximation of Arbogast et al. (1997). Addition of the second part with \(rs = 1/12\) yields a finite-difference approximation with errors that are more isotropic. For constant tensor coefficients \(a = d_{11}, b = d_{12}, c = d_{22}\) this stencil is consistent with the three examples in Figures 9.

For spatially varying tensor coefficients, the various sums and differences in this 9-point stencil maintain the important SPS property of our finite-difference approximation. As shown above, a simpler way to maintain this SPS property is to gather and scatter consistently, as illustrated by the following program fragment:

```c
for(int i2=1; i2<n2; ++i2) {
    for(int i1=1; i1<n1; ++i1) {
        float a = 0.5f*d11[i1][i1];
        float b = 0.5f*d12[i1][i1];
        float c = 0.5f*d22[i1][i1];
        float t = 2.0f*rs*(a+c);
        float fpp = f[i2][i1-1][i1-1];
        float fpm = f[i2][i1][i1-1];
        float fmp = f[i2-1][i1][i1];
        float fmm = f[i2-1][i1-1][i1-1];
        float appm = (a-t)*(fpp-fpm);
        float anpm = (a-t)*(fpm-fmm);
        float bppm = (b+t)*(fpp-fmp);
        float bpmn = (b+t)*(fpm-fmm);
        float cppm = (c-t)*(fpp-fmp);
        float cpmn = (c-t)*(fpm-fmm);
        g[i12][i1][i2] = appm+bppm+cppm;
        g[i12][i1-1][i1-1] += appm+bpmn+cpmm;
    }
}
```

Note that this implementation requires for each image sample only one evaluation of tensor elements.

## 4 3-D LAPLACIANS

The methodology used above to derive a finite-difference approximation to a 2-D anisotropic Laplacian extends naturally to three dimensions. The two important steps are

(i) Find approximations \(G_1, G_2,\) and \(G_3\) to components of the gradient such that \(G^T G = G_1^2 G_1 + G_2^2 G_2 + G_3^2 G_3\) has (to second order) isotropic discretization errors.

(ii) Average all possible combinations of these components in approximations \(G^T DG\) to the anisotropic Laplacian.

The combinations \(G^T DG\) will have the form

\[
G^T DG = \begin{bmatrix} G_1^2 & G_2^2 & G_3^2 \\ D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix} \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix}
\]

(20)

Compact \(2 \times 2 \times 2\) stencils for \(G_1, G_2,\) and \(G_3\) are rotated versions of each other:

\[
\begin{align*}
G_1 &= \begin{bmatrix} -t & -s & -s \\ t & s & r \\ 0 & 0 & 0 \end{bmatrix} \\
G_2 &= \begin{bmatrix} -t & t & -s \\ -s & s & -r \\ 0 & 0 & 0 \end{bmatrix} \\
G_3 &= \begin{bmatrix} -t & -s & 0 \\ -s & r & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{align*}
\]

(21)

In this notation, the back part of each stencil is specified left of the front part. The condition \(rt = s^2\) follows from desired symmetries in \(G_1^2 G_1, G_2^2 G_2\) and \(G_3^2 G_3\). The condition \(r + 2s + t = 1\) ensures that \(G_1^2 G_1, G_2^2 G_2\) and \(G_3^2 G_3\) are 2nd-order approximations.

These stencils for \(G_1, G_2,\) and \(G_3\) lead to compact \(3 \times 3 \times 3\) 27-point stencils for \(G^T DG\). In the isotropic case where \(D = I\), we have

\[
G^T G = G_1^2 G_1 + G_2^2 G_2 + G_3^2 G_3
\]

(22)

For this case Patra and Karttunen (2005) give conditions for which compact 3-D stencils have (to 2nd order) isotropic discretization errors, and they list examples of stencils that meet those conditions. In three dimensions, unlike two dimensions, more than one such stencil is possible.

However, none of the isotropic stencils they cite can be expressed in the form of equation 22, in terms of approximations to components of gradients. In other words, they do not correspond to approximations \(G_1, G_2,\) and \(G_3\) that we could use in the anisotropic form of equation 20.

By carefully choosing the coefficients \(r, s\) and \(t\) subject to the conditions cited above, I found a new finite-
difference approximation with the desired form. The coefficients are

\[ t = 5/12 - 1/\sqrt{6}, \quad r = (1 - \sqrt{1})^2, \quad s = \sqrt{rt}. \]

These coefficients lead to the following 27-point stencil:

\[
G^T G = \begin{bmatrix}
    c_1 & c_2 & c_1 & c_2 & c_1 & c_2 & c_1 \\
    c_2 & c_3 & c_2 & c_3 & c_2 & c_3 & c_2 \\
    c_1 & c_2 & c_1 & c_2 & c_1 & c_2 & c_1 \\
    c_2 & c_3 & c_2 & c_3 & c_2 & c_3 & c_2 \\
    c_1 & c_2 & c_1 & c_2 & c_1 & c_2 & c_1 \\
    c_2 & c_3 & c_2 & c_3 & c_2 & c_3 & c_2 \\
    c_1 & c_2 & c_1 & c_2 & c_1 & c_2 & c_1
\end{bmatrix},
\]

\[ c_1 = -1/48, \quad c_2 = -1/8, \quad c_3 = -5/12, \quad c_4 = 25/6. \] (23)

This stencil meets the conditions for isotropic discretization errors specified by Patra and Karttunen (2005), who cite a different 27-point stencil proposed by Spotz and Carey (1995):

- \[ c_1 = -1/30, \quad c_2 = -1/10, \quad c_3 = -7/15, \quad c_4 = 64/15. \]

As noted above, this different stencil does not have corresponding approximations \( G_1, G_2 \) and \( G_3 \) that we can use in equation 20.

Before inserting the stencils \( G_1, G_2 \) and \( G_3 \) of equations 21 into equation 20, we first recognize that each of these 3-D stencils can be written in four different ways. Those four ways are analogous to the two different ways we expressed the 2-D stencils \( G_1 \) and \( G_2 \) in equations 18.

Since each of the 3-D stencils \( G_1, G_2 \) and \( G_3 \) can be written four different ways, we have a total of \( 64 = 4 \times 4 \times 4 \) combinations of the form of equation 20. By averaging all 64 combinations, we obtain a composite 3-D finite-difference approximation for an anisotropic Laplacian.

This large number of combinations to average can be reduced if we consider only pairwise combinations such as \( G_1^T D_1 G_1 \) (four ways) or \( G_1^T D_2 G_2 \) (sixteen ways). In any case, the averaging can again be performed analytically and, although too lengthy to be included in this paper, the result is analogous to that obtained above by averaging 2-D approximations.

5 CONCLUSION

The method described here for finite-difference approximation of anisotropic Laplacians is straightforward. We begin with compact finite-difference approximations to components of the gradient. We then average symmetric positive-semidefinite combinations of those gradient approximations to obtain the desired approximations to anisotropic Laplacians. Finite-difference approximations obtained in this way are guaranteed to be symmetric and positive-semidefinite.

This design method yields 9-point stencils for 2-D approximations and 27-point stencils for 3-D approximations of anisotropic Laplacians. Our 2-D approximation is a generalization of that proposed by Arbogast al. (1997), in that for one particular gradient approximation we obtain the coefficients of their 9-point stencil. However, an alternative gradient approximation in our method leads to a 9-point stencil with isotropic discretization errors.

Arbogast et al. (1997) also proposed a 19-point stencil for a 3-D approximation, and our 27-point stencil is not a generalization of theirs. Indeed, I have not found a way to modify the design proposed here to obtain such a 19-point stencil. A 19-point stencil is attractive because it implies lower computational costs.

In both 2-D and 3-D approximations, required symmetry in the resulting 9-point or 27-point finite-difference stencils implies only one free parameter in the corresponding gradient approximations. I choose this parameter to obtain stencils with isotropic discretization errors.

All finite-difference methods exhibit discretization errors. By choosing methods for which those errors are isotropic to leading order, we may reduce artifacts associated with sampling grids in applications such as image smoothing and image painting.

REFERENCES


Effective reflection coefficients for curved interfaces in TI media

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ABSTRACT

Plane-wave reflection coefficients (PWRC) are routinely used in amplitude-variation-with-offset (AVO) analysis and for generating boundary data in Kirchhoff modeling. However, the geometrical-seisimics approximation based on PWRC becomes inadequate in describing reflected wavefields at near- and post-critical incidence angles. Also, PWRC are derived for plane interfaces and breakdown in the presence of significant reflector curvature. Here, we discuss so-called effective reflection coefficients (ERC) designed to overcome the limitations of PWRC for multicomponent data from heterogeneous anisotropic media.

We show that the reflected wavefield in the immediate vicinity of a curved interface can be represented by a generalized plane-wave decomposition, which approximately reduces to the conventional Weyl-type integral computed for an “apparent” source location. The ERC is then obtained as the ratio of the reflected and incident wavefields at each point of the interface. To carry out diffraction modeling, we combine ERC with the tip-wave superposition method (TWSM) extended to elastic media. This methodology is implemented for curved interfaces separating an isotropic incidence halfspace and a transversely isotropic (TI) medium with the symmetry axis orthogonal to the reflector.

If the interface is plane, ERC represent the exact solution sensitive to the anisotropy parameters and source-receiver geometry. Numerical tests demonstrate that the difference between ERC and PWRC for typical TI models can be significant, especially at low frequencies and in the post-critical domain. For curved interfaces, ERC provide a practical approximate tool to compute the reflected wavefield. We analyze the dependence of ERC on reflector shape and demonstrate their advantages over PWRC in 3D diffraction modeling of PP and PS reflection data.

Key words: reflection coefficient, spherical wave, Kirchhoff modeling, AVO analysis, anisotropy, transverse isotropy, P-waves, converted waves, curved interfaces

INTRODUCTION

Plane-wave reflection and transmission coefficients provide the basis for ray-theory treatment of seismic wavefields in layered media. In the geometrical-seismics approximation, which represents the leading term of the ray-series expansion, the amplitude of any wave mode is proportional to the product of the reflection/transmission coefficients along the raypath (Brekhovskikh, 1980; Červený, 2001). For example, the well-known geometrical-seismics expression for a wave reflected from the bottom of a homogeneous layer includes the plane-wave reflection coefficient (PWRC)
multiplied by the source-radiation function and divided by the geometrical-spreading factor.

Geometrical seismics, however, becomes inaccurate for near- and post-critical incidence angles or when the source and/or receiver is located close (compared to the predominant wavelength) to the reflector (Brekhovskikh, 1980; Tsvankin, 1995). Deviations from the geometrical-seisimcs approximation become much more pronounced in the presence of even moderate seismic anisotropy (Tsvankin, 2005). Also, since PWRC are derived for plane interfaces, they cannot be used for ray-theory modeling in the presence of significant reflector curvature.

The limitations of the geometrical-seismics approximation pose serious problems for dynamic ray tracing and Kirchhoff integral modeling techniques (Frazer and Sen, 1985; Hanuga and Helle, 1995; Ursin and Tygel, 1997; Červený, 2001; Ursin, 2004). In particular, the boundary data used in conventional Kirchhoff modeling are obtained by simply multiplying the amplitude of the incident wave (which generally has a curved wavefront) with the PWRC. This approach produces artificial diffractions on synthetic data due to the discontinuous slope of the PWRC at the critical angle (Kampmann, 1988; Wenzel et al., 1990; Sen and Frazer, 1991).

Another practically important method based on geometrical seismics is amplitude-variation-with-offset (AVO) analysis, which operates with PWRC estimated from surface reflection data. Furthermore, because of the complexity of exact reflection coefficients, PWRC used in AVO processing are often linearized in the velocity and density contrasts across the reflector. The weak-contrast approximation of PWRC is given by Shuey (1985) for isotropic media and extended by Thomson (1993) and Rüger (1997) to VTI (transversely isotropic with a vertical symmetry axis) models. The VTI expressions involve an additional linearization in the anisotropy parameters on both sides of the interface, which helps to separate the reflection coefficient into isotropic and anisotropic terms. Rüger (1997, 2002) generalizes the weak-contrast, weak-anisotropy PWRC equations for azimuthally anisotropic models and discusses their application in fracture characterization using wide-azimuth reflection data.

Whereas conventional PWRC are defined through the magnitude of the displacement vector, Schleicher et al. (2001) introduce linearized reflection coefficients obtained from the ratio of the energy flux for the reflected and incident waves. They show that application of the flux-normalized coefficients in Kirchhoff modeling produces reciprocal reflected wavefields. The flux-normalized reflection coefficients are extended to viscoelastic VTI media by Stovas and Ursin (2003).

The linearized approximations, however, lose accuracy with increasing incidence angle and break down near the critical ray. To overcome this problem, Down- ton and Ursenbach (2006) express the reflection coefficient as a function of the averaged incidence and transmission angles and develop an analytic continuation of the linearized PWRC in the post-critical domain. For weak parameter contrasts across the interface, their approximation remains close to the exact PWRC for post-critical angles.

Still, even exact PWRC employed in the geometrical-seismics approximation cannot describe the post-critical reflected wavefield, which includes the interfering head and reflected waves. To make PWRC suitable for amplitude analysis in the post-critical domain, van der Baan and Smith (2006) propose to apply the $\tau - p$ transform to wide-angle reflection data. Although the transformed wavefield exhibits a better fit to the corresponding PWRC, the $\tau - p$ technique does not properly account for head waves and is limited to laterally homogeneous models.

In an earlier publication (Ayzenberg et al., 2007), we introduce so-called effective reflection coefficients (ERC) for acoustic wave propagation and demonstrate their advantages in Kirchhoff modeling. ERC are designed to generalize PWRC for wavefields from point sources at curved interfaces, and are not limited to small incidence angles and weak parameter contrasts across the reflector. In particular, Kirchhoff-type modeling with ERC removes the critical-angle artifacts mentioned above and correctly reproduces the amplitudes of the head and reflected waves.

The goal of this paper is to extend ERC to curved reflectors in heterogeneous anisotropic models and implement the new formalism for an interface between isotropic and TI media. We begin by defining ERC through a generalized plane-wave decomposition similar to the one proposed by Klem-Musatov et al. (2004) for the acoustic problem. Although this solution involves integration over a curved reflecting surface, ERC can be approximately obtained from Weyl-type integrals computed for locally plane interface segments. By conducting numerical tests, we evaluate the difference between ERC and PWRC for a plane interface and study the dependence of ERC on the anisotropy parameters, frequency and local reflector shape. Finally, using the tip-wave superposition method (TWSM), we implement ERC in 3D elastic diffraction modeling. Tests for curved interfaces of different shape confirm the ability of our algorithm to model reflection wavefields in the presence of multipathing and caustics.

**EFFECTIVE REFLECTION COEFFICIENTS FOR ANISOTROPIC MEDIA**

**Wavefield representation using surface integrals**

We consider the wavefield reflected from a smooth curved interface $S$, which separates homogeneous isotropic and transversely isotropic (TI) halfspaces (Figure 1). The symmetry axis of the TI medium is assumed...
Effective reflection coefficients 87

where the index $j$ corresponds to a surface element, $l_{P[j]}(x) = \nabla g_{P}(x_{j}, x)/|\nabla g_{P}(x_{j}, x)|$, and $g_{P}(x', x)$ is the scalar P-wave Green’s function. $\Delta B_{P_{P[j]}}(x)$ is the scalar contribution of the $j$-th element given by

$$
\Delta B_{P_{P[j]}}(x) = \int \int_{\Delta u_{[j]}} \phi \left. \frac{\partial g_{P}(x', x)}{\partial n'} d_{1, P_{P}}(x') \right|_{dS'} + g_{P}(x', x) d_{2, P_{P}}(x') \right|_{dS'},
$$

where $d_{1, P_{P}}(x')$ and $d_{2, P_{P}}(x')$ are the scalar boundary values of the reflected PP-wave at the interface. Equation F-13 expresses the boundary data $d_{1, P_{P}}$ and $d_{2, P_{P}}$ through the incident wavefield and the PP-wave effective reflection coefficient (ERC) introduced below.

We also show in Appendix F that the reflected PS-wavefield can be represented as the sum of the tip-wave beams described by equations F-21 and F-22:

$$
u_{PS}(x) \approx \sum_{j} \frac{i \rho_{s}^{b}}{v_{s}^{b}} l_{S[j]}(x) \times \Delta B_{PS[j]}(x),
$$

where $l_{S[j]}(x) = \nabla g_{S}(x_{j}, x)/|\nabla g_{S}(x_{j}, x)|$, $g_{S}(x', x)$ is the scalar S-wave Green’s function, and $\Delta B_{PS[j]}(x)$ is the vector contribution of the $j$-th surface element:

$$
\Delta B_{PS[j]}(x) = \int \int_{\Delta u_{[j]}} \phi \left. \frac{\partial g_{S}(x', x)}{\partial n'} \right|_{dS'} + g_{S}(x', x) \right|_{dS'},
$$

$d_{1, PS}(x')$ and $d_{2, PS}(x')$ are the vector boundary values of the reflected PS-wavefield at the interface expressed through the corresponding ERC in equation F-20. To evaluate integrals 3 and 5, we use the far-field approximation developed by Ayyenberg et al. (2007).

### Wavefield at the interface in terms of ERC

In conventional Kirchhoff modeling, it is assumed that the reflected wavefield $u_{PQ}(x')$ can be approximately written as

$$
u_{PQ}(x') \approx R_{PQ}(\theta(x')) \left[ \frac{i \omega}{v_{P}} \hat{h}_{P}(x') \right] \hat{h}_{Q}(x'),
$$

where $R_{PQ}(\theta(x'))$ is the plane-wave reflection coefficient (PWRC), $\theta(x')$ is the incidence angle, and $\hat{h}_{P}(x')$ and $\hat{h}_{Q}(x')$ are the unit polarization vectors of the incident P-wave and reflected PQ-wave, respectively. This approach, which is based on the geometrical-seismics approximation, assumes that the wavefront curvature at the reflector can be ignored, the reflector is plane, and the medium near the reflector homogeneous. However, equation 6 is adequate only for sub-critical incidence angles and causes artificial diffractions due to the discontinuous slope of the PWRC at the critical angle (Kampmann, 1988; Wenzel et al., 1990; Sen and Frazer, 1991).
For a plane interface between homogeneous media, the assumption about the wavefront curvature can be relaxed by representing the incident wave in the form of the Weyl integral over plane waves (Aki and Richards, 2002; Tsvankin, 1995). Each elementary plane wave in the integrand is multiplied with the PWRC to obtain an exact integral expression for the reflected wavefield. To handle curved reflectors in heterogeneous media, Klem-Musatov et al. (2004) introduced a rigorous theory of reflection and transmission for interfaces of arbitrary shape in acoustic models. They showed that the boundary data in the acoustic Kirchhoff integral can be represented by a generalized plane-wave decomposition called the "reflection operator." For curved interfaces, the decomposition is local and has to be evaluated separately for each individual point at the interface. Ayzenberg et al. (2007) proved that the exact action of the reflection operator upon the incident wavefield may be approximately described by multiplication of the incident wavefield and the corresponding effective reflection coefficient (ERC) for each point at the interface. This formalism is not limited to small incidence angles and weak parameter contrasts across the interface. It also incorporates the local interface curvature into the reflection response.

Here we generalize the reflection operator for curved interfaces between isotropic and TI media. In Appendix A we demonstrate that in the immediate vicinity of a curved interface there exist local exponentials of the wave equation with variable coefficients in the form of generalized plane waves. Using these solutions as the basis, in Appendix B we introduce spectral integrals describing the decomposition of the displacement field into the generalized plane P-, S1- and S2-waves propagating to and from the interface. In the special case of a plane reflector separating two homogeneous half-spaces, the generalized spectral integral reduces to the known Weyl-type decomposition over conventional plane waves. For curved reflectors, the generalized spectral integrals satisfy the boundary conditions (i.e., the continuity of displacement and traction across the interface) and are invariant with respect to the interface shape. In Appendix C, we rewrite the boundary conditions in the form of reflection and transmission operators for anisotropic media. Here we concentrate on the reflected displacement field, \( \mathbf{u}_{\text{RP}}(\mathbf{x}') \). The traction \( \mathbf{t}_{\text{RP}}(\mathbf{x}') \) can be eliminated, which simplifies synthetic modeling and reduces computing time.

As shown in Appendix C, the generalized plane-wave decomposition for the displacement component \( j \) of the PQ-mode reflected from a curved interface can be represented as

\[
\mathbf{u}_{\text{RP},j}(s_1, s_2, 0; \mathbf{x}') = \frac{\omega^2}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} R_{\text{RP}}(p; \mathbf{x}') \frac{h_{p,j}(\mathbf{x}')}{h_{p,j}(\mathbf{x})} \times u_{\text{inc}}^{\text{PQ},j}(p_1, p_2, 0; \mathbf{x}') e^{-i\omega(p_1s_1+p_2s_2)} dp_1 dp_2,
\]

where \((s_1, s_2)\) are the curvilinear Chebyshev coordinates covering the interface \( S_1(p_1, p_2) \) are the projections of the slowness vector onto the plane tangential to the interface at point \( \mathbf{x}' \), \( p = \sqrt{p_1^2 + p_2^2} \), \( R_{\text{RP}}(p; \mathbf{x}') \) is the PWRC at point \( \mathbf{x}' \), and \( h_{p,j}(\mathbf{x}') \) and \( h_{p,j}(\mathbf{x}) \) are the components of the unit polarization vectors of the incident P-wave and reflected PQ-wave, respectively. For arbitrary interface geometry, the spectrum \( u_{\text{inc}}^{\text{PQ},j}(p_1, p_2, 0; \mathbf{x}') \) of the incident wave has to be evaluated using the Fourier transform in the Chebyshev coordinates \((s_1, s_2)\):

\[
\mathbf{u}_{\text{inc}}^{\text{PQ},j}(p_1, p_2, 0; \mathbf{x}') = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathbf{u}_{\text{inc}}^{\text{PQ},j}(s_1, s_2, 0; \mathbf{x}') \times e^{-i\omega(p_1s_1+p_2s_2)} ds_1 ds_2.
\]

The generalized plane-wave decomposition is local and has to be computed at each point \( \mathbf{x}' \). It is valid within an infinitely thin layer near the interface, and can be used only for calculation of the reflection response in the immediate vicinity of the reflector.

The exact PWRC \( R_{\text{RP}}(p; \mathbf{x}') \) for a plane interface between two VTI media can be found in Graeber (1992) and Rüger (2002). In Appendix D we reproduce the derivation of PWRC in our notation and correct types in the published solutions.

In the special case of a plane interface, the decompositions reduce to the known Weyl-type integrals over conventional plane waves (Aki and Richards, 2002; Tsvankin, 1995). For a horizontal reflector, the curvilinear coordinates \((s_1, s_2)\) coincide with the ordinary Cartesian coordinates \((x_1, x_2)\). Also, the spectrum \( u_{\text{inc}}^{\text{PQ},j}(p_1, p_2, 0; \mathbf{x}') \) in 8 does not depend on positon \( \mathbf{x}' \) and is represented by a known analytic function.

The integral formula 7 is much more complicated than the geometrical-seismics solution 6. In particular, evaluation of the spectrum \( u_{\text{inc}}^{\text{PQ},j}(p_1, p_2, 0; \mathbf{x}') \) from equation 8 involves extremely time-consuming integration over the whole interface. However, the generalized decomposition 7 can be approximately reduced to the following form similar to equation 6:

\[
\mathbf{u}_{\text{RP}}(\mathbf{x}') \approx \chi_{\text{RP}}(\theta(\mathbf{x}'), L(\mathbf{x}')) \left[ \mathbf{u}_{\text{inc}}^{\text{PQ},j}(\mathbf{x}') \cdot \mathbf{h}_{\text{RP}}(\mathbf{x}') \right] \mathbf{h}_{\text{Q}}(\mathbf{x}'),
\]

where \( \chi_{\text{RP}}(\theta(\mathbf{x}'), L(\mathbf{x}')) \) is the ERC and \( L(\mathbf{x}') = \omega R^*(\mathbf{x}')/v_P^{(1)} \) is a dimensionless frequency-dependent parameter,

\[
R^*(\mathbf{x}') = R(\mathbf{x}') \frac{2 - \sin^2 \theta(\mathbf{x}')}{2 - \sin^2 \theta(\mathbf{x}') - 2R(\mathbf{x}')H(\mathbf{x}') \cos \theta(\mathbf{x}')).
\]

Here, \( R(\mathbf{x}') \) is the distance between the source and point \( \mathbf{x}' \) at the interface and \( H(\mathbf{x}') \) is the mean interface curvature (Ayzenberg et al., 2007). The parameter \( R^*(\mathbf{x}') \), which depends on the actual incidence angle \( \theta(\mathbf{x}') \) and the local interface curvature \( H(\mathbf{x}') \), has the meaning of...
the distance between the source of an “apparent” incident spherical P-wave and the interface.

In Appendix E we express the ERC in equation 9 through the Fourier-Bessel integrals for the reflected wavefield (Brekhovskikh, 1980; Aki and Richards, 2002):

\[
\chi_{PP}(\theta(x'), L(x')) = \frac{u^+_{PP, \text{norm}}(x') \cos \theta(x') + u^+_{PP, \text{tan}}(x') \sin \theta(x')}{(ik_p - \frac{1}{2}) \omega \rho_p R^*},
\]

\[
\chi_{PS}(\theta(x'), L(x')) = -\frac{u^+_{PS, \text{norm}}(x') \sin \theta(x') + u^+_{PS, \text{tan}}(x') \cos \theta(x')}{(ik_p - \frac{1}{2}) \omega \rho_p R^*},
\]

where

\[
u^+_{PP, \text{norm}}(x') = \omega^2 \int_0^{+\infty} R_{PP}(p) \frac{h^+_{P, \text{norm}}(x') e^{i \omega p}}{h^+_{P, \text{norm}}} J_0(\omega p) p dp,
\]

\[
u^+_{PP, \text{tan}}(x') = -\omega^2 \int_0^{+\infty} R_{PP}(p) \frac{h^+_{P, \text{tan}}(x') e^{i \omega p}}{h^+_{P, \text{tan}}} J_1(\omega p) p dp.
\]

The reflection S-wave angle \(\theta_S(x')\) is obtained from Snell’s law as

\[
\theta_S(x') = \sin^{-1}\left(\frac{(v_S^{(1)})/v_P^{(1)}}{\sin \theta(x')}\right),
\]

\(J_0(\omega p)\) and \(J_1(\omega p)\) are the zero-order and first-order Bessel functions,

\[p_{P3} = \sqrt{\left(v_P^{(1)}\right)^2 - p^2}\]

is the vertical P-wave slowness,

\[l = R^*(x') \cos \theta(x'),\]

and

\[r = R^*(x') \sin \theta(x').\]

For the reflected PP-wave,

\[h^+_{P, \text{norm}} / h^+_{P, \text{norm}} = -1\]

and

\[h^+_{P, \text{tan}} / h^+_{P, \text{tan}} = 1.\]

For the PS-wave,

\[h^+_{S, \text{norm}} / h^+_{P, \text{norm}} = (v_S^{(1)} / v_P^{(1)}) (v_S^{(1)} / v_P^{(1)})\]

and

\[h^+_{S, \text{tan}} / h^+_{P, \text{tan}} = (v_S^{(1)} / v_P^{(1)}) (v_S^{(1)} / v_P^{(1)}) p,
\]

where

\[p_{S3} = \sqrt{\left(v_S^{(1)}\right)^2 - p^2}\]

is the vertical S-wave slowness.

The ERC in equation 11 is defined as the ratio of the magnitude of the reflected PQ-wave and the incident P-wave. Therefore, they generalize the PWRC from equation 6 by taking into account the curvatures of both the incident wavefront and the reflector. While PWRC depend on the stiffness and density contrasts across the boundary and the incidence angle \(\theta(x')\), ERC are controlled by one more dimensionless parameter, \(L(x')\), which incorporates the interface curvature. In the stationary-phase approximation, the ERC reduce to the corresponding PWRC. In contrast to PWRC, ERC correctly describe reflection phenomena at near-critical and post-critical incidence angles.

Equation 10 shows how the local reflector curvature is incorporated into ERC. If the reflector is locally plane, then \(H(x') = 0\), and the distance \(R^*(x')\) reduces to \(R(x')\). For particular parameter combinations, \(R^*(x')\) may go to infinity, which means that the incident P-wave appears to be locally plane; in that case, the ERC reduces to the PWRC. For certain values of the product \(R(x') H(x')\), \(R^*(x')\) may become negative. Then the apparent source represents the focus of an apparent converging spherical wave, and the ERC becomes complex conjugate.

### PARAMETER SENSITIVITY STUDY AND 3D DIFFRACTION MODELING

#### Numerical study of effective reflection coefficients

As follows from the formalism discussed above, ERC represent the exact reflection response for a plane reflector and incident spherical P-wave. When the reflecting interface is curved, ERC provide a practical approximate tool to compute the reflected wavefield. Here, we study the ERC for an interface between isotropic and TI media as a function of the parameter \(L\), Thomsen anisotropy parameters of the reflecting halfspace, and the local interface geometry incorporated into the distance \(R^*\). If the reflected wavefield is well-described by geometrical seismics, ERC reduces to the corresponding PWRC. Therefore, the difference between the effective and plane-wave reflection coefficients helps to estimate the error of the geometrical-seismics approximation.

#### Influence of the parameter \(L\)

First, we examine the dependence of ERC computed for a plane interface on the parameter \(L = \omega R^*/v_P^{(1)}\) (\(\omega\) is the angular frequency and \(R^*\) is the distance from the apparent source to point \(x'\) at the interface). Figure 2 shows comparison of the ERC for PP- and PS-waves computed for a wide range of \(L\) with the corresponding PWRC. For both modes, the difference between the ERC and PWRC decreases for larger values of \(L\) (i.e.,
Figure 2. Dependence of the magnitude of the (a) PP-wave and (b) PS-wave effective reflection coefficients (ERC) on the parameter $L$. The corresponding plane-wave reflection coefficients (PWRC) are shown for comparison. The reflector is a horizontal plane located 1 km below the source. The parameters of the incidence isotropic medium are $v_P^{(1)} = 2$ km/s, $v_S^{(1)} = 1.2$ km/s, and $\rho^{(1)} = 2.15$ g/cm$^3$; for the reflecting TI medium, $v_P^{(2)} = 2.4$ km/s, $v_S^{(2)} = 1.4$ km/s, $\rho^{(2)} = 2.35$ g/cm$^3$, $\varepsilon = 0.2$, and $\delta = 0.1$.

Figure 3. Dependence of the PP-wave ERC on the anisotropy parameters. (a) $\delta = 0.1$ and $\varepsilon = 0, 0.1, 0.2$; (b) $\varepsilon = 0.2$ and $\delta = -0.1, 0.1, 0.3$. The interface is a horizontal plane located 1 km below the source. The medium parameters are $v_P^{(1)} = 2$ km/s, $v_S^{(1)} = 1.2$ km/s, $\rho^{(1)} = 2.15$ g/cm$^3$, $v_P^{(2)} = 2.4$ km/s, $v_S^{(2)} = 1.4$ km/s, and $\rho^{(2)} = 2.35$ g/cm$^3$; the frequency $f = 32$ Hz.

For larger frequency or distance $R^*$), however, in contrast to PWRC, ERC oscillate in the post-critical domain even for $L = 10^3$ due to the interference of the reflected and head waves.

For the relatively small $L = 10$, the ERC (especially the one for PS-waves) substantially deviate from the PWRC even at sub-critical incidence angles. This means that for low values of $L$ geometrical seismics can be used only for near-vertical incidence (i.e., small source-receiver offsets). Indeed, it is well known that the accuracy of the geometrical-seismics approximation strongly depends on the source-interface distance normalized by the predominant wavelength (Tsvankin, 1995). If the source (in our case, the apparent source) is located close to the interface, the reflected wavefield is influenced by the curvature of the incident wavefront and cannot be accurately described by geometrical seismics.

Influence of the anisotropy parameters

The anisotropy parameters $\varepsilon$ and $\delta$ contribute to the ERC for the PP- and PS-waves mostly at near- and post-critical incidence angles (Figures 3 and 4). The critical angle is controlled by the horizontal P-wave velocity in the TI medium that depends on $\varepsilon$ ($v_P^{(2)}(90) = v_P^{(2)}(1)\sqrt{1 + 2\varepsilon}$). Figures 3a and 4a confirm that the critical angle decreases for larger values of $\varepsilon$, which causes a horizontal shift of the ERC curves. Also, the PS-
Effective reflection coefficients

Figure 4. Dependence of the PS-wave ERC on the anisotropy parameters for the model from Figure 3.

The PS-wave ERC in the post-critical domain substantially increases with $\varepsilon$. In general, the reflectivity of PS-waves is more sensitive to the anisotropy parameters than that of PP-waves, likely because shear-wave signatures are controlled primarily by the relatively large parameter $\sigma$ ($\sigma = \left[ \frac{v_{PS}^{(2)}}{v_{SO}^{(2)}} \right]^2 (\varepsilon - \delta)$). The magnitude of $\sigma$ typically is much larger than that of $\varepsilon$ and $\delta$; in our model, $\sigma$ varies from -2.94 to 2.94.

Since ERC at post-critical incidence angles include the contributions of both the head and reflected waves, Figures 3 and 4 do not provide enough information to predict the influence of $\varepsilon$ and $\delta$ on the time-domain wavefield. The long-offset synthetic seismograms discussed below help to separate the head and reflected waves and evaluate their dependence on the anisotropy of the reflecting medium.

Figure 5. (a) Model with a curved reflector. (b) The corresponding apparent distance $R^*$. The source is placed at the surface, and an array of 101 receivers is located at a depth of 585 m with a step of 50 m. The reflector is described by the equation $x_3 = -1.185 + \Delta z \tanh \left[ 2\pi (x_1 - 0.75) \right]$; the parameter $\Delta z$ is marked on the plot.

Influence of the reflector shape

Here, we generate ERC for a curved interface that has a flexural shape governed by the parameter $\Delta z$ (Figure 5). When the reflector degenerates into a horizontal plane ($\Delta z = 0$), the apparent distance $R^*$ reduces to the actual source-reflector distance $R$, which has no singular points. The offset dependence of $R^*$ becomes more complicated with increasing reflector curvature (Figure 5b).

The ERC for both PP- and PS-waves are displayed in Figure 6 for three values of $\Delta z$. We observe a rapid change in both ERC near an offset of 0.75 km, where the distance $R^*$ exhibits sharp spikes associated with the flexural segment of the reflector. The offset of the post-critical reflection on the left side of the model increases with reflector depth, which is controlled by $\Delta z$. 
Tip-wave superposition method for elastic media

To model reflected wavefields for curved interfaces, we need to evaluate the surface integral 1. We obtain the high-frequency (or far-field) approximation of the integral using the tip-wave superposition method (TWSM) (Klem-Musatov and Aizenberg, 1985; Klem-Musatov et al., 1993, 1994). This published version of the method is designed for modeling 3D wavefields in layered models with complex interface geometries. The main assumption of the method is that the source-interface, receivers-interface and interface-interface distances obey the Rayleigh principle (i.e., they are on the order of several wavelengths or larger).

Because the upper halfspace in our model is isotropic, in Appendix F we extend TWSM to isotropic elastic media. We show that TWSM generates the reflection response by the superposition of tip-diffacted waves (which explains the name of the method) excited at the reflector in accordance with the Huygens principle.

Our implementation of TWSM involves splitting the reflector into rhombic elements that conform to the Chebyshev coordinates introduced earlier. Each element acts as a secondary source emitting a tip-wave beam towards the receiver array, and the beams form what we call the receiver matrix. We compute the boundary data using either the ERC or PWRC, and form the source matrix for all rhombic elements at the interface. Then the two matrices are multiplied element-by-element to generate the reflected wavefield and sum the reflection responses at each receiver. The superposition of the tip-wave beams in TWSM produces the correct reflection travel-times at the receivers, but the amplitudes may be somewhat distorted by the high-frequency approximations applied in the computation of the ERC and the surface integral.

The TWSM with PWRC is computationally inexpensive, but requires storing large matrices containing information about tip waves. Although the data storage may present a logistical problem, minor changes of the model can be incorporated without recalculating all tip-wave beams. This advantage of TWSM becomes particularly valuable for layered models and in survey design. Application of ERC in TWSM involves computation of the Fourier-Bessel integrals for the entire frequency range of the initial wavelet instead of the simple closed-form PWRC expressions. Also, the disk-space requirements become even more demanding because the tip-wave matrices have to be stored separately for each frequency.

Modeling results

As illustrated by the numerical tests above, effective reflection coefficients are sensitive to the elastic parameters and the shape of the interface. Here, we combine ERC with the tip-wave superposition method to generate the time-domain wavefield and analyze its behavior for two different reflector shapes.

Influence of the anisotropy parameters

The seismograms in Figures 7–10 are computed for a curved reflector described by the function \( x_3 = -1 + 0.3 \exp(-8x_1^2 - 8x_2^2) \). The reflection traveltimes of both PP- and PS-waves exhibit a wide triplication (cusp) at the far offsets, which corresponds to the caustic produced at the antinodal part of the Gaussian-cap reflector.

In agreement with the ERC in Figure 3a, the PP-wave reflection amplitude at long offsets rapidly increases with \( \varepsilon \) (Figure 7). The amplitude at the largest offset (2.5 km) is approximately four times higher for
Figure 7. Influence of $\varepsilon$ on the vertical displacement of the PP-wave reflected from a curved interface. The source and an array of 101 receivers are placed at the surface. The reflector is described by $x_3 = -1 + 0.3\exp(-8x_1^2 - 8x_2^2)$, so that the cap of the Gaussian anticline is located at a depth of 0.7 km below the source. The medium parameters are $v_P^{(1)} = 2$ km/s, $v_S^{(1)} = 1.2$ km/s, $\mu^{(1)} = 2.15$ g/cm$^3$, $v_P^{(2)} = 2.4$ km/s, $v_S^{(2)} = 1.4$ km/s, and $\rho^{(2)} = 2.35$ g/cm$^3$; the values of $\varepsilon$ and $\delta$ are marked on the plots.

Figure 8. Influence of $\delta$ on the vertical PP-wave displacement for the model from Figure 7.
Figure 9. Influence of $\varepsilon$ on the vertical PS-wave displacement for the model from Figure 7.

Figure 10. Influence of $\delta$ on the vertical PS-wave displacement for the model from Figure 7.
Effective reflection coefficients

$\varepsilon = 0.2$ than for $\varepsilon = 0$. In contrast, the near-offset reflections are weakly sensitive to $\varepsilon$. The influence of $\delta$ on PP-wave amplitudes is most visible at moderate offsets between 1.5 km and 1.7 km (Figure 8). For the maximum offset, the amplitude becomes about 20% higher when $\delta$ increases by 0.2.

The PS wavefield for a range of $\varepsilon$ and $\delta$ values is shown in Figures 9 and 10. The influence of both anisotropy parameters on the reflected wave can be generally predicted from the corresponding ERC in Figure 4. In particular, the moderate- and far-offset reflection amplitudes noticeably increase with $\varepsilon$. The amplitude at the largest offset doubles when $\varepsilon$ changes from zero to 0.2. It is interesting that the amplitude of the PPS head wave (marked with an arrow for the rightmost receiver) for the same change in $\varepsilon$ decreases by only 12%. Although the influence of $\delta$ is less pronounced, a 0.2 increase in $\delta$ reduces the maximum-offset amplitude of the reflected PS-wave by about 30%. The head-wave amplitude, however, is practically independent of $\delta$.

**Influence of the reflector shape**

Synthetic PP-wave seismograms computed for a flexural reflector with variable mean curvature (Figure 5) is displayed in Figure 11. The isotropic 2D version of this model has been used for testing finite-difference modeling and generalized ray tracing (Hanyga and Helle, 1995). As the value of $\Delta z$ increases, the flexure produces a strong caustic loop formed near zero offset. The head waves cannot be clearly identified due to the limited length of the receiver array, which extends only up to the interference zone of the reflected and head waves.

For a plane reflector ($\Delta z = 0$), we compared our modeling results with the exact wavefield computed by the reflectivity method. As expected, the elastic version of TWSM based on the superposition of tip-wave beams accurately reproduces traveltimes for the whole offset range. The amplitudes in Figure 11 are only a few percents higher than those produced by the reflectivity algorithm.

To evaluate the errors of the conventional Kirchhoff modeling technique, we also computed the wavefield using the plane-wave reflection coefficient in TWSM (Figure 12). The discontinuous slope of the PWRC at the critical angles causes artificial diffractions for both plane ($\Delta z = 0$) and curved reflectors. Additionally, the reflection amplitudes for near-critical and post-critical offsets are higher than those obtained with the ERC in Figure 11.

Similar conclusions can be drawn from the PS-wave seismograms for the same model in Figures 13 and 14. The PS reflection also exhibits a caustic loop that becomes more prominent for $\Delta z = 0.2$ km. The critical offset for the converted (PPS) head wave is smaller than that for the corresponding PPP-wave, which explains the separation of the head wave (marked with an ar-
Figure 12. Vertical PP-wave displacement computed with the PWRC for the model from Figure 11.

Figure 13. Vertical PS-wave displacement computed with the ERC for the model from Figure 11.
Effective reflection coefficients (ERC) provide a practical tool for modeling near- and post-critical reflected wavefields and for taking the interface curvature into account. By extending a formalism suggested previously for the acoustic problem, we gave a complete analytic description of ERC for curved reflectors in anisotropic media. The reflected wavefield can be expressed through a generalized plane-wave decomposition, which includes the local spatial spectrum of the incident wave expressed through an integral over the whole interface.

Although this decomposition gives an accurate wavefield representation near a reflector of arbitrary shape, its computational cost for 3D anisotropic models is prohibitive. Therefore, we suggested to approximately obtain the reflected wavefield from the conventional Weyl-type integral computed for an “apparent” source location, which depends on the incidence angle and the mean reflector curvature. Then the ratio of the reflected and incident wavefields yields the spatially varying ERC along the reflector. To incorporate ERC in 3D diffraction modeling, we employed the tip-wave superposition method (TWSM) generalized for elastic wave propagation. The superposition of the tip-wave beams corresponding to rhombic interface segments produces correct reflection traveltimes, while the accuracy of amplitudes depends on the validity of the high-frequency approximation used both in TWSM and in the computation of ERC. TWSM is also capable of modeling multipathing and caustics produced by curved segments of the reflector.

We implemented this formalism and studied the properties of ERC for an interface separating isotropic and TI media. The symmetry axis in the reflecting TI halfspace was assumed to be orthogonal to the reflector, which is typical for anisotropic shale layers. For the special case of a plane interface, the ERC represents the frequency-dependent exact wavefield governed by the velocity and density contrasts, Thomsen anisotropy parameters, and source-receiver geometry. Numerical tests show that the ERC for PP-waves at post-critical incidence angles is particularly sensitive to the parameter \( \varepsilon \) responsible for near-horizontal P-wave propagation in the TI halfspace.

The ERC substantially deviates from the corresponding plane-wave reflection coefficient (PWRC) in the post-critical domain, where the displacement field is influenced by the head wave. At low frequencies, the difference between the ERC and PWRC may be significant even for sub-critical incidence angles. These results confirm the limitations of the geometrical-seismics approximation, which is based on PWRC, in describing point-source radiation in layered media.

We also presented synthetic examples illustrating the importance of properly accounting for the reflector

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**Figure 14.** Vertical PS-wave displacement computed with the PWRC for the model from Figure 11.

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row for the left-most receiver) and reflected wave at the far offsets in Figure 13. Although the artificial diffractions caused by the PWRC in Figure 14 are not as pronounced as those for PP-waves, application of the ERC (Figure 13) yields a cleaner gather.

Our 3D modeling results obtained with TWSM agree well in the kinematic sense with the wavefields computed by finite differences and generalized ray tracing for the corresponding isotropic 2D model (Hanyga and Helle, 1995). The amplitudes, however, are not the same because of a different geometrical spreading in 2D and 3D and the influence of anisotropy in our model.

**DISCUSSION AND CONCLUSIONS**

Effective reflection coefficients (ERC) provide a practical tool for modeling near- and post-critical reflected wavefields and for taking the interface curvature into account. By extending a formalism suggested previously for the acoustic problem, we gave a complete analytic description of ERC for curved reflectors in anisotropic media. The reflected wavefield can be expressed through a generalized plane-wave decomposition, which includes the local spatial spectrum of the incident wave expressed through an integral over the whole interface.

Although this decomposition gives an accurate wavefield representation near a reflector of arbitrary shape, its computational cost for 3D anisotropic models is prohibitive. Therefore, we suggested to approximately obtain the reflected wavefield from the conventional Weyl-type integral computed for an “apparent” source location, which depends on the incidence angle and the mean reflector curvature. Then the ratio of the reflected and incident wavefields yields the spatially varying ERC along the reflector. To incorporate ERC in 3D diffraction modeling, we employed the tip-wave superposition method (TWSM) generalized for elastic wave propagation. The superposition of the tip-wave beams corresponding to rhombic interface segments produces correct reflection traveltimes, while the accuracy of amplitudes depends on the validity of the high-frequency approximation used both in TWSM and in the computation of ERC. TWSM is also capable of modeling multipathing and caustics produced by curved segments of the reflector.

We implemented this formalism and studied the properties of ERC for an interface separating isotropic and TI media. The symmetry axis in the reflecting TI halfspace was assumed to be orthogonal to the reflector, which is typical for anisotropic shale layers. For the special case of a plane interface, the ERC represents the frequency-dependent exact wavefield governed by the velocity and density contrasts, Thomsen anisotropy parameters, and source-receiver geometry. Numerical tests show that the ERC for PP-waves at post-critical incidence angles is particularly sensitive to the parameter \( \varepsilon \) responsible for near-horizontal P-wave propagation in the TI halfspace.

The ERC substantially deviates from the corresponding plane-wave reflection coefficient (PWRC) in the post-critical domain, where the displacement field is influenced by the head wave. At low frequencies, the difference between the ERC and PWRC may be significant even for sub-critical incidence angles. These results confirm the limitations of the geometrical-seismics approximation, which is based on PWRC, in describing point-source radiation in layered media.

We also presented synthetic examples illustrating the importance of properly accounting for the reflector
curvature in the computation of ERC. When the reflector is curved, the ERC may change rapidly along the interface in accordance with variations of the local interface shape, thus influencing synthetic modeling.

The methodology developed here can be used to generate accurate boundary data for 3D Kirchhoff-type modeling in anisotropic media. In particular, our synthetic examples confirm that ERC eliminate the artifacts produced by PWRC and provide more accurate amplitudes for large incidence angles and in the presence of significant reflector curvature. Our results can be also applied in anisotropic AVO analysis of long-offset PP and PS reflection data.

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APPENDIX A: GENERALIZED PLANE WAVES

The conventional plane-wave decomposition of point-source radiation (i.e., the Weyl integral) can be used to obtain the reflected or transmitted wavefield for a plane interface between two homogeneous media. Here, we define generalized plane waves, which help to extend the principle of plane-wave decomposition to interfaces of arbitrary shape and to account for local heterogeneity.

Let us consider wave propagation in a medium with a smooth curved interface S, which separates two heterogeneous, arbitrarily anisotropic halfspaces $D^{(1)}$ and $D^{(2)}$. Each medium (superscript $m$) is described by the stiffness tensor $C^{(m)}(\mathbf{x}) = \left[ c_{ijkl}^{(m)}(\mathbf{x}) \right]$ and density $\rho^{(m)}$; the unit vector $n$ normal to the interface points toward $D^{(1)}$.

We define the curvilinear coordinates $(s_1, s_2, s_3)$ in the immediate vicinity of the interface $S$ inside $D^{(m)}$, such that $(s_1, s_2)$ form the Chebyshev coordinate mesh along the interface, and the axis $s_3$ is normal to the interface and points inside $D^{(m)}$. Additionally, we define the local Cartesian coordinates $(y_1, y_2, y_3)$ with the origin at point $\mathbf{x}'$. The axis $y_3$ coincides with $s_3$, while $y_1$ and $y_2$ are tangential to the curves $s_1$ and $s_2$ at $\mathbf{x}'$.

In the vicinity of point $\mathbf{x}'$, the Chebyshev and local Cartesian coordinates are related as (Weatherborn, 1930; do Carmo, 1976; Klem-Musatov et al., 2004; Ayzenberg et al., 2007):

\[
s_1(y_1, y_2, y_3) = y_1 + O(y^3),
\]
\[
s_2(y_1, y_2, y_3) = y_2 + O(y^3),
\]
\[
s_3(y_1, y_2, y_3) = y_3 - \frac{1}{2} [C_1(\mathbf{x}')y_1^2 + C_2(\mathbf{x}')y_2^2] + O(y^3),
\]

where $C_1(\mathbf{x}')$ and $C_2(\mathbf{x}')$ are the local curvatures of the interface at $s_1$ and $s_2$. The local and global Cartesian coordinates are related by the linear transform:

\[
y_j(x_1, x_2, x_3) = b_{ij}(\mathbf{x}')x_j,
\]

where $b_{ij}(\mathbf{x}')$ are the elements of the linear transform matrix, which is specified, for example, in Červený (2001).

We introduce a generalized plane wave in the vicinity of the interface as

\[
\mathbf{u}^{(m)}(s_1, s_2, s_3) = a^{(m)} \left[ \mathbf{h}^{(m)} + iv^{(m)} \frac{\omega^2}{2} \right] e^{i\omega(p_1s_1 + p_2s_2 + p_3s_3)},
\]

where $p_1$ and $p_2$ can be treated as the components of the slowness vector tangential to the interface. The normal slowness $p_3$, amplitude factor $a^{(m)}$ and polarization vector $\mathbf{h}^{(m)}$ along with its perturbation $v^{(m)}$ have to be found. At the interface, where $s_3 = 0$ and the term proportional $s_3^2$ vanishes, equation A-3 describes a conventional plane wave (Červený, 2001).

The unknown parameters of the generalized plane wave can be determined by substituting equation A-3 for a point $\mathbf{x}'$ into the wave equation in the frequency domain (the “stationary” wave equation). First, we rewrite the stationary wave equation in the two-index notation $C^{(m)}_{jl}(\mathbf{x}') = \left[ c_{ijkl}^{(m)}(\mathbf{x}') \right]$ (Kennett, 1994):

\[
C^{(m)}_{jl}(\mathbf{x}') \frac{\partial^2 \mathbf{u}^{(m)}(\mathbf{x}')}{\partial x_j \partial x_l} + \frac{\partial C^{(m)}_{jl}(\mathbf{x}')}{\partial x_j} \frac{\partial \mathbf{u}^{(m)}(\mathbf{x}')}{\partial x_l} + \rho \omega^2 \mathbf{u}^{(m)}(\mathbf{x}') = 0.
\]

Substituting the generalized plane wave A-3 into equation A-4 and taking the coordinate transformations A-1 and A-2 into account yields

\[
-\omega^2 \left[ C_{ik}^{(m)}(\mathbf{x}') p_j p_k - \rho^{(m)} \mathbf{i} \right] \mathbf{h}^{(m)} - i \left[ \omega \mathbf{D}^{(m)}(\mathbf{x}') \mathbf{h}^{(m)} + C_{33}^{(m)}(\mathbf{x}') v^{(m)} \right] = 0,
\]

where $C_{ik}^{(m)}(\mathbf{x}') = b_{ij}(\mathbf{x}')b_{ik}(\mathbf{x}') C_{jl}^{(m)}(\mathbf{x}')$ is the local stiffness tensor, and $D^{(m)}(\mathbf{x}') = p_3 \left[ C_1(\mathbf{x}') C_{11}^{(m)}(\mathbf{x}') + C_2(\mathbf{x}') C_{22}^{(m)}(\mathbf{x}') \right] - p_l \frac{\partial C_{jl}^{(m)}(\mathbf{x}')}{\partial y_l}$ is the matrix that contains information about the local interface curvature. Both the real and imaginary parts of the left-hand side of equation A-5 have to go to zero. The real part of equation A-5 reduces to the well-known Christoffel equation (Červený, 2001)

\[
\left[ C_{ik}^{(m)}(\mathbf{x}') p_j p_k - \rho^{(m)} \mathbf{i} \right] \mathbf{h}^{(m)} = 0.
\]
The slowness components $p_{Q3}^{(m)}(p_1, p_2; x')$ of waves $Q = P, S_1$ and $S_2$ are obtained from the equation det $\left[ \tilde{C}_{ik}^{(m)}(x') \rho_{ik} - \rho^{(m)}i \right] = 0$. By substituting $p_{Q3}^{(m)}(p_1, p_2; x')$ into equation A-6, we find the mutually orthogonal unit polarization vectors $h_Q^{(m)}(x')$. Note that the slownesses $p_{Q3}^{(m)}(p_1, p_2; x')$ and polarization vectors $h_Q^{(m)}(x')$ are functions of the medium parameters at point $x'$, but do not depend on the local interface curvature.

The imaginary part of equation A-5 constrains the perturbation vectors:

$$v_Q^{(m)}(x') = \omega \left[ \tilde{C}_{Q3}^{(m)}(x') \right]^{-1} D^{(m)}(x') h_Q^{(m)}(x').$$

In the special case of a plane interface and homogeneous media, the derivatives $\frac{\partial \tilde{C}_{kl}^{(m)}(x')}{\partial y}$ and curvatures $C_1(x')$ and $C_2(x')$ are equal to zero. Then the term $D^{(m)}(x')$ and the perturbation $v_Q^{(m)}(x')$ vanish.

To solve the reflection/transmission problem, it is necessary to separate waves traveling towards the interface $(u_Q^{(m)+}(s_1, s_2, s_3))$ from those traveling away from it $(u_Q^{(m)}(s_1, s_2, s_3))$. Červený (2001; Aki and Richards, 2002). We assume that sorting is done according to the orientation of the group velocity vector. If the slownesses $p_{Q3}^{(m)-}$ and $p_{Q3}^{(m)+}$ correspond to waves traveling towards and away from the interface (respectively), the generalized plane wave $A-3$ can be represented as

$$u_Q^{(m)}(s_1, s_2, s_3; x') = a_Q^{(m)} \begin{bmatrix} h_Q^{(m)}(x') + iv_Q^{(m)}(x') \frac{\omega}{\gamma} \end{bmatrix} e^{i\omega(p_{1s1} + p_{2s2} + p_{Q3}^{(m)-}(s'x))}. \quad (A-8)$$

**APPENDIX B: GENERALIZED PLANE-WAVE DECOMPOSITION AT THE INTERFACE**

Here we introduce the generalized spectral integrals designed to decompose the displacement at the interface into the generalized plane $P$, $S_1$- and $S_2$-waves described in Appendix A. The total displacement inside $D^{(m)}$ can be expressed as the sum of the waves traveling towards and away from the interface (equation A-8):

$$u^{(m)}(s_1, s_2, s_3) = u^{(m)+}(s_1, s_2, s_3; x') + u^{(m)-}(s_1, s_2, s_3; x'), \quad (B-1)$$

with the displacements represented by the generalized plane-wave decomposition,

$$u^{(m)}(s_1, s_2, s_3; x') = \frac{\omega}{C_m} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \begin{bmatrix} H^{(m)}(x') + iV^{(m)}(x') \frac{\omega}{\gamma} \end{bmatrix} e^{i\omega(p_{1s1} + p_{2s2})} dp_1 dp_2. \quad (B-2)$$

Equation B-2 is a generalization of the conventional Weyl-type integral for curved interfaces and locally heterogeneous media. Whereas the Weyl-type decomposition is valid everywhere in the halfspace $D^{(m)}$, the generalized expression B-2 is restricted to an infinitely thin layer covering the interface. Therefore, our formalism can be used for calculation of the reflection response only in the immediate vicinity of the reflector.

The orthogonal polarization matrices $H^{(m)}$ are similar to those introduced by Červený (2001) in his equation 5.4.110,

$$H^{(m)}(x') = \begin{bmatrix} h_p^{(m)}(x'); & h_{S1}^{(m)}(x'); & h_{S2}^{(m)}(x') \end{bmatrix}. \quad (B-3)$$

The vectors $a^{(m)+} = (a_p^{(m)+}, a_{S1}^{(m)+}, a_{S2}^{(m)+})^T$ contain the unknown amplitudes of the generalized plane waves.

The generalized plane-wave decomposition B-2 is valid for interfaces of arbitrary shape in heterogeneous anisotropic media. If the interface is plane, the curvatures $C_1(x')$ and $C_2(x')$ go to zero, and the curvilinear coordinates $(s_1, s_2, s_3)$ coincide with the local Cartesian coordinate system. If, in addition, the medium near the interface is homogeneous, the normal components of the slownesses and polarization vectors do not depend on the reference point.
\( x' \). Then integral B-2 reduces to the well-known Weyl-type decomposition over conventional plane waves (Červený, 2001; Aki and Richards, 2002; Tsvankin, 1995, 2005).

At the interface \((s_3 = 0)\) equation B-2 reduces to the inverse Fourier integral,

\[
\mathbf{u}^{(m)\pm}(s_1, s_2, 0; x') = \frac{\omega^2}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} H^{(m)\pm} \mathbf{a}^{(m)\pm} e^{i\omega(p_1 s_1 + p_2 s_2)} dp_1 dp_2.
\]  

(B-3)

**APPENDIX C: REFLECTION AND TRANSMISSION OPERATORS IN ANISOTROPIC MEDIA**

The results of Appendix B make it possible to introduce the generalized plane-wave representation of the reflected wavefield at the interface. We assume that a point dislocation source is located in the upper halfspace and there are no sources in the lower halfspace \(D^{(2)}\). Then equations B-1 and B-3 can be written for \(D^{(1)}\) as

\[
\mathbf{u}^{(1)\pm}(s_1, s_2, 0) = \mathbf{u}^{(1)\pm}(s_1, s_2, 0; x') + \mathbf{u}^{(1)\pm}(s_1, s_2, 0; x'),
\]  

(C-1)

where \(\mathbf{u}^{(1)\pm}_0(s_1, s_2, 0; x')\) and \(\mathbf{u}^{(1)\pm}(s_1, s_2, 0; x')\) may be considered as the incident and reflected wavefields (respectively) at the interface. The reflected displacement \(\mathbf{u}^{(1)\pm}(s_1, s_2, 0; x')\) is represented by the generalized spectral integral,

\[
\mathbf{u}^{(1)\pm}(s_1, s_2, 0; x') = \frac{\omega^2}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathbf{H}^{(1)\pm} \mathbf{a}^{(1)\pm} e^{i\omega(p_1 s_1 + p_2 s_2)} dp_1 dp_2.
\]  

(C-2)

The amplitudes of the reflected \((\mathbf{a}^{(1)\pm})\) and incident \((\mathbf{a}^{(1)\pm})\) waves are related by the matrix \(\mathbf{R}(p; x')\) of the generalized plane-wave reflection and transmission coefficients:

\[
\mathbf{a}^{(1)\pm} = \mathbf{R}(p; x') \mathbf{a}^{(1)\pm},
\]  

(C-3)

where \(p = \sqrt{p_1^2 + p_2^2}\), and

\[
\mathbf{R}(p; x') = \begin{bmatrix} R_{PP} & R_{S_1P} & R_{S_2P} \\ R_{P_{S_1}} & R_{S_1S_1} & R_{S_2S_1} \\ R_{P_{S_2}} & R_{S_1S_2} & R_{S_2S_2} \end{bmatrix}.
\]  

(C-4)

The matrix C-4 coincides with the one introduced by Červený (2001), if the stiffness coefficients are fixed at location \(x'\), and the plane interface is tangential to the actual reflector at \(x'\).

Because the matrix \(\mathbf{H}^{(1)\pm}\) is orthogonal, it satisfies the equality \([\mathbf{H}^{(1)\pm}]^{-1} = [\mathbf{H}^{(1)\pm}]^T\). From equation B-3 it follows that \(\mathbf{u}^{(1)\pm}(p_1, p_2, 0; x') = \mathbf{H}^{(1)\pm} \mathbf{a}^{(1)\pm}\), which allows us to obtain the amplitude vector of the incident wave in the form

\[
\mathbf{a}^{(1)\pm} = [\mathbf{H}^{(1)\pm}]^T \mathbf{u}^{(1)\pm}(p_1, p_2, 0; x').
\]  

(C-5)

Taking into account equations C-3 and C-5, the reflected wavefield C-2 can be represented as

\[
\mathbf{u}^{(1)\pm}(s_1, s_2, 0; x') = \frac{\omega^2}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathbf{H}^{(1)\pm} \mathbf{R}(p; x') [\mathbf{H}^{(1)\pm}]^T \mathbf{u}^{(1)\pm}(p_1, p_2, 0; x') e^{i\omega(p_1 s_1 + p_2 s_2)} dp_1 dp_2,
\]  

(C-6)

where the spatial spectrum of the incident wavefield is expressed by the generalized Fourier integral over the curved interface:

\[
\mathbf{u}^{(1)\pm}(p_1, p_2, 0; x') = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathbf{u}^{(1)\pm}(s_1, s_2, 0; x') e^{-i\omega(p_1 s_1 + p_2 s_2)} ds_1 ds_2.
\]  

(C-7)
Effective reflection coefficients

For the incident spherical P-wave excited by a point source, \( u^{(1)-}(s_1, s_2, 0; \mathbf{x}') = u^{(1)-}_m(s_1, s_2, 0; \mathbf{x}') \). The polarization matrix \( \mathbf{H}^{(1)+} \) can be separated into the matrices for P- and S-waves:

\[
\mathbf{H}^{(1)+}(\mathbf{x}') = \mathbf{H}_P^{(1)+}(\mathbf{x}') + \mathbf{H}_S^{(1)+}(\mathbf{x}'),
\]

\[
\mathbf{H}_P^{(1)+}(\mathbf{x}') = [h_{pm}^{(m)+}(\mathbf{x}') \quad 0 \quad 0], \quad \mathbf{H}_S^{(1)+}(\mathbf{x}') = [0 \quad h_{S1}^{(m)+}(\mathbf{x}') \quad h_{S2}^{(m)+}(\mathbf{x}')] .
\]

The reflected wavefield \( C-6 \) can be decomposed into the displacements of PP-waves and split PS-waves. The spectral representation for PP-waves (\( Q=P \)) or converted PQ-waves (\( Q=S \)) at the interface is given by

\[
u_{PQ}^{(1)+}(s_1, s_2, 0; \mathbf{x}') = \frac{c^2}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} H_{PQ}(p; \mathbf{x}') H_{Q}^{(1)-}(p; x') H_{P}^{(1)+}(\mathbf{x}') \times u_{P}^{(1)-}(p_1, p_2, 0; \mathbf{x}') e^{i(\omega(p_1+p_2^2)dp_1 dp_2}.
\]

\[
\nu_{PQ,norm}^{(1)+}(s_1, s_2, 0; \mathbf{x}') = \frac{c^2}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} R_{PQ}(p; \mathbf{x}') H_{P}^{(1)+}(\mathbf{x}') H_{Q,norm}^{(1)-}(x') \times u_{P,norm}^{(1)}(p_1, p_2, 0; \mathbf{x}') e^{i(\omega(p_1+p_2^2)dp_1 dp_2}.
\]

For the two displacement components \( (j = 1, 2) \) tangential to the interface, we have

\[
u_{PQ,j}^{(1)+}(s_1, s_2, 0; \mathbf{x}') = \frac{c^2}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} R_{PQ}(p; \mathbf{x}') H_{P,j}^{(1)+}(\mathbf{x}') H_{Q}^{(1)-}(p; x') \times u_{P,j}^{(1)}(p_1, p_2, 0; \mathbf{x}') e^{i(\omega(p_1+p_2^2)dp_1 dp_2}.
\]

**APPENDIX D: PLANE-WAVE REFLECTION COEFFICIENTS FOR VTI MEDIA**

The symmetry axis of the reflecting TI medium in our model is assumed to be orthogonal to the interface. Therefore, the plane-wave reflection coefficients in equations C-10–C-12 coincide with those for a horizontal interface between isotropic and VTI media. Also, for purposes of computing the reflection coefficient, the slowness vectors of the incident, reflected, and transmitted waves can be confined to the \( (x_1, x_2) \)-plane. The vertical slowness components \( q^{(m)} \) are obtained from the eigenvalues of the Christoffel equation,

\[
\text{det} \left( c_{13}^{(m)} p^2 + c_{55}^{(m)} (q^{(m)})^2 - p^{(m)} \right) c_{33}^{(m)} (q^{(m)})^2 + c_{55}^{(m)} p^2 - p^{(m)} = 0 .
\]

The vertical slownesses of P- and SV-waves are given by

\[
q_P^{(m)} = \frac{1}{\sqrt{2}} \sqrt{\frac{K_1^{(m)}}{K_1^{(m)} + p^2 - \frac{c_{55}^{(m)}}{c_{33}^{(m)}}}}, \\
q_S^{(m)} = \frac{1}{\sqrt{2}} \sqrt{\frac{K_1^{(m)}}{K_1^{(m)} + p^2 - \frac{c_{55}^{(m)}}{c_{33}^{(m)}}}},
\]

where

\[
K_1^{(m)} = \frac{c_{33}^{(m)}}{c_{55}^{(m)}} p^2 + \frac{c_{13}^{(m)}}{c_{55}^{(m)}} - \frac{c_{13}^{(m)} + c_{55}^{(m)} p^2}{c_{33}^{(m)}}, \\
K_2^{(m)} = \frac{c_{33}^{(m)}}{c_{55}^{(m)}} p^2 - \frac{c_{33}^{(m)}}{c_{55}^{(m)}}, \\
K_3^{(m)} = p^2 - \frac{c_{55}^{(m)}}{c_{33}^{(m)}} .
\]

The eigenvectors of the Christoffel equation D-1 yield the directional cosines of the polarization vectors:
Then the plane-wave reflection coefficients \( R \) are normalized stiffnesses
\( 10^4 \). Ayzenberg et al.

The cofactors of the matrix \( M \) are

\[
\begin{align*}
M_{11} &= m_{22}(m_{33}m_{44} - m_{34}m_{43}) - m_{23}(m_{32}m_{44} - m_{34}m_{42}) + m_{24}(m_{32}m_{43} - m_{33}m_{42}) , \\
M_{12} &= -m_{12}(m_{32}m_{44} - m_{34}m_{42}) + m_{13}(m_{32}m_{44} - m_{34}m_{43}) - m_{14}(m_{32}m_{43} - m_{33}m_{42}), \\
M_{13} &= m_{12}(m_{32}m_{44} - m_{34}m_{43}) - m_{13}(m_{32}m_{44} - m_{34}m_{42}) + m_{14}(m_{32}m_{43} - m_{33}m_{42}), \\
M_{14} &= -m_{12}(m_{32}m_{43} - m_{34}m_{42}) + m_{13}(m_{32}m_{43} - m_{34}m_{43}) - m_{14}(m_{32}m_{43} - m_{33}m_{43}), \\
M_{21} &= -m_{21}(m_{32}m_{44} - m_{34}m_{42}) + m_{23}(m_{32}m_{44} - m_{34}m_{43}) - m_{24}(m_{32}m_{43} - m_{33}m_{42}), \\
M_{22} &= m_{21}(m_{32}m_{44} - m_{34}m_{43}) - m_{23}(m_{32}m_{44} - m_{34}m_{42}) + m_{24}(m_{32}m_{43} - m_{33}m_{42}), \\
M_{23} &= -m_{21}(m_{32}m_{43} - m_{34}m_{43}) + m_{23}(m_{32}m_{43} - m_{34}m_{42}) - m_{24}(m_{32}m_{43} - m_{33}m_{43}), \\
M_{24} &= m_{21}(m_{32}m_{43} - m_{34}m_{42}) - m_{23}(m_{32}m_{43} - m_{34}m_{43}) + m_{24}(m_{32}m_{43} - m_{33}m_{43}) , \\
M_{31} &= -m_{31}(m_{32}m_{44} - m_{34}m_{42}) + m_{33}(m_{32}m_{44} - m_{34}m_{43}) - m_{34}(m_{32}m_{43} - m_{33}m_{42}), \\
M_{32} &= m_{31}(m_{32}m_{44} - m_{34}m_{43}) - m_{33}(m_{32}m_{44} - m_{34}m_{42}) + m_{34}(m_{32}m_{43} - m_{33}m_{42}), \\
M_{33} &= -m_{31}(m_{32}m_{43} - m_{34}m_{43}) + m_{33}(m_{32}m_{43} - m_{34}m_{42}) - m_{34}(m_{32}m_{43} - m_{33}m_{43}), \\
M_{34} &= m_{31}(m_{32}m_{43} - m_{34}m_{42}) - m_{33}(m_{32}m_{43} - m_{34}m_{43}) + m_{34}(m_{32}m_{43} - m_{33}m_{43}) .
\end{align*}
\]

Then the plane-wave reflection coefficients \( R_{PP}(p) \) and \( R_{PS}(p) \) can be found as

\[
\begin{align*}
R_{PP}(p) &= \frac{-m_{11}M_{11} - m_{21}M_{12} + m_{31}M_{13} + m_{41}M_{14}}{m_{11}M_{11} + m_{12}M_{12} + m_{13}M_{13} + m_{14}M_{14}} , \\
R_{PS}(p) &= \frac{-m_{11}M_{12} - m_{21}M_{22} + m_{31}M_{32} + m_{41}M_{42}}{m_{11}M_{11} + m_{12}M_{12} + m_{13}M_{13} + m_{14}M_{14}} .
\end{align*}
\]

**APPENDIX E: EFFECTIVE REFLECTION COEFFICIENTS FOR CURVED INTERFACES**

For arbitrary interface geometry and heterogeneity, evaluation of integral 7 becomes complicated because it involves generating the curvilinear mesh \( (s_1, s_2) \) and applying it in the computation of the spectrum \( u_{m}(p, 0; x') \) by means of the Fourier transform 8. However, the integration in equation 7 is performed over the tangential slowness plane \( (p_1, p_2) \) and is not explicitly related to the geometry of the mesh \( (s_1, s_2) \). This fact can be used to represent these integrals in the form similar to equation 6:

\[
u_{PQ}(x') = \left[ \chi'_{PQ}(x') \mathbf{h}^+_P(x') + \epsilon_{PQ}(x') \mathbf{e}_Q(x') \right] \left[ u_0^{(1)}(x') \cdot \mathbf{h}_P(x') \right] ,
\]

where \( \chi'_{PQ}(x') \) are the effective reflection coefficients (ERC), \( \epsilon_{PQ}(x') \) are the “spurious” reflection coefficients, and \( \mathbf{e}_Q(x') \) are the unit vectors orthogonal to the polarization vectors \( \mathbf{h}_Q(x') \).

We define the effective and spurious reflection coefficients as
where \( H(\mathbf{x}') \) computed in the dominant-frequency approximation for an "apparent" source location and a plane interface tangential of the reflected wavefield. Spurious reflection coefficients represent diffraction corrections, which are much smaller in magnitude and can be neglected in equation E-1.

For acoustic wave propagation, integrals similar to those in equations C-11 and C-12 can be approximately computed in the dominant-frequency approximation for an "apparent" source location and a plane interface tangential to the actual reflector at point \( \mathbf{x}' \) (Ayzenberg et al., 2007). Then the problem reduces to the evaluation of Fourier-Bessel integrals similar to the ones for a plane interface. The same approach can be applied to elastic media because it is entirely based on the geometry of the incident P-wave. While the incidence angle \( \theta(\mathbf{x}') \) stays the same, the actual source moves along the ray to a new position located at the distance \( R'(\mathbf{x}') \) from the plane interface:

\[
R'(\mathbf{x}') = R(\mathbf{x}') \frac{2 - \sin^2 \theta(\mathbf{x}')} {2 - \sin^2 \theta(\mathbf{x}') - 2R(\mathbf{x}')H(\mathbf{x}') \cos \theta(\mathbf{x}')},
\]

where \( H(\mathbf{x}') \) is the mean curvature of the interface. If the reflector is locally plane and \( H(\mathbf{x}') = 0 \), the distance \( R'(\mathbf{x}') \) reduces to \( R(\mathbf{x}') \).

Adapting the results by Ayzenberg et al. (2007) for scalar integrals similar to 7, we replace the actual incident P-wave \( \mathbf{u}_{P}^{inc}(s_1, s_2, 0; \mathbf{x}') \) in equation 8 by an apparent spherical wave \( \mathbf{u}_{P}^{app}(s_1, s_2, 0; \mathbf{x}') \) and assume that the mesh \( (s_1, s_2) \) belongs to the plane tangential to the actual reflector at point \( \mathbf{x}' \). Then the ERC in equation E-2 becomes

\[
\chi_{PQ}(\mathbf{x}') \simeq \chi_{PQ}(\theta(\mathbf{x}'), L(\mathbf{x}')) = \frac{\mathbf{u}_{P}^{app}(\mathbf{x}') \cdot \mathbf{h}_{Q}^{+}(\mathbf{x}')} {\mathbf{u}_{P}^{inc}(\mathbf{x}') \cdot \mathbf{h}_{P}^{+}(\mathbf{x}')},
\]

where \( L(\mathbf{x}') = \omega R'(\mathbf{x}')/v_{P}^{(1)} \) is a dimensionless frequency-dependent parameter. In contrast to integral 8, equation E-5 does not involve integration over the curvilinear mesh. For each point \( \mathbf{x}' \) at the curved reflector, the displacement \( \mathbf{u}_{P}^{app}(\mathbf{x}') \) is given by the conventional Weyl-type integral, while \( \mathbf{u}_{P}^{inc}(\mathbf{x}') \) describes the apparent incident P-wave in the plane tangential to the reflector at point \( \mathbf{x}' \).

Neglecting the term containing \( \epsilon_{PQ}(\mathbf{x}') \), we rewrite equation E-1 as

\[
\mathbf{u}_{P}^{inc}(\mathbf{x}') \simeq \chi_{PQ}(\theta(\mathbf{x}'), L(\mathbf{x}')) \left[ \mathbf{u}_{P}^{inc}(\mathbf{x}') \cdot \mathbf{h}_{P}^{+}(\mathbf{x}') \right] \mathbf{h}_{Q}^{+}(\mathbf{x}').
\]

The apparent incident P-wave is described by

\[
\mathbf{u}_{P}^{app}(s_1, s_2, s_3; \mathbf{x}') = \text{grad} e^{ik_{P} R^{*}} = \left( ik_{P} - \frac{1}{R^{*}} \right) \frac{e^{ik_{P} R^{*}}}{R^{*}} \left( x_{1}^{s_{1} - s_{1}}^{*}, x_{2}^{s_{2} - s_{2}}^{*}, x_{3}^{s_{3} - s_{3}}^{*} \right)^{T},
\]

where \( x_{i}^{s_{i}} = (x_{i}^{s_{i}}, x_{2}^{s_{i}}, x_{3}^{s_{i}}) \) are the apparent source coordinates in the global Cartesian system, \( R^{*} = \sqrt{l^{2} + r^{2}} \), \( l = |x_{3}^{s_{3}} - s_{3}| \), and \( r = \sqrt{(x_{1}^{s_{1}} - s_{1})^{2} + (x_{2}^{s_{2}} - s_{2})^{2}} \). Hereafter in this appendix, \((s_1, s_2)\) are the local Cartesian coordinates in the plane tangential to the actual reflector at point \( \mathbf{x}' \). Note that the product \( \mathbf{u}_{P}^{inc}(\mathbf{x}') \cdot \mathbf{h}_{P}^{+}(\mathbf{x}') \) from E-5 is

\[
\mathbf{u}_{P}^{inc}(s_1, s_2, s_3; \mathbf{x}') \cdot \mathbf{h}_{P}^{+}(\mathbf{x}') = \left( ik_{P} - \frac{1}{R^{*}} \right) e^{ik_{P} R^{*}} R^{*}.
\]

The plane-wave decomposition of the displacement of the apparent incident P-wave has the form (Aki and Richards, 2002)
\[
\mathbf{u}_P^*(s_1, s_2, s_3; \mathbf{x}') = \text{grad} \left[ \frac{\omega}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{i\omega p(1)} e^{i\omega(p_1 s_1 + p_2 s_2)} dp_1 dp_2 \right].
\] (E-9)

Interchanging the order of differentiation and integration and setting \(s_3 = 0\), we obtain:

\[
\mathbf{u}_P^*(p_1, p_2, 0; \mathbf{x}') = -\omega e^{i\omega p(1)} \left( p_1, p_2, -p_{P, 3}^{(1)} \right)^T.
\] (E-10)

Thus, the unit polarization vectors of the incident P-wave (\(\mathbf{h}_{P, 1}^{(1)}\)) and reflected PP-wave (\(\mathbf{h}_{P, 2}^{(1)}\)) are given by

\[
\mathbf{h}_{P, 1}^{(1)} = v_P^{(1)} \left( p_1, p_2, -p_{P, 3}^{(1)} \right)^T = v_P^{(1)} \left( p \cos \psi, p \sin \psi, -p_{P, 3}^{(1)} \right)^T,
\]

\[
\mathbf{h}_{P, 2}^{(1)} = v_P^{(1)} \left( p \cos \psi, p \sin \psi, -p_{P, 3}^{(1)} \right)^T,
\]

where \(\psi\) is the polar angle in the plane \((p_1, p_2)\). It is straightforward to show that the polarization of the converted PS-wave is

\[
\mathbf{h}_{S, 1}^{(1)} = v_S^{(1)} \left( p_{S, 1} \cos \psi, p_{S, 2} \sin \psi, -p \right)^T.
\]

Hence, for the PP-wave, \(h_{P, \text{norm}}^{+}/h_{P, \text{norm}}^{-} = -1\) and \(h_{P, \text{tan}}^{+}/h_{P, \text{tan}}^{-} = 1\). For the PS-wave, \(h_{S, \text{norm}}^{+}/h_{P, \text{norm}}^{-} = (v_{S, 1}^{(1)}/v_{P, 1}^{(1)})\) and \(h_{S, \text{tan}}^{+}/h_{P, \text{tan}}^{-} = (v_{S, 2}^{(1)}/v_{P, 3}^{(1)})\).

Using equations E-7 and 7, we find the normal to the interface component of the displacement vector of the reflected PQ-mode:

\[
u_{PQ, \text{norm}}^*(s_1, s_2, 0; \mathbf{x}') = \frac{2}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} R_{PQ}(p; \mathbf{x}') \frac{h_{Q, \text{norm}}^{(1)}(\mathbf{x}')}{h_{P, \text{norm}}^{(1)}(\mathbf{x}')} e^{i\omega p(1)} e^{i\omega(p_1 s_1 + p_2 s_2)} dp_1 dp_2,
\] (E-11)

In the polar coordinates \((p, \psi)\) and \((r, \varphi)\), equation E-11 reduces to the Fourier-Bessel integral:

\[
u_{PQ, \text{norm}}^*(s_1, s_2, 0; \mathbf{x}') = \frac{2}{2\pi} \omega^2 \int_{0}^{+\infty} R_{PQ}(p; \mathbf{x}') \frac{h_{Q, \text{norm}}^{(1)}(\mathbf{x}')}{h_{P, \text{norm}}^{(1)}(\mathbf{x}')} e^{i\omega p(1)} J_0(r \omega p) dp,
\] (E-12)

where \(J_0\) is the zero-order Bessel function,

\[J_0(r \omega p) = \frac{1}{2\pi} \int_{0}^{2\pi} e^{i\varphi} \cos(\psi - \varphi) d\psi.\]

As follows from equation 7, the two tangential displacement components of the reflected PQ-wave are:

\[
u_{PQ, \text{tan}}^*(s_1, s_2, 0; \mathbf{x}') = -\frac{2}{2\pi} \omega^2 \int_{0}^{+\infty} \int_{0}^{+\infty} R_{PQ}(p; \mathbf{x}') \frac{h_{Q, \text{norm}}^{(1)}(\mathbf{x}')}{h_{P, \text{norm}}^{(1)}(\mathbf{x}')} e^{i\omega p(1)} e^{i\omega(p_1 s_1 + p_2 s_2)} dp_1 dp_2.
\] (E-13)

In the polar coordinates \((r, \varphi)\),

\[
u_{PQ, \text{tan}}^*(\mathbf{x}') = u_{PQ, 1}(\mathbf{x}') \cos \varphi + u_{PQ, 2}(\mathbf{x}') \sin \varphi,
\]

and
Effective reflection coefficients

\[ u_{PQ, \tan}(s_1, s_2, 0; \mathbf{x}') = -\omega^2 \int_{-\infty}^{+\infty} f_{PQ}(p; \mathbf{x}') \frac{h_{P,j}^{(1)+}(\mathbf{x}')}{h_{P,j}^{(1)}(\mathbf{x})} \times \frac{1}{p_{P3}} p \cos(\psi - \varphi) \exp(2i\gamma p_{11} + 2i\gamma p_{22}) dp_1 dp_2. \]  

(E-14)

Equation E-14 can also be reduced to the Fourier-Bessel integral:

\[ u_{PQ, \tan}(s_1, s_2, 0; \mathbf{x}') = -\omega^2 \int_{0}^{+\infty} R_{PQ}(p; \mathbf{x}') \frac{h_{P,j}^{(1)+}(\mathbf{x}')}{h_{P,j}^{(1)}(\mathbf{x})} \frac{ie^{i\gamma p_{11}}}{p_{P3}} J_1(p \omega p) dp, \]  

where \( J_1 \) is the first-order Bessel function:

\[ J_1(p \omega p) = -\frac{i}{2\pi} \int_{0}^{2\pi} \cos(\psi - \varphi) e^{ir \omega p \cos(\psi - \varphi)} d\psi. \]

The normal and tangential to the reflector components of the polarization vectors can be written as \( h_{P,\text{norm}}^{(1)+} = \cos \theta(\mathbf{x}'), h_{P,\tan}^{(1)+} = \sin \theta(\mathbf{x}'), h_{S,\text{norm}}^{(1)+} = -\sin \theta_S(\mathbf{x}'), \) and \( h_{S,\tan}^{(1)+} = \cos \theta_S(\mathbf{x}'), \) where \( \theta(\mathbf{x}') \) is the P-wave incidence angle and \( \theta_S(\mathbf{x}') \) is the S-wave reflection angle determined from Snell’s law as \( \theta_S(\mathbf{x}') = \sin^{-1} \left( \frac{v_S^{(1)}/v_P^{(1)}}{v_P^{(1)/v_P^{(1)}}} \sin \theta(\mathbf{x}') \right). \)

Finally, substitution of the Fourier-Bessel integrals E-12 and E-15 and the polarization components into the definition E-5 of the ERC yields

\[ \chi_{PP}(\theta(\mathbf{x}'), L(\mathbf{x}')) = \frac{u_{P,\text{norm}}(\mathbf{x}')}{u_{P,\tan}(\mathbf{x}')} \sin \theta(\mathbf{x}'), \]

\[ \chi_{PS}(\theta(\mathbf{x}'), L(\mathbf{x}')) = -\frac{u_{S,\text{norm}}(\mathbf{x}')}{u_{S,\tan}(\mathbf{x}')} \cos \theta_S(\mathbf{x}'). \]  

(E-16)

APPENDIX F: TIP-WAVE SUPERPOSITION METHOD FOR ISOTROPIC ELASTIC MEDIA

Here, we generalize the tip-wave superposition method (TWSM) for elastic media to obtain the PP- and PS-wavefields reflected from a curved interface. If the medium is homogeneous, it is possible to avoid evaluation of the traction vector \( \mathbf{t}(\mathbf{x}') \) and traction tensor \( \mathbf{T}(\mathbf{x}', \mathbf{x}) \) in the conventional wavefield representation 1 (Morse and Feshbach, 1953).

We start by rewriting integral 1 in a form similar to equation 20 of Pao and Varatharajulu (1976):

\[ \mathbf{u}(\mathbf{x}) = \rho^{(1)} V_P^{(1)} \int \int \mathbf{G}(\mathbf{u} \cdot \mathbf{n}') - (\mathbf{G} \cdot \mathbf{n}')(\mathbf{\nabla}' \cdot \mathbf{u}) \]  

\[ - \rho^{(1)} V_S^{(1)} \int \int \mathbf{G}\nabla \mathbf{u} \cdot (\mathbf{\nabla}' \mathbf{G}) - (\mathbf{n}' \times \mathbf{\nabla}' \mathbf{u}) \cdot \mathbf{G} \]  

\[ dS(\mathbf{x}'). \]  

(F-1)

The reflected displacement field can be separated into the PP- and PS-modes (Ben-Menahem and Singh, 1998):

\[ \mathbf{u}(\mathbf{x}) = \mathbf{u}_{PP}(\mathbf{x}) + \mathbf{u}_{PS}(\mathbf{x}), \]  

(F-2)

which satisfy the equations

\[ \left[ V_P^{(1)} \right]^2 \nabla \cdot \mathbf{u}_{PP}(\mathbf{x}) + \omega^2 \mathbf{u}_{PP}(\mathbf{x}) = 0, \]  

(F-3)

\[ \left[ V_S^{(1)} \right]^2 \nabla \times \mathbf{u}_{PS}(\mathbf{x}) + \omega^2 \mathbf{u}_{PS}(\mathbf{x}) = 0. \]

Likewise, the Green’s displacement tensor can be split into the P- and S-wave components:

\[ \mathbf{G}(\mathbf{x}', \mathbf{x}) = \mathbf{G}_P(\mathbf{x}', \mathbf{x}) + \mathbf{G}_S(\mathbf{x}', \mathbf{x}), \]  

(F-4)

where
Substituting equations F-9 and F-10 into equation F-7 yield

\[ G_P(x', x) = \frac{1}{\rho^{(1)} \omega^2} \nabla g_P(x', x) \nabla', \]

\[ G_S(x', x) = \frac{1}{\rho^{(1)} \omega^2} \nabla \times [g_S(x', x) \mathbf{I}] \times \nabla' \]

\[ = \frac{1}{\rho^{(1)} \omega^2} \left[ \frac{\omega^2}{c_0^2} g_S(x', x) \mathbf{I} - \nabla g_S(x', x) \nabla' \right], \tag{F-5} \]

and

\[ g_Q(x', x) = \frac{e^{i \omega R/c_0^2}}{4\pi R}, \quad R = |x - x'|, \quad Q = P, S. \tag{F-6} \]

Substituting equations F-2 and F-4 in F-1 and dropping the zero-value surface integrals, we obtain the reflected PP-wavefield as

\[ u_{PP}(x) = \rho^{(1)} \left[ v_P^{(1)} \right]^2 \int_{S} \left[ (\nabla' \cdot G_P) (u_{PP} \cdot n') - (G_P \cdot n')(\nabla' \cdot u_{PP}) \right] dS'. \tag{F-7} \]

For the PS-wavefield,

\[ u_{PS}(x) = -\rho^{(1)} \left[ v_S^{(1)} \right]^2 \int_{S} \left( u_{PS} \times n' \right) \cdot (\nabla' \times G_S) - (n' \times \nabla' \times u_{PS}) \cdot G_S \right] dS'. \tag{F-8} \]

Next, we rewrite the terms involving \( G_P \) in equation F-7:

\[ \nabla' \cdot G_P = -\nabla \cdot G_P = -\frac{1}{\rho^{(1)} \omega^2} \Delta g_P \nabla' \]

\[ = \frac{1}{\rho^{(1)} \left[ v_P^{(1)} \right]^2} g_P \nabla' = \frac{1}{\rho^{(1)} \left[ v_P^{(1)} \right]^2} \nabla g_P, \tag{F-9} \]

and

\[ G_P \cdot n' = \frac{1}{\rho^{(1)} \omega^2} \nabla g_P \nabla' \cdot n' = \frac{1}{\rho^{(1)} \left[ v_P^{(1)} \right]^2} \nabla [ (n' \cdot \nabla') g_P ]. \tag{F-10} \]

Substituting equations F-9 and F-10 into equation F-7 yields

\[ u_{PP}(x) = \nabla \int \int_{S} \left[ \frac{\partial g_P(x', x)}{\partial n'} \right] d_{1,PP}(x') - g_P(x', x) d_{2,PP}(x') \right] dS(x'), \tag{F-11} \]

where

\[ d_{1,PP}(x') = -\left[ \frac{v_P^{(1)}}{\omega^2} \right]^2 \nabla' \cdot u_{PP}(x'), \quad d_{2,PP}(x') = u_{PP}(x') \cdot n'. \tag{F-12} \]

The parameters \( d_{1,PP} \) and \( d_{2,PP} \) can be expressed through the incident wavefield and effective reflection coefficient \( \chi_{PP} \) using approximation E-6:

\[ d_{1,PP}(x') \simeq -\left[ \frac{v_P^{(1)}}{\omega^2} \right]^2 \chi_{PP}(x') \nabla' \cdot \left[ [u_P^{inc}(x') \cdot h_P(x')] h_P(x') \right] \]

\[ \simeq \chi_{PP}(x') g_P^{inc}(x', x), \tag{F-13} \]

\[ d_{2,PP}(x') \simeq \chi_{PP}(x') [u_P^{inc}(x') \cdot h_P(x')] [h_P(x') \cdot n'], \]

where \( u_P^{inc}(x') = \nabla' g_P^{inc}(x', x) \).
Because the surface integral in equation F-11 coincides with the acoustic surface integral 7 analyzed in Ayzenberg et al. (2007), we can use their methodology (the tip-wave superposition method, or TWSM) to split the reflector into small rhombic elements. To extend TWSM to elastic media, we represent the PP-wavefield \( F_{PP} \) in a form similar to equations 11 and 12 from Ayzenberg et al. (2007), where

\[
\mathbf{u}_{PP}(\mathbf{x}) \simeq \sum_j \frac{i \omega}{v_{PS}^2} I_{P[j]}(\mathbf{x}) \Delta B_{PP[j]}(\mathbf{x}),
\]

where the index \( j \) corresponds to a surface element, \( I_{P[j]}(\mathbf{x}) = \nabla g_P(\mathbf{x}, \mathbf{x}) / |\nabla g_P(\mathbf{x}, \mathbf{x})| \), and \( \Delta B_{PP[j]}(\mathbf{x}) \) is the scalar contribution of the \( j \)-th element:

\[
\Delta B_{PP[j]}(\mathbf{x}) = \int_{\Delta \Pi[j]} \left[ \frac{\partial g_P(x', x)}{\partial n'} d_1,PP(x') - g_P(x', x) d_2,PP(x') \right] dS'.
\]

To develop a similar expression for the PS-wavefield \( F_{PS} \), we rewrite the terms involving \( G_S \):

\[
\nabla' \times G_S = -\nabla \times G_S
\]

\[
= -\nabla \times \left[ \frac{1}{\mu^{(1)} v_{PS}^2} \left[ \frac{\omega^2}{v_{PS}^2} g_S \mathbf{I} - \nabla g_S \nabla' \right] \right]
\]

\[
= -\frac{1}{\mu^{(1)} v_{PS}^2} \mathbf{I} \times \nabla g_S,
\]

\[
-\rho^{(1)} \left[ \frac{v_{PS}^2}{\mu^{(1)}} \right]^2 (\mathbf{u}_{PS} \times \mathbf{n}') \cdot (\nabla' \times G_S) = (\mathbf{u}_{PS} \times \mathbf{n}') \cdot (\mathbf{I} \times \nabla g_S)
\]

\[
= [(\mathbf{u}_{PS} \times \mathbf{n}') \cdot \mathbf{I}] \times \nabla g_S = (\mathbf{u}_{PS} \times \mathbf{n'}) \times \nabla g_S
\]

\[
= -\nabla g_S \times (\mathbf{u}_{PS} \times \mathbf{n'}) = \nabla \times [-g_S(\mathbf{u}_{PS} \times \mathbf{n'})],
\]

and

\[
\rho^{(1)} \left[ \frac{v_{PS}^2}{\mu^{(1)}} \right]^2 (\mathbf{n} \times \nabla' \times \mathbf{u}_{PS}) \cdot G_S = \rho^{(1)} \left[ \frac{v_{PS}^2}{\mu^{(1)}} \right]^2 [G_S \times \mathbf{n}] \cdot [\nabla' \times \mathbf{u}_{PS}]
\]

\[
= \nabla \times \left[ \frac{\partial g_S(x', x)}{\partial n'} d_{1,PS}(x') \right].
\]

Substituting equations F-16 and F-17 into equation F-8, we find:

\[
\mathbf{u}_{PS}(\mathbf{x}) = \nabla \times \int_S \left[ \frac{\partial g_S(x', x)}{\partial n'} d_{1,PS}(x') - g_S(x', x) d_{2,PS}(x') \right] dS(x'),
\]

where

\[
d_{1,PS}(x') = \frac{\left[ \frac{v_{PS}^2}{\mu^{(1)}} \right]^2}{\omega^2} [\nabla' \times \mathbf{u}_{PS}(x')], \quad d_{2,PS}(x') = \mathbf{u}_{PS}(x') \times \mathbf{n}'.
\]

Approximation E-6 allows us to express the boundary data through the ERC \( \chi_{PS} \) for PS-waves:

\[
d_{1,PS}(x') \simeq -\frac{\left[ \frac{v_{PS}^2}{\mu^{(1)}} \right]^2}{\omega^2} \chi_{PS}(x') \nabla' \times \left[ [\mathbf{u}_{PP}^{inc}(x') \cdot \mathbf{h}_P(x')] \mathbf{h}_S^+ (x') \right],
\]

\[
d_{2,PS}(x') \simeq \chi_{PS}(x') \left[ \mathbf{u}_{PP}^{inc}(x') \cdot \mathbf{h}_P(x') \right] \mathbf{h}_S^+ (x') \times \mathbf{n}'.
\]
The vector surface integral in equation F-18 is similar to the acoustic integral 7 in Ayzenberg et al. (2007), but the boundary values $d_{1,PS}(x')$ and $d_{2,PS}(x')$ become vectors. Therefore we can adapt equations 11 and 12 from Ayzenberg et al. (2007) to obtain the following TWSM representation of the PS-wavefield F-18:

$$u_{PS}(x) \simeq \sum_j \frac{i\omega}{v_S} l_{S[j]}(x) \times \Delta B_{PS[j]}(x), \quad (F-21)$$

where $l_{S[j]}(x) = \nabla g_S(x'_j, x)/|\nabla g_S(x'_j, x)|$, and $\Delta B_{PS[j]}(x)$ is the vector contribution of the $j$-th surface element:

$$\Delta B_{PS[j]}(x) = \int \int_{\Delta \Omega[j]} \left[ \frac{\partial g_S(x', x)}{\partial n'} d_{1,PS}(x') - g_S(x', x) d_{2,PP}(x') \right] dS'. \quad (F-22)$$

To evaluate integrals F-15 and F-22, we use the far-field approximation 16 of Ayzenberg et al. (2007).
Reflection coefficients in attenuative anisotropic media

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ABSTRACT
Reservoir rocks such as heavy oils are characterized by significant attenuation and, in some cases, attenuation anisotropy. Most existing attenuation studies are focused on plane-wave attenuation coefficients, which determine the amplitude decay along the raypath of seismic waves. Here, we discuss the influence of attenuation on PP- and PS-wave reflection coefficients for anisotropic media with main emphasis on models with VTI (transversely isotropic with a vertical symmetry axis) symmetry. Concise analytic solutions obtained by linearizing the exact plane-wave reflection coefficients are verified by numerical modeling. To make a substantial contribution to reflection coefficients, attenuation has to be strong, with the quality factor $Q$ not exceeding 10. For such highly attenuative media, it is also necessary to take attenuation anisotropy into account if the magnitude of the Thomsen-style attenuation-anisotropy parameters is relatively large. In general, the linearized reflection coefficients in attenuative media include velocity-anisotropy parameters but have almost “isotropic” dependence on attenuation. Our formalism also helps to evaluate the influence of the inhomogeneity angle (the angle between the real and imaginary parts of the slowness vector) on the reflection coefficients. A nonzero inhomogeneity angle of the incident wave introduces additional terms into the PP- and PS-wave reflection coefficients, making conventional AVO (amplitude-variation-with-offset) analysis inadequate for strongly attenuative media. It is interesting that an incident P-wave with a nonzero inhomogeneity angle generates a mode-converted PS-wave at normal incidence, even if both halfspaces have a horizontal symmetry plane. This phenomenon can provide an alternative explanation for substantial PS-wave energy at zero offset observed on field data. The linearized solutions developed here can be used in AVO inversion for highly attenuative (e.g., gas-sand and heavy-oil) reservoirs.

Key words: Attenuation anisotropy, quality factor, AVO, reflection coefficient, inhomogeneity angle

1 INTRODUCTION

Traditionally AVO analysis has been carried out assuming the medium to be purely elastic. It is common knowledge, however, that subsurface formations are attenuative. Direct measurements using vertical seismic profiling (VSP) (Hauge, 1981; Hedlin et al., 2001), well logs (Schmitt, 1999), and rock samples (Behura et al., 2007; Winkler & Nur, 1982) show that attenuation and velocity dispersion can be significant, especially within hydrocarbon-saturated zones. Therefore, some failures of conventional AVO analysis can be attributed to the influence of attenuation (Luh, 1988; Samec et al., 1990).

Physical-modeling experiments (Hosten et al., 1987; Maultzsch et al., 2003; Zhu et al., 2007), rock-physics studies (Behura et al., 2006; Prasad & Nur, 2003; Tao & King, 1990), and analysis of field data (Liu et al., 1993; Lynn et al., 1999; Vasconcelos & Jenner, 2005) show that attenuation can be directionally dependent and attenuation anisotropy often is more significant than velocity anisotropy (Arts & Rasolofosaon, 1992; Hosten et al., 1987; Zhu et al., 2007). Therefore, it is important to evaluate the influence of attenuation
and attenuation anisotropy on plane-wave reflection coefficients.

Reflection coefficients in isotropic as well as anisotropic attenuative media have been analyzed both analytically and using numerical modeling. For example, Krebes (1983) and Ursin & Stovas (2002) derive closed-form expressions for reflection/transmission coefficients in isotropic attenuative media, while Nechtschein & Hron (1997) and Hearn & Krebes (1990) study them numerically. For attenuative VTI media, numerical analysis of reflection/transmission coefficients are presented by Sidler & Carcione (2007). Existing results for anisotropic media, however, do not provide physical insight into the dependence of reflection/transmission coefficients on the medium properties, in particular on the anisotropy parameters.

Here, using the Thomsen-style notation introduced by Zhu & Tsvankin (2006), we develop linearized approximations for PP- and PS-wave reflection coefficients in VTI media. In particular, our analytic solutions help to evaluate the influence of the inhomogeneity angle (the angle between the real and the imaginary parts of the slowness vector) of the incident wave on the reflection coefficients. Then we compute exact reflection coefficients for a realistic range of the anisotropy parameters and assess the accuracy of the linearized expressions.

\section{2 Perturbation Analysis of Reflection/Transmission Coefficients}

For a welded contact between two arbitrary anisotropic attenuative halfspaces, the boundary conditions of the continuity of traction and displacement result in the following system of linear equations (e.g., Vavrycuk & Pšenčík, 1998):

\[ \tilde{\mathbf{C}} \cdot \tilde{\mathbf{U}} = \tilde{\mathbf{B}}, \]

where the accent ` denotes a complex quantity, \( \tilde{\mathbf{C}} \) corresponds to the displacement-stress matrix for the reflected and transmitted plane waves \( P, S_1 \) and \( S_2, \tilde{\mathbf{B}} \) is the displacement-stress vector of the incident wave, and \( \tilde{\mathbf{U}} \) is the vector of the reflection (\( R \)) and transmission (\( T \)) coefficients. \( \tilde{\mathbf{C}}, \tilde{\mathbf{U}}, \) and \( \tilde{\mathbf{B}} \) are complex quantities because the stiffness tensor in attenuative media is complex. Exact reflection/transmission coefficients (\( \tilde{\mathbf{U}} \)) can be computed by solving the system of equations 1 numerically.

Following Vavrycuk & Pšenčík (1998) and Jílek (2002a,b), we apply the first-order perturbation theory to a homogeneous attenuative isotropic background medium. Linearization of the boundary conditions (equation 1) yields the perturbation \( \delta \tilde{\mathbf{U}} \) in the form

\[ \delta \tilde{\mathbf{U}} = (\tilde{\mathbf{C}}^0)^{-1} (\delta \tilde{\mathbf{B}} - \delta \tilde{\mathbf{C}} \cdot \tilde{\mathbf{U}}^0). \]

Here \( \tilde{\mathbf{C}}^0 \) is the displacement-stress matrix for the reflected/transmitted waves in the background medium and \( \delta \tilde{\mathbf{C}} \) represents the perturbation of \( \tilde{\mathbf{C}}^0 \). Similarly, \( \delta \tilde{\mathbf{B}} \) is the perturbation of the displacement-stress vector of the incident wave. \( \tilde{\mathbf{U}}^0 \), the reflection/transmission vector in the homogeneous background, is given by

\[ \tilde{\mathbf{U}}^0 = [0, 0, 0, 0, 0]^T. \]

A similar perturbation approach is adopted by Ursin & Stovas (2002) to derive reflection/transmission coefficients for isotropic attenuative media. (Their formalism introduces a weak contrast in parameters across the interface while keeping the perturbed upper and lower halfspaces isotropic.) The change in the slownesses and polarizations of the scattered waves, required for obtaining \( \delta \tilde{\mathbf{C}} \) and \( \delta \tilde{\mathbf{B}} \) in equation 2, can be computed by perturbing the isotropic background medium (Jech & Pšenčík, 1989). We extend their method, developed for purely elastic media, to attenuative models by taking attenuation in the background into account.

The density-normalized complex stiffness tensor of the perturbed medium \( a_{ijkl} \) can be written as

\[ a_{ijkl} = a_{ijkl}^0 + \delta a_{ijkl}, \]

where \( a_{ijkl}^0 \) corresponds to the background medium and \( \delta a_{ijkl} \) is the perturbation. The tensor \( \delta a_{ijkl} \) is responsible for both the velocity and attenuation anisotropy of the perturbed medium. The background tensor \( a_{ijkl}^0 \) is complex:

\[ a_{ijkl}^0 = a_{ijkl}^0 + i a_{ijkl}^I, \]

where \( a_{ijkl}^0 \) and \( a_{ijkl}^I \) are the real and imaginary parts of \( a_{ijkl}^0 \), respectively.

The quality-factor (\( Q \)) matrix (in the two-index Voigt notation) is defined as (e.g., Carcione, 2007)

\[ Q_{ij} = \frac{a_{ij}}{a_{ij}^0}. \]

For an isotropic medium, the \( Q \)-matrix takes the form

\[ Q = \begin{bmatrix} Q_{P0} & Q_{13} & Q_{13} & 0 & 0 & 0 \\
Q_{13} & Q_{P0} & Q_{13} & 0 & 0 & 0 \\
Q_{13} & Q_{13} & Q_{P0} & 0 & 0 & 0 \\
0 & 0 & 0 & Q_{S0} & 0 & 0 \\
0 & 0 & 0 & 0 & Q_{S0} & 0 \\
0 & 0 & 0 & 0 & 0 & Q_{S0} \end{bmatrix}. \]

where \( Q_{P0} \) and \( Q_{S0} \) control the P- and the S-wave attenuation, respectively, and \( Q_{13} \) is the following function of \( Q_{P0} \) and \( Q_{S0} \) (Zhu & Tsvankin, 2006):

\[ Q_{13} = Q_{P0} - \frac{2a_{33} - 2a_{35} - 2a_{53}}{a_{33} - 2a_{35}}. \]

The Christoffel equation, which describes plane-wave propagation, can be written as

\[ (\tilde{a}_{ijkl} \tilde{k}^2 n_i n_l - \omega \delta_{jk}) \tilde{g}_i = 0; \]

\( n \) is the unit slowness vector, \( \tilde{k} \) is the wave vector, \( \omega \) is the frequency, and \( \tilde{g} \) is the polarization vector. In
Reflection coefficients in attenuative anisotropic media

\[ \tilde{k} = k - i k', \quad (10) \]

where \( k \) controls the velocity and \( k' \) the attenuation in the direction \( \mathbf{n} \). The ratio of \( k' \) to \( k \) yields the normalized attenuation coefficient \( A \), which defines the rate of amplitude decay per wavelength (Zhu & Tsvankin, 2006):

\[ A = k' / k. \quad (11) \]

When attenuation is weak (\( 1/Q \ll 1 \)) and isotropic, \( A \approx 1 / 2Q \). \( (12) \)

In general, the real (\( k \)) and imaginary (\( k' \)) parts of the wave vector can have different orientations, and the angle \( \xi \) between them is usually called the inhomogeneity angle (Figure 1b). For \( \xi = 0 \), the phase direction coincides with the direction of maximum attenuation (Figure 1a), which corresponds to so called “homogeneous wave propagation.”

The perturbations of the wave (\( \delta \tilde{k} \)) and polarization (\( \delta \tilde{g} \)) vectors, obtained by substituting the perturbed tensor \( \tilde{a}_{ijkl} \) (equation 4) into the Christoffel equation 9, are used in equation 2 to derive \( \delta \tilde{U} \) (Jech & Pšenčík, 1989; Vavrycuk & Pšenčík, 1998; Jílek, 2002b). Note that the vector \( \mathbf{n} \) in equation 9 is assumed to be real, so the perturbation analysis used here for computing \( \delta \tilde{k} \) and \( \delta \tilde{g} \) is strictly valid only for plane waves with zero inhomogeneity angle. Nevertheless, our results are applicable for small inhomogeneity angles (\( < 30^\circ \)) even if attenuation is strong (\( < 10 \)), as will be shown later.

The complex P- and S-wave velocities (\( \tilde{V}_{P0} \) and \( \tilde{V}_{S0} \)) in the background attenuative isotropic medium have the form

\[ \tilde{V}_{P0} = \frac{\omega}{k_{P0}} \approx V_{P0} (1 + i A_{P0}), \quad (13) \]

\[ \tilde{V}_{S0} = \frac{\omega}{k_{S0}} \approx V_{S0} (1 + i A_{S0}), \quad (14) \]

where \( V_{P0} \) and \( V_{S0} \) are the phase velocities and \( A_{P0} \) and \( A_{S0} \) are the normalized attenuation coefficients of P- and S-waves, respectively. In equations 13 and 14, terms of the second and higher order in \( 1/Q \) are neglected.

3 INCIDENT P-WAVE WITH ZERO INHOMOGENEITY ANGLE

If the inhomogeneity angle is set to zero, all terms in equation 2 coincide with those given in Vavrycuk & Pšenčík (1998) and Jílek (2002a,b) for non-attenuative media, but they become complex quantities. Hence, the linearized reflection coefficients for P-waves (Vavrycuk & Pšenčík, 1998) and PS-waves (Jílek, 2002a) can be adapted in a straightforward way for attenuative media.

\[ R_{PP}^{H} = \frac{\Delta \rho}{2 \rho_{0}} + \frac{\Delta \tilde{\sigma}_{33}}{4 V_{P0}^{2}} \]

\[ + \left( \frac{\Delta \tilde{\sigma}_{13}}{2 V_{P0}^{2}} - \frac{\Delta \tilde{\sigma}_{33}}{4 V_{P0}^{2}} \frac{\Delta \tilde{\sigma}_{55}}{V_{P0}^{2}} - \frac{2 V_{S0}^{2} \Delta \rho}{V_{P0}^{2} \rho_{0}} \right) \sin^{2} \theta \]

\[ + \frac{\Delta \tilde{\sigma}_{11}}{4 V_{P0}^{2}} \sin^{4} \theta, \quad (15) \]

Figure 1. Incident plane wave with (a) zero inhomogeneity angle and (b) nonzero inhomogeneity angle \( \xi \). \( \tilde{k} \) and \( \tilde{k}' \) are the real and imaginary components (respectively) of the wave vector, and \( \theta \) is the incidence phase angle.

3.1 PP-wave reflection coefficient

3.1.1 Arbitrarily anisotropic media

The linearized PP-wave reflection coefficient in arbitrarily anisotropic media obtained from equation 2 for an incident wave with zero inhomogeneity angle is given by
where the superscript "H" ("homogeneous") denotes an incident wave with $Q = 0\degree$, $\Delta$ is the contrast in a certain parameter across the interface, $\tilde{\rho}$ is the density-normalized complex stiffness coefficients in Voigt notation (i.e., the stiffness matrix), and $\theta$ the incidence angle (i.e., the angle between the slowness vector of the incident wave and the interface normal; Figure 1a). Equation 15 is derived under the assumption that the contrasts in the medium properties across the interface are small: $|\Delta a_{ijkl}| \ll |\tilde{a}_{ijkl}|$, $|\Delta \rho| \ll \rho$.

The linearized reflection coefficient in equation 15 reduces to that in purely elastic media if all parameters are made real. Thus, linearized reflection coefficients for attenuative media can be derived from those for purely elastic media by simply making the stiffnesses complex. Although equation 15 is strictly valid only for zero inhomogeneity angle (for all waves), it remains sufficiently accurate for an arbitrary inhomogeneity angle, unless the medium is strongly attenuative (as shown below).

3.1.2 VTI media

Next, we analyze equation 15 for the special case of attenuative VTI media. It is convenient to express the reflection coefficients in terms of the velocity-anisotropy and attenuation-anisotropy parameters using Thomsen-style notation. Here, we use the anisotropy parameters $A_{P0}$, $A_{S0}$, $\epsilon_Q$, $\delta_Q$, and $\gamma_Q$ for attenuative TI media introduced by Zhu & Tsvankin (2006). $A_{P0}$ and $A_{S0}$ are the normalized symmetry-direction attenuation coefficients of P- and S-waves, respectively, $\epsilon_Q$ and $\delta_Q$ control the angular variation of the P- and SV-wave attenuation coefficients, and $\gamma_Q$ governs SH-wave attenuation anisotropy.

To simplify equation 15, it is convenient to assume that terms proportional to $1/Q^2$ (but not to $1/Q$) are small. If we retain terms such as $\Delta V_{P0}/(V_P Q_{P0})$ and $\Delta A_{P0}/Q_{P0}$ but drop those proportional to $1/Q_{P0}$ and $1/Q_{S0}$, equation 15 takes the form

$$R_{PP}^H = R_{PP}^H(0) + \frac{\Delta \rho}{2 \rho_0} + \frac{\Delta V_{P0}}{2 V_{P0}} + \frac{\Delta A_{P0}}{2} (i + \frac{1}{Q_{P0}}),$$

$$C_{PP}^H = \frac{-2 \Delta \rho}{g^2} + \frac{\Delta V_{P0}}{2 V_{P0}} - 4 \frac{A_{P0}}{g V_{S0}} + \frac{\Delta \delta_Q}{2}$$

$$+ i \left( \frac{\Delta A_{P0}}{2} - \frac{4 \Delta A_{S0}}{g^2} \right) + \frac{i}{Q_{P0}} \left( \frac{2 \Delta \rho}{g^2} \right) + \frac{4 \Delta V_{S0}}{g V_{S0}} - \frac{i}{2} \frac{\Delta A_{P0}}{Q_{S0} g^2} + \frac{4 i}{g^2} A_{S0} + \frac{\Delta \delta_Q}{4} \right),$$

and

$$G_{PP}^H = \frac{\Delta V_{P0}}{2 V_{P0}} + \frac{\Delta \epsilon_Q}{2} + \frac{1}{Q_{P0}} \left( \frac{\Delta A_{P0}}{2} + \frac{i}{4} \Delta \gamma_Q \right);$$

$g \equiv V_{P0}/V_{S0}$. The contribution of the terms scaled by $1/Q_{P0}$ and $1/Q_{S0}$ in equations 17–19 is of the second order, unless attenuation is uncommonly strong.

Eliminating the influence of attenuation on $R_{PP}^H(0)$, $G_{PP}^H$, and $C_{PP}^H$ in equations 17–19 reduces them to well-known expressions for the PP-wave intercept, gradient, and curvature (respectively) for purely elastic VTI media (R"uger, 2002). Since the attenuation coefficient $A \sim 1/2Q$, it is clear from equations 17–19 that the influence of attenuation on the reflection coefficient is comparable to that of the velocity and density contrasts only if the medium is strongly anelastic (i.e., $Q_{P0}, Q_{S0} < 10$). This conclusion is confirmed by the
test in Figure 3 with the model parameters simulating an interface between shale and oil sand (the shale is non-attenuative). When the oil sand is moderately attenuative \( Q_{P,2} = Q_{S,2} = 50 \), the reflection coefficient is almost identical to those in the elastic case. Even a \( Q \)-value of 10 does not significantly change the reflection coefficient. However, when the attenuation is extremely strong \( Q_{P,2} = Q_{S,2} = 2.5 \) or 5), the reflection coefficient substantially deviates from that for the purely elastic model.

Note that \( \Delta A_{P,0} \) in equation 17 is responsible for the influence of attenuation on the normal-incidence reflection coefficient. In fact, the “isotropic” parameter \( \Delta A_{P,0} \) makes a more significant contribution to \( G_{PP}^H \) and \( C_{PP}^H \) than do \( \epsilon_Q \) and \( \delta_Q \), because the attenuation-anisotropy parameters in equations 18 and 19 are scaled by \( 1/Q_{P,0} \).

In purely elastic anisotropic media, the linearized AVO gradient is sensitive to the velocity-anisotropy parameters (e.g., to the coefficient \( \delta \) for P-waves in VTI media; see equation 18). The AVO gradient in equation 18, however, is weakly dependent on the attenuation-anisotropy parameters, unless attenuation is uncommonly strong. Although the parameter \( \delta_Q \) governs the P-wave attenuation near the symmetry axis, its influence on \( G_{PP}^H \) is less significant than that of \( \delta \) because \( \Delta \delta_Q \) is scaled by \( 1/Q_{P,0} \). Similarly, the contribution of \( \Delta \epsilon_Q \) to \( C_{PP}^H \) (equation 19) is smaller than that of \( \Delta \epsilon \). Because the influence of the parameter \( \epsilon_Q \) increases with the incidence angle, it does not contribute to the AVO gradient.* Therefore, the reflection coefficient in media with \( Q > 10 \) is more influenced by velocity anisotropy than by attenuation anisotropy.

This conclusion is confirmed by Figure 4 where the model is similar to that in Figure 3, but the oil sand exhibits attenuation anisotropy. When attenuation is weak \( Q = 50 \), the AVO gradient barely varies with the attenuation-anisotropy parameter \( \delta_Q \). However, as the magnitude of attenuation increases \( Q \leq 10 \), the influence of attenuation anisotropy becomes pronounced. Indeed, strong attenuation can even change the sign of the AVO gradient. Our results confirm the common view that moderate attenuation does not substantially distort reflection coefficients. For highly attenuative media

\*The parameters \( \gamma \) and \( \gamma_Q \) only control the anisotropy of SH-waves, which are decoupled from P- and SV-waves analyzed here.

### Table 1.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Fig. 2</th>
<th>Fig. 3</th>
<th>Fig. 4</th>
<th>Figs. 5 &amp; 6</th>
<th>Fig. 8</th>
<th>Fig. 9</th>
</tr>
</thead>
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Medium parameters used in the numerical tests. For all models, the symmetry-direction velocities \( V_{P0} \) and \( V_{S0} \) are in km/s and density \( \rho \) is in gm/cm\(^3\).
where the coefficients $B_{PS}^H$ and $K_{PS}^H$ (the gradient and curvature terms, respectively, in conventional PS-wave AVO analysis) are given by

$$B_{PS}^H = -\frac{2 + g \Delta \rho}{2g \rho_0} - \frac{2 \Delta V_{SS}}{g V_{SS}} + \frac{g}{2(1 + g)} \Delta \delta$$

$$-i \frac{2}{g} \Delta A_{SO} - i \frac{Q}{Q_{SO}} f_1 - i \frac{Q}{Q_{SO}} f_2,$$  \hspace{1em} (21)

$$K_{PS}^H = (3 + 2g) \frac{\Delta \rho}{4g^2 \rho_0} + 2 + \frac{g}{V_{SS}} \Delta V_{SO}$$

$$+ \frac{1 - 4g}{2(1 + g)} \Delta \delta + \frac{g}{1 + g} \Delta \epsilon$$

$$+ i \frac{2 + g}{g^2} \Delta A_{SO} = \frac{i}{2Q_{PS} f_3 + i}{2Q_{SS}} f_4.$$  \hspace{1em} (22)

Here, $f_1$, $f_2$, $f_3$, and $f_4$ are linear combinations of the parameter contrasts across the interface listed in Appendix A. The contribution of $f_{1,3,4}$ to the reflection coefficient is of the second order because they are scaled by $1/Q_{PS}$ or $1/Q_{SO}$.

The real part of the reflection coefficient in equation 20 coincides with the corresponding linearized expression for PS-waves in a purely elastic VTI medium. Most conclusions drawn above for $P$-waves remain valid for the wave in the reflecting halfspace. The model is similar to that in Figure 3, but the attenuation in the oil sand is anisotropic (Table 1).

with $Q < 10$, however, it is necessary to take not just attenuation, but also attenuation anisotropy into account.

### 4 INCIDENT P-WAVE WITH A NONZERO INHOMOGENEITY ANGLE

If the upper halfspace is attenuative, the incident $P$-wave can have a nonzero inhomogeneity angle (Figure 1b). This situation may be typical, for example, for the bottom of an attenuative reservoir. Since $\xi$ is determined by the medium properties along the whole raypath, the imaginary part $k'$ of the wave vector of the incident wave may even deviate from the incidence plane. However, for simplicity, we assume that both the real and imaginary parts of the wave vector are confined to the incidence plane.

For a nonzero inhomogeneity angle, the Christoffel equation becomes

$$(\tilde{a}_{ijkl} \tilde{k}_i \tilde{k}_j - \omega \delta_{jk}) \tilde{g}_j = 0.$$  \hspace{1em} (23)

Here, the real ($k$) and imaginary ($k'$) parts of $\tilde{K}$ point in different directions (Figure 1b).

Although the perturbation analysis of Jech & Pšenčík (1989) is not strictly valid for equation 23, it
Figure 5: Magnitude of the exact (solid lines) and approximate (dashed lines, equation 25) PP-wave reflection coefficient as a VTI/isotropic interface for different inhomogeneity angles. The P- and S-wave attenuation coefficients in the vertical (symmetry-axis) direction are identical and do not change across the interface ($Q = Q_{P0,1} = Q_{P0,2} = Q_{S0,1} = Q_{S0,2}$). (a) $\xi = 0^\circ, Q = 25$, (b) $\xi = 0^\circ, Q = 5$, (c) $\xi = 0^\circ, Q = 2.5$, (d) $\xi = 10^\circ, Q = 25$, (e) $\xi = 10^\circ, Q = 5$, (f) $\xi = 10^\circ, Q = 2.5$, (g) $\xi = 25^\circ, Q = 25$, (h) $\xi = 25^\circ, Q = 5$, and (i) $\xi = 25^\circ, Q = 2.5$. The other model parameters are listed in Table 1.
remains sufficiently accurate for small values of $\xi$, especially when attenuation is not strong (Zhu & Tsvankin, 2006). Therefore, the formulation of Vavrycuk & Psenčík (1998) and Jílek (2002a,b) can be applied in a straightforward way to linearize the reflection coefficients for an incident wave with nonzero $\xi$. The numerical results below confirm that this approach yields an accurate approximation for reflection coefficients for most plausible attenuative models.

4.1 PP-wave reflection coefficient

The linearized PP-wave reflection coefficient for arbitrarily anisotropic media depends on the following parameters:

\[ R_{\text{PP}} = f_0(\rho^0, V_{P0}, V_{S0}, \Delta a_{11}, \Delta a_{13}, \Delta a_{15}) \]

\[ \Delta a_{33}, \Delta a_{35}, \Delta a_{55}, \theta, \xi, \]...

where $f_0$ is a linear function. Due to the complicated form of $f_0$, it is not shown here explicitly. The reflection coefficient in equation 24 depends on three additional stiffness contrasts ($\Delta a_{11}, \Delta a_{13}, \Delta a_{35}$) compared to that for $\xi = 0^\circ$ (equation 15).

Dropping cubic and higher-order terms in $\sin \theta$ and $\sin \xi$, we simplify the perturbation result 24 to

\[ R_{\text{PP}} = R_{\text{PP}}^0 + B_{\text{PP}} \sin \theta + G_{\text{PP}} \sin^2 \theta, \]

where

\[ R_{\text{PP}}^0 = R_{\text{PP}}^0(0) + \frac{\sin^2 \xi}{Q_{P0}} f_5, \]

\[ B_{\text{PP}} = -i \frac{\sin \xi}{Q_{P0}} f_6, \]

\[ G_{\text{PP}} = G_{\text{PP}}^0 + i \frac{\sin^2 \xi}{8Q_{P0}} f_7. \]

Here, $R_{\text{PP}}^0(0)$ and $G_{\text{PP}}^0$ are the solutions for $\xi = 0^\circ$ (superscript “H”) given by equations 17 and 18, respectively, and $f_5, f_6,$ and $f_7$ are linear functions listed in Appendix A.

As illustrated by Figure 5, equation 25 remains accurate for the inhomogeneity angle as large as $25^\circ$. Even for $Q = 2.5$ and $\xi = 25^\circ$, (Figure 5i), approximation 25 deviates from the exact reflection coefficient by less than 10%.

In contrast to the conventional AVO equation for non-converted waves, which represents an even function of $\theta$ (e.g., equation 16), equation 25 includes the $\sin \theta$-term. Therefore, if attenuation is strong and the inhomogeneity angle $\xi$ is non-negligible, the basic equation of conventional PP-wave AVO analysis breaks down, which may have significant implications for AVO inversion and interpretation.

However, since the inhomogeneity angle $\xi$ is associated with the terms $f_5, f_6,$ and $f_7$, which are scaled by

Figure 6. Magnitude of the exact PP-wave reflection coefficient at a VTI/isotropic interface for different inhomogeneity angles. As in Figure 5, $Q = Q_{P0,1} = Q_{P0,2} = Q_{S0,1} = Q_{S0,2}$; (a) $Q = 50$, (b) $Q = 5$, and (c) $Q = 2.5$. The other model parameters are listed in Table 1.
Reflection coefficients in attenuative anisotropic media

Figure 7. PP-wave reflection coefficient may become asymmetric with respect to \( \theta = 0^\circ \) for a nonzero inhomogeneity angle \( \xi \). As before, \( k \) and \( k' \) are the real and imaginary parts, respectively, of the wave vector of the incident P-wave.

As \( 1/Q \), its influence becomes pronounced only in strongly attenuative media. As illustrated in Figure 6, the variation of the inhomogeneity angle from 0\(^\circ\) to 50\(^\circ\) does not significantly change the reflection coefficient even for \( Q = 5 \) (Figure 6b). Only when \( Q = 2.5 \) and the inhomogeneity angle exceeds 30\(^\circ\), its contribution to the reflection coefficient (in particular, to the term \( B_{PP}^{IH} \)) becomes substantial (Figure 6c).

The asymmetry of the reflection coefficient with respect to \( \theta = 0^\circ \) (Figures 6b and 6c), which increases with the inhomogeneity angle, is explained in Figure 7. In our modeling, the inhomogeneity angle of the incident wave is fixed, which implies that the imaginary part \( k' \) of the wave vector makes different angles with the vertical for the incidence angles \( \theta \) and \( -\theta \). As a result, the reflection coefficient for positive and negative incidence angles is not the same.

In reality, it is unlikely for the incident wave to have a constant inhomogeneity angle for a wide range of \( \theta \). A more plausible scenario is depicted in Figure 8a. The model includes an attenuative reservoir beneath a purely elastic cap rock. Because the cap rock is purely non-attenuative, the wave incident upon the reservoir has a real wave vector. According to Snell’s law, the horizontal slowness (and the horizontal component of the wave vector) has to be preserved during reflection and transmission. Therefore, the vector \( k' \) (the imaginary part of \( k \)) in the reservoir cannot have a horizontal component, and the inhomogeneity angle of the transmitted wave is equal to the transmission angle \( \theta_T \) (Figure 8a). For the reflection from the bottom of the reservoir, \( \theta_T \) becomes the incidence angle. Therefore, the wave vectors \( k \) and \( k' \) for the angles \( \theta_T \) and \( -\theta_T \) are symmetric with respect to the reflector normal, and the PP-wave reflection coefficient is an even function of \( \theta \) (Figure 8b, gray line). However, for more complicated overburden models, the inhomogeneity angle can be different from the incidence angle, which makes the reflection coefficient asymmetric with respect to \( \theta \) (Figure 8b, black line; \( \xi = 50^\circ \) was held constant).

4.2 PS-wave reflection coefficient

As is the case for PP-waves, the inhomogeneity angle of the incident P-wave changes the conventional PS-wave AVO equation. The linearized PS-wave coefficient takes the form

\[
R_{PS}^{IH} = R_{PS}^{IH}(0) + B_{PS}^{IH} \sin \theta + G_{PS}^{IH} \sin^2 \theta,
\]

where

\[
R_{PS}^{IH}(0) = \frac{\sin \xi}{Q \rho_0} f_k, \quad B_{PS}^{IH} = B_{PS}^{IH},
\]
\( G_{PS}^{IH} = -i \frac{\sin \xi}{Q_{P0}} f_0. \) (32)

In equations 29-32, cubic and higher-order terms in \( \sin \theta \) and \( \sin \xi \) are neglected. \( B_{PS}^{IH} \) is the PS-wave AVO gradient for an incident wave with zero inhomogeneity angle (equation 21), and the terms \( f_0 \) and \( f_3 \) are linear combinations of the parameter contrasts across the interface (Appendix A).

Equation 29 is different from equation 20 for \( \xi = 0^\circ \), in which only the coefficients of odd powers in \( \sin \theta \) are nonzero (i.e., the reflection coefficient is an odd function of \( \theta \)). The deviation of equation 29 from the conventional PS-wave AVO equation is illustrated in Figure 9, where the magnitude of the PS-wave reflection coefficient in strongly attenuative media (\( Q = 2.5 \)) for \( \xi = 50^\circ \) is visibly asymmetric with respect to \( \theta = 0^\circ \). Also, for \( Q < 10 \) \( B_{PS}^{IH} \) significantly deviates from the reflection coefficient for a purely elastic medium, which almost coincides with that for \( Q = 50 \).

Because the linearized AVO gradient \( B_{PS}^{IH} \) is independent of \( \xi \) (for small \( \xi \)), the inhomogeneity angle has a greater influence on \( B_{PS}^{IH}(0) \) and \( G_{PS}^{IH} \) than on \( B_{PP}^{IH} \). For zero inhomogeneity angle of the incident wave, \( R_{PS}(0) \) and \( G_{PS}^{IH} \) vanish and equation 29 reduces to equation 20 (ignoring the cubic term in \( \sin \theta \)) discussed above.

For \( \xi \neq 0 \), the normal-incidence PS-wave reflection coefficient \( R_{PS} \) in strongly attenuative media can attain substantial values (Figure 9). Nonzero PS-wave amplitude at normal incidence can also be caused by such factors as lateral heterogeneity, the influence of additional terms of the ray-series expansion on point-source radiation (Tsvankin, 1995), and the deviation of the reflector from the symmetry planes of the model (Behura & Tsvankin, 2006). However, we consider only plane-wave reflection coefficients and the model in Figure 9 is composed of homogeneous VTI halfspaces with a common horizontal symmetry plane. A nonzero inhomogeneity angle of the vertically travelling P-wave makes its wave vector asymmetric with respect to the reflector normal, which generates the PS conversion.

5 CONCLUSIONS

We analyzed the PP- and PS-wave reflection coefficients in attenuative anisotropic media using linearized approximations verified by exact numerical modeling. For an incident P-wave with zero inhomogeneity angle, the form of the linearized PP- and PS-wave reflection coefficients is the same as that in purely elastic media, but all terms become complex. The general solutions for arbitrarily anisotropic media were simplified for VTI symmetry to obtain simple closed-form expressions in Thomsen-style notation.

Both analytic and numerical results show that only in the presence of strong attenuation (\( Q < 10 \)) does the contribution of the imaginary part of the stiffness tensor (which is responsible for attenuation) become comparable to that of the real part. In particular, the influence of the attenuation-anisotropy parameters \( \epsilon_Q \) and \( \delta_Q \) on reflection coefficients typically is much weaker than that of the velocity-anisotropy parameters \( \epsilon \) and \( \delta \). As expected from the parameter definitions, \( \delta_Q \) contributes to the AVO gradient (i.e., to the reflection coefficient at small incidence angles), while the influence of \( \epsilon_Q \) increases with incidence angle. The largest attenuation terms in the reflection coefficients for both PP- and PS-waves are proportional to the contrasts in the normalized symmetry-direction attenuation coefficients \( A_{P0} \) and \( A_{S0} \) because the contrasts in the attenuation-anisotropy parameters are scaled by \( 1/Q_{P0} \). Therefore, for AVO analysis in strongly attenuative media (\( Q < 10 \)), it is sufficient to take the influence of \( A_{P0} \) and \( A_{S0} \) into account, while in media with exceptionally strong attenuation (\( Q < 5 \)), it is necessary to consider the influence of \( \epsilon_Q \) and \( \delta_Q \) as well.

If the incident wave has a nonzero inhomogeneity angle \( \xi \), the form of the linearized reflection coefficients is different from the conventional AVO expression. In particular, the PP-wave reflection coefficient depends on \( \sin \theta \) and \( \sin^3 \theta \) and is no longer an even function of \( \theta \). Likewise, the PS-wave reflection coefficient at normal incidence does not vanish for \( \xi \neq 0^\circ \) and may even attain substantial values. However, the contribution of the inhomogeneity angle to the AVO response becomes significant only in media with anomalously high attenuation (such as heavy-oil-saturated rocks) with \( Q < 5 \).

Despite the presence of attenuation-related terms, our linearized AVO equations have an easily interpretable form that provides useful physical insight into the reflectivity of anisotropic attenuative media. Their application can help to avoid errors in AVO analysis.
Reflection coefficients in attenuative anisotropic media

6 ACKNOWLEDGMENTS
We are grateful to members of the A(nisotropy)-Team of the Center for Wave Phenomena (CWP), Colorado School of Mines for helpful discussions. The support for this work was provided by the Consortium Project on Seismic Inverse Methods for Complex Structures at CWP and the Petroleum Research Fund of the American Chemical Society.

References
APPENDIX A: LINEAR FUNCTIONS IN THE APPROXIMATE REFLECTION COEFFICIENTS

Here, we give explicit expressions for the linear functions $f_i$ in the approximate equations for the reflection coefficients.

The functions $f_1$, $f_2$, $f_3$, and $f_4$ in equations 21 and 22 have the form

$$f_1 = \frac{1}{2g} \frac{\Delta \rho}{\rho_0} + \frac{1}{g} \frac{\Delta V_{S0}}{V_{S0}} + \frac{g}{4(1+g)^2} \Delta \delta$$

$$+ \frac{i}{g} \Delta A_{S0} + \frac{g}{4(1+g)} \Delta \delta_Q,$$

(A1)

$$f_2 = \frac{1}{2g} \frac{\Delta \rho}{\rho_0} + \frac{1}{g} \frac{\Delta V_{S0}}{V_{S0}} + \frac{g}{4(1+g)^2} \Delta \delta$$

$$+ \frac{i}{g} \Delta A_{S0},$$

(A2)

$$f_3 = \frac{3 + g}{2g^2} \frac{\Delta \rho}{\rho_0} + \frac{4 + g}{g^2} \frac{\Delta V_{S0}}{V_{S0}}$$

$$- \frac{g}{(1+g)^2} \Delta \epsilon + \frac{5g}{4(1+g)^2} \Delta \delta + \frac{4+g}{g^2} \Delta A_{S0}$$

$$- \frac{g}{(1+g)^2} \Delta \epsilon_Q + \frac{4g-1}{4(1+g)} \Delta \delta_Q,$$

(A3)

$$f_4 = \frac{3 + g}{2g^2} \frac{\Delta \rho}{\rho_0} + \frac{4 + g}{g^2} \frac{\Delta V_{S0}}{V_{S0}} - \frac{g}{(1+g)^2} \Delta \epsilon$$

$$+ \frac{5g}{4(1+g)^2} \Delta \delta + \frac{1}{g^2} \Delta A_{S0},$$

(A4)

where $g \equiv V_{P0}/V_{S0}$.

The functions $f_5$, $f_6$, and $f_7$ in equations 26–28 are given by

$$f_5 = -i \frac{\Delta V_{P0}}{V_{P0}} + \Delta A_{P0},$$

(A5)
Estimation of interval anisotropic attenuation from reflection data

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ABSTRACT

Knowledge of interval attenuation can be highly beneficial in reservoir characterization and lithology discrimination. Here, we combine the spectral-ratio method with velocity-independent layer-stripping to develop a technique for estimation of the interval phase attenuation coefficient from reflection seismic data. The algorithm is designed for arbitrarily anisotropic target layers, but the overburden is assumed to be laterally homogeneous with a horizontal symmetry plane. Although no velocity information about the overburden is needed, interpretation of the anisotropic attenuation coefficient in the target layer requires estimation of the velocity function.

The interval phase attenuation in a reservoir can be used to predict the presence and distribution of hydrocarbons. For example, our method can help to distinguish between steam and heat fronts in heavy-oil reservoirs. Azimuthal variation of the interval attenuation in fractured formations can provide sensitive attributes for fracture characterization and reconstruction of the stress field.

Key words: Attenuation anisotropy, interval attenuation, spectral-ratio method, layer stripping

1 INTRODUCTION

Attenuation analysis can provide valuable information about lithology and physical properties of subsurface rocks. Laboratory measurements (e.g., Winkler & Nur, 1982) show that attenuation is closely related to the presence of fluids. In particular, attenuation may serve as an indicator of permeability, mobility of fluids, and fluid saturation (e.g., Batzle et al., 2006; Behura et al., 2007).

A number of laboratory measurements (Behura et al., 2006; Hosten et al., 1987; Prasad & Nur, 2003; Tao & King, 1990) and field case studies (Ganley & Kanasewich, 1980; Liu et al., 2007; Maultzsch et al., 2007) indicate that attenuation can be strongly anisotropic (directionally-dependent) because of the preferential alignment of fractures, interbedding of thin attenuative layers, and nonhydrostatic stress. The field study of Maultzsch et al. (2007) finds the symmetry of attenuation anisotropy to be different from that of velocity anisotropy. Therefore, measurements of attenuation anisotropy may provide additional information about the fluid properties of fractured reservoirs (Liu et al., 2007).

Existing attenuation estimates from reflection data (e.g., Vasconcelos & Jenner, 2005) are obtained for the whole section above the reflecting interface. Dasgupta & Clark (1998) introduce a technique for estimating interval attenuation from reflection data based on the spectral ratio method. This algorithm, however, is restricted to zero-offset attenuation and requires the knowledge of the source signature. Moreover, they apply the NMO stretch prior to attenuation analysis, which may distort the estimated attenuation values.

Here, we present a method for computing the interval attenuation coefficients using an extension of the layer-stripping technique originally introduced by Dewangan & Tsvankin (2006) for reflection traveltimes. Our algorithm reconstructs the offset-dependent interval attenuation of an arbitrarily anisotropic heterogeneous target layer without knowledge of the velocity and attenuation in the overburden. Synthetic examples for layered VTI (transversely isotropic with a vert-
Fig. 1. 2D ray diagram of the layer-stripping algorithm. Points B and E are located at the bottom of the overburden. The target reflection \( ABCEG \) and the reflection \( ABD \) from the bottom of the overburden share the same downgoing leg \( AB \). The upgoing leg of the target event \( EG \) coincides with a leg of another overburden reflection (\( GEF \)).

2 METHODOLOGY

Although our technique of estimating interval attenuation can be applied in 3D, this work is restricted to 2D models. Let us consider a pure-mode reflection in a medium that consists of an anisotropic, heterogeneous target layer under a laterally homogeneous overburden with a horizontal symmetry plane (Figure 1).

To make wave propagation two-dimensional, the vertical incidence plane has to be a plane of symmetry in all layers. This restriction can be relaxed in the 3D version of the method operating with wide-azimuth data.

The exact interval traveltime-offset function in the target layer can be constructed by combining the target event with reflections from the bottom of the overburden (Figure 1). Dewangan & Tsvankin (2006) show that any reflection \( ABC \) can be constructed by combining the target event \( ABCEG \) with the reflection \( ABD \) and \( GEF \) and the parameters \( g_{ABD} \), \( g_{GEF} \), and \( g_{ABCEG} \) include the source/receiver radiation patterns as well as geometrical spreading along the corresponding raypaths.

Then the interval attenuation coefficient in the target layer can be computed using the following combination of amplitudes:

\[
\ln \left( \frac{|U_{ABCD}(\omega)|^2}{|U_{ABD}(\omega)| |U_{GEF}(\omega)|} \right) = \ln(g) - 2k_g l_{BD} + l_{CE}
\]

where \( t_{ABCEG}, t_{ABD}, \) and \( t_{GEF} \) are the traveltimes along the raypaths \( ABCEG, ABD, \) and \( GEF \) respectively.

Here, we extend this layer-stripping technique to attenuation analysis by applying the spectral-ratio method to the frequency-domain amplitudes of the target and overburden events. The amplitude of the overburden reflections can be written as

\[
|U_{ABD}(\omega)| = S(\omega) g_{ABD} e^{-k_g l_{BD} + l_{BD}}
\]

and

\[
|U_{GEF}(\omega)| = S(\omega) g_{GEF} e^{-k_g l_{GE} + l_{CE}}.
\]

while for the target reflection \( U_{ABCEG}(\omega) \), we have

\[
|U_{ABCEG}(\omega)| = S(\omega) g_{ABCEG} e^{-k_g l_{BD} + l_{CE}}
\]

where \( S(\omega) \) is the spectrum of the source wavelet, \( I_X \) represents the distance along the raypath \( XY \); \( k_g \) and \( k_g \) (\( "T" \) stands for the imaginary part of the wavenumber) are the average group attenuation coefficients in the target layer and overburden, respectively, and the parameters \( g_{ABD}, g_{GEF}, \) and \( g_{ABCEG} \) include the source/receiver radiation patterns as well as the reflection/transmission coefficients.

Therefore, application of equation 6 helps to remove the influence of the attenuation in the overburden.

For anisotropic attenuative media, the group attenuation coefficient \( k_g \) generally differs from the phase attenuation coefficient \( k_g \). If the inhomogeneity angle (the angle between the real and imaginary parts of the complex wave vector) is small, the group and phase attenuation coefficients are related by \( k_g = k_g \cos \psi \), where
ψ is the angle between the group- and phase-velocity vectors (Zhu, 2006).

For an arbitrarily heterogeneous target layer, the attenuation coefficient varies along the ray, and \( k_{l,T} \) in equation 6 represents the average value along the ray-path BCE. However, if the target layer is horizontal, homogeneous, and has a horizontal symmetry plane (or is purely isotropic), then \( t_{BC} + t_{CE} = V_t t_{BCE} \), where \( V_t \) is the group velocity. Then equation 6 can be written as

\[
\ln \left( \frac{|U_{ABCEG}(\omega)|^2}{|U_{ABD}(\omega)||U_{GDF}(\omega)|} \right) = -2k l \cos \psi V_t t_{BCE}. \tag{7}
\]

Expressing \( V_t \) through the phase velocity \( V \) (\( V_t = V / \cos \psi \)), we represent the attenuation-related term in equation 7 as follows:

\[
k l \cos \psi V_t t_{BCE} = \omega \frac{k l}{k} t_{BCE} = \omega A t_{BCE}. \tag{8}
\]

\( A = k l / k \) (\( k = \omega / V \)) is the normalized phase attenuation coefficient (Zhu & Tsvankin, 2006). Therefore, equation 7 can be rewritten as

\[
\ln \left( \frac{|U_{ABCEG}(\omega)|^2}{|U_{ABD}(\omega)||U_{GDF}(\omega)|} \right) = \ln(\tilde{G}) - 2\omega A t_{BCE}. \tag{9}
\]

In the above derivation, the inhomogeneity angle \( \xi \) (i.e., the angle between the real and imaginary parts of the wave vector) is assumed to be zero (Figure 2a). However, as discussed below, equation 9 remains valid for a wide range of inhomogeneity angles.

The slope of the logarithmic spectral ratio in equation 9 expressed as a function of \( \omega \) yields the product \( 2A t_{BCE} \). Since the interval traveltime \( t_{BCE} \) can be obtained from the layer-stripping algorithm (equation 1), equation 9 can be used to estimate the phase attenuation coefficient \( A \). To invert \( A \) for the attenuation-anisotropy parameters (Zhu et al., 2007), it is necessary to estimate the phase direction. Since the influence of attenuation on velocity is typically of the second order (Zhu & Tsvankin, 2006), velocity analysis for the target layer can be performed prior to attenuation processing using the interval traveltme. Note that the inversion for the parameters of an anisotropic target layer typically requires a priori information (e.g., the layer thickness).

Computation of the interval values of \( A \) for different source-receiver pairs can help to evaluate both the anisotropy and lateral variation of attenuation. Although we described the methodology only for 2D models, it can be readily extended to 3D wide-azimuth data using the 3D version of the layer-stripping algorithm of Dewangan & Tsvankin (2006).

![Figure 2](image)

**Figure 2.** Incident plane wave with (a) zero inhomogeneity angle and (b) nonzero inhomogeneity angle \( \xi \). \( k \) and \( k' \) are the real and imaginary components (respectively) of the wave vector, and \( \theta \) is the incidence phase angle.

### 3 SYNTHETIC EXAMPLE

The layer-stripping method introduced above was tested on synthetic PP-reflection data generated for a layered VTI model (Figure 3). A shot gather computed by an anisotropic reflectivity code is shown in Figure 4a. Note that the reflections from the bottom of the attenuative layers have much lower frequency content than the water-bottom event (Figure 4b).

Attenuation in VTI media can be conveniently characterized using the Thomsen-style notation (\( A_{P0} \approx 1/2Q_{P0}, A_{S0} \approx 1/2Q_{S0}, \epsilon_Q, \delta_Q, \gamma_Q \)) introduced by Zhu & Tsvankin (2006). \( A_{P0} \) and \( A_{S0} \) are the normalized symmetry-direction attenuation coefficients of the P- and S-waves, respectively, \( \epsilon_Q \) and \( \delta_Q \) control the angular variation of the P- and SV-wave attenuation coefficients, and \( \gamma_Q \) governs SH-wave attenuation anisotropy. Although the second layer has uncommonly strong attenuation (\( Q_{P0} = 10 \)), the estimated interval attenuation coefficient is close to the exact values for a wide range of angles (Figure 5a). Attenuation coefficients for large phase angles (outside the range of \( \pm60^\circ \)) are missing because of the limited acquisition aperture. The interval attenuation coefficient computed for the third layer (Figure 5b) is accurate only up to \( \pm30^\circ \). For phase angles exceeding \( 30^\circ \), the target reflection interferes...
Figure 3. Model used to test the attenuation layer-stripping algorithm. The first layer is water (purely elastic and isotropic) with the P-wave velocity \( V_P = 1500 \text{ m/s} \) and thickness \( d = 1000 \text{ m} \); the other three layers are VTI. For the second layer, the parameters are as follows: the vertical P- and S-wave velocities are \( V_{P0} = 1600 \text{ m/s} \) and \( V_{S0} = 200 \text{ m/s} \), \( d = 300 \text{ m} \), and Thomsen velocity-anisotropy parameters are \( \epsilon = 0.3 \) and \( \delta = -0.2 \); the attenuation parameters are \( Q_{P0} = 10 \), \( Q_{S0} = 10 \), \( \epsilon_Q = -0.5 \), and \( \delta_Q = -1.0 \). The third layer has \( V_{P0} = 2000 \text{ m/s} \), \( V_{S0} = 1000 \text{ m/s} \), \( d = 1000 \text{ m} \), \( \epsilon = 0.1 \), \( \delta = 0.6 \), \( Q_{P0} = 200 \), \( Q_{S0} = 200 \), \( \epsilon_Q = -0.3 \), and \( \delta_Q = 1.0 \). For the bottom halfspace, \( V_{P0} = 2200 \text{ m/s} \), \( V_{S0} = 1100 \text{ m/s} \), \( \epsilon = 0 \), \( \delta = -0.2 \), \( Q_{P0} = 100 \), \( Q_{S0} = 100 \), \( \epsilon_Q = 0.5 \), and \( \delta_Q = 0.5 \).

with the direct arrival and other events, making the spectral-ratio method inadequate. Interference-related distortions can be mitigated by operating in the \( \tau - p \) domain or suppressing the direct arrival and ground roll before layer stripping. This and other tests (not shown here) performed for a wide range of attenuative anisotropic models confirm the accuracy and efficiency of our method.

4 DISCUSSION

Although the above methodology is developed for zero inhomogeneity angle \( \xi \) (Figure 2a), our analysis supported by exact numerical modeling shows that it remains accurate for a wide range of \( \xi \). Application of equations 7 and 8 for nonzero inhomogeneity angles yields the normalized phase attenuation coefficient \( A \) corresponding to \( \xi = 0^\circ \). This conclusion remains valid even for large inhomogeneity angles (\( \xi < 80^\circ \)) and media with \( Q \) as low as 10. For this reason, the phase attenuation coefficient in off-symmetry directions in Figure 5 is computed accurately, even though the inhomogeneity angle reaches \( 45^\circ \) for long offsets.

The attenuation-anisotropy parameters can be obtained by fitting the estimated interval phase attenuation coefficient to the following approximate expression

![Figure 4.](image-url) (a) Shot gather computed for the model in Figure 3 and (b) its blow-up showing reflections with different frequency content.)
and monitoring. The 3D version of the method can be used to estimate azimuthally varying interval attenuation coefficients for fracture-characterization purposes.

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Figure 5. Normalized interval phase attenuation coefficient \( A \) (solid black lines) for the second (a) and third (b) layers of the model from Figure 3 computed as a function of phase angle. The dashed gray lines mark the exact values of \( A \).

\[
A = A_{P0} \left( 1 + \delta_Q \sin^2 \theta \cos^2 \theta + \epsilon_Q \sin^4 \theta \right),
\]

(10)

(\text{Zhu} \& \text{Tsvankin, 2006}):

where \( \theta \) is the phase angle with the symmetry axis.

5 CONCLUSIONS

We extended velocity-independent layer stripping to amplitude analysis and employed the spectral-ratio method to obtain the interval attenuation coefficient from reflection data. While the target layer can be arbitrarily anisotropic and heterogeneous, the overburden has to be laterally homogeneous with a horizontal symmetry plane. It should be emphasized that our attenuation layer stripping does not require knowledge of the overburden velocity and attenuation parameters.

Numerical examples for layered VTI media confirm that the method yields accurate interval phase attenuation coefficients even for models with uncommonly strong attenuation and substantial velocity and attenuation anisotropy. The algorithm is designed to process isolated overburden and target reflections, so the results may be distorted by interference with other events. As any other layer-stripping technique, our method may become inaccurate for thin attenuative layers.

Interval attenuation measurements may provide important information for reservoir characterization and monitoring. The 3D version of the method can be used to estimate azimuthally varying interval attenuation coefficients for fracture-characterization purposes.

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Angle-domain elastic reverse-time migration

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ABSTRACT

Multi-component data are not usually processed with specifically designed procedures, but with procedures analogous to the ones used for single-component data. Commonly, the vertical and horizontal components of the data are taken as proxies for P and S wave modes which are imaged independently with the acoustic wave equation. This procedure works only if the vertical and horizontal component accurately represent P and S wave modes, which is not true in general. Therefore, multi-component images constructed with this procedure exhibit artifacts caused by the incorrect wave mode separation at the surface.

An alternative procedure for elastic imaging uses the vector fields for wavefield reconstruction and imaging. The wavefields are reconstructed using the multi-component data as boundary conditions for a numerical solution to the elastic wave equation. The key component for wavefield migration is the imaging condition that evaluates the match between wavefields reconstructed from sources and receivers. For vector wavefields, a simple component-by-component cross-correlation between two wavefields leads to artifacts caused by cross-talk between the unseparated wave modes. An alternative method is to separate elastic wavefields after reconstruction in the subsurface and implement the imaging condition as cross-correlation of pure wave modes instead of the Cartesian components of the displacement wavefield. This approach leads to images that are easier to interpret, since they describe reflectivity of specified wave modes at interfaces of physical properties.

As for imaging with acoustic wavefields, the elastic imaging condition can be formulated conventionally (cross-correlation with zero lag in space and time), as well as extended to non-zero space and time lags. The elastic images produced by an extended imaging condition can be used for angle decomposition of primary (PP or SS) and converted (PS or SP) reflectivity. Angle gathers constructed with this procedure have applications for migration velocity analysis and amplitude versus angle analysis.

Key words: imaging, elastic, reverse time

1 INTRODUCTION

Seismic processing is usually based on acoustic wave equations, which assume that the Earth represents a liquid that propagates only compressional waves. Although useful in practice, this assumption is not theoretically valid. Earth materials allow for both primary and shear wave propagation in the subsurface. Shear waves, either generated at the source or converted from compressional waves at various interfaces in the subsurface, are recorded by multi-component receivers. Shear waves are usually stronger at large incidence and reflection angles, often corresponding to large offsets. However, for complex geological structures near the surface, shear waves can be quite significant even at small offsets. Conventional single-component imaging ignores shear wave modes, which often leads to incorrect characterization of wave-propagation, incomplete illumination of the subsurface and poor amplitude characterization, etc.

Even when multi-component data are used for imaging, they are usually not processed with specifically designed procedures. Instead, those data are processed with ad-hoc procedures borrowed from acoustic wave equation imaging algorithms. A typical assumption is that the recorded vertical and horizontal components are good approximations for the P and S wave modes, respectively, which can be imaged independently. This assumption is not always correct, leading to errors
and noise in the images, since P and S wave modes are normally mixed on all recorded components. Also, since P and S modes are mixed on all components, true-amplitude imaging is questionable no matter how accurate the wavefield reconstruction and imaging condition are.

Multi-component imaging has long been an active research area for exploration geophysicists. Techniques proposed in the literature perform imaging in the time domain, e.g. by Kirchhoff migration (Kuo & Dai, 1984; Hokstad, 2000) and reverse time migration (Chang & McMechan, 1986, 1994) adapted for multi-component data. The reason for working in the time domain, as opposed to the depth domain, is that the coupling of displacements in different directions in elastic wave equations makes it difficult to derive a dispersion relation that can be used to extrapolate wavefields in depth.

Early attempts at multi-component imaging use the Kirchhoff framework. Kuo & Dai (1984) perform shot-profile elastic Kirchhoff migration and Hokstad (2000) performs survey-sinking elastic Kirchhoff migration. Although these techniques represent different migration procedures, they compute travel-times for both PP and PS reflections, and sum data along these travel time trajectories. This approach is equivalent to distinguishing between PP reflection and PS reflections, and applying acoustic Kirchhoff migration for each mode separately. When geology is complex, the elastic Kirchhoff migration technique suffers from drawbacks similar to those of acoustic Kirchhoff migration because ray theory breaks down (Gray et al., 2001).

There are two main difficulties with independently imaging P and S wave modes separated on the surface. The first is that conventional elastic migration techniques either consider vertical and horizontal components of recorded data as P and S modes, which is not always accurate, or separate these wave modes on the recording surface using approximations, e.g. polarization (Pestana et al., 1989) or elastic potentials (Etgen, 1988; Zhe & Greenhalgh, 1997). Other elastic reverse time migration techniques do not separate wave modes on the surface and reconstruct vector fields, but use imaging conditions based on ray tracing (Chang & McMechan, 1986, 1994) that are not always robust in complex geology. The second difficulty is that images produced independently from P and S modes are hard to interpret together, since often they do not line-up consistently, thus requiring image post processing, e.g. by manual or automatic registration of the images (Fomel & Backus, 2003; Nickel & Sonneland, 2004).

We advocate an alternative procedure for imaging elastic wavefield data. Instead of separating wavefields into scalar wave modes on the acquisition surface followed by scalar imaging of each mode independently, we use the entire vector wavefields for wavefield reconstruction and imaging. The vector wavefields are reconstructed using the multi-component vector data as boundary conditions for a numerical solution to the elastic wave equation. The key component of such a migration procedure is the imaging condition which evaluates the match between wavefields reconstructed from the source and receiver. For vector wavefields, a simple component-by-component cross-correlation between the two wavefields leads to artifacts caused by cross-talk between the unseparated wave modes, i.e. all P and S modes from the source wavefield correlate with all P and S modes from the receiver wavefield. This problem can be alleviated by using separated wavefields, with the imaging condition implemented as cross-correlation of wave modes instead of cross-correlation of the Cartesian components of the wavefield. This approach leads to images that are cleaner and easier to interpret since they represent reflections of single wave modes at interfaces of physical properties.

As for imaging with acoustic wavefields, the elastic imaging condition can be formulated conventionally (cross-correlation with zero lag in space and time), as well as extended to non-zero space lags. The elastic images produced by extended imaging condition can be used for angle decomposition of PP and PS reflectivity. Angle gathers have many applications, including migration velocity analysis (MVA) and amplitude versus angle (AVA) analysis.

The advantage of imaging with multi-component seismic data is that the physics of wave propagation is better represented, and resulting seismic images more accurately characterize the subsurface. Multi-component images have many applications. For example they can be used to provide reflection images where the P-wave reflectivity is small, image through gas clouds where the P-wave signal is attenuated, detect fractures through shear-wave splitting, validate bright spot reflections and provide parameter estimation for this media, and Poisson’s ratio estimates (Simmons & Backus, 2003). Assuming no attenuation in the subsurface, converted wave images also have higher resolution than pure-mode images in shallow parts of sections, because S waves have shorter wavelengths than P waves. Modeling and migrating multi-component data with elastic migration algorithms, enables us to make full use of information provided by elastic data, and correctly positions geologic structures.

We begin by summarizing wavefield imaging methodology, focusing on reverse-time migration for wavefield multi-component migration. Then, we describe different options for wavefield multi-component imaging conditions, e.g. based on vector displacements and vector potentials. Finally, we describe the application of extended imaging condition to multi-component data and corresponding angle decomposition. We illustrate the wavefield imaging techniques using data simulated on the Marmousi II model (Martin et al., 2002).

2 WAVEFIELD IMAGING

Seismic imaging is based on numerical solutions to wave equations, which can be classified into ray-based (integral) solutions and wavefield-based (differential) solutions. Kirchhoff migration is a typical ray-based imaging procedure which is computationally efficient but often fails in areas of complex geology, such as sub-salt, because the wavefield is severely distorted by lateral velocity variations leading to complex multipathing. Wavefield imaging works better for complex geology, but is more expensive than Kirchhoff migration. Depending on computational time constraints and available re-
sources, different levels of approximation are applied to accelerate imaging, i.e. one-way vs. two-way, acoustic vs. elastic, isotropic vs. anisotropic, etc.

Despite the complexity of various types of wavefield migration algorithms, any wavefield imaging can be separated into two parts: wavefield reconstruction followed by the application of an imaging condition. For prestack depth migration, source and receiver wavefields have to be reconstructed at all locations in the subsurface. The wavefield reconstruction can be done in both depth and time domain, and with different modeling approaches, such as finite-difference (Dablain, 1986; Alford et al., 1990), finite-element (Bolt & Smith, 1976), spectral method (Seriani & Priolo, 1991; Seriani et al., 1992; Dai & Cheadle, 1996), etc. After reconstructing wavefields with the recorded data as boundary conditions into the subsurface, an imaging condition must be applied at all locations in the subsurface in order to obtain a seismic image. The simplest types of imaging condition are based on cross-correlation or deconvolution of the reconstructed wavefields (Claerbout, 1971). These imaging conditions can be implemented in the time or frequency domains, depending on the domain in which wavefields have been reconstructed. Here, we concentrate on reverse-time migration with wavefield reconstruction and imaging condition implemented in the time domain.

2.1 Reverse-time migration

Reverse-time migration reconstructs the source wavefield forward in time and the receiver wavefield backward in time. It then applies an imaging condition to extract reflectivity information out of the reconstructed wavefields. The advantages of reverse-time migration over other depth migration techniques are that the extrapolation in time does not involve evanescent energy, and no dip limitations exist for the imaged structures. Although conceptually simple, reverse-time migration has not been used extensively in practice due to its high computational cost. However, the algorithm is becoming more and more attractive to the industry because of its robustness in imaging complex geology, e.g. sub-salt Jones et al. (2007); Boechat et al. (2007).

Whitmore (1983) and Baysal et al. (1983) first used reverse-time migration for poststack or zero-offset data. The procedure of poststack reverse-time migration is the following: first, reverse the recorded data in time; second, use these reversed data as sources along the recording surface to propagate the wavefields in the subsurface; third, extract the image at zero time, e.g. apply an imaging condition. The principle of poststack reverse-time migration is that the subsurface reflectors work as exploding reflectors and that the wave equation used to propagate data can be applied either forward or backward in time by simply reversing the time axis (Levin, 1984).

Chang & McMechan (1986) apply reverse-time migration to prestack data. Prestack reverse-time migration reconstructs source and receiver wavefields. The source wavefield is reconstructed forward in time, and the receiver wavefield is reconstructed backward in time. Chang & McMechan (1986, 1994) use a so-called excitation-time imaging condition, where images are formed by extracting the receiver wavefield at the time taken by a wave to travel from the source to the image point. This imaging condition is a special case of the cross-correlation imaging condition of Claerbout (1971).

2.2 Elastic imaging vs. acoustic imaging

Multi-component elastic data are often recorded in land or marine (ocean-bottom) seismic experiments. However, as mentioned earlier, elastic vector wavefields are not usually processed by specifically designed imaging procedures, but rather by extensions of techniques used for scalar wavefields. Thus, seismic data processing does not take full advantage of the information contained by elastic wavefields. In other words, it does not fully illuminate complex geology or correctly preserve imaging amplitudes and estimate model parameters, etc.

Elastic wave propagation in a homogeneous and arbitrarily anisotropic medium is characterized by the wave equation

$$\rho \frac{\partial^2 u_i}{\partial t^2} - c_{ijkl} \frac{\partial^2 u_k}{\partial x_j \partial x_l} = f_i,$$

where $u_i$ is the component $i$ of the displacement vector $\mathbf{u}$, $\rho$ is the density, $c_{ijkl}$ is the stiffness tensor and $f_i$ are the Cartesian components of the vector source $\mathbf{f}(t)$. This wave equation assumes a slowly varying stiffness tensor over the imaging space. The coupling of displacements for different directions in the wave equation makes it difficult to derive a dispersion relation that can be used to extrapolate wavefields in depth. This makes it natural to process elastic wavefields in time. There are two main options for elastic imaging in the time domain: Kirchhoff migration and reverse-time migration.

Acoustic Kirchhoff migration is based on diffraction summation, which accumulates the data along diffraction curves in the data space and maps them onto the image space. For multi-component elastic data, Kuo & Dai (1984) discuss Kirchhoff migration for shot-record data. Here, identified PP and PS reflections can be migrated by computing source and receiver traveltimes using P wave velocity for the source rays, and P and S wave velocities for the receiver rays. Hokstad (2000) performs multi-component anisotropic Kirchhoff migration for multi-shot, multi-receiver experiments, where pure-mode and converted mode images are obtained by downward continuation of visco-elastic vector wavefields and application of survey-sinking imaging condition to the reconstructed vector wavefields. The wavefield separation is effectively done by the Kirchhoff integral which handles both P- and S-waves, although this technique fails in areas of complex geology where ray theory breaks down.

Elastic reverse-time migration has the same two components as acoustic reverse-time migration: reconstruction of source and receiver wavefield and application of an imaging condition. The source and receiver wavefields are reconstructed by forward and backward propagation in time with various modeling approaches. For acoustic reverse-time migration, wavefield reconstruction is done with the acoustic wave-equation using the recorded scalar data as boundary con-
dition. In contrast, for elastic reverse-time migration, wavefield reconstruction is done with the elastic wave-equation using the recorded vector data as boundary condition.

Since pure-mode and converted-mode reflections are mixed on all components of recorded data, images produced with reconstructed elastic wavefields are characterized by cross-talk due to the interference of various wave modes. In order to obtain images with clear physical meanings, most imaging conditions separate wave modes, because they are mixed in all components of the data. There are two potential approaches to separate wavefields and image elastic seismic wavefields. The first option is to separate P and S modes on the acquisition surface from the recorded elastic wavefields. This procedure involves either approximations for the propagation path and polarization direction of the recorded data, or reconstruction of the seismic wavefields in the vicinity of the acquisition surface by a numerical solution of the elastic wave equation, followed by wavefield separation of scalar and vector potentials using Helmholtz decomposition (Etgen, 1988; Zhe & Greenhalgh, 1997). An alternative data decomposition using P and S potentials is to reconstruct wavefields in the subsurface using the elastic wave equation, then decompose the wavefields in P and S wave modes. This is followed by forward extrapolating the separated wavefields back to the surface using the acoustic wave equation with the appropriate propagation velocity for the various wave modes (Sun et al., 2006) by conventional procedures used for scalar wavefields.

The second option is to extrapolate wavefields in the subsurface using a numeric solution to the elastic wave equation and then apply an imaging condition that extracts reflectivity information from the source and receiver wavefields. In case extrapolation is implemented by finite-difference methods (Chang & McMechan, 1986, 1994), this procedure is known as elastic reverse-time migration, and is conceptually similar to acoustic reverse-time migration (Baysal et al., 1983), which is more frequently used in seismic imaging.

Many imaging conditions can be used for reverse-time migration. The elastic imaging condition is more complex than acoustic imaging condition because both source and receiver wavefields are vector fields. Different elastic imaging conditions have been proposed for extracting reflectivity information from reconstructed elastic wavefields. For example, Hokstad et al. (1998) use elastic reverse-time migration with Lamé potential methods. Chang & McMechan (1986) use the excitation-time imaging condition which extracts reflectivity information from extrapolated wavefields at travel times from the source to image positions computed by ray tracing, etc. Ultimately, these imaging conditions represent special cases of a more general type of imaging condition that involves time cross-correlation or deconvolution of source and receiver wavefields at every location in the subsurface.

3 CONVENTIONAL ELASTIC IMAGING CONDITIONS

For vector elastic wavefields, the cross-correlation imaging condition needs to be implemented on all components of the displacement field. The problem with this type of imaging condition is that the source and receiver wavefields contain a mix of P and S wave modes which cross-correlate independently, thus hampering interpretation of migrated images. An alternative to this type of imaging is to perform wavefield separation of scalar and vector potentials after wavefield reconstruction in the imaging volume, but prior to the imaging condition and then cross-correlate pure modes from the source and receiver wavefields, as suggested by Dellinger & Etgen (1990) and illustrated by Cunha Filho (1992).

3.1 Imaging with scalar wavefields

As mentioned earlier, assuming single scattering in the Earth (Born approximation), a conventional imaging procedure consists of two components: wavefield extrapolation and imaging. Wavefield extrapolation is used to reconstruct in the imaging volume the seismic wavefield using the recorded data on the acquisition surface as boundary condition, and imaging is used to extract reflectivity information from the extrapolated source and receiver wavefields.

Assuming scalar recorded data, wavefield extrapolation using a scalar wave equation reconstructs scalar source and receiver wavefields, $u_s(x,t)$ and $u_r(x,t)$, at every location in the subsurface. Using the extrapolated scalar wavefields, a conventional imaging condition (Claerbout, 1985) can be implemented as cross-correlation at zero-lag time:

$$I(x) = \int u_s(x,t) u_r(x,t) \, dt. \quad (2)$$

Here, $I(x)$ denotes a scalar image obtained from scalar wavefields $u_s(x,t)$ and $u_r(x,t)$, $x = \{x,y,z\}$ represent Cartesian space coordinates, and $t$ represents time.

3.2 Imaging with vector displacements

Assuming vector recorded data, wavefield extrapolation using a vector wave equation reconstructs source and receiver wavefields $u_s(x,t)$ and $u_r(x,t)$ at every location in the subsurface. Here, $u_s$ and $u_r$ represent displacement fields reconstructed from data recorded by multi-component geophones at the surface boundary. Using the vector extrapolated wavefields $u_s = \{u_{s,x}, u_{s,y}, u_{s,z}\}$ and $u_r = \{u_{r,x}, u_{r,y}, u_{r,z}\}$, an imaging condition can be formulated as a straightforward extension of 2 by cross-correlating all combinations of components of the source and receiver wavefields. Such an imaging condition for vector displacements can be formulated mathematically as

$$I_{ij}(x) = \int u_{si}(x,t) u_{rj}(x,t) \, dt, \quad (3)$$

where the quantities $u_i$ and $u_j$ stand for the Cartesian components $x, y, z$ of the vector source and receiver wavefields, $u_i(x,t)$. For example, $I_{xz}(x)$ represents the image component produced by cross-correlation of the $z$ components of the source and receiver wavefields, $I_{xz}(x)$ the image component produced by cross-correlation of the $x$ component of the source wavefield with the $x$ component of the receiver
wavefield, etc. In general, an image produced with this procedure has 9 components at every location in space.

The main drawback of applying this type of imaging condition is that the wavefield used for imaging contain a combination of P and S wave modes. Those wavefield vectors interfere with one-another in the imaging condition, since the P and S components are not separated in the extrapolated wavefields. The cross-talk between various components of the wavefield creates artifacts and makes it difficult to interpret the images in terms of pure wave modes, e.g. PP or PS reflections. This situation is similar to the case of imaging with acoustic data contaminated by multiples or other types of coherent noise which are mapped in the subsurface using incorrect velocity.

3.3 Imaging with scalar and vector potentials

An alternative to the elastic imaging condition 3 is to separate the extrapolated wavefield in P and S potentials after extrapolation and imaging using cross-correlations of vector and scalar potentials (Dellinger & Etgen, 1990). Separation of scalar and vector potentials can be achieved by Helmholtz decomposition, which is applicable to any vector field \( \mathbf{u}(\mathbf{x}, t) \):

\[
\mathbf{u} = \nabla \Phi + \nabla \times \Psi ,
\]

where \( \Phi(\mathbf{x}, t) \) represents the scalar potential of the wavefield \( \mathbf{u}(\mathbf{x}, t) \) and \( \Psi(\mathbf{x}, t) \) represents the vector potential of the wavefield \( \mathbf{u}(\mathbf{x}, t) \). For isotropic elastic wavefields, equation 4 is not used directly in practice, but the scalar and vector components are obtained indirectly by the application of the divergence \( \nabla \cdot \) and curl \( \nabla \times \) operators to the extrapolated elastic wavefield \( \mathbf{u}(\mathbf{x}, t) \):

\[
P = \nabla \cdot \mathbf{u} = \nabla \cdot \Phi,
\]

\[
S = \nabla \times \mathbf{u} = -\nabla \times \Psi .
\]

For isotropic elastic fields far from the source, quantities \( P \) and \( S \) describe compressional and shear components of the wavefield, respectively (Aki & Richards, 2002).

Using the separated scalar and vector components, we can formulate an imaging condition that combines various incident and reflected wave modes. The imaging condition for vector potentials can be formulated mathematically as

\[
I_{ij}(\mathbf{x}) = \int \alpha_i(\mathbf{x}, t) \alpha_j(\mathbf{x}, t) \, dt ,
\]

where the quantities \( \alpha_i \) and \( \alpha_j \) stand for the various wave modes \( \alpha = \{ P, S_1, S_2 \} \) of the vector source and receiver wavefields, \( \mathbf{u}(\mathbf{x}, t) \). For example, \( I_{PP}(\mathbf{x}) \) represents the image component produced by cross-correlation of the \( P \) wave mode of the source and receiver wavefields, \( I_{PS_1}(\mathbf{x}) \) represents the image component produced by cross-correlation of the \( P \) wave mode of the source wavefield with the \( S_1 \) wave-mode of the receiver wavefield, etc. In general, an image produced with this procedure also has 9 components at every location in space, similarly to the image produced by the cross-correlation of the various Cartesian components of the vector displacements. However, in this case, the images correspond to various combinations of incident P or S and reflected P or S waves, thus having clear physical meaning and being easier to interpret for physical properties.

4 EXTENDED ELASTIC IMAGING CONDITIONS

The conventional imaging condition 2 discussed in the preceding section uses the zero lag of the cross-correlation between the source and receiver wavefields. This imaging condition represents a special case of a more general form of imaging condition, sometimes referred-to as an extended imaging condition (Sava & Fomel, 2006b)

\[
I_{ij}(\mathbf{x}, \lambda, \tau) = \int u_i(\mathbf{x} - \lambda, t - \tau) u_j(\mathbf{x} + \lambda, t + \tau) \, dt ,
\]

where \( \lambda = \{ \lambda_x, \lambda_y, \lambda_z \} \) and \( \tau \) stand for cross-correlation lags in space and time, respectively. The imaging condition 2 is equivalent with the extended imaging condition 8 for \( \lambda = 0 \) and \( \tau = 0 \).

The extended imaging condition has two main uses. First, the extended imaging condition characterizes wavefield reconstruction errors, since for incorrectly reconstructed wavefields, the cross-correlation energy does not focus completely at zero lags in space and time. Sources of wavefield reconstruction error include inaccurate numeric solutions to the wave-equation, inaccurate (velocity) models used for wavefield reconstruction, inadequate wavefield sampling on the acquisition surface, uneven illumination of the subsurface, etc. Typically, all these causes of inaccurate wavefield reconstruction occur simultaneously and it is difficult to separate them after imaging. Second, assuming accurate wavefield reconstruction, the extended imaging condition can be used for angle decomposition. This leads to representations of reflectivity function of angles of incidence and reflection at all points in the imaged volume (Sava & Fomel, 2003). Here, we assume that wavefield reconstruction is accurate and concentrate on further extensions of the imaging condition, such as angle decomposition.

4.1 Imaging with vector displacements

For the case of imaging with vector wavefields, the extended imaging condition 8 can be applied directly to the various components of the reconstructed source and receiver wavefields, similarly to the conventional imaging procedure described in the preceding section. Therefore, an extended image constructed from vector displacement wavefields is

\[
I_{ij}(\mathbf{x}, \lambda, \tau) = \int u_{ij}(\mathbf{x} - \lambda, t - \tau) u_{ij}(\mathbf{x} + \lambda, t + \tau) \, dt ,
\]

where the quantities \( u_{ij} \) and \( u_{ij} \) stand for the Cartesian components \( x, y, z \) of the vector source and receiver wavefields, and \( \lambda \) and \( \tau \) stand for cross-correlation lags in space and time, respectively. This imaging condition suffers from the same drawbacks described for the similar conventional imaging condition applied to the Cartesian components of the reconstructed wavefields, i.e. cross-talk between the unseparated wave modes, etc.
where α based on the relation wave modes can be constructed after imaging using mapping reflection angle corresponding to incidence and reflection of P for the case of imaging with the acoustic wave equation, the angle decomposition for vector (elastic) wavefields. Thus, we can distinguish angle decomposition for scalar (acoustic) wavefields and type of wavefields involved in imaging. Thus, we can distinguish angle decomposition takes different forms corresponding to the procedure which is often referred to as angle decomposition. An- components corresponding to various angles of incidence, pro-

4.2 Imaging with scalar and vector potentials

An extended imaging condition can also be designed for elastic wavefields decomposed in scalar and vector potentials, similarly to the conventional imaging procedure described in the preceding section. Therefore, an extended image constructed from scalar and vector potentials is

\[ I_{ij}(x, \lambda, \tau) = \int \alpha_{si}(x - \lambda, t - \tau) \alpha_{rj}(x + \lambda, t + \tau) \, dt, \]

where the quantities \( \alpha_{si} \) and \( \alpha_{rj} \) stand for the various wave modes \( \alpha = \{ P, S_1, S_2 \} \) of the source and receiver wavefields, and \( \lambda \) and \( \tau \) stand for cross-correlation lags in space and time, respectively.

5 ANGLE DECOMPOSITION

As indicated earlier, the main uses of images constructed using extended imaging conditions are migration velocity analysis (MVA) and amplitude versus angle analysis (AVA). Such analysis, however, require that the images be decomposed in components corresponding to various angles of incidence, procedure which is often referred to as angle decomposition. Angle decomposition takes different forms corresponding to the type of wavefields involved in imaging. Thus, we can distinguish angle decomposition for scalar (acoustic) wavefields and angle decomposition for vector (elastic) wavefields.

5.1 Scalar wavefields

For the case of imaging with the acoustic wave equation, the reflection angle corresponding to incidence and reflection of P wave modes can be constructed after imaging using mapping based on the relation

\[ \tan \theta_s = \frac{k_\lambda}{k_s}, \]

where \( \theta_s \) is the incidence angle and \( k_\lambda = k_r + k_a \) and \( k_s = k_r - k_a \) are defined using the source and receiver wavenumbers, \( k_a \) and \( k_r \). The information required for decomposition of the reconstructed wavefields as a function of wavenumbers \( k_s \) and \( k_\lambda \) is readily available in the images \( I(x, \lambda, \tau) \) constructed by extended imaging conditions 9 or 10. Following angle decomposition, the image \( I(x, \theta, \phi) \) represents a mapping of the image \( I(x, \lambda, \tau) \) from offsets to angles. In other words, all information for characterizing angle-dependent reflectivity is already available in the image obtained by the extended imaging conditions.

5.2 Vector wavefields

A similar approach can be used for decomposition of the reflectivity as a function of incidence and reflection angles for elastic wavefields imaged with extended imaging conditions 9 or 10. The angle \( \theta_c \) characterizing the average angle between incidence and reflected rays can be computed using the expression:

\[ \tan^2 \theta_c = \frac{(1 + \gamma) |k_\lambda|^2 - (1 - \gamma) |k_s|^2}{(1 + \gamma) |k_s|^2 - (1 - \gamma) |k_\lambda|^2}, \]

where \( \gamma \) is the velocity ratio of the incident and reflected waves, e.g. \( v_r/v_p \) ratio for incident P mode and reflected S mode. Figure 1 shows the schematic and the notations used in the above formula. The angle decomposition equation 12 designed for PS reflections reduces to equation 11 for PP reflections when \( \gamma = 1 \).

Angle decomposition using equation 12 requires computation of extended imaging condition with 3-D space lags \( (\lambda_x, \lambda_y, \lambda_z) \), which is computationally costly. Faster computation can be done if we avoid computing the vertical lag \( \lambda_z \), in which case the angle decomposition can be done using the expression (Sava & Fomel, 2006a):

\[ \tan \theta_c = \frac{(1 + \gamma) (a_{\lambda_x} + b_x)}{2 \gamma k_z + \sqrt{4 \gamma^2 k_z^2 + (\gamma^2 - 1) (a_{\lambda_x} + b_x) (a_x + b_\lambda)}}. \]

where \( a_{\lambda_x} = (1 + \gamma) k_{\lambda_x}, \) \( a_x = (1 + \gamma) k_x, \) \( b_{\lambda_x} = (1 - \gamma) k_{\lambda_x}, \) and \( b_x = (1 - \gamma) k_x. \) Figure 2 shows a model of five reflectors and the extracted angle gathers for these re-
Figure 2. (a) Model showing one shot over multiple reflectors dipping at 0°, 15°, 30°, 45°, and 60°. The vertical dashed line shows a CIG location. Incidence ray is vertically down and P to S conversions are marked by arrowed lines pointing away from reflectors. (b) Converted wave angle gather obtained from algorithm described by Sava & Fomel (2006a). Notice that converted wave angles are always smaller than incidence angle (in this case, the dip of the reflector) except for normal incidence.

6 EXAMPLES

We test the different imaging conditions discussed in the preceding sections with data simulated on a modified subset of the Marmousi II model (Martin et al., 2002). The section is chosen to be at the left side of the entire model which is relatively simple and therefore easier to examine the quality of the images.

6.1 Imaging with vector displacements

Consider the images obtained for the model depicted in Figures 3(a) & 3(b). Figure 3(a) depicts the P-wave velocity (smooth function between 1.6 - 3.2 km/s), and Figure 3(b) shows the density (variable between 1 - 2 g/cm³). The S-wave velocity is a scaled version of the P-wave velocity with \( v_P/v_s = 2 \). Data are modeled and migrated in the smooth velocity background to avoid back-scattering and the migration density is constant throughout the model. Elastic data, Figures 4(a) & 4(b), are simulated using a space-time staggered-grid finite-difference solution to the isotropic elastic wave equation (Virieux, 1984, 1986; Mora, 1987, 1988). We simulate data for a source located at position \( x = 6.75 \) km and \( z = 0.5 \) km. Since we are using an explosive source and the background velocity is smooth, the simulated wavefield is represented mainly by P-wave incident energy and the receiver wavefield is represented by a combination of P- and S-wave reflected energy. The data contain a mix of P and S modes, as can be seen by comparing the vertical and horizontal displacement components, shown in Figures 4(a) & 4(b), with the separated P and S wave modes, shown in Figures 4(c) & 4(d).

Imaging the data shown in Figures 4(a) & 4(b) using the imaging condition 3, we obtain the images depicted in Figures 5(a) & 5(d). Since the input data do not represent separated wave modes, the images produced with the imaging condition based on vector displacements do not separate PP and PS reflectivity. Thus, the images are hard to interpret, since it is not clear what incident and reflected wave mode the reflections correspond to. In reality, reflections corresponding to all wave modes are present in all panels.

6.2 Imaging with scalar and vector potentials

Consider the images (Figures 6) obtained for imaging condition equations 7 applied to the data (Figures 4(a) and 4(b)) used for the preceding example. Given the explosive source used in our simulation, the source wavefield contains mostly
P-wave energy, while the receiver wavefield contains P- and S-wave mode energy. Helmholtz decomposition after extrapolation but prior to imaging isolates P and S wavefield components. Therefore, migration produces images of reflectivity corresponding to PP and PS reflections, Figures 6(a) and 6(b), but not reflectivity corresponding to SP or SS reflections, Figures 6(c) and 6(d). The illumination regions are different between PP and PS images, due to different illumination angles of the two propagation modes for the given acquisition geometry. The PS image, Figure 6(b), also shows the usual polarity reversal for positive and negative angles of incidence measured relative to the reflector normal.

6.3 Angle decomposition

The images shown in the preceding subsection correspond to the conventional imaging conditions 3 and 7. We can construct other images using the extended imaging conditions 9 and 10, which can be used for angle decomposition after imaging. Then, we can use equation 13 to compute angle gathers from horizontal space cross-correlation lags.

Figures 7(a) & 7(c) and Figures 7(b) & 7(d) show, respectively, the PP and PS horizontal lags and angle gathers for the common image gather (CIG) location in the middle of the reflectivity model, given a single source at $x = 6.75$ km. PP and PS horizontal lags are lines dipping at angles related to the incidence angle at the CIG location. PP angles are larger than PS angles at all reflectors, as illustrated on the simple synthetic example shown in Figure 2.

Figures 8(a) & 8(c) and Figures 8(b) & 8(d) show, respectively, the PP and PS horizontal lags and angle gathers for the same CIG location, given many sources from $x = 5.5$ to 7.5 km. The horizontal space cross-correlation lags are focused around $\lambda = 0$, which justifies the use of conventional imaging condition extracting the cross-correlation of the source and receiver wavefields at zero lag in space and time. Thus, the zero lag of the images obtained by extended imaging condition represent the image at the particular CIG location. The PP and PS gathers for many sources are flat, since the migration was done with correct migration velocity. The PS angle gather, depicted in Figure 8(d), shows a polarity reversal at $\theta = 0$, which is consistent with the fact that PS images change polarity at normal incidence.

7 CONCLUSIONS

The elastic reverse time migration and related imaging conditions can be used for imaging and constructing angle domain common image gathers. Those techniques can be utilized in processing of land, ocean-bottom (OBC) and VSP multicomponent data.

The elastic reverse-time migration imaging condition can be formulated using decomposition of extrapolated wavefields in P and S wave modes. The formed images separate reflections corresponding to forward-propagating P or S modes and backward propagating P or S modes. In contrast, images formed by simple cross-correlation of displacement wavefield components mix contributions from P and S reflections and are harder to interpret. Artifacts caused by back-propagating the recorded data with displacement sources are present in both types of images, although they are easier to distinguish and attenuate on the images constructed with elastic components separated prior to imaging.

Extended imaging conditions can be applied to multicomponent images constructed using vector potentials. Angle decomposition allows for separation of reflectivity function of incidence and reflection angles. Angle-dependent reflectivity can be used for migration velocity analysis (MVA) and amplitude versus angle (AVA) analysis, which are extensions that fall outside the scope of this paper.

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Figure 4. Elastic data simulated in model 3(a) and 3(b) with a source at $x = 6.75$ km and $z = 0.5$ km, and receivers at $z = 0.5$ km: (a) vertical component, (b) horizontal component, (c) scalar potential and (d) vector potential of the elastic wavefield. Both vertical and horizontal components, panels (a) and (b), contain a mix of P and S modes, as seen by comparison with panels (c) and (d).
Figure 5. Images produced with the displacement components imaging condition, equation 3. (a), (b), (c) and (d) correspond to the cross-correlation of the vertical and horizontal components of the source wavefield with the vertical and horizontal components of the receiver wavefield, respectively. The image corresponds to one shot at position \( x = 6.75 \) km and \( z = 0.5 \) km. Receivers are located at all locations at \( z = 0.5 \) km.

Figure 6. Images produced with the scalar and vector potentials imaging condition, equation 7. (a), (b), (c) and (d) correspond to the cross-correlation of the P and S components of the source wavefield with the P and S components of the receiver wavefield, respectively. The image corresponds to one shot at position \( x = 6.75 \) km and \( z = 0.5 \) km. Receivers are located at all locations at \( z = 0.5 \) km.
Figure 7. Horizontal cross-correlation lags for (a) PP and (c) PS reflections for model in Figures 3(a) and 3(b). The source is at $x = 6.75$ km, and the CIG is located at $x = 6.5$ km. Panels (b) and (d) depict PP and PS angle gathers decomposed from the horizontal lag gathers in panels (a) and (c), respectively. As expected, PS angles are smaller than PP angles for a particular reflector due to smaller reflection angles.
Figure 8. Horizontal cross-correlation lags for PP (a) and PS (c) reflections for model in Figures 3(a) and 3(b). These CIGs correspond to 81 sources from $x = 5.5$ to $7.5$ km at $z = 0.5$ km. The CIG is located at $x = 6.5$ km. Panels (b) and (d) depict PP and PS angle gathers decomposed from the horizontal lag gathers in panels (a) and (c), respectively. Since the velocity used for imaging is correct, the PP and PS gathers are flat. The PP angle gathers do not change polarity at normal incidence, but the PS angle gathers change polarity at normal incidence.
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Elastic reverse-time migration 141


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Elastic wavefield separation for VTI media

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ABSTRACT

The separation of wave modes from isotropic elastic wavefields is typically done using Helmholtz decomposition. However, Helmholtz decomposition using conventional divergence and curl operators in anisotropic media does not give satisfactory results and leaves the different wave modes only partially separated. The separation of anisotropic wavefields requires the use of more sophisticated operators which depend on local material parameters. We construct operators for anisotropic wavefield separation based on the polarization vectors evaluated at each point of the medium by solving the Christoffel equation using local medium parameters. These polarization vectors can be represented in the space domain as localized convolutional operators, which resemble conventional derivative operators. The spatially-variable “pseudo” derivative operators perform well in heterogeneous VTI media even at places where the media are changing rapidly. Synthetic results indicate that the operators can be used to separate wavefields for VTI media with arbitrary degree of anisotropy. This methodology is applicable for elastic reverse time migration (RTM) in heterogeneous anisotropic media.

Key words: elastic, imaging, anisotropic

1 INTRODUCTION

Wave equation migration for elastic data usually consists of two steps. The first step is wavefield reconstruction in the subsurface from data recorded at the surface. The second step is an imaging condition which extracts reflectivity information from the reconstructed wavefields.

The elastic wave equation migration can be implemented in two ways. The first approach is to separate recorded elastic data into compressional and transverse (P and S) modes, and use these separated modes for acoustic wave equation migration respectively. This acoustic imaging approach to elastic waves is more frequently done, but it is based on the assumption that P and S data can be successfully separated on the surface, which is not always true (Etgen, 1988; Zhe & Greenhalgh, 1997). The second approach is to not separate P and S modes on the surface, extrapolate the entire elastic wavefield at once, then separate wave the modes prior to applying an imaging condition. The reconstruction of elastic wavefields can be implemented using various techniques, including reconstruction by time reversal (RTM) (Chang & McMechan, 1986, 1994) or by Kirchhoff integral techniques (Hokstad, 2000). The imaging condition (I.C.) applied to the reconstructed vector wavefields directly determines the quality of the images. Conventional cross-correlation imaging condition does not separate the wave modes and cross-correlates the Cartesian components of the elastic wave components. In general, the various wave modes (P and S) are mixed on all wavefield components, and cause crosstalk and image artifacts. Yan & Sava (2007) suggest using imaging conditions based on elastic potentials, which require cross-correlation of separated modes. Potential-based imaging condition creates images that have clear physical meanings, in contrast with images obtained with wavefield components, thus justifying the need for wave mode separation.

As the need for anisotropic imaging increases, more and more processing and migration are performed based on anisotropic acoustic one-way wave equations (Alkhalifah, 1998; Shan, 2006; Shan & Biondi, 2005). However, much less research has been done on anisotropic elastic migration based on two-way wave equations. Elastic Kirchhoff migration (Hokstad, 2000) obtains pure-mode and converted mode images by downward continuation of elastic vector wavefields with a visco-elastic wave equation. The wavefield separation is effectively done with elastic Kirchhoff integration, which handles both P- and S-waves. However, Kirchhoff migration does no perform well in areas of complex geology where ray theory breaks down (Gray et al., 2001), thus requiring migration with more accurate methods, e.g., reverse time migration.

One of the complexities that impedes elastic wave equa-
ion anisotropic migration is the difficulty to separate anisotropic wavefields into different wave modes. However, the proper separation of anisotropic wave modes is as important for anisotropic elastic migration as is the separation of isotropic wave modes for isotropic elastic migration. The main difference between anisotropic and isotropic wavefield separation is that Helmholtz decomposition is only suitable for the separation of isotropic wavefields, and does not work well for anisotropic wavefields.

In this abstract, we show how to construct wavefield separators for VTI (vertical transverse isotropy) media applicable to models with spatially varying parameters. We apply these operators to anisotropic elastic wavefields and show that they successfully separate anisotropic wave modes, even for extremely anisotropic media.

2 SEPARATION METHOD

Separation of scalar and vector potentials can be achieved by Helmholtz decomposition, which is applicable to any vector field \( \mathbf{W}(x, y, z) \). By definition, the vector wavefield \( \mathbf{W} \) can be decomposed into a curl-free scalar potential \( \Theta \) and a divergence-free vector potential \( \Psi \) according to the relation:

\[
\mathbf{W} = \nabla \Theta + \nabla \times \Psi .
\]  

Equation 1 is not used directly in practice, but the scalar and vector components are obtained indirectly by the application of the \( \nabla \cdot \) and \( \nabla \times \) operators to the extrapolated elastic wavefield:

\[
P = \nabla \cdot \mathbf{W} ,
\]

\[
S = \nabla \times \mathbf{W} .
\]  

For isotropic elastic fields far from the source, quantities \( P \) and \( S \) describe compressional and transverse wave modes, respectively (Aki & Richards, 2002).

Equations 2 and 3 allows us to understand why \( \nabla \cdot \mathbf{W} \) and \( \nabla \times \mathbf{W} \) pass compressional and transverse wave modes, respectively. In the space domain, we can write:

\[
P = \nabla \cdot \mathbf{W} = D_x \ast W_x + D_y \ast W_y + D_z \ast W_z ,
\]

where \( D_x, D_y \) and \( D_z \) represent derivatives in \( x, y \) and \( z \) directions and \( \ast \) represents spatial convolution. In the Fourier domain, we can represent the operators \( D_x, D_y \) and \( D_z \) by \( ik_x, ik_y \) and \( ik_z \), therefore, we can write an equivalent expression to equation 4 as:

\[
P = i k \cdot \mathbf{W} = i k_x \widehat{W}_x + i k_y \widehat{W}_y + i k_z \widehat{W}_z ,
\]  

where \( k = \{k_x, k_y, k_z\} \) represents the wave vectors, and \( \mathbf{W}(k_x, k_y, k_z) \) is the 3D Fourier transform of the wavefield \( \mathbf{W}(x, y, z) \). We see that in this domain, the operator \( i k \) essentially projects the wavefield \( \mathbf{W} \) onto the wave vector \( k \), which represents the polarization direction for \( P \) waves. Similarly, the operator \( \nabla \times \) projects the wavefield onto the direction orthogonal to the wave vector \( k \), which represents the polarization direction for \( S \) waves (Dellinger & Etgen, 1990). For illustration, Figure 1(a) shows the polarization vectors of the \( P \) mode of an isotropic model as a function of normalized \( k_x \) and \( k_z \) ranging from 0 to 0.5 cycles. The polarization vectors are radial because \( P \) waves in an isotropic medium are polarized in the same directions as the wave vectors.

For isotropic and anisotropic media, the polarization vectors are different. Figure 1(b) shows the polarization vectors of the \( S \) mode for a VTI model with \( V_P = 3 \) km/s and \( V_S = 1.5 \) km/s, \( \epsilon = 0.25 \) and \( \delta = -0.29 \).

\( \epsilon \) is the ellipticity of the wave vector and \( \delta \) is the delay in the wave vector. The polarization vectors are not radial because \( S \) waves in an anisotropic medium are polarized in different directions for the \( P \) and \( S \) modes. This requires that we modify the wave equation for anisotropic media.
Anisotropic wave-mode separation

Figure 2. A 1.2 km x 1.2 km model that has parameters (a) \( V_{S0} = 3 \text{ km/s} \) except for a low Gaussian anomaly at \( x = 0.65 \text{ km} \) and \( z = 0.65 \text{ km} \), (b) \( V_{20} = 1.5 \text{ km/s} \) except for a low Gaussian anomaly at \( x = 0.65 \text{ km} \) and \( z = 0.65 \text{ km} \), (c) \( \rho = 1 \text{ g/cm}^3 \) in the top layer and 2 g/cm\(^3\) in the bottom layer, (d) \( \epsilon \) smoothly varying from 0 to 0.25 from top to bottom, (e) \( \delta \) smoothly varying from 0 to −0.29 from left to right. The source is located at \( x = 0.6 \text{ km} \) and \( z = 0.6 \text{ km} \).

We can extend the procedure described in the preceding section to heterogeneous media by computing a different set of waveforms at every grid point. In the symmetry planes of VTI media, equation 8 becomes

\[
\left[ G - \rho V^2 I \right] U = 0 ,
\]

which allows us to compute the components of the polarization vector \( U \) (the eigenvectors of the matrix \( G \)) of a given wave mode given the stiffness tensor and density at every location of the medium.

We can extend the procedure described in the preceding section to heterogeneous media by computing a different set of waveforms at every grid point. In the symmetry planes of VTI media, the operators are 2D and depend on the local values of the stiffness coefficients. For each point, we pre-compute...
the polarization vectors as a function of the local medium parameters, and transform them to the space domain to obtain the wave mode separators. If we represent the stiffness coefficients using Thomsen parameters (Thomsen, 1986), then the pseudo derivative operators \( L_x \) and \( L_z \) depend on \( \epsilon \), \( \delta \), \( V_P^0 \) and \( V_S^0 \), which are in general spatially dependent parameters. We can compute and store the operators for each grid point in the medium, and then use those operators to separate P and S modes from reconstructed elastic wavefields. Thus, wavefield separation in VTI media can be achieved simply by nonstationary convolution with operators \( L_x \) and \( L_z \).

3 EXAMPLES

We consider a 2D isotropic model characterized by the \( V_P \), \( V_S \) and density shown in Figures 2(a)–2(c). The model contains negative P and S velocity anomalies that triplicate the wavefields. The source is located in the center of the model. Figure 4(a) shows the vertical and horizontal components of the simulated elastic data. Figure 4(b) shows the separation to P and S modes using \( \nabla \cdot \) and \( \nabla \times \) operators, and Figure 4(c) shows the mode separation obtained using the pseudo operators dependent on the medium parameters. A comparison of Figures 4(b) and 4(c) indicates that the \( \nabla \cdot \) and \( \nabla \times \) operators and the pseudo operators work identically well for isotropic media.

We consider a 2D anisotropic model analogous to the one described by the parameters shown in Figures 2(a)–2(c) (with \( V_P \), \( V_S \) representing the vertical P and S wave velocities), and additionally characterized by the parameters \( \epsilon \) and \( \delta \) shown in Figures 2(d) and 2(e). The parameters \( \epsilon \) and \( \delta \) vary gradually from top to bottom and left to right, respectively. The upper left part of the medium is isotropic and the lower right part is highly anisotropic. Since the difference of \( \epsilon \) and \( \delta \) is large at the bottom part of the model, the S waves in that region are severely triplicated due to this strong anisotropy.

Figure 3 illustrates the pseudo derivative operators obtained at different locations in the model defined by the intersections of \( x \) coordinates 0.3, 0.6, 0.9 km and \( z \) coordinates 0.3, 0.6, 0.9 km. Since the operators correspond to different combination of the parameters \( \epsilon \) and \( \delta \), they have different forms. The isotropic operators are purely vertical and horizontal, while the anisotropic operators have tails radiating from the center.

Figure 5(a) shows the vertical and horizontal components of the simulated elastic anisotropic data, Figure 5(b) shows the separation to P and S modes using conventional \( \nabla \cdot \) and \( \nabla \times \) operators, and Figure 5(c) shows the mode separation obtained using the pseudo operators constructed using the local medium parameters. A comparison of Figure 4(b) and 4(c) in-
Figure 4. (a) Isotropic wavefield modeled with a vertical source at $x=0.6 \text{ km}$ and $z=0.6 \text{ km}$, isotropic wave modes separated by (b) $\nabla \cdot$ and $\nabla \times$ and (c) pseudo derivative operators.
Figure 5. (a) Anisotropic wavefield modeled with a vertical source at $x=0.6$ km and $z=0.6$ km, anisotropic wave modes separated by (b) $\nabla \cdot$ and $\nabla \times$ and (c) pseudo derivative operators.
4 CONCLUSIONS

We present a method of obtaining spatially-varying pseudo derivative operators with application to wave mode separation in anisotropic media. The main idea is to utilize polarization vectors constructed in the wavenumber domain using the local media parameters at specific locations and then transform back to the space domain. The main advantage of applying the pseudo derivative operators in the space domain is that they can be used for heterogeneous media. The wave mode separators obtained using the method described in this abstract are spatially-variable convolutional operators and can be used to separate wavefields in VTI media with arbitrary degree of anisotropy. This methodology is applicable for elastic RTM in heterogeneous anisotropic media.

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Seismic imaging with Wigner distribution functions

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ABSTRACT

The fidelity of depth seismic imaging depends on the accuracy of the velocity models used for wavefield reconstruction. Models can be decomposed into two components corresponding to large scale and small scale variations. In practice, the large scale velocity model component can be estimated with high accuracy using repeated migration/tomography cycles, but the small scale component cannot. When the Earth has significant small-scale velocity components, wavefield reconstruction does not completely describe the recorded data and migrated images are perturbed by artifacts.

There are two possible ways to address this problem: improve wavefield reconstruction by estimating more accurate velocity models and image using conventional techniques (e.g. wavefield cross-correlation), or reconstruct wavefields with conventional methods using the known background velocity model, but improve the imaging condition to alleviate the artifacts caused by the imprecise reconstruction, which is what we suggest in this paper.

We describe the unknown component of the velocity model as a random function with local spatial correlations. Imaging data perturbed by such random variations is characterized by statistical instability, i.e. various wavefield components image at wrong locations that depend on the actual realization of the random model. Statistical stability can be achieved by pre-processing the reconstructed wavefields prior to the imaging condition. We employ Wigner distribution functions to attenuate the random noise present in the reconstructed wavefields, parametrized as a function of image coordinates. Wavefield filtering using Wigner distribution functions and conventional imaging can be lumped-together into a new form of imaging condition which we call an “interferometric imaging condition” due to its similarity to concepts from recent work on interferometry. The interferometric imaging condition can be formulated both for zero-offset and for multi-offset data, leading to robust and efficient imaging procedures that are effective in attenuating imaging artifacts due to unknown velocity models.

Key words: wave-equation, imaging, Wigner distribution functions

1 INTRODUCTION

Seismic imaging in complex media requires accurate knowledge of the medium velocity. Assuming single scattering (Born approximation), imaging requires propagation of the recorded wavefields from the acquisition surface, followed by the application of an imaging condition highlighting locations where backscattering occurs, i.e. where reflectors are present. Typically, this is achieved with simple image processing techniques, e.g. cross-correlation of wavefields reconstructed from sources and receivers.

The main requirement for good-quality imaging is accurate knowledge of the velocity model. Errors in the model used for imaging lead to inaccurate reconstruction of the seismic wavefields and to distortions of the migrated images. In any realistic seismic field experiment the velocity model is never known exactly. Migration velocity analysis estimates large scale approximations of the model, but some fine scale variations always remain elusive. For example, when geology includes complicated stratigraphic structures or complex salt/carbonate bodies, the rapid velocity variations on the scale of the seismic wavelength and smaller cannot be estimated correctly by kinematic methods. Therefore, even if the broad kinematics of the seismic wavefields are reconstructed cor-
rectly, the extrapolated wavefields also contain phase and amplitude distortions that lead to image artifacts obstructing the image of the geologic structure under consideration. While it is certainly true that even the recovery of a long-wave background may prove to be a challenge in some circumstances, we do not attempt to address that issue in this paper. Instead, we concentrate solely on the problem of dealing with the effect of a small scale random variations not estimated by conventional methods.

There are two ways in which we can approach this problem. The first option is to improve our velocity analysis methods to estimate the small-scale variations in the model. Such techniques take advantage of all information contained in seismic wavefields and are not limited to kinematic information of selected events picked from the data. Examples of techniques in this category are waveform inversion (Tarantola, 1987; Pratt & Worthington, 1990; Pratt, 1990; Sipghe & Pratt, 2004), wave-equation tomography (Woodward, 1992) or wave-equation migration velocity analysis (Sava & Biondi, 2004a,b; Shen et al., 2005). A more accurate velocity model allows for more accurate wavefield reconstruction. Then, wavefields can be used for imaging using conventional procedures, e.g. cross-correlation. The second option is to concentrate on the problem of dealing with the effect of random small-scale variations not estimated by conventional procedures, e.g. cross-correlation. The goal is to design an imaging condition that alleviates artifacts caused by those random fluctuations. Conventional imaging consists of cross-correlations of extrapolated source and receiver wavefields at image locations. Since wavefield extrapolation is performed using an approximation of the true model, the wavefields contain random time delays, or equivalently random phases, which lead to imaging artifacts.

One way of mitigating the effects of the random model on the quality of the resulting image is to use techniques based on acoustic time reversal (Fink, 1999). Under certain assumptions, a signal sent through a random medium, recorded by a receiver array, time reversed and sent back through the same medium, refracts at the source location in a statistically stable fashion. Statistical stability means that the refocusing properties (i.e. image quality) are independent of the actual realization of the random medium (Papanicolaou et al., 2004; Fouque et al., 2005).

We investigate an alternative way of increasing imaging statistical stability. Instead of imaging the reconstructed wavefields directly, we first apply a transformation based on Wigner distribution functions (Wigner, 1932) to the reconstructed wavefields. We consider a special case of the Wigner distribution function (WDF) which has the property that it attenuates random fluctuations from the wavefields after extrapolation with conventional techniques. The idea for this method is borrowed from image processing where WDFs are used for filtering of random noise. Here, we apply WDFs to the reconstructed wavefields, prior to the imaging condition. This is in contrast to data filtering prior to wavefield reconstruction or to image filtering after the application of an imaging condition.

Our procedure closely resembles conventional imaging procedures where wavefields are extrapolated in the image volume and then cross-correlated in time at every image location. Our method uses WDFs defined in three-dimensional windows around image locations which makes it both robust and efficient. From an implementation and computational cost point of view, our technique is similar to conventional imaging, but its statistical properties are improved. Although conceptually separate, we can lump-together the WDF transformation and conventional imaging into a new form of imaging condition which resembles interferometric techniques (Papanicolaou et al., 2004; Fouque et al., 2005). Therefore, we use the name interferometric imaging condition for our technique to contrast it with the conventional imaging condition.

A related method discussed in the literature is known under the name of coherent interferometric imaging (Borcea et al., 2006a,b,c). This method uses similar local cross-correlations and averaging, but unlike our method, it parametrizes reconstructed wavefields as a function of receiver coordinates. Thus, the coherent interferometric imaging functional requires separate wavefield reconstruction from every receiver position, which makes this technique prohibitively expensive and probably unusable in practice on large-scale seismic imaging projects. In contrast, the imaging technique advocated in this paper achieves similar statistical stability properties as coherent interferometric imaging, but at an affordable computational cost since we apply wavefield reconstruction only once for all receiver locations corresponding to a given seismic experiment, typically a “shot”.

2 IMAGING CONDITIONS

2.1 Conventional imaging condition

Let $D(x,t)$ be the data recorded at time $t$ at receivers located at coordinates $x$ for a seismic experiment with buried sources (Figure 1(a)) also known as an exploding reflector seismic experiment (Loewenthal et al., 1976). A conventional imaging procedure for this type of data consists of two steps (Claerbout, 1985): wavefield reconstruction at image coordinates $y$ from data recorded at receiver coordinates $x$, followed by an imaging condition taking the reconstructed wavefield at time $t = 0$ as the seismic image.

Mathematically, we can represent the wavefield $V$ reconstructed at coordinates $y$ from data $D$ recorded at coordinates...
\[ V(\mathbf{x}, y, t) = D(\mathbf{x}, t) \ast_t G(\mathbf{x}, y, t) \]  

The total wavefield \( U \) reconstructed at \( y \) from all receivers is the superposition of the wavefields \( V \) reconstructed from individual traces:

\[ U(\mathbf{y}, t) = \int d\mathbf{x} \, V(\mathbf{x}, \mathbf{y}, t) \]  

where the integral over \( \mathbf{x} \) spans the entire receiver space. As stated, the imaging condition extracts the image \( R(\mathbf{y}) \) from the wavefield \( U(\mathbf{y}, t) \) at time \( t = 0 \), i.e.

\[ R(\mathbf{y}) = U(\mathbf{y}, t = 0) \]  

The Green’s function used in the procedure described by equation 1 can be implemented in different ways. For our purposes, the actual method used for computing Green’s functions is not relevant. Any procedure can be used, although different procedures will be appropriate in different situations, with different cost of implementation. We assume that a satisfactory procedure exists and is appropriate for the respective velocity models used for the simulations. In our examples, we compute Green’s functions with time-domain finite-difference solutions to the acoustic wave-equation, similar to reverse-time migration (Baysal et al., 1983).

Consider the velocity model depicted in Figure 2(a) and the imaging target depicted in Figure 2(b). We assume that the model with random fluctuations (Figure 3(b)) represents the real subsurface velocity and use this model to simulate data. We consider the background model (Figure 3(a)) to represent the migration velocity and use this model to migrate the data simulated in the random model. We consider one source located in the subsurface at coordinates \( z = 8 \) km and \( x = 13.5 \) km, and receivers located close to the top of the model at discrete horizontal positions and depth \( z = 0.0762 \) km. Figures 3(c)-3(d) show snapshots of the simulated wavefields at a later time. The panels on the left correspond to modeling in the background model, while the panels on the right correspond to modeling in the random model.

Figure 3(f) shows the data recorded on the surface. The direct wavefield arrival from the seismic source is easily identified in the data, although the wavefronts are distorted by the random perturbations in the medium. For comparison, Figure 3(e) shows data simulated in the background model, which do not show random fluctuations.

Conventional imaging using the procedure described above implicitly states that data generated in the random model, Figure 3(b), are processed as if they were generated in the background model, Figure 3(a). Thus, the random phase variations in the data are not properly compensated during the imaging procedure causing artifacts in the image. Figure 6(b) shows the image obtained by migrating data from Figure 3(f) using the model from Figure 3(a). For comparison, Figure 6(a) shows the image obtained by migrating the data from Figure 3(e) using the same model from Figure 3(a).

Ignoring aperture effects, the artifacts observed in the im-

---

**Figure 1.** Zero-offset seismic experiment sketch (a) and multi-offset seismic experiment sketch (b). Coordinates \( \mathbf{x} = \{x, y\} \) characterize receiver positions on the surface and coordinates \( \mathbf{y} = \{x, y, z\} \) characterize reflector positions in the subsurface.
ages are caused only by the fact that the velocity models used for modeling and migration are not the same. Small artifacts caused by truncation of the data on the acquisition surface can also be observed, but those artifacts are well-known (i.e. truncation butterflies) and are not the subject of our analysis. In this example, the wavefield reconstruction procedure is the same for both modeling and migration (i.e. time-domain finite-difference solution to the acoustic wave-equation), thus it is not causing artifacts in the image. We can conclude that the migration artifacts are simply due to the phase errors between the Green’s functions used for modeling (with the random velocity) and the Green’s functions used for migration (with the background velocity). The main challenge for imaging in media with random variations is to design procedures that attenuate the random phase delays introduced in the recorded data by the unknown variations of the medium without damaging the real reflections present in the data.

The random phase fluctuations observed in recorded data (Figure 3(f)) are preserved during wavefield reconstruction using the background velocity model. We can observe the randomness in the extrapolated wavefields in two ways, by reconstructing wavefields using individual data traces separately, or by reconstructing wavefields using all data traces at once.

The first option is to reconstruct the seismic wavefield at all image locations \( y \) from \textit{individual} receiver positions on the surface. Of course, this is not conventionally done in reverse-time imaging, but we describe this concept just for illustration purposes. Figure 4(a) shows the wavefield reconstructed separately from individual data traces depicted in Figure 3(f) using the background model depicted in Figure 3(b). In Figure 4(a), the horizontal axis corresponds to receiver positions on the surface, i.e. coordinates \( x \), and the vertical axis represents time. According to the notations used in this paper, Figure 4(a) corresponds to wavefield \( V(x, y, t) \) reconstructed from data at all receiver coordinates \( x \) to image coordinates \( y \). The reconstructed wavefield does not focus completely at the image coordinate and time \( t = 0 \) indicating that the input data contains random phase delays that are not compensated during wavefield reconstruction using the background velocity.

The second option is to reconstruct the seismic wavefield at all image locations \( y \) from \textit{all} receiver positions on the surface at once. This is a conventional procedure for reverse-time imaging. Figure 5(a) shows the wavefield reconstructed from all data traces depicted in Figure 3(f) using the background model depicted in Figure 3(b). In Figure 5(a), the vertical and horizontal axes correspond to depth and horizontal positions around the source, i.e. coordinates \( y \), and the third cube axis represents time. According to the notations used in this paper, Figure 5(a) corresponds to wavefield \( U(y, t) \) reconstructed from data at all receiver coordinates \( x \) to image coordinates \( y \). The reconstructed wavefield does not focus completely at the image coordinate and time \( t = 0 \) indicating that the input data contains random phase delays that are not compensated during wavefield reconstruction using the background velocity.

2.2 Wigner distribution functions

One possible way to address the problem of random fluctuations in reconstructed wavefields is to use Wigner distribution functions (Wigner, 1932) to pre-process the wavefields prior to the application of the imaging condition. Appendix C provides a brief introduction for readers unfamiliar with Wigner distribution functions. More details about this topic are presented by Cohen (1995).

Wigner distribution functions (WDF) are bi-linear representations of multi-dimensional signals defined in phase space, i.e. they depend simultaneously on position-wavenumber \((y - k)\) and time-frequency \((t - \omega)\). Wigner (1932) developed these concepts in the context of quantum physics as probability functions for the simultaneous description of coordinates and momenta of a given wave function. WDFs were introduced to signal processing by Ville (1948) and have since found many applications in signal and image processing, speech recognition, optics, etc.

A variation of WDFs, called \textit{pseudo Wigner distribution functions} are constructed using small windows localized in space and/or time (Appendix C). Pseudo WDFs are
Figure 3. Seismic snapshots of acoustic wavefields simulated in the background velocity model (a)-(c), and in the random velocity model (b)-(d). Data recorded at the surface from simulation in the background velocity model (e) and from the simulation in the random velocity model (f).
simple transformations with efficient application to multi-dimensional signals. In this paper, we apply the pseudo WDF transformation to multi-dimensional seismic wavefields obtained by reconstruction from recorded seismic data. We use pseudo WDFs for decomposition and filtering of extrapolated space-time signals as a function of their local wavenumber-frequency. In particular, pseudo WDFs can filter reconstructed wavefields to retain their coherent components by removing high-frequency noise associated with random fluctuations in the wavefields due to random fluctuations in the model.

The idea for our method is simple: instead of imaging the reconstructed wavefields directly, we first filter them using pseudo WDFs to attenuate the random phase noise, and then proceed to imaging using a conventional or an extended imaging conditions. Wavefield filtering occurs during the application of the zero-frequency end-member of the pseudo WDF transformation, which reduces the random character of the field. For the rest of the paper, we use the abbreviation WDF to denote this special case of pseudo Wigner distribution functions, and not its general form.

As we described earlier, we can distinguish two options. The first option is to use wavefield parametrization as a function of data coordinates \( x \). In this case, we can write the pseudo WDF of the reconstructed wavefield \( V(x, y, t) \) as

\[
V_x(x, y, t) = \int_{[x_h] \leq X} \int_{[y_h] \leq Y} V\left(x - \frac{x_h}{2}, y, t - \frac{t_h}{2}\right) V\left(x + \frac{x_h}{2}, y, t + \frac{t_h}{2}\right),
\]

where \( x_h \) and \( t_h \) are variables spanning space and time intervals of total extent \( X \) and \( T \), respectively. For 3D surface acquisition geometry, the 2D variable \( x_h \) is defined on the acquisition surface. The second option is to use wavefield parametrization as a function of image coordinates \( y \). In this case, we can write the pseudo WDF of the reconstructed wavefield \( U(y, t) \) as

\[
U_y(y, t) = \int_{[y_h] \leq Y} \int_{[y_h] \leq Y} U\left(y - \frac{y_h}{2}, t - \frac{t_h}{2}\right) U\left(y + \frac{y_h}{2}, t + \frac{t_h}{2}\right),
\]

where \( y_h \) and \( t_h \) are variables spanning space and time intervals of total extent \( Y \) and \( T \), respectively. For 3D surface acquisition geometry, the 3D variable \( y_h \) is defined around image positions.

For the examples used in this section, we employ 41 grid points for the interval \( X \) centered around a particular receiver position, 5 \times 5 grid points for the interval \( Y \) centered around a particular image point, and 21 grid points for the interval \( T \) centered around a particular time. These parameters are not necessarily optimal for the transformation, since they characterize the local WDF windows and depend on the specific implementation of the pseudo WDF transformation. The main criterion used for selecting the size of the space-time window for the pseudo WDF transformation is that of avoiding cross-talk between nearby events, e.g. reflections. Finding the optimal size of this window is an important consideration for our method, although its complete treatment falls outside the scope of the current paper and we leave it for future research. Preliminary results on optimal window selection are discussed by Borcea et al. (2006a).

Figure 4(b) depicts the results of applying the pseudo WDF transformation to the reconstructed wavefield in Figure 4(a). For the case of modeling in the random model and reconstruction in the background model, the pseudo WDF attenuates the random character of the wavefield significantly, Figure 4(b). The random character of the reconstructed wavefield is reduced and the main events cluster more closely around time \( t = 0 \). Similarly, Figure 5(b) depicts the results of applying the pseudo WDF transformation to the reconstructed wavefields in Figure 5(a). For the case of modeling in the random model and reconstruction in the background model, the pseudo WDF also attenuates the random character of the wavefield significantly, Figure 5(b). The random character of the reconstructed wavefields is also reduced and the main events focus at the correct image location at time \( t = 0 \).

### 2.3 Zero-offset interferometric imaging condition

After filtering the reconstructed wavefields with pseudo WDFs, we can perform imaging with normal procedures. For the case of wavefields parametrized as a function of data coordinates, we obtain the total wavefield at image coordinates by summing over receiver coordinates \( x \)

\[
W_x(y, t) = \int_x V_x(x, y, t),
\]

followed by a conventional imaging condition extracting time \( t = 0 \) from the pseudo WDF of the reconstructed wavefields:

\[
R_x(y) = W_x(y, t = 0).
\]

The image obtained with this imaging procedure is shown in Figure 6(c). As expected, the artifacts caused by the unknown random fluctuations in the model are reduced, leaving a cleaner image of the source.

Similarly, for the case of wavefields parametrized as a function of image coordinates, we obtain the image by application of the conventional imaging condition extracting time \( t = 0 \) from the pseudo WDF of the reconstructed wavefield:

\[
R_y(y) = W_y(y, t = 0).
\]

The image obtained with this imaging procedure is shown in Figure 6(d). As in the preceding case, the artifacts caused by the unknown random fluctuations in the model are reduced, producing a cleaner image of the source, comparable with the one in Figure 6(c).

### 2.4 Multi-offset interferometric imaging condition

The imaging procedure in equations 5-8 can be generalized for imaging prestack (multi-offset) data (Figure 1(b)). The con-
conventional imaging procedure for this type of data consists of two steps (Claerbout, 1985): wavefield simulation from the source location to the image coordinates \( y \) and wavefield reconstruction at image coordinates \( y \) from data recorded at receiver coordinates \( x \), followed by an imaging condition evaluating the match between the simulated and reconstructed wavefields.

Let \( U_{S}(y, t) \) be the source wavefield constructed from the location of the seismic source and \( U_{R}(y, t) \) the receiver wavefield reconstructed from the receiver locations. A conventional imaging procedure produces a seismic image as the zero-lag of the time cross-correlation between the source and receiver wavefields. Mathematically, we can represent this operation as

\[
R(y) = \int dt \ U_{S}(y, t) \ U_{R}(y, t) ,
\]

where \( R(y) \) represents the seismic image for a particular seismic experiment at coordinates \( y \). When multiple seismic experiments are processed, a complete image is obtained by summation of the images constructed for individual experiments. The actual reconstruction methods used to produce the wavefields \( U_{S}(y, t) \) and \( U_{R}(y, t) \) are irrelevant for the present discussion. As in the zero-offset/explosive reflector case, we use time-domain finite-differences to the acoustic wave-equation, but any other reconstruction technique can be applied without changing the imaging approach.

When imaging in random media, the data recorded at the surface incorporates phase delays caused by the velocity variations encountered while waves propagate in the subsurface. In a typical seismic experiment, random phase delays accumulate both on the way from the source to the reflectors, as well as on the way from the reflectors to the receivers. Therefore, the receiver wavefield reconstructed using the background velocity model is characterized by random fluctuations, similar to the ones seen for wavefields reconstructed in the zero-offset situation. In contrast, the source wavefield is simulated in the background medium from a known source position and, therefore, it is not affected by random fluctuations. However, the zero-lag of the cross-correlations between the source wavefields (without random fluctuations) and the receiver wavefield (with random fluctuations), still generates image artifacts similar to the ones encountered in the zero-offset case.

Statistically stable imaging using pseudo WDFs can be obtained in this case, too. What we need to do is attenuate the phase errors in the reconstructed receiver wavefield and then apply a conventional imaging condition. Therefore, a multi-offset interferometric imaging condition can be formulated as

\[
R(y) = \int dt \ U_{S}(y, t) \ W_{R}(y, t) ,
\]

where \( W_{R}(y, t) \) represents the pseudo WDF of the receiver wavefield \( U_{R}(y, t) \) which can be constructed, in principle, either with parametrization relative to data coordinates, according to equations 4-6, or relative to image coordinates, according to equation 5. Of course, our choice is to use image-space parametrization for computational efficiency reasons.

\section{5 Discussion}

The strategies described in the preceding section have notable similarities and differences. The imaging procedures 4-6-7 and 5-8 are similar in that they employ wavefields reconstructed from the surface data in similar ways. Neither method uses the surface recorded data directly, but they use wavefields reconstructed from those data as boundary conditions to numerical solutions of the acoustic wave-equation. The actual wavefield reconstruction procedure is identical in both cases.

The techniques are different because imaging with equations 4-6-7 employs independent wavefield reconstruction from receiver locations \( x \) to image locations \( y \). In practice, this requires separately solving the acoustic wave-equation, e.g. by time-domain finite-differences, from all receiver locations on the surface. Such computational effort is often prohibitive in practice. In contrast, imaging with equations 5-8 is similar to conventional imaging because it requires only one wavefield reconstruction using all recorded data at once, i.e. only one solution to the acoustic wave-equation, similar to conventional shot-record migration.

The techniques 4-6-7 and 5-8 are similar in that they both employ noise suppression using pseudo WDF distribution functions. However, the methods are parametrized differently, the former relative to data coordinates with 2D local space averaging and the later relative to image coordinates with 3D local space averaging.

The imaging functionals presented in this paper are described as functions of space coordinates, \( x \) or \( y \), and time,
t. As suggested in Appendix C, pseudo WDFs can be implemented either in time or frequency, so potentially the imaging conditions discussed in this paper can also be implemented in the frequency-domain. However, we restrict our attention in this paper to the time-domain implementation and leave the frequency-domain implementation subject to future study.

Equations 4-6-7 can be collected into the zero-offset imaging functional

$$R_{CINT}(y) = \int dt \int \delta(t) \int dt_h \int dx \int dx_h \int dx_h,$$

where the temporal \(\delta\) function implements the zero time imaging condition. A similar form can be written for the multi-offset case. Equation 11 corresponds to the time-domain version of the coherent interferometric functional proposed by Borcea et al. (2006a,b,c). Consistent with the preceding discussion, the cost required to implement this imaging functional is often prohibitive for practical application to seismic imaging problems.

3 STATISTICAL STABILITY

The interferometric imaging condition described in the preceding section is used to reduce imaging artifacts by attenuating the incoherent energy corresponding to velocity errors, as illustrated in Figures 6(b) and 6(d). The random model used for this example corresponds to the weak fluctuation regime, as explained in Appendix A (characteristic wavelength of similar scale with the random fluctuations in the medium and fluctuations with small magnitude).

By statistical instability we mean that images obtained for different realizations of random models with the identical statistics are different. Figures 7(a)-7(c) illustrate data modeled for different realizations of the random model in Figure 3(b). The general kinematics of the data are the same, but subtle differences exist between the various datasets due to the random model variations. Migration using a conventional imaging condition leads to the images in Figures 7(d)-7(f) which also show variations from one realization to another. In contrast, Figures 7(g)-7(i) show images obtained by the interferometric imaging condition in equations 5-8, which are more similar to one-another since many of the artifacts have been attenuated.

In typical seismic imaging problems, we cannot ensure that random velocity fluctuations are small (e.g. \(\sigma \leq 5\%\)). It is desirable that imaging remains statistically stable even in cases when velocity varies with larger magnitude. We investigate the statistical properties of the imaging functional in equations 5-8 using numerical experiments similar to the one used earlier. We describe the random noise present in the velocity models using the following parameters explained in Appendix A: seismic spatial wavelength \(\lambda = 76.2\) m, wavelet central frequency \(\omega = 20\) Hz, random fluctuations parameters: \(r_a = 0.0762\), \(r_c = 0.0762\), \(\alpha = 2\), and random noise magnitude \(\sigma\) between 15% and 45%. This numerical experiment simulates a situation that mixes the theoretical regimes explained in Appendix B: random model fluctuations of comparable scale with the seismic wavelength lead to destruction of the wavefronts, as suggested by the “weak fluctuations” regime; large magnitude of the random noise leads to diffusion of the wavefronts,
as suggested by the “diffusion approximation” regime. This combination of parameters could be regarded as a worst-case-scenario from a theoretical standpoint.

Figures 8(a)-8(c) show data simulated in models similar to the one depicted in Figure 3(b), but where the random noise component is described by $\sigma = 15, 30, 45\%$, respectively. As expected, the wavefronts recorded at the surface are increasingly distorted to the point where some of the later arrival are not even visible in the data.

Migration using a conventional imaging condition leads to the images in Figures 8(d)-8(f). As expected, the images show stronger artifacts due to the larger defocusing caused by the unknown random fluctuations in the model. However, migration using the interferometric imaging condition leads to the images in Figures 8(g)-8(i). Artifacts are significantly reduced and the images are much better focused.

4 MULTI-OFFSET IMAGING EXAMPLES

There are many potential applications for this interferometric imaging functional. One application we illustrate in this paper is imaging of complex stratigraphy through a medium characterized by unknown random variations. In this situation, accurate imaging using conventional methods requires velocity models that incorporate the small scale (random, as we view them) velocity variations. However, practical migration velocity analysis does not produce models of this level of accuracy, but approximates them with smooth, large-scale fluctuations one order of magnitude larger than that of the typical seismic wavelength. Here, we study the impact of the unknown (random) component of the velocity model on the images and whether interferometric imaging increases the statistical stability of the image.

For all our examples, we extrapolate wavefields using time-domain finite-differences both for modeling and for migration. Thus, we simulate a reverse-time imaging procedure, although the theoretical results derived in this paper apply equally well to other wavefield reconstruction techniques, e.g. downward continuation, Kirchhoff integral methods, etc. The parameters used in our examples, explained in Appendix A, are: seismic spatial wavelength $\lambda = 76.2$ m, wavelet central frequency $\omega = 20$ Hz, random fluctuations parameters: $r_a = 76.2$ m, $r_c = 76.2$ m, $\alpha = 2$, and random noise magnitude $\sigma = 20\%$.

Consider the model depicted in Figures 10(a)-10(d). As in the preceding example, the left panels depict the known smooth velocity $v_0$, and the right panels depict the model with random variations. The imaging target is represented by the oblique lines, Figure 9(b), located around $z = 8$ km, which simulates a cross-section of a stratigraphic model.

We model data with a random velocity model and image using the smooth model. Figures 10(a)-10(d) show wavefield snapshots in the two models for different propagation times, one before the source wavefields interact with the target reflectors and one after this interaction. The propagating waves are affected the the random fluctuations in the model both before and after their interaction with the reflectors. Figures 10(e) and 10(f) show the corresponding recorded data on the acquisition surface located at $z = \lambda$, where $\lambda$ represents the wavelength of the source pulse.

Migration with a conventional imaging condition of the data simulated in the background model using the same velocity produces the image in Figure 11(a). The targets are well imaged, although the image also shows artifacts due to truncation of the data on the acquisition surface. In contrast, migration with the conventional imaging condition of the data simulated in the random model using the background velocity produces the image in Figure 11(b). This image is distorted by the random variations in the model that are not accounted for in the background migration velocity. The targets are harder to discern since they overlap with many truncation and defocusing artifacts caused by the inaccurate migration velocity.

Finally, Figure 11(c) shows the migrated image using the interferometric imaging condition applied to the wavefields reconstructed in the background model from the data simulated in the random model. Many of the artifacts caused by the inaccurate velocity model are suppressed and the imaging targets are more clearly visible and easier to interpret. Furthermore, the general patterns of amplitude variation along the imaged reflectors are similar between Figures 11(b) and 11(c).

We note that the reflectors are not as well imaged as the ones obtained when the velocity is perfectly known. This is because the interferometric imaging condition described in this
Figure 7. Illustration of statistical stability for the interferometric imaging condition in presence of random model variations. Data modeled using velocity with random variations of magnitude $\sigma = 30\%$, for different realizations of the noise model $n$. Images obtained by conventional imaging (d)-(f) and images obtained by interferometric imaging (g)-(i).
Figure 8. Illustration of interferometric imaging condition robustness in presence of random model variations. Data modeled using velocity with random variations of magnitudes $\sigma = 15\%$ (a), $\sigma = 30\%$ (b), and $\sigma = 45\%$ (c). Images obtained by conventional imaging (d)-(f) and images obtained by interferometric imaging (g)-(i).
paper does not correct kinematic errors due to inaccurate velocity. It only acts on the extrapolated wavefields to reduce wavefield incoherence and add statistical stability to the imaging process. Further extensions to the interferometric imaging condition can improve focusing and enhance the images by correcting wavefields prior to imaging. However, this topic falls outside the scope of this paper and we do not elaborate on it further.

5 CONCLUSIONS

We extend the conventional seismic imaging condition based on wavefield cross-correlations to achieve statistical stability for models with rapid, small-scale velocity variation. We assume that the random velocity variations are correlated with the seismic wavelength and modeled by correlated Gaussian distributions. Our proposed interferometric imaging condition achieves statistical stability by applying conventional imaging to the Wigner distribution functions of the reconstructed seismic wavefields. The interferometric imaging condition is a natural extension of the cross-correlation imaging condition and adds minimally to the cost of migration. The resulting condition is a natural extension of the conventional imaging condition to the Wigner distribution functions of the reconstructed seismic wavefields. The interferometric imaging condition is a natural extension of the conventional imaging condition and adds minimally to the cost of migration. The main characteristic of the method is that it operates on extrapolated wavefields at image positions (thus the name interferometric imaging condition), in contrast with costlier alternative approaches using interferometry parametrized as a function of receiver coordinates.

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7 APPENDIX A: NOISE MODEL

Consider a medium whose behavior is completely defined by the acoustic velocity, i.e. assume that the density \( \rho \) is constant and the velocity \( v(x, y, z) \) fluctuates around a homogenized value \( v_0(x, y, z) \) according to the relation

\[
\frac{1}{v^2(x, y, z)} = \frac{1 + \sigma m(x, y, z)}{v_0^2(x, y, z)},
\]

where the parameter \( m \) characterizes the type of random fluctuations present in the velocity model, and \( \sigma \) denotes their strength.

Consider the covariance orientation vectors

\[
a = (a_x, a_y, a_z) \in \mathbb{R}^3 \quad (A-2)
b = (b_x, b_y, b_z) \in \mathbb{R}^3 \quad (A-3)
c = (c_x, c_y, c_z) \in \mathbb{R}^3 \quad (A-4)
\]
defining a coordinate system of arbitrary orientation in space. Let \( r_a, r_b, r_c > 0 \) be the covariance range parameters in the directions \( a, b, \) and \( c, \) respectively.

We define a covariance function
\[
\text{cov}(x, y, z) = \exp[-l^2(x, y, z)] ,
\]
where \( \alpha \in [0, 2] \) is a distribution shape parameter and
\[
l(x, y, z) = \sqrt{(a \cdot r^2_a + (b \cdot r^2_b)} + (c \cdot r^2_c)^2
\]
(A-5)
is the distance from a point at coordinates \( r = (x, y, z) \) to the origin in the coordinate system defined by \( \{r_a, r_b, r_c\} \).

Given the IID Gaussian noise field \( n(x, y, z) \), we obtain the random noise \( m(x, y, z) \) according to the relation
\[
m(x, y, z) = F^{-1}\left[\sqrt{\text{cov}(k_x, k_y, k_z)} \hat{n}(k_x, k_y, k_z)\right] ,
\]
where \( k_x, k_y, k_z \) are wavenumbers associated with the spatial coordinates \( x, y, z \), respectively. Here,
\[
\hat{n} = F[n] \quad \text{(A-7)}
\]
are Fourier transforms of the covariance function \( \text{cov} \) and the noise \( n, F[\cdot] \) denotes Fourier transform, and \( F^{-1}[\cdot] \) denotes inverse Fourier transform. The parameter \( \alpha \) controls the visual pattern of the field, and \( a, b, c, r_a, r_b, r_c \) control the size and orientation of a typical random inhomogeneity.

\section{APPENDIX B: WAVE PROPAGATION AND SCALE REGIMES}

Acoustic waves characterized by pressure \( p(x, y, z, t) \) propagate according to the second order acoustic wave-equation for constant density
\[
\frac{\partial^2 p}{\partial t^2} = v^2 \nabla^2 p + F_\lambda(t) ,
\]
(B-1)
where \( F_\lambda(t) \) is a wavelet of characteristic wavelength \( \lambda \).

Given the parameters \( l \) (size of inhomogeneities), \( \lambda \) (wavelength size), \( L \) (propagation distance) and \( \sigma \) (noise strength), we can define several propagation regimes.

The weak fluctuations regime characterized by waves with wavelength of size comparable to that of typical inhomogeneities propagating over a medium with small fluctuations to a distance of many wavelengths. This regime is characterized by negligible back scattering, and the randomness impacts the propagating waves through forward multipathing. The relevant length parameters are related by
\[
l \sim \lambda \ll L ,
\]
and the noise strength is assumed small
\[
\sigma \ll 1 .
\]

The diffusion approximation regime characterized by waves with wavelength much larger than that of typical inhomogeneities propagate over a medium with strong fluctuations to a distance of many wavelengths. This regime is characterized by traveling waves that are statistically stable but diffuse with time. Back propagation of such waves in a medium without random fluctuations results in loss of resolution. The relevant length parameters are related by
\[
l \ll \lambda \ll L ,
\]
and the noise strength is not assumed small
\[
\sigma \sim 1 .
\]

\section{APPENDIX C: WIGNER DISTRIBUTION FUNCTIONS}

Consider the complex signal \( u(t) \) which depends on time \( t \). By definition, its Wigner distribution function (WDF) is \( W(t, \omega) \) (Wigner, 1932):
\[
W(t, \omega) = \frac{1}{2\pi} \int u^*(t - \frac{\tau}{2}) u(t + \frac{\tau}{2}) e^{-i\omega\tau} d\tau ,
\]
(C-1)
where \( \omega \) denotes temporal frequency, \( \tau \) denotes relative time shift of the considered signal relative to a reference time \( t \), and the sign \( * \) denotes complex conjugation of complex signal
Figure 10. Seismic snapshots of acoustic wavefields simulated in the background velocity model (a)-(c), and in the random velocity model (b)-(d). Data recorded at the surface from the simulation in the background velocity model (e) and from the simulation in the random velocity model (f).
Figure 11. Image produced using the conventional imaging condition from data simulated in the background model (a) and from data simulated in the random model (b). Image produced using the interferometric imaging condition from data simulated in the random model (c).

$s$. The same WDF can be obtained in terms of the spectrum $U(\omega)$ of the signal $u(t)$:

$$W(t, \omega) = \frac{1}{2\pi} \int U^* \left( \omega + \frac{\omega_h}{2} \right) U \left( \omega - \frac{\omega_h}{2} \right) e^{-it \omega_h} d\omega_h ,$$

(C-2)

where $\omega_h$ denotes the relative frequency shift of the considered spectrum relative to a reference frequency $\omega$. The time integral in equation C-1 or the frequency integral in equation C-2 spans the entire domain of time and frequency, respectively. When the interval is limited to a region around the reference value, the transformation is known as pseudo Wigner distribution function.

A special subset of the transformation equation C-1 corresponds to zero temporal frequency. For input signal $u(t)$, we obtain the output Wigner distribution function $W(t)$ as

$$W(t) = \frac{1}{2\pi} \int u^* \left( t - \frac{\tau_h}{2} \right) u \left( t + \frac{\tau_h}{2} \right) d\tau_h .$$

(C-3)

The WDF transformation can be generalized to multidimensional signals of space and time. For example, for 2D
real signals function of space, \( u(x, y) \), the zero-wavenumber pseudo WDF can be formulated as

\[
W(x, y) = \frac{1}{4\pi^2} \int_{|x_x| \leq X} \int_{|y_y| \leq Y} u \left( x - \frac{x_x}{2}, y - \frac{y_y}{2} \right) \\
\times u \left( x + \frac{x_x}{2}, y + \frac{y_y}{2} \right) dx_x dy_y,
\]

where \( x_x \) and \( y_y \) denote relative shift of the signal \( s \) relative to positions \( x \) and \( y \), respectively. In this particular form, the pseudo WDF transformation has the property that it filters the input of random fluctuations preserving in the output image the spatially coherent components in a noise-free background.

For illustration, consider the model depicted in Figure 1(a). This model consists of a smoothly-varying background with 25% random fluctuations. The acoustic seismic wavefield corresponding to a source located in the middle of the model is depicted in Figure 1(b). This wavefield snapshot can be considered as the random “image”. The application of the 2D pseudo WDF transformation to images shown in Figure 1(b) produces the image shown in Figure 1(c). We can make three observations on this image: first, the random noise is strongly attenuated; second, the output wavelet is different from the input wavelet, as a result of the bi-linear nature of the pseudo WDF transformations; third, the transformation is isotropic, i.e. it operates identically in all directions. The pseudo WDF applied to this image uses \( 11 \times 11 \) grid points in the vertical and horizontal directions. As indicated in the body of the paper, we do not discuss here the optimal selection of the WDF window. Further details of Wigner distribution functions and related transformations are discussed by Cohen (1995).
Micro-earthquake monitoring with sparsely-sampled data

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ABSTRACT

Micro-seismicity can be used to monitor the migration of fluids during reservoir production and hydro-fracturing operations in brittle formations or for studies of naturally occurring earthquakes in fault zones. Micro-earthquake locations can be inferred using wave-equation imaging under the exploding reflector model, assuming densely sampled data and known velocity. Seismicity is usually monitored with sparse networks of seismic sensors, for example located in boreholes. The sparsity of the sensor network itself degrades the accuracy of the estimated locations, even when the velocity model is accurately known, thus limiting the resolution at which fluid pathways can be inferred. Wavefields reconstructed in known velocity using data recorded with sparse arrays can be described as having a random character due to the incomplete interference of wave components. Similarly, wavefields reconstructed in unknown velocity using data recorded with dense arrays can be described as having a random character due to the inconsistent interference of wave components. In both cases, the reconstructed wavefields are characterized by random fluctuations which obstruct focusing that occurs at source locations. This situation can be improved using interferometry in the imaging process. Reverse-time imaging with an interferometric imaging condition attenuates random fluctuations, thus producing crisper images which support the process of robust automatic micro-earthquake location. The similarity of random wavefield fluctuations due to model fluctuations and sparse acquisition, as well as the applicability of conventional and interferometric imaging techniques, are illustrated in this paper with a complex synthetic example.

Key words: wave-equation, imaging, micro-earthquakes

1 INTRODUCTION

Seismic imaging based on the single scattering assumption, also known as Born approximation, consists of two main steps: wavefield reconstruction which serves the purpose of propagating recorded data from the acquisition surface back into the subsurface, followed by an imaging condition which serves the purpose of highlighting locations where scattering occurs. This framework holds both when the source of seismic waves is located in the subsurface and the imaging target consists of locating this source, as well as when the source of seismic waves is located on the acquisition surface and the imaging target consists of locating the places in the subsurface where scattering or reflection occurs. In this paper, I concentrate on the case of imaging seismic sources located in the subsurface, although the methodology discussed here applies equally well for the more conventional imaging with artificial sources.

An example of seismic source located in the subsurface is represented by micro-earthquakes triggered by natural causes or by fluid injection during reservoir production or fracturing. One application of micro-earthquake location is monitoring of fluid injection in brittle reservoirs when micro-earthquake evolution in time correlates with fluid movement in reservoir formations. Micro-earthquakes can be located using several methods including double-difference algorithms (Waldhauser & Ellsworth, 2000), Gaussian-beam migration (Rentisch et al., 2004), diffraction stacking (Gajewski et al., 2007) or reverse-time migration (Gajewski & Tessmer, 2005; Vasconcelos et al., 2007).

Micro-earthquake location using reverse-time migration, which is the technique advocated in this paper, follows the same general pattern mentioned in the preceding paragraph: wavefield-reconstruction backward in time followed by an imaging condition extracting the image, i.e. the location of the
source. The main difficulty with this procedure is that the onset of the micro-earthquake is unknown, i.e. time \( t = 0 \) is unknown, so the imaging condition cannot be simply applied as it is usually done in zero-offset migration. Instead, an automatic search needs to be performed in the back-propagated wavefield to identify the locations where wavefield energy focuses. This process is difficult and often ambiguous since false focusing locations might overlap with locations of wavefield focusing. This is particularly true when imaging an approximate model which does not explain all random fluctuations observed in the recorded data. This problem is further complicated if the acquisition array is sparse, e.g. when receivers are located in a borehole. In this case, the sparsity of the array itself leads to artifacts in the reconstructed wavefield which makes the automatic picking of focused events even harder.

The process by which sampling artifacts are generated is explained in Figures 1(a)-1(c). Each segment in Figure 1(a) corresponds to a wavefront reconstructed from a receiver. For dense and uniform receiver coverage and reconstruction using accurate velocity, the wavefronts overlap at the source position. However, if the velocity used for wavefield reconstruction is inaccurate, then the wavefronts do not all overlap at the source position, Figure 1(b), thus leading to imaging artifacts. Likewise, if receiver sampling is sparse, reconstruction at the source position is incomplete, Figure 1(c), even if the velocity used for reconstruction is accurate. The cartoons depicted in Figures 1(a)-1(c) represent an ideal situation with receivers surrounding the seismic source, which is not typical for seismic experiments. In those cases, source illumination is limited to a range which correlates with the receiver coordinates.

In general, artifacts caused by unknown velocity fluctuations and receiver sampling overlap and, although the two phenomena are not equivalent, their effect on the reconstructed wavefields are analogous. As illustrated in the following sections, the general character of those artifacts is that of random wavefield fluctuations. Ideally, the imaging procedure should attenuate those random wavefield fluctuations irrespective of their cause in order to support automatic source identification.

2 CONVENTIONAL IMAGING CONDITION

Assuming data \( D(\mathbf{x}, t) \) acquired at coordinates \( \mathbf{x} \) function of time \( t \) (e.g. in a borehole) we can reconstruct the wavefield \( V(\mathbf{x}, \mathbf{y}, t) \) at coordinates \( \mathbf{y} \) in the imaging volume using an appropriate Green’s function \( G(\mathbf{x}, \mathbf{y}, t) \) corresponding to the locations \( \mathbf{x} \) and \( \mathbf{y} \) (Figure 2)

\[
V(\mathbf{x}, \mathbf{y}, t) = D(\mathbf{x}, t) *_t G(\mathbf{x}, \mathbf{y}, t) ,
\]

where the symbol \(*_t\) indicates time convolution. The total wavefield \( U(\mathbf{y}, t) \) at coordinates \( \mathbf{y} \) due to data recorded at all receivers located at coordinates \( \mathbf{x} \) is represented by the superposition of the reconstructed wavefields \( V(\mathbf{x}, \mathbf{y}, t) \):

\[
U(\mathbf{y}, t) = \int d\mathbf{x} V(\mathbf{x}, \mathbf{y}, t) .
\]

A conventional imaging condition (CIC) applied to this reconstructed wavefield extracts the image \( R_{\text{CIC}}(\mathbf{y}) \) as the wavefield at time \( t = 0 \)

\[
R_{\text{CIC}}(\mathbf{y}) = U(\mathbf{y}, t = 0) .
\]

This imaging procedure succeeds if several assumptions are fulfilled: first, the velocity model used for imaging has to be accurate; second, the numeric solution to the wave-equation used for wavefield reconstruction has to be accurate; third, the data need to be sampled densely and uniformly on the acquisition surface. In this paper, I assume that the first and third assumptions are not fulfilled. In these cases, the imaging is not accurate because contributions to the reconstructed wavefield from the receiver coordinates do not interfere constructively, thus leading to imaging artifacts. As indicated earlier, this situation is analogous to the case of imaging with an inaccurate velocity model, e.g. imaging with a smooth velocity of data corresponding to geology characterized by rapid velocity variations.

Different image processing procedures can be employed to reduce the random wavefield fluctuations. The procedure advocated in this paper uses interferometry for noise cancella-
Interferometric procedures can be formulated in various frameworks, e.g. coherent interferometric imaging (Borcea et al., 2006) or wave-equation migration with an interferometric imaging condition (Sava & Poliannikov, 2008).

3 INTERFEROMETRIC IMAGING CONDITION
Migration with an interferometric imaging condition (IIC) uses the same generic framework as the one used for the conventional imaging condition, i.e. wavefield reconstruction followed by an imaging condition. However, the difference is that the imaging condition is not applied to the reconstructed wavefield directly, but it is applied to the wavefield which has been transformed using pseudo Wigner distribution functions (WDF) (Wigner, 1932). By definition, the zero frequency pseudo WDF of the reconstructed wavefield $U(y, t)$ is

$$W(y, t) = \int_{|y_h| \leq Y} \int_{|t_h| \leq T} d y_h \, d t_h \, U_y \frac{y_h}{2}, t - \frac{t_h}{2} \right) U_y \frac{y_h}{2}, t + \frac{t_h}{2},$$

(4)

where $Y$ and $T$ denote averaging windows in space and time, respectively. In general, $Y$ is three dimensional and $T$ is one dimensional. Then, the image $R_{IIC}(y)$ is obtained by extracting the time $t = 0$ from the pseudo WDF $W(y, t)$, of the wavefield $U(y, t)$:

$$R_{IIC}(y) = W(y, t = 0).$$

(5)

The interferometric imaging condition represented by equations 4 and 5 effectively reduces the artifacts caused by the random fluctuations in the wavefield by filtering out its rapidly varying components (Sava & Poliannikov, 2008). In this paper, I use this imaging condition to attenuate noise caused by sparse data sampling or noise caused by random velocity variations. As suggested earlier, the interferometric imaging condition attenuates both types of noise at once, since it does not explicitly distinguish between the various causes of random fluctuations.

The parameters $Y$ and $T$ defining the local window of the pseudo WDF are selected according to two criteria (Cohen, 1995). First, the windows have to be large enough to enclose a representative portion of the wavefield which captures the random fluctuation of the wavefield. Second, the window has to be small enough to limit the possibility of cross-talk between various events present in the wavefield. Furthermore, cross-talk can be attenuated by selecting windows with different shapes, for example Gaussian or exponentially-decaying. For simplicity, in all examples presented in this paper, the space and time windows are rectangular with no tapering and the size is selected assuming that micro-earthquakes occur sufficiently sparse, i.e. the various sources are located at least twice as far in space and time relative to the wavenumber and frequency of the considered seismic event. Typical window sizes used here are 11 grid points in space and 5 grid points in time. A complete discussion of optimal window selection falls outside the scope of this paper.

4 EXAMPLE
I exemplify the interferometric imaging condition method with a synthetic example simulating the acquisition geometry of the passive seismic experiment performed at the San Andreas Fault Observatory at Depth (Chavarría et al., 2003). This numeric experiment simulates waves propagating from three micro-earthquake sources located in the fault zone, Figure 3, which are recorded in a deviated well located at a distance from the fault. For the imaging procedure described in this paper, the micro-earthquakes represent the seismic sources. This experiment uses acoustic waves, corresponding to the situation in which we use the P-wave mode recorded by the three-component receivers located in the borehole, Figures 4(b) and 5(b). The three sources are triggered 40 ms apart and the trig...
Figure 3. Geometry of the sources used in the numeric experiment. The horizontal and vertical separation between sources is 250 m. The sources are triggered with 40 ms delays in the order indicated by their numbers. Time \( t = 0 \) is conventionally set to the triggering moment of source 2.

The geometry of the experiment is shown in the figure. The horizontal and vertical separation between sources is 250 m. The sources are triggered with 40 ms delays in the order indicated by their numbers. Time \( t = 0 \) is conventionally set to the triggering moment of source 2.

The goal of this experiment is to locate the source positions by focusing data recorded using dense acquisition in media with random fluctuations or by focusing data recorded using sparse acquisition arrays in media without random fluctuations. In the first case, the imaging artifacts are caused by the fact that data are imaged with a velocity model that does not incorporate all random fluctuations of the model used for data simulation, while in the second case, the imaging artifacts are caused by the fact that the data are sampled sparsely in the borehole array.

Figures 6(a)-6(b) and 7(a)-7(b) show the wavefields reconstructed in reverse time around the target location. From left to right, the panels represent the wavefield at different times. As indicated earlier, the time at which source 2 focuses is selected as time \( t = 0 \), although this convention is not relevant for the experiment and any other time could be selected as reference. The experiment depicted in Figures 6(a)-6(b) corresponds to modeling in a model with random fluctuations and migration in a smooth background model. In this experiment, the data used for imaging are densely-sampled in the borehole. In contrast, the experiment depicted in 7(a)-7(b) corresponds to modeling and migration in the smooth background model. In this experiment, the data are sparsely-sampled in the borehole. In both cases, panels (a) correspond to imaging with a conventional imaging condition (CIC), i.e. simply select the reconstructed wavefield at various times, and panels (b) correspond to imaging with the interferometric imaging condition, i.e. select various times from the wavefield transformed with a pseudo-WDF of 11 grid points in space and 5 grid points in time. For this example, WDF window corresponds to 44 m in space and 2 ms in time.

Figure 6(a) shows significant random fluctuations caused by wavefield reconstruction using an inaccurate velocity model. The fluctuations caused by the random velocity and encoded in the recorded data are not corrected during wavefield reconstruction and they remain present in the model. Likewise, Figure 7(a) shows significant random fluctuations caused by reconstruction using the sparse borehole data. However, the zero-frequency pseudo WDF applied to the reconstructed wavefields attenuates the rapid wavefield fluctuations and leads to sparser, better focused images that are easier to use for source location. This conclusion applies equally well for the experiments depicted in Figures 6(a)-6(b) or 7(a)-7(b).

Finally, I note that the 2D imaging results from this example show better focusing than what would be expected in 3D. This is simply because the 1D acquisition in the borehole cannot constrain the 3D location of the micro-earthquakes, i.e. the azimuthal resolution is poor, especially if scatterers are not present in the model used for imaging. This situation can be improved by using data acquired in several boreholes or by using additional information extracted from the wavefields, e.g. polarization of multicomponent data.

5 CONCLUSIONS

The interferometric imaging condition used in conjunction with reverse-time migration reduces the artifacts caused by random velocity fluctuations that are unaccounted-for in imaging and by the sparse wavefield sampling on the acquisition array. The images produced by this procedure are crisper and support automatic picking of micro-earthquake locations. The interferometric imaging procedure has a similar structure to conventional imaging and the moderate cost increase is proportional to the size of the windows used by the pseudo Wigner distribution functions. The source positions obtained using this procedure can be used to monitor fluid injection or for studies of naturally occurring earthquakes in fault zones.
Figure 4. (a) Wavefields simulated in random media and (b) data acquired with a dense receiver array. Overlaid on the model and wavefield are the positions of the sources and borehole receivers. The hashed area corresponds to the images depicted in Figures 6(a)-6(b).

6 ACKNOWLEDGMENT

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Figure 5. (a) Wavefields simulated in smooth media and (b) data acquired with a sparse receiver array. Overlain on the model and wavefield are the positions of the sources and borehole receivers. The hashed area corresponds to the images depicted in Figures 7(a)-7(b).

Figure 6. Images corresponding to migration of the densely-sampled data modeled in the random velocity by (a) conventional I.C. and (b) interferometric I.C. using the background velocity. The left-most panel shows focusing at source 1, the middle panel shows focusing at source 2, and the right-most panel shows focusing at source 3. The overlain dots represent the exact source positions.

Figure 7. Images corresponding to migration of the sparsely-sampled data modeled in the background velocity by (a) conventional I.C. and (b) interferometric I.C. using the background velocity. The left-most panel shows focusing at source 1, the middle panel shows focusing at source 2, and the right-most panel shows focusing at source 3. The overlain dots represent the exact source positions.
Range and resolution analysis of wide-azimuth angle decomposition

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ABSTRACT

Imaging in complex media benefits from uniform illumination of the target from all possible directions. Moreover, it is desirable to recover reflectivity from seismic data as a function of incidence and azimuthal angles at every location over the reflector. Applications of angle-dependent reflectivity include velocity and anisotropy estimation and amplitude versus angle (AVA) analysis. One way of constructing angle-dependent reflectivity is to apply an extended imaging condition to the extrapolated source and receiver wavefields. This imaging condition allows one to construct images as a function not only of three-dimensional position but also space-lags of the source/receiver wavefield cross-correlation. The information in the space-lag domain can be mapped into the angle-domain, defined by the reflection and azimuthal angles, at every image point. The relationship between sampling parameters in the angle and space-lag domain, together with the equations used to perform the mapping, show that the sample interval in the space-lag domain controls the range of angles that can be accurately recovered from the image in this domain. This is the range of angles for which the energy is well focused, at the depth of the reflector, in the angle-domain common-image gathers. From the amplitude spectrum of the lag-domain common-image gathers, we can calculate the maximum angle that limits this range. For angles greater than this upper bound, the image energy starts to spread away from the depth of the reflector. This analysis is important for the case where the angle-domain common-image gathers are employed for amplitude versus angle analysis. In this case, the amplitudes corresponding to the angles outside of this range would not be reliable, regardless of how accurate wavefield reconstruction is.

Key words: angle-decomposition, common-image gathers, extended imaging condition, wide-azimuth

1 INTRODUCTION

Many hydrocarbonate reservoirs are embedded in complex media characterized by strong velocity heterogeneity and discontinuities associated with salt bodies. In the presence of complex subsurface structures, it is crucial to form a reflector map (image) with well focused energy for efficient development and exploration of reservoirs located in such areas. The right choice of procedures, from acquisition to data processing, is of fundamental importance since they all influence the image quality.

In order to produce accurate images it is necessary that the data contain enough information about the area of interest. Even with the use of the most sophisticated processing tools and techniques, it is impossible to achieve the desired image quality from narrow-azimuth data in complex media because these data do not contain all the necessary information needed to create good images.

In the wide-azimuth acquisition context there are two relevant parameters: the azimuth of the source/receiver line (surface azimuth) and the distance between source and receiver (surface offset). The surface azimuth is the angle of the line connecting source and receiver in relation to a reference direction on the acquisition surface. In wide-azimuth acquisition the surface azimuth varies a wide range. For the rest of the paper, data from wide-azimuth survey are referred to as wide-azimuth
Data and data from narrow-azimuth surveys are referred to as narrow-azimuth data.

Data acquired at different surface azimuths illuminate different parts of the target. In complex media, where velocity varies with direction, it is desirable to illuminate each location along the reflector from many directions (at the image point) in order to study how reflectivity changes with direction. Some applications of wide-azimuth reflectivity are velocity analysis, amplitude versus angle analysis, and anisotropy estimation.

Salt bodies cause irregular illumination of targets located beneath them. This not only makes it difficult to properly image these shadow zones, but also creates spurious events in the images. Recently published work shows that it is possible to significantly improve image quality by acquiring and processing wide-azimuth data (Regone, 2006). The wide-azimuth data helps to improve resolution in velocity model (Michell et al., 2006; LaDart et al., 2006; Shoshitaishvili et al., 2006), identify directional fractures and attenuate coherent noise (Shirui et al., 2006), and consequently improve the image.

We characterize the data by two surface parameters, namely, surface offset (source(receiver) offset) and surface azimuth. In general, a reflection phenomenon can be characterized by two parameters as well: reflection angle and azimuthal angle at the image point. These angles are defined locally at the image point and, in general, are not simply related to the surface parameters, as explain next.

The reflection plane is the plane containing the incident ray from the source to the reflection point, the normal to this point, and the reflected ray from the reflection point to the receiver, all measured at the image point. This plane is defined in the vicinity of the image point. The angle between the incident or reflected ray and the normal is the reflection angle. The angle that characterizes the orientation of the reflection plane in 3D, in relation to a determined direction is the azimuthal angle $A_z$. When the subsurface is a homogeneous isotropic medium, the surface offset and azimuth are simply related to the reflection and azimuthal angles, respectively, at the reflection point (Figure 1(a)), independent of reflector orientation. Changes in velocity cause the incident and reflected rays to deviate from straight lines to more complicated paths (Figure 1(b)). Thus, when the medium is complex, the surface azimuth is not directly related to the azimuthal angle of the reflection plane at the image point, nor is the surface offset related to the reflection angle. Even though this relationship is not direct, a wide-azimuth acquisition is likely to increase the range of the azimuthal angles of the reflection planes at the image point.

To obtain accurate images in complex geology with large and sharp velocity variation, it is necessary to perform wavefield reconstruction using methods capable of treating lateral velocity variations, phenomena such as multipathing and finite bandwidth propagation effects (Gray et al., 2001). Wavefield extrapolation by downward-continuation methods (Gazdag (1985); Gazdag & Sguazzero (1985); Stoffa et al. (1990); Freire & Stoffa (1986); Kessinger (1992); Cockshott & Jakubowicz (1996)), is one possible way to perform accurate wavefield reconstruction in such media, since it is capable to handle these issues. In this paper, we use the Split Step Fourier downward continuation method (Stoffa et al., 1990).

Reflectivity information can be obtained by applying an imaging condition to the reconstructed source and receiver wavefields. A conventional imaging condition is a cross-correlation type imaging condition (Claerbout, 1971) that creates an image function of three-dimensional position.

Migration velocity analysis (MVA), applied after migration by downward continuation, has often been performed using lag-domain common-image gathers (Biondi et al., 2003; Biondi & Symes, 2004). These gathers are usually referred to as offset-domain common-image gathers (ODCIG). However, in this paper, we use the term space-lag to refer to this domain and use the word offset for the source-receiver separation on the surface. The lag gathers can be constructed from images obtained by extended imaging conditions, that preserve the orthogonal space-lags from the source(receiver) wavefield cross-correlation (Sava & Fomel, 2006). Events in the lag gathers may suffer from ambiguity on the reflection position due to multipathing, which is a common phenomenon for propagation in complex media (Nolan & Symes, 1996). Reflector position ambiguity in the lag gathers, in turn, may lead to velocity update ambiguities. To alleviate this problem,
the space-lag common-image gathers can be transformed as a function of the reflection angle at the image point. Events in angle-domain common-image gathers (ADCGs) uniquely represent energy scattered from specific locations in the subsurface. These events are usually associated with pairs of incident and reflected rays which, in turn, uniquely define the reflector (Prucha et al., 1999; Stolk & Symes, 2002).

As indicated earlier, one way of constructing angle-dependent reflectivity is through migration by wavefield extrapolation combined with an extended imaging condition. The space-lags images can be decomposed at every image point into components that depend on the reflection angle (Sava & Fomel, 2005, 2003; Mosher & Foster, 2000; Prucha et al., 1999). It is also possible to construct angle-dependent reflectivity by integral methods (Xu et al., 1998; Brandsberg-Dahl et al., 1999; Bleistein et al., 2005). For wide-azimuth data, it is possible to decompose the space-lags in reflection and azimuthal angle (Sava & Fomel, 2003; Biondi & Tissot, 2004; Sava & Fomel, 2005).

In this paper, we construct angle-dependent reflectivity by performing angle decomposition based on the geometry of ray vectors at the image point (Sava & Fomel, 2005). We analyze resolution in the angle-domain by means of the range of angles that can be accurately recovered from the space-lag domain, and of the energy focusing for reflections corresponding to these angles in the angle gathers. We can obtain the reciprocal resolution/range relationship between lag and angle-domain from the equations used to perform the lag to angle-domain mapping, together with the sampling parameters in both domains. We perform the analysis for two-dimensional (2D) and wide-azimuth three-dimensional (3D) geometries. First, this relationship shows that the sample interval in the space-lag domain controls the range of angles that can be recovered from the image in the space-lag domain. In 2D, we find that there is an angle that limits the range of reflection angles for which the energy focus, at the depth of the reflector, in the angle gathers; for angles greater than this one, the energy starts to spread. In 3D, this range varies with the azimuthal angle. Also, the range in the space-lag domain controls the sample interval in the angle-domain. Using these relationships, one can choose the parameters in both domains according to the desired resolution and range on angles for the ADCIGs. We show synthetic examples that illustrate the sampling relationship between space-lag and angle-domain as well as an example of the angle decomposition.

2 ANGLE DECOMPOSITION

Migration of seismic data consists mainly of two parts: a wavefield reconstruction procedure that creates a source wavefield \( U_s \) and a receiver (scattered) wavefield \( U_r \), at all space locations and all times (or frequencies) given a presumed velocity field, and an imaging condition that extracts an image \( I \) (reflectivity information) from the reconstructed source and receiver wavefields, \( U_s \) and \( U_r \).

From the reconstructed source and receiver wavefields, one can extract an image \( I \) by applying a conventional correlation imaging condition (Claerbout, 1971),

\[
I(x) = \sum_{\omega} U_r(x, \omega)U^*_s(x, \omega),
\]

where \( x = (x, y, z) \) are the space coordinates, \( \omega \) is the temporal frequency, \( U^*_s(x, \omega) \) is the complex conjugate of the source wavefield, \( U_r(x, \omega) \) is the receiver wavefield, and \( I(x) \) is the image, which depends on the particular choice of source and receiver positions. This image contains some information for velocity analysis, such as the presence of non focused diffracted energy. It is possible, however, to obtain additional velocity information, by applying a more general imaging condition, sometimes referred to as an extended imaging condition (Sava & Fomel, 2005).

One can define an extended imaging condition by space and time cross-correlation that preserves the space and/or time cross-correlation lags. Here, we employ an extended imaging condition that preserves the cross-correlation space-lags (which are, for simplicity, referred to as space-lags) to obtain an image function of the space coordinates and space-lags,

\[
I(x, \lambda) = \sum_{\omega} U_r(x + \lambda, \omega)U^*_s(x - \lambda, \omega).
\]

The image \( I(x, \lambda) \), function of space \( x \) and cross-correlation space-lag, contains information for velocity analysis aimed at improving on the presumed velocity model. When the velocity model used for migration is wrong the energy does not map to the zero-lag, so we can get velocity information by looking at the energy in the image \( I(x, \lambda) \) for the other lags different from zero. The conventional image \( I(x) \) from equation 1 is equivalent to the extended image \( I(x, \lambda) \) at \( \lambda = 0 \).

An image function of space, reflection angle \( \theta \), and azimuthal angle \( \phi \) can be obtained by mapping the information from the lag-domain into the angle-domain (Sava & Fomel, 2005)

\[
I(x, \lambda) \rightarrow I(x, \theta, \phi).
\]

The angle between the incident or reflected ray and the reflector normal (at the image point) is the reflection angle, \( \theta \). The angle that characterizes the orientation of the reflection plane in 3D, in relation to a specified direction, is the azimuthal angle, \( \phi \). As mentioned earlier, these angles are related to a local geometry of source and receiver ray vectors at the image point.

The relationship between the space-lags \( \lambda \), and the reflection and azimuthal angles, \( \theta \) and \( \phi \), in general, involves \( (k_x, k_y, k_z) \), the wavenumbers related to the space coordinates \( (x, y, z) \), and \( (k_{\lambda_x}, k_{\lambda_y}, k_{\lambda_z}) \), the wavenumbers related to the space-lags \( (\lambda_x, \lambda_y, \lambda_z) \) (equations A-14 and A-18). It is possible to avoid the vertical space-lag \( \lambda_z \) by taking advantage of the fact that the wavenumber vectors \( k_r \) and \( k_\lambda \) are orthogonal (equations A-19 and A-20). This observation allows us to reduce the image \( I(x, \lambda) \) from six to five dimensions. The mapping equations used in this work are relations A-21 and A-22 derived in the Appendix.

Implementation of equations A-21 and A-22 seemingly requires a five-dimensional Fourier transform to create AD-
select the horizontal CIG location
apply a one-dimensional Inverse Fourier Transform on axes
apply a three-dimensional Fourier Transform on space wavenumbers
perform the mapping

where position described by relations 6 and 7 involves the following
location.
be stored and make computations independent for each CIG
tions A-21 and A-22. This reduces the size of FFTs from
size of migrated images obtained with extended imaging condition?
Once the relationships between angle-domain and space-lag domain sampling parameters are established, it is possible to
to establish the maximum angle range
that could potentially be used for AVA and MVA.

3 RESOLUTION AND RANGE IN THE ANGLE-DOMAIN

Angle-domain common-image gathers are commonly used to perform migration velocity analysis. Since these angle-domain
common-image gathers are constructed from the space-lag domain common-image gathers, it is important to understand the
relation between sampling in the angle and space-lag domains. This section is devoted to analyzing angle-domain resolution and range, as a functions of the sampling parameters
in the space-lag domain. The resolution study involves understanding how to choose the sampling parameters in the space-
and angle domains such that the angle-domain be uniformly sampled and not aliased. The range analysis is meant to answer the question “What angle range can be accurately recovered for a given range and sampling in the lag-domain of
migrated images obtained with extended imaging condition?”

In this manner, we eliminate $k_x$ and $k_y$ from the equations A-21 and A-22. This reduces the size of FFTs from five to three dimensions and reduces the amount of data to be stored and make computations independent for each CIG location.

Construction of one CIG by applying the angle decomposition described by relations 6 and 7 involves the following steps:

- select the horizontal CIG location $(x_0, y_0)$ and obtain the lag-domain CIG $I(z, \lambda_x, \lambda_y) = I(x = x_0, y = y_0, z, \lambda_x, \lambda_y, \lambda_z = 0)$ from the image $I(x, \lambda)$;
- apply a three-dimensional Fourier Transform on axes $z, \lambda_x, \lambda_y$
  \[ I(z, \lambda_x, \lambda_y) \rightarrow I(k_z, k_{\lambda_x}, k_{\lambda_y}) \]  
- perform the mapping
  \[ I(k_z, k_{\lambda_x}, k_{\lambda_y}) \rightarrow I(k_z, \theta, \phi) \]

according to relations 6 and 7;

- apply a one-dimensional Inverse Fourier Transform on the depth axis
  \[ I(k_z, \theta, \phi) \rightarrow I(z, \theta, \phi) \]

to obtain the angle domain CIG.

3.1 2D range and resolution analysis

The two-dimensional image obtained from the extended imaging condition is, in general, a function of four coordinates: $I = I(x, z, \lambda_x, \lambda_z)$. As discussed earlier, we can use geometrical relationships between the wavenumber vectors to eliminate $\lambda_z$ and reduce the necessary image size from four to three dimensions, i.e., $I = I(x, z, \lambda_x)$.

Given a space-lag common-image gather $I(z, \lambda_x)$ at coordinates $x$, we can construct angle-domain gathers using the mapping

\[ \tan \theta = \frac{k_{\lambda_x}}{k_z}, \]  

which requires that the space-lag CIG be transformed to the Fourier domain.

In discussing angle-domain resolution, we consider the following parameters: origin, sample interval, and number of points on the depth axis, $(\alpha_x, d_x, n_x)$, and on the space lag axis, $(\alpha_{\lambda_x}, d_{\lambda_x}, n_{\lambda_x})$; The corresponding parameters in the wavenumber-domain are: origin, sample interval, and number of points on the depth wavenumber axis, $(\alpha_{k_z}, d_{k_z}, n_{k_z})$, and on the space-lag wavenumber axis, $(\alpha_{k_{\lambda_x}}, d_{k_{\lambda_x}}, n_{k_{\lambda_x}})$.

Using conventional discrete Fourier transform theory, the wavenumber sampling parameters $k_z$ and $k_{\lambda_x}$ can be expressed in terms of the model parameters as

\[ \alpha_{k_z} = -\frac{1}{2d_z}, \quad d_{k_z} = \frac{1}{n_z d_z}, \]  

and

\[ \alpha_{k_{\lambda_x}} = -\frac{1}{2d_{\lambda_x}}, \quad d_{k_{\lambda_x}} = \frac{1}{n_{\lambda_x} d_{\lambda_x}}. \]

The sampling parameters for the angle $\theta$ are: origin,
Wide-azimuth angle decomposition analysis

Figure 2. Example illustrating mapping from a lag gather with a single spike located at $z = 1135$ m and $\lambda_x = 0$ m: lag gather $I(z, \lambda_x)$ with one single spike (a); 2D Fourier transform $I(k_z, k_{\lambda_x})$ of the lag gather (b); mapping from $k_{\lambda_x}$ to $\theta$, $I(k_z, \theta)$ (c); angle-domain common-image gather $I(z, \theta)$ (d).

sample interval, and number of points on the depth axis, $(\delta z, \delta \theta, n_\theta)$. We can analyze the behavior of $d_\theta$ as a function of $k_{\lambda_x}$, i.e., as a function of $n_{\lambda_x}$ and $d_{\lambda_x}$. Using equation 11 and the relationships 13 and 12, we have $\theta = \theta(k_z, k_{\lambda_x}) = \theta(n_{\lambda_x}, d_z, n_{\lambda_x} d_{\lambda_x})$. Isolating $\theta$ from equation 11,

$$\theta = \arctan \frac{k_{\lambda_x}}{k_z}, \quad (14)$$

we can differentiate the above equation and obtain an expression for $d_\theta$

$$d_\theta = \frac{k_z}{k_z^2 + k_{\lambda_x}^2} \frac{1}{n_{\lambda_x} d_{\lambda_x}}, \quad (15)$$

Noting that $n_{\lambda_x} d_{\lambda_x} = \lambda_{\text{max}}$, where $\lambda_{\text{max}}$ defines the range of the space-lag domain, we find

$$d_\theta = \frac{k_z}{k_z^2 + k_{\lambda_x}^2} \frac{1}{\lambda_{\text{max}}}, \quad (16)$$

Thus, $d_\theta$ is inversely proportional to $n_{\lambda_x} d_{\lambda_x} = \lambda_{\text{max}}$ so the parameter $\lambda_{\text{max}}$, which is the range in space-lag domain, controls the resolution in the angle-domain: The larger the range in the space-lag domain, the better the resolution in the angle-domain.

According to equation 16, $d_\theta$ differs for each $(k_z, k_{\lambda_x})$, i.e., the angle-domain is non uniformly sampled. For processing reasons, we would like to have the angle-domain uniformly sampled. One possible way to achieve that is to sample the angle-domain uniformly using a $d_\theta$ chosen such as to avoid aliasing. The minimum value of $d_\theta$, in order to avoid aliasing, is

$$d_{\theta_{\text{min}}} = \frac{k_{\lambda_{\text{max}}}}{k_z^2 + k_{\lambda_{\text{max}}}^2} \frac{1}{\lambda_{\text{max}}} \quad (17)$$

$$= \frac{1}{(\frac{1}{2d_z})^2 + (\frac{1}{2d_{\lambda_x}})^2} \frac{1}{\lambda_{\text{max}}}. \quad (18)$$

Note that, $d_\theta$ is greater for low frequencies than for high fre-
Figure 3. Example illustrating mapping from a lag gather with a wavelet located at $z = 1135$ m and $\lambda_x = 0$ m: lag gather $I(z, \lambda_x)$ with one single spike (a); 2D Fourier transform $I(k_z, k_{\lambda_x})$ of the lag gather (b); mapping from $k_{\lambda_x}$ to $\theta$, $I(k_z, \theta)$ (c); angle-domain common-image gather $I(z, \theta)$ (d).

The angle corresponding to $k_{z_{\text{max}}}$ is given by

$$\theta_F = \arctan \left( \frac{k_{k_{z_{\text{max}}}}}{k_{z_{\text{max}}}} \right) = \arctan \left( \frac{1}{2d_{\lambda_x} k_{z_{\text{max}}}} \right).$$

Thus, not only $k_{z_{\text{max}}}$ but also the sample interval in the $\lambda_x$ dimension, controls $\theta_F$. The angle $\theta_F$ defines the range of angles that can be accurately recovered from the lag-domain. For reflection angles greater than $\theta_F$, we observe energy defocusing and spreading away from the depth of the reflector. This phenomenon becomes more pronounced for increasing angles. Therefore, even if there is information in the lag gather corresponding to reflection at $90^\circ$ in the angle gather, when mapped to the angle-domain, the energy corresponding to this event is defocused as discussed next.

In order to analyze and visualize this transformation from lag to angle-domain, we consider the idealized case of a lag gather that consists of a spike at depth $z = 1135$ m. Ide-
Wide-azimuth angle decomposition analysis

Figure 4. Example illustrating mapping from a 3D lag gather with a single spike located at \( z = 1135 \) m and \( \lambda_x = \lambda_y = 0 \) m: lag gather \( I(z, \lambda_x, \lambda_y) \) with one single spike (a); 2D Fourier transform of the lag gather \( I(k_z, k_{\lambda_x}, k_{\lambda_y}) \) (b); mapping from \( (k_{\lambda_x}, k_{\lambda_y}) \) to \( (\theta, \phi) \), \( I(k_z, \theta, \phi) \) (c); angle-domain common-image gather \( I(z, \theta, \phi) \) (d).

ally, the angle-domain common-image gather obtained from this space-lag domain gather should contain reflections for the whole range of \( \theta \), from zero to 90°.

The example in Figures 2(a) – 2(d) illustrates the relationships between angle and space-lag domain parameters, as discussed in the preceding section. Figure 2(a) represents the synthetic lag gather \( I(z, \lambda_x) \) with one single spike located at \( z = 1135 \) and \( \lambda_x = 0 \) m. Figure 2(b) is the 2D Fourier transform \( I(k_z, k_{\lambda_x}) \) of the lag gather. Figure 2(c) shows the result after applying the mapping \( k_{\lambda_x} \rightarrow \theta \), according to relation 11, to obtain \( I(k_z, \theta) \). As discussed earlier, as \( k_z \) increases, the mapping from equation 11 leads to a decreasing range of \( \tan \theta \) for each fixed value of \( k_z \), which in turn leads to decreasing values of \( \theta \). The angle gather \( I(z, \theta) \) is shown on Figure 2(d). In this example, \( d_z = d_{\lambda_x} = 5 \) m so, \( \theta_F = 45° \). Up to angle \( \theta_F \), the energy is well focused at the true depth of the reflector, and after this point is starts to defocus. In \( I(k_z, \theta) \), Figure 2(c), up to the value \( \tan \theta_F = 45 \), information exists at all values of \( k_z \) and beyond that there is a decrease in information associated with higher absolute values of \( k_z \). One can think of the data in \( I(k_z, \theta) \) as the data from \( I(k_z, k_{\lambda_x}) \) multiplied by a boxcar filter of different width for each \( k_{\lambda_x} \). Therefore, after applying the inverse Fourier transform on the axis \( k_z \) to obtain \( I(z, \theta) \), the angles corresponding the the boxcar filtered data appear convolved with a sinc function, and the energy corresponding to these angles defocuses, spreading away from the depth of the reflector, as seen in Figure 2(d). Thus, the analysis of angle-domain range involves finding \( \theta_F \) which is the maximum angle of energy that is well focused at the reflector depth. The relevance of calculating \( \theta_F \) is that if one wants to use ADCIGs to perform amplitude analysis, the amplitudes for angles greater than \( \theta_F \) would not be reliable because of the energy spreading effect caused by the angle-mapping transformations.

Figures 3(a) – 3(d) are similar to Figures 2(a) – 2(d). All subplots have the same meaning as the ones in the previous example. Since the lag gather image is band-limited, it does not contain the zero depth wavenumber \( k_z = 0 \), therefore, for this
Figure 5. Example illustrating mapping from a 3D lag gather with a wavelet located at \( z = 1135 \, \text{m} \) and \( \lambda_x = \lambda_y = 0 \, \text{m} \): lag gather \( I(z, \lambda_x, \lambda_y) \) with one single spike (a); 2D Fourier transform of the lag gather \( I(k_z, k_{\lambda_x}, k_{\lambda_y}) \) (b); mapping from \( (k_{\lambda_x}, k_{\lambda_y}) \) to \( (\theta, \phi) \), \( I(k_z, \theta, \phi) \) (c); angle-domain common-image gather \( I(z, \theta, \phi) \) (d).

From the examples in Figures 2(a) – 2(d) and Figures 3(a) – 3(d), the reflection energy is not well focused at the reflector depth for the angles greater than \( \theta_F \). This angle, is the ratio between the maximum lag wavenumber and maximum depth wavenumber in the lag gather. In the first example we do not have events corresponding to up to 90° in the angle gather because of the energy defocusing that occurs for angles after \( \theta_F \).

For a 3D horizontal reflector, the reflection angle \( \theta \) can be computed using the relation

\[
\tan \theta = \frac{\sqrt{k_{\lambda_x}^2 + k_{\lambda_y}^2}}{k_z},
\]

where \( k_{\lambda_x}, k_{\lambda_y}, k_z \) are the wavenumbers on the space-lag axis \( \lambda_x, \lambda_y \), and on the depth wavenumber axis \( \lambda_z \), respectively.

3. 2 3D range and resolution analysis

A similar analysis to the one performed in the preceding section can be done for the more general 3D case. Here, we limit our analysis to the case of a horizontal reflector. In this case, we are discussing about wide-azimuth angle-decomposition so, there are two angles for which we need to perform the range and resolution analysis: the reflection angle \( \theta \) and the azimuthal angle \( \phi \).

A 3D lag-domain common-image gather can be described as function of two space-lags: \( \lambda_x \) and \( \lambda_y \). The additional 3D model parameters are: origin, sample interval and number of points on the space-lag axis \( \lambda_x, (\alpha_{\lambda_x}, d_{\lambda_x}, n_{\lambda_x}) \), on the wavenumber axis \( k_{\lambda_x}, (\alpha_{k_{\lambda_x}}, d_{k_{\lambda_x}}, n_{k_{\lambda_x}}) \), and on the azimuthal angle axis \( \phi, (\alpha_\phi, d_\phi, n_\phi) \).

For a 3D horizontal reflector, the reflection angle \( \theta \) can be computed using the relation

\[
\tan \theta = \frac{\sqrt{k_{\lambda_x}^2 + k_{\lambda_y}^2}}{k_z},
\]
Similarly to the 2D case, we can analyze the behavior of \(d_\theta \) and \(d_\phi \) as functions of \(k_{\lambda_x} \) and \(k_{\lambda_y} \), i.e., as a function of \(n_{\lambda_x} \) and \(d_{\lambda_x} \), and \(n_{\lambda_y} \) and \(d_{\lambda_y} \). We rewrite equations 20 and 21 as

\[
\begin{align*}
\theta &= \arctan \left( \frac{k_{\lambda_x}}{k_{\lambda_x} + k_{\lambda_y}} \right), \\
\phi &= \arccos \left( \frac{k_{\lambda_y}}{k_{\lambda_x} + k_{\lambda_y}} \right).
\end{align*}
\]

To study how \(d_\theta \) changes with \(k_{\lambda_x} \) and \(k_{\lambda_y} \), we differentiate equation 22 first with respect to \(k_{\lambda_x} \)

\[
d_\theta = \frac{k_{\lambda_x} k_{\lambda_y}}{(k_x^2 + k_{\lambda_x}^2 + k_{\lambda_y}^2) \sqrt{k_{\lambda_x}^2 + k_{\lambda_y}^2}} \frac{1}{n_{\lambda_x} d_{\lambda_x}},
\]

and then with respect to \(k_{\lambda_y} \)

\[
d_\theta = \frac{k_{\lambda_y} k_{\lambda_x}}{(k_x^2 + k_{\lambda_x}^2 + k_{\lambda_y}^2) \sqrt{k_{\lambda_x}^2 + k_{\lambda_y}^2}} \frac{1}{n_{\lambda_y} d_{\lambda_y}}.
\]

Similarly, to study how \(d_\phi \) changes with \(k_{\lambda_x} \) and \(k_{\lambda_y} \), we differentiate equation 23 first with respect to \(k_{\lambda_x} \)

\[
d_\phi = \frac{1}{\sqrt{1 + \frac{k_{\lambda_y}^2}{k_{\lambda_x}^2 + k_{\lambda_y}^2}}} \frac{k_{\lambda_y} k_{\lambda_x}}{(k_x^2 + k_{\lambda_x}^2 + k_{\lambda_y}^2)^{3/2}} \frac{1}{n_{\lambda_x} d_{\lambda_x}}
\]

and then with respect to \(k_{\lambda_y} \)

\[
d_\phi = \sqrt{1 + \frac{k_{\lambda_y}^2}{k_{\lambda_x}^2 + k_{\lambda_y}^2}} \frac{1}{(k_x^2 + k_{\lambda_x}^2 + k_{\lambda_y}^2)^{3/2}} \frac{1}{n_{\lambda_y} d_{\lambda_y}}.
\]

Equations 24, 25, 26, and 27 show that \(d_\theta \) and \(d_\phi \) are both inversely proportional to \(n_{\lambda_x} d_{\lambda_x} = \lambda_{r_{max}} \) and \(n_{\lambda_y} d_{\lambda_y} = \lambda_{y_{max}} \), where the parameters \(\lambda_{r_{max}} \) and \(\lambda_{y_{max}} \) are the ranges in space-lag domain, which control the resolution in the angle-domain. As in the preceding section, we conclude that the larger the range in the space-lag domain, the better the resolution in the angle-domain.

Using similar arguments to the ones employed for the 2D analysis, to uniformly sample the angle-domain and avoiding aliasing, the parameters \(d_\theta \) and \(d_\phi \) need to be lesser or equal to their minimum possible values given by substituting the wavenumbers in equations 24, 25, 26, and 27 by their maximum values (Nyquist). This way, there are two possible values for both \(d_\theta \) and \(d_\phi \), and the minimum of the two should be used in each case.

From equation 22, for each \(k_{\lambda_y} \) there is an angle that can be mapped from the lag-domain, given by \(k_{\lambda_y_{max}} \) and \(k_{\lambda_y_{max}} \) (as defined in previous section), that limits the range of angles for which there is information at all values of \(k_x \). We define \(\theta_F \) in 3D as the smallest of these angles, which corresponds to \(k_{\lambda_y} = 0 \) in equation 22. This angle \(\theta_F \) defines the range of reflection angles that can be accurately recovered from the lag-domain, because this range of angles contains information for all depth wavenumbers \(k_z \) for all azimuths. The maximum \(\theta \) for which there is information at all wavenumbers \(k_z \) changes with \(\phi \). Outside the range defined by \(\theta_F \), due to the fact that this maximum \(\theta \) changes with \(\phi \), there are an uneven information distribution along the \(\phi \) dimension, with respect to \(k_z \). The azimuthal angles 45, 135, 225, and 315° have more information at more wavenumbers \(k_z \) than the other angles, and there is a decrease on this wavenumbers as angles goes away from these four ones, reaching a minimum at 0, 90, 180, and 270°. Thus, reflections occurring at these four azimuthal angles where there is more information, or in their vicinity, have better focused events in the angle gathers than at other azimuthal angles.

Similarly to the 2D case, we construct synthetic 3D lag-gathers (first with a spike and then with a band-limited wavelet at zero lag in \(x \) and \(y \)) and study the steps involved in the implementation of the mappings described in equations 20 and 21.

For the first example, we construct a lag gather \(I(\lambda, \lambda_x, \lambda_y) \) with a spike at \((z = 1135, \lambda_x = 0, \lambda_y = 0) \). Then, we apply a 3D Fourier transform on the lag gather to obtain \(I(k_z, k_{\lambda_x}, k_{\lambda_y}) \). Next, we apply the mappings in equations 20 and 21 to get \(I(k_z, \theta, \phi) \). And, finally, we inverse Fourier transform the axis \(k_z \) to create the angle gather \(I(\theta, \phi) \). Figures 4(a) to 4(d) shows four 3D plots representing the image in these four domains. In each of the 3D plots, the cross lines show the position of the sections displayed on the surfaces. Figure 4(a) is the lag gather, \(I(\lambda, \lambda_x, \lambda_y) \), with one single spike. Figure 4(b) is the 3D Fourier transform, \(I(k_z, k_{\lambda_x}, k_{\lambda_y}) \), of the lag gather. Figure 4(c) shows both mappings from equations 20 and 21 applied to \(I(k_z, k_{\lambda_x}, k_{\lambda_y}) \) to obtain \(I(k_z, \theta, \phi) \). For this wide-band example \(\theta_F = 45 \) because the lag gather has information for all wavenumbers (wide-band) and the sample interval in all dimensions are equal. In the \(k_z, \theta \) cross-section in Figure 4(c), for angles \(\theta \) up to 45° information exists at all wavenumbers \(k_z \), and for angles greater than 45° there is an uneven information distribution along the \(\phi \) dimension, with respect to \(k_z \). The azimuthal angles 45, 135, 225, and 315° have more information at more wavenumbers \(k_z \) than the other angles, and there is a decrease on this wavenumbers as angles goes away from these four ones, reaching a minimum at 0, 90, 180, and 360°. The same pattern can be seen in the \((k_z, \phi) \) cross-section in this same Figure. For the reflection angles in the interval \(|\theta| < 45 \), the energy is well focused in the angle gathers for all azimuthal angles.

As in the 2D case, for the next example, we construct a 3D lag gather with a wavelet rather than a spike, with the intent of simulating a more realistic situation (a band-limited lag-gather). Figures 5(a) – 5(d) shows, respectively, \(I(\lambda, \lambda_x, \lambda_y) \), \(I(k_z, k_{\lambda_x}, k_{\lambda_y}) \), \(I(k_z, \theta, \phi) \), \(I(\lambda, \theta, \phi) \). The wavelet in the 3D lag gather is band limited and the maximum depth wavenum-
ber is smaller than the Nyquist frequency \( k_{z, \text{max}} \). That causes
the angle \( \theta_{F} \) for this case to be greater than for the previous
example. Therefore, in this case, the range of angles for which
the energy is well focused in the 3D angle-domain common-
image gather, for all azimuthal angles, is larger than for the
spike lag example (around 63°). As for the preceding example,
the angle \( \theta_{F} \) changes with azimuth reaching maximum
values at \( \phi = 45, 135, 225, \) and \( 315^\circ \).

4 EXAMPLES

The following two examples are meant to illustrate the rela-
tionships discussed in preceding section. We create two ex-
amples with equal geometry but different velocity models. In
each examples the velocity varies with depth, \( v = v(z) \), but
with negative and positive gradients, respectively. We con-
struct CIGs for the same locations in both cases, as illustrated
in Figure 6. For the first example, the reflection angles for
these CIG locations are within the range of well focused en-
ergy defined in the previous section, i.e., the reflection
angles at these locations are smaller than \( \theta_{F} \) for each gather. In
this case, the energy in the angle gathers are well focused. In
the second example, the reflection angles in all gathers but one
are greater than \( \theta_{F} \), and in this case the energy is less well
focused.

The reflectivity model is a horizontal reflector embed-
ded in a \( v = v(z) \) medium. The model parameters are:
\((\alpha_{z}, d_{x}, n_{z}) = (0, 30, 180), (\alpha_{y}, d_{y}, n_{y}) = (0, 30, 180), \) and
\((\alpha_{z}, d_{z}, n_{z}) = (0, 5, 180) \). There is one source located in
the center of the horizontal plane. The reflector is located at
depth \( z = 700 \) m. The velocity model for the first exam-
ple is \( v(z) = 4500 - 4z \). The synthetic data (due to one
source) \( D(x, y, t) \) and migrated image \( I(x) \) is shown in Fig-
ure 7. There, we can see the flat reflector located at depth
\( z_{r} = 700 \) m.

As discussed earlier, we construct ADCIGs just for some
locations over the reflector. From the hypercube image \( I(x, \lambda) \)
we can select a location in the horizontal space, \((x, y) = (x_{0}, y_{0}) \), where we want to construct a ADCIG, and
obtain the three-dimensional subset \( I_{h} = I(x = x_{0}, y = y_{0}, z, \lambda_{x}, \lambda_{y}, \lambda_{z} = 0) \). Figure 8 shows lag gathers for nine dif-
ferent locations at the reflector depth (Figure 6). Each gather
is obtained by selecting the depth of the reflector, thus, each
gather represents the set \( I_{h} = I(x = x_{0}, y = y_{0}, z = z_{r}, \lambda_{x}, \lambda_{y}, \lambda_{z} = 0) \). The gather in the center has the same hori-
zontal location as the source. The gathers around the central
one are located 300 m away from it in one or both horizontal
directions. For each panel in Figure 8, the horizontal axis is \( \lambda_{x} \)
and the vertical is \( \lambda_{y} \).

We create an angle gather for each lag gather. To better vi-
ualize an ADCIG, we make use of polar plots. Figure 9 helps
in locating and understanding how an event appears on a polar
plot. The azimuthal angle \( \phi \) is zero at the positive hori-
zontal direction and increases in the counter-clockwise direction.
The distance from the center of the circles to the event rep-
resents the angle \( \theta \), which has a minimum value of zero for
events appearing in the center of the Figure, and a maximum
value of 90 for events locateded on the outer circle. The angle
 gathers obtained from the lag gathers in Figure 8 are shown in
Figure 10.

As the medium is homogeneous and the reflector is hori-
izontal, the azimuth of the reflection plane coincides with the
source-receiver azimuth on the surface. As can be seen in Fig-
ure 10, the azimuth of the reflection planes coincides with the
azimuth of the source-receiver direction. The central polar plot
corresponds to the image point right under the source and, as
expected, we have an event at \( \theta = 0 \), i.e., we have normal
incidence at this point. We can see in these gathers that the en-
ergy is very focused, as expected, since the reflection angles
are within the complete energy focusing range.

The velocity model for the second example is \( v(z) = 1000 + 4z \). The synthetic data \( D(x, y, t) \) and migrated image
\( I(x) \) is shown in Figure 11. There, we can see the flat reflector
located at depth \( z_{r} = 700 \) m.

Figure 12 shows nine lag gathers for the same nine loca-
tions in the previous example. Each gather is also obtained by
selecting the depth of the reflector in \( I_{h} = I(x = x_{0}, y = y_{0}, z = z_{r}, \lambda_{x}, \lambda_{y}, \lambda_{z} = 0) \). The angle gathers obtained from
the lag gathers in Figure 12 are shown in Figure 13.

As can be seen in Figure 13 the energy is less well fo-
cused compared to Figure 10. That is because the reflection
angles are outside the range of well focused energy defined by
the angle \( \theta_{F} \) explained in the preceding sections. The central
CIG looks defocused even though the reflection angle is zero
at this location. This is due to a low clip value used to create
the plots in that case. We could see through these two exam-
pies how the energy focuses well or not depending on whether
the reflection angles are inside or outside the range of com-
plete energy focusing. The angles \( \theta \) and \( \phi \) in the images are
validated by theoretical expressions found in Dobrin (1988).

5 CONCLUSIONS

One can extract angle information from the image \( I(x, \lambda) \) just
by using simple relationships between the ray parameters vec-
tors. Using this angle information we can construct angle de-
pendent reflectivity. The model parameters control the range
of reflection angles that can be recovered from the image with
well focused events in the angle-domain common-image gath-
erers. In general, the angle that limits this range is proportional
to the ratio between maximum depth wavenumber and maxi-
mum space-lag wavenumber. It may not be possible to change
the sample intervals in the lag-domain because this is usually
the same as the ones in characterizing the space-domain co-
oordinates. Conventional implementations of extended imaging
condition keep the space and space-lag samplings the same.
The largest wavenumbers of the lag gather limit the range of
angles for which one can “trust” the amplitudes in the angle
gather. This is particularly important if these angle gatherers are
being employed to perform amplitude versus angle analysis or
migration velocity analysis.
Figure 6. CIG locations. Each circle is a CIG location and the cross is the source location. The central CIG horizontal location is the same as the source location. The CIGs around the central one are 300 meters apart in both $x$ and $y$ directions.

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Figure 7. Data for the one shot located at center of x-axis (a), and image of horizontal reflector located at $z_r = 700$ m (b).

The two-way traveltime $t = t(x, \lambda)$ from a source to a receiver location is a function of $x$ and $\lambda$. The traveltime can be regarded as function of $s$ and $r$, $t = t \left( \frac{r + s}{2}, \frac{r - s}{2} \right)$. Changes in the traveltime with respect to change in $x$, $y$ and $z$ are, respectively,

$$\frac{\partial t}{\partial x} = \frac{\partial t}{\partial r_x} \frac{\partial r_x}{\partial x} + \frac{\partial t}{\partial s_x} \frac{\partial s_x}{\partial x}, \quad (A-3)$$

and

$$\frac{\partial t}{\partial y} = \frac{\partial t}{\partial r_y} \frac{\partial r_y}{\partial y} + \frac{\partial t}{\partial s_y} \frac{\partial s_y}{\partial y}, \quad (A-4)$$

and

$$\frac{\partial t}{\partial z} = \frac{\partial t}{\partial r_z} \frac{\partial r_z}{\partial z} + \frac{\partial t}{\partial s_z} \frac{\partial s_z}{\partial z}, \quad (A-5)$$

Similarly, changes in the traveltime with respect to $\lambda_x$, $\lambda_y$ and $\lambda_z$ are, respectively,

$$\frac{\partial t}{\partial \lambda_x} = \frac{\partial t}{\partial r_x} \frac{\partial r_x}{\partial \lambda_x} + \frac{\partial t}{\partial s_x} \frac{\partial s_x}{\partial \lambda_x}, \quad (A-6)$$

and

$$\frac{\partial t}{\partial \lambda_y} = \frac{\partial t}{\partial r_y} \frac{\partial r_y}{\partial \lambda_y} + \frac{\partial t}{\partial s_y} \frac{\partial s_y}{\partial \lambda_y}, \quad (A-7)$$
Figure 8. Space-lag domain common-image gathers: each individual plot is a lag gather for a different horizontal location over the reflector. In each Figure, the horizontal axis is $\lambda_x$ and the vertical is $\lambda_y$. The gather in the center has the same horizontal location as the source. The gathers around the central one are located 300 m away from it in one or both horizontal directions.

and

$$\frac{\partial t}{\partial \lambda_z} = \frac{\partial t}{\partial r_z} + \frac{\partial s}{\partial s} \frac{\partial t}{\partial \lambda_z},$$  \hspace{1cm} (A-8)

Since $p = \nabla t$, we can relate the wavenumber vectors as

$$p_x = p_r + p_s,$$  \hspace{1cm} (A-9)

$$p_\lambda = p_r - p_s,$$  \hspace{1cm} (A-10)

where $p_x$, $p_\lambda$, $p_s$, and $p_r$ are the wavenumber vectors relative to $x$, $\lambda$, $s$, and $r$, respectively.

We can define the azimuthal angle as the angle between any vector that rotates along with the reflection plane and an arbitrary direction. The vector $p_x \times p_\lambda$ is contained in the reflecting plane and rotates $360^\circ$ with the reflection plane. Let $v$ be an arbitrary direction, then the azimuthal angle between $p_x \times p_\lambda$ and $v$ is given by

$$\cos \phi = \frac{(v) \cdot (p_x \times p_\lambda)}{|v||p_x \times p_\lambda|}.$$  \hspace{1cm} (A-15)

Here, we choose to define the $x$-axis (inline) direction as the azimuthal reference direction $v = (1, 0, 0)$, and the azimuthal angle as the angle between the line of intersection line between the reflection plane and the horizontal $x$-$y$ plane and the $x$-axis. One advantage of this choice of reference is that it allows one to study local media property variations in relation to the same direction. One disadvantage of this choice is that this definition breaks down for vertical reflectors because the intersection between all possible reflection planes, at a fixed image point location, and the horizontal plane all have the same direction.

Let vector $d$ be the cross-product if the vectors $p_\lambda$ and $p_x$.

$$d = p_\lambda \times p_x,$$  \hspace{1cm} (A-16)

As indicated earlier, $d$ is orthogonal to the reflection plane and it is contained in the reflecting plane. As the reflection plane rotates, the vector $d$ also rotates, so we can associate the orientation of $d$ with the orientation of the reflection plane. Let $d_h = (d_x, d_y, 0)$ be the horizontal projection of $d$ in the $x$-$y$
Figure 9. Polar representation of reflection energy function of reflection and azimuthal angle: The azimuthal angle $\phi$ is zero at the positive horizontal direction and increases in the counter-clockwise direction. The distance from the center of the circles to the event represents the angle $\theta$, which has a minimum value of zero for events appearing in the center of the Figure, and a maximum value of 90 for events located on the outer circle. The filled dot represents a reflection event where $\theta = 45^\circ$ and $\phi = 60^\circ$.

plane. We define the azimuthal angle $\phi$ as the angle between $d$ and the inline direction, so

$$
\cos\phi = \frac{k_{\lambda x}k_x - k_{\lambda y}k_y}{\sqrt{(k_{\lambda y}k_z - k_{\lambda z}k_y)^2 + (k_{\lambda z}k_x - k_{\lambda x}k_z)^2}}.
$$
(A-18)

Since $k_\lambda$ and $k_x$ are orthogonal to each other

$$
k_\lambda \cdot k_x = k_{\lambda x}k_x + k_{\lambda y}k_y + k_{\lambda z}k_z = 0,
$$
(A-19)

and

$$
k_{\lambda x} = \frac{-k_{\lambda y}k_z - k_{\lambda z}k_y}{k_x}.
$$
(A-20)

Thus, we can eliminate $k_{\lambda x}$ from equations A-14 and A-18 and rewrite these equations as

$$
\tan\theta = \frac{\sqrt{(k_{\lambda y}^2 + k_{\lambda z}^2)k_x^2 + (-k_xk_{\lambda y} - k_yk_{\lambda z})^2}}{\sqrt{k_x^2(k_x^2 + k_{\lambda y}^2 + k_{\lambda z}^2))}},
$$
(A-21)

and

$$
\cos\phi = \frac{k_{\lambda x}k_y + k_{\lambda y}k_x + k_{\lambda z}k_z}{\sqrt{(k_x^2k_{\lambda y} + k_y^2k_{\lambda z} + k_z^2k_{\lambda x})^2 + (-k_x^2k_{\lambda y} - k_y^2k_{\lambda z} - k_z^2k_{\lambda x})^2}}.
$$
(A-22)
Figure 10. Angle-domain common-image gathers: each individual plot is the angle gather (CIG) obtained from the lag gathers in Figure 8. The gathers are displayed in a polar plot format. Refer to Figure 9 for a polar plot interpretation.

Figure 11. Data for the one shot located at center of x-axis (a), and image of horizontal reflector located at $z_r = 700$ m (b).
Figure 12. Space-lag domain common-image gathers: each individual plot is a lag gather for a different horizontal location over the reflector. In each Figure, the horizontal axis is $\lambda_x$ and the vertical is $\lambda_y$. The gather in the center has the same horizontal location as the source. The gathers around the central one are located 300 m away from it in one or both horizontal directions.

Figure 13. Angle-domain common-image gathers: each individual plot is the angle gather (CIG) obtained from the lag gathers in Figure 12. The gathers are displayed in a polar plot format.
Figure A-1. Geometrical relations between ray parameter vectors.
Modeling and imaging with isochron rays

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\textbf{ABSTRACT}

Isochron rays are lines perpendicular to isochrons, which represent surfaces of constant two-way traveltime. The image of a temporal sequence of seismic impulses is a sequence of isochrons in depth. The later the time impulse the deeper the isochron. The term isochron ray arises from an analogy between the isochron “movement” and the wave propagation. While isochrons behave as wavefronts, its perpendicular lines can be regarded as rays. The speed of the isochron movement depends on the medium velocity and the source-receiver position. We introduce the term equivalent-velocity to refer to the speed of isochron movement. In the particular case of zero-offset data, the equivalent velocity is half of the medium velocity. We use the concepts of isochron-rays and equivalent velocity to extend the application of the exploding reflector model to non-zero offset imaging problems. In particular, we employ these concepts to extend the use of zero-offset wave-equation algorithms for modeling and imaging common-offset sections. In our imaging approach, the common-offset migration is implemented as a trace-by-trace algorithm in three steps: 1) equivalent velocity computation, 2) data-conditioning for zero-offset migration, and 3) zero-offset wave-equation migration. We apply this methodology for modeling and imaging synthetic common-offset sections using two kinds of algorithms: finite-difference and split-step wavefield extrapolation. We illustrate the isochron-ray imaging methodology with a field-data example and compare the results with conventional common-offset Kirchhoff migration. This methodology is attractive because (1) it permits depth migration of common-offset sections or just pieces of that by using wave-equation algorithms, (2) it extends the use of robust zero-offset algorithms, (3) it presents favorable features for parallel processing, (4) it permits the creation of hybrid migration algorithms, and (5) it is appropriate for migration velocity analysis.

\textbf{Key words:} wave equation, imaging, modeling, isochron, isochron-rays

1 INTRODUCTION

Isochron rays are curves associated with propagating isochrons, that is, with surfaces that are related to seismic reflections with the same two-way traveltime. Isochron surfaces play an important role in seismic imaging because they are closely related to the impulse response of depth migration. Hubral \textit{et al.} (1996) showed how a weighted Kirchhoff-type isochron-stack integral can be applied to true-amplitude algorithms for both modeling and data transformation. The general theory of data mapping, presented by Bleistein \textit{et al.} (2000), emphasizes the importance of isochrons in the establishment of integral formulas for inversion.

Iversen (2004) introduced the term isochron ray for trajectories associated with surfaces of equal two-way time, i.e. isochron surfaces, and suggested the potential use of isochron rays in future implementations of prestack depth migration. Here we exploit the idea and present a methodology that makes use of the isochron ray concept to perform prestack depth migration. We consider as isochron rays the curves that are perpendicular to the isochrons associated with the image produced by the migration of a single finite-offset seismic trace. That is, isochron rays are the orthogonal trajectories to isochrons. This concept differs from the one introduced by Iversen (2004), which involves non-orthogonal trajectories.

Our imaging approach consists of a trace-by-trace al-
algorithm, wherein each finite-offset input trace is first conditioned for zero-offset extrapolation and then migrated using an equivalent-velocity model. We present two imaging strategies. In the first approach the prestack depth migration is achieved by performing the downward continuation of the conditioned data along the isochron rays, followed by the application of the zero-offset imaging condition. The second imaging approach consists of a reverse-time migration algorithm, where the conditioned data is reversed and injected into the equivalent velocity model along an isochron defined by the time-shift previously applied on the input trace.

The main purpose of this research is the development of methodology to apply zero-offset wave-equation algorithms to solve finite offset problems. In particular, this methodology can be useful in the implementation of migration velocity analysis methods based on offset continuation, see Silva (2005). Also, the presented methodology is attractive because (1) it permits depth migration of common-offset gathers using wave-equation algorithms, (2) it extends the use of robust zero-offset algorithms to the common-offset case, (3) it is based on algorithms that are appropriate for parallel processing, and (4) it permits to combine different imaging algorithms.

2 THE ISOCHRON RAY CONCEPT

Given a source-receiver pair and a fixed reflection traveltime, the related isochron is the surface that answers the question: Where are the possible reflection points located? In other words, an isochron surface is the image of points with the same reflection time. An isochron surface can be viewed as a hypothetical reflector whose reflections from the source are recorded simultaneously at the receiver R. For any point M belonging to an isochron surface, the traveltime measured along the path SMR does not vary.

The isochron surfaces play an important role in seismic imaging, especially in prestack depth migration. For a single trace composed of a sequence of impulses, the image produced by depth migration is represented by a set of isochrons. The longer the reflection time, the greater the distance between the source-receiver pair and the isochron. The shape of the isochrons depends on the velocity field, the reflection time, and the spatial location of the source and receiver. In a homogeneous medium, every isochron has an ellipsoidal shape whose focus points are located at the source and receiver position, while the eccentricity is defined by the seismic wave velocity and the reflection time. The shallowest isochron surface tends to collapse into a straight line connecting the source-receiver pair.

Consider a sequence of depth migrated images where the input data consist of a sequence of seismic impulses with varying reflection time. Notice in Figure 1(a) as the impulse time increases by the same amount with the first being just little longer than the direct arrival traveltime. The initial surface (first isochron) observed in the first image moves in depth, changing its shape and acting as a propagating wavefront. If a point of the initial surface is selected and followed during the sequence, its trajectory will define a curve. We refer to this curve as an isochron ray because it acts as a ray, while the moving isochron plays the role of a propagating wavefront.

2.1 Equivalent velocity media

The propagating isochron "moves" through the model with a speed that is different from the wave propagation velocity. The isochron propagation velocity depends on the source-receiver location, and the medium velocity and it varies even in isotropic-homogeneous media.

For a given source-receiver pair, we can imagine a hypothetical medium with the velocity distribution identical to that of the isochron velocity medium. We refer to this as the equivalent velocity medium. There are two features that characterize the equivalent velocity field. First, isochron propagation velocity depends on the isochron ray direction, which means it can be multivalued in the presence of caustics. Second, equivalent velocity fields present a singularity in the line connecting the source-receiver pair, which corresponds to the hypothetical starting isochron.

Depending on the complexity of the original velocity model, we distinguish two cases of determining equivalent velocity and isochron ray tracing. One is based on the assumption of the absence of caustics, and another is the general case, where no restrictions are imposed on the velocity model.

Let us assume a smooth seismic velocity model with no caustics. Consider a source-receiver pair located in a horizontal plane where the velocity does not vary in a small slab between the source and the receiver. In this situation, the isochron-field can be reproduced by a hypothetical experiment where the seismic source is a horizontal segment connecting the source-receiver pair, and the seismic velocity is the equivalent velocity medium. All the isochron rays are perpendicular to the line source. In the vertical plane that contains the source-receiver pair, all the isochron rays are vertical at their starting point. The isochron rays obey Snell’s law and the wavefronts are the surfaces of constant traveltime in space satisfying the eikonal equation:

\[
\frac{\partial t_{sr}}{\partial x} \frac{\partial t_{sr}}{\partial x} + \frac{\partial t_{sr}}{\partial y} \frac{\partial t_{sr}}{\partial y} + \frac{\partial t_{sr}}{\partial z} \frac{\partial t_{sr}}{\partial z} = \frac{1}{V_{eq}(x, y, z)}. \tag{1}
\]

In the absence of triplications, the equivalent velocity medium can be directly determined by applying the eikonal equation on the two-way traveltime map. This velocity field can be used to migrate the conditioned data using conventional zero-offset migration algorithms. In the presence of triplications, equivalent velocity media are multivalued and cannot be derived from traveltime maps. In this case, the isochron propagation velocity should be represented in isochron ray coordinates and has to be computed using a proper isochron ray-tracing algorithm. Also, the migration should use wave-field extrapolation in isochron ray coordinates. This case falls outside the scope of this paper and remains subject to future research.
2.2 Isochron ray-tracing without media restriction

The isochron propagation can be performed by applying basic principles of wave propagation. For simplicity, consider an isochron in a 2D media. In Figure 2(a), the point \( M \) is the intersection of three curves:

- \( z = \zeta_s(x, S, t_s) \) is the wavefront that comes from the source position, \( S = (x_s, z_s) \), at the time \( t_s \).
- \( z = \zeta_r(x, R, t_r) \) is the wavefront that comes from the receiver position, \( R = (x_r, z_r) \), at the time \( t_r \).
- \( z = \zeta(x, S, R, t_s + t_r) \) is the isochron corresponding to the source-receiver pair \( SR \) at the two-way traveltime \( t_{sr} = t_s + t_r \).

Figure 2(b) shows the isochrons after a propagation time of \( \delta t/2 \), the wavefront \( z = \zeta_s(x, S, t_s) \) moves to \( z = \zeta_s(x, S, t_s + \delta t/2) \), the wavefront \( z = \zeta_r(x, R, t_r) \) moves to \( z = \zeta_r(x, R, t_r + \delta t/2) \), the isochron \( z = \zeta(x, S, R, t_s + t_r) \) moves to \( z = \zeta(x, S, R, t_s + t_r + \delta t) \), and the intersection point moves to \( M' \). In 2D models, the triple intersection point can be found by merely locating the intersection between the source and receiver wavefronts. In this case, an isochron ray can be traced just by mapping the intersection points step by step.

In the case of 3D, the intersection between the source and receiver wavefronts is a curve instead of a point. Consequently the use of the unknown isochron is needed for determining the intersection point. The isochron is unknown but it can be defined as the envelope of intersection lines between source and receiver wavefronts whose total traveltime is constant.

2.3 The exploding reflector model

The exploding reflector model (Lowenthal et al., 1985) is widely applied in both seismic modeling and imaging algorithms. Although it is an approximation that cannot be reproduced by a physical experiment, it leads to simple, robust and efficient algorithms. Zero-offset seismic data can be modeled and migrated by a large number of approaches, such as Kirchhoff, finite-differences, and Gaussian beams.

For zero-offset data, the two-way traveltime of primary reflections with normal-incidence angle can be computed by tracing normal rays in a half-velocity medium.

The isochron-rays play a role analogous to normal rays, i.e., they are perpendicular to the reflectors, and the traveltime measured along them is the time on the two-way path: source-reflector-receiver. While the normal rays can be traced using the half-velocity medium, the isochron-rays need an equivalent velocity medium that depends on the source and receiver location. Therefore we need to define an equivalent velocity medium for each source-receiver pair. Another important dif-
difference between normal and isochron rays is the take-off (or emergence) angle. While normal rays can assume any direction at the recording surface even when the first layer is homogeneous, the isochron rays are always perpendicular to it, see figure 3(b). Because of the analogy described above, we suggest the expression "Equivalent exploding reflector model" to refer to the association of isochron rays and equivalent velocity model.

3 IMAGING USING ISOCHRON RAYS

3.1 Modeling

In this section, we present two examples in which we extend the use of zero-offset algorithms to finite offset data by applying the concepts of isochron rays, equivalent velocity, and exploding reflector model. We generate three common-offset sections with the methods: finite-difference, split-step wavefield extrapolation and Kirchhoff. The third dataset was generated by conventional Kirchhoff modeling to be used as benchmark.

The 2D seismic model consists of six interfaces immersed in a smooth velocity field, where the wave velocity propagation varies from 1500 m/s at the shallow part, to 3000 m/s at the bottom. The Figure 4 shows the velocity model and the interfaces. For all cases, the source-receiver pair is located on the horizontal plane at z=0.

For Kirchhoff modeling, a Ricker wavelet with dominant frequency of 20 Hz was used. The integration was performed using a spatial interval of 1 m without any special care about the dynamic aspects. Figure 5(a) shows the synthetic Kirchhoff common-offset section for the half-offset \( h = 500 \) m.

For both finite-difference and wavefield extrapolation modeling, the common-offset section was constructed trace-by-trace using an equivalent velocity medium for each CMP position. The equivalent velocity media were defined following the steps:

- Build the travel-time map \( t_s(x, z) \) from the source location to all points of the model using an eikonal solver algorithm.
- Do the same from the receiver location, getting the travel-time map \( t_r(x, z) \).
- Add the two maps to obtain the two-way travel-time \( t_{sr} = t_s + t_r \).
- Find the spatial partial derivatives of the two-way travel-time \( t_{sr} \).
- Apply the eikonal equation to determine the equivalent velocity for every grid point.

The equivalent velocity media has a singularity between the source-receiver pair, where the velocity goes to infinity. Thus we need to adopt special procedures to avoid numerical problems in this region. The applied procedures are different for each case. However both are based on an analytical solution for the isochron propagation in the vicinity of the source-receiver pair. In Figure 6 we present the equivalent-velocity field for the central position of the seismic model presented in Figure 4.

For finite difference modeling, we avoid numerical instability by clipping the equivalent velocity field and placing the receivers along an isochron located away from the singularity. A good choice for locating this recording isochron is a region where the wave propagation is constant because it is an ellipse in this case.

The recorded isochron-field contains information from all directions, but only information that travels along isochron rays should be considered. In other words, we have to sum the amplitudes along isochrons, which is equivalent to a stack of the information collected in the recording isochron along the time. Figure 5(b) is the common-offset gather modeled by the finite-difference approach. Each trace of this gather is the result of the stacking of all traces recorded along an isochron whose midpoint between the source and receiver corresponds to the trace location. Figure 7(a) corresponds to the seismogram recorded at the central position of the seismic model.

For modeling by wave-field extrapolation, the adopted procedure consists of recording the wave-field (isochron-field)
at a horizontal plane below the singularity. Since the source-receiver pair is located at the surface \( z = 0 \), the singularity is taken out of the equivalent velocity model just by excluding a tiny slab with a thickness corresponding to the vertical sample interval. The wave-field is extrapolated from the bottom to the one-sample deep surface where it is recorded, see Figure 7(b). After applying a time-shift to compensate for the slab removing, the data are stacked to generate the modeled trace. The initial time of the modeled trace corresponds to the direct arrival traveltime. The procedure described above is independently applied for all desired output positions of the modeled common-offset gather. Alternatively, the problem of avoiding the singularity can be addressed by redatuming the data from the surface to a deeper plane.

3. 2 Migration

In principle, all of the zero-offset migration methods based on the exploding reflector model can have their use extended to finite-offset gathers by making use of isochron rays and equivalent velocity media. In this section, we discuss two cases: migration by wave-field extrapolation (WEM) and reverse time migration (RTM). In both cases, the isochron ray migration can be implemented in a trace-by-trace algorithm. For each trace, the following steps are carried out: 1) computation of the equivalent velocity model, 2) creation of the conditioned data for wave-field reconstruction, 3) migration of the conditioned data by a zero-offset algorithm using the equivalent velocity model, and 4) addition of the migration result to the image.

The conditioning data procedure is not the same for WEM and RTM, but in both cases a half-derivative followed by a time shift is applied to the input trace. For the WEM case, the input trace is time-shifted by a negative amount that corresponds to the traveltime measured along the raypath connecting the source and receiver. The conditioned data gather for WEM is obtained by repeating the shifted trace for each trace position located between the source-receiver, while the remaining positions are filled with zeros. In the RTM case the time-shift is also negative and it corresponds to the time of an isochron where the reverse-data is injected. This isochron should be as far as possible from the singularity. A good choice would be an ellipse when the source-receiver pair is located in a homogeneous region. In addition to the required steps described above, linear spatial tapering is applied to the conditioned data to avoid the presence of artifacts at the final image. Another important difference between RTM and WEM using the equivalent exploding reflector model is the imaging condition. In the RTM case, the partial image gather (the result of the migration of a single trace) corresponds to the last snapshot of the wave-field, while this gather is obtained after applying a zero-offset image condition in WEM case.

Figure 8(a) corresponds to the zero-offset image obtained by conventional wave-field extrapolation migration. Figures 8(b) and 8(c) are the common-offset WEM and RTM images, respectively. In both cases, the synthetic Kirchhoff common-offset gather was used as input.

4 APPLICATION TO FIELD-DATA

The WEM using isochron rays was applied in a pseudo 2.5D dataset, which consists of twenty-two common-offset gathers extracted from a 3D dataset via the following sequence. First, the input traces were organized in twenty-two groups, using as sorting criteria the source-receiver offset; second a 3D Kirchhoff time migration algorithm was applied; finally, a 2.5D Kirchhoff time demigration procedure was applied to each image. In the sorting procedure of the first step, each input trace was multiplied by an areal factor in order to compensate for the effect of acquisition irregularities. The weight function used in the 3D time-migration algorithm produces a true-amplitude image gather when the medium velocity is constant, i.e the output amplitudes are proportional to the reflection coefficients. Also, the applied demigration program uses a true-amplitude weight function, that produces a 2D common-offset gather whose amplitudes are affected by a 3D geometrical spreading factor, which is correct when the medium is homogeneous.
The minimum offset is 160 m and the increment between offsets is 200 m. Each common-offset gather has 1351 traces and the distance between them is 18.75 m. The traces are 5.0 s long and the time sampling interval is 4 ms.

The same smoothed velocity model (figure 11) was used to migrate four common-offset gathers (from 1560 m to 2160 m) using the wave-field extrapolation approach and a traditional common-offset Kirchhoff program. The WEM image from the gather with a larger offset is presented in the figure 9(a), while the figure 9(b) shows the Kirchhoff result. As expected, there are no significant differences between the images from these migration methods. The result is basically the same because we use the same velocity and the same approach to compute traveltime maps for Kirchhoff migration and to calculate the equivalent velocity in wave-field extrapolation. A similar result is observed when we stack the four migrated sections: compare Figure 10(a) and Figure 10(b). A significant difference between the Kirchhoff method and the WEM isochron ray migration should be expected in the presence of caustics if we determine the equivalent velocity medium by means of an isochron ray-tracing algorithm.

5 DISCUSSION

The computational cost of modeling or migrating an individual seismic trace using the isochron ray approach as presented above is close to the cost of modeling or migrating a zero-offset section. Besides the high computational cost, the described methodology is restricted to smooth velocity models, which reduces the attractiveness of this approach. The computational cost can be reduced by using beams, redatuming, and limited aperture. Silva & Sava (2008) show that the combination of these procedures can drastically decrease the processing time, especially for greater offsets. Problems due to triplication can be eliminated by representing the equivalent velocity in isochron ray coordinates.

Future research includes the development of an effective isochron ray tracing algorithm without making any assumption about the medium. This algorithm might be based in the algorithm described in the section isochron ray-tracing, which should be implemented by making use of the paraxial ray theory. The isochron ray-tracing algorithm could be used to define equivalent velocity media in isochron ray coordinates, which can be used to extrapolate the isochron-field in this coordinate system (Sava & Fomel, 2005).

6 CONCLUSION

We apply, for the first time, the concept of isochron rays to modeling and imaging of seismic data. We introduced the concept of equivalent velocity to extend the use of the exploding reflector model to non-zero-offset data. We show how to use this concept for modeling and imaging of single finite-offset traces in 2-D media using two kind of algorithms: finite difference and wave-field extrapolation.

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Figure 5. Common-offset gathers: a) Conventional Kirchhoff modeling, b) isochron-ray finite-difference modeling, and c) isochron-ray wavefield extrapolation modeling.

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Figure 6. Equivalent velocity for the central position of the seismic model clipped at 4.5 m/s.

Figure 7. a) Finite-difference recorded isochron field, b) Extrapolated isochron field recorded at $z = 0.005km$. 
Figure 8. a) Zero-offset wave-field extrapolation migration, b) Common-offset wave-field extrapolation migration, c) Common-offset reverse time migration.
Figure 9. Common-offset images: a) wavefield isochron-ray migration, b) Kirchhoff migration
Figure 10. Stack of common-offset images: a) wavefield isochron-ray migration, b) Kirchhoff migration
Figure 11. Velocity model for the field-data example
Accelerating isochron-ray wavefield extrapolation migration

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\textbf{ABSTRACT}

Wavefield extrapolation isochron-ray migration (WEIM) is a wave-equation depth migration method for common-offset sections. The method uses the concepts of isochron-rays and equivalent-velocity media to extend the application of the exploding-reflector model to finite-offset imaging. WEIM is implemented as a trace-by-trace algorithm in three steps: 1) equivalent-velocity computation, 2) data-conditioning for zero-offset migration, and 3) wavefield-extrapolation migration. WEIM is attractive for migration velocity analysis (MVA) because it permits the migration of common-offset sections using standard zero-offset wave-equation algorithms, which, in turn, is favorable for development of algorithms for parallel processing. The problem, however, is that WEIM is computationally expensive. The objective here is to present strategies to improve the computational performance of WEIM, perhaps making the method feasible for MVA. The reduction in computation of WEIM is achieved by redatuming and migration of beams rather than single traces, and by reducing the migration aperture. The procedures of beam formation and redatuming differ from the conventional processes described in the literature and are simultaneously applied in the data-conditioning step. The use of beams and redatuming improves the computational performance of WEIM particularly for larger offsets. In a successful application of these strategies to a field-data example, the effective cost reduction obtained is about seven times.

\textbf{Key words:} wave equation, imaging, beams, redatuming, isochron-rays

1 INTRODUCTION

Isochron rays are lines associated with propagating isochrons, which are surfaces related to seismic reflections all having the same two-way traveltime. Iversen (2004) introduced the term \textit{isochron ray} for trajectories associated with surfaces of equal two-way time, i.e., isochron surfaces and suggested the potential use of the isochron rays in future implementations of prestack depth migration.

Silva & Sava (2008a) exploit this idea in a methodology that uses the isochron ray concept to perform prestack depth migration. This methodology uses the additional concept of \textit{equivalent velocity} to extend the use of zero-offset algorithms to finite offset. The method is implemented in a trace-by-trace algorithm in three steps: equivalent-velocity computation, data conditioning, and migration.

Wavefield extrapolation isochron-ray migration (WEIM) is attractive for several reasons, among them we can list:

- It extends the use of zero-offset algorithms;
- It permits migrating a single finite-offset trace using a 3D wave-equation algorithm, which is a favorable feature for the development of parallel processing algorithms; and
- It permits the migration of small pieces of common-offset gathers using the wave equation, which is attractive for migration-velocity analysis.

In this article, we are focused on reducing the computational cost of common-offset isochron-ray migration by wavefield extrapolation, using the acronym WEIM to refer to this common-offset approach exclusively. The cost of WEIM is reduced by adopting the following procedures: migrating beams instead of single traces, performing the redatuming during the data-conditioning step, and using limited aperture.

Mulder (2005) defines redatuming as an operation on seismic data that in effect translates the positions of sources or receivers, or both. Here, we perform this operation by applying additional static time-shifts to the input trace during the data-conditioning step. For each position where the input trace
is repeated, the time-shift corresponds to the difference of two-way travel times for this position and the initial position of the corresponding isochron rays. For homogeneous media, time-shifts are analytically calculated.

Although we apply Gaussian tapering to form beams, the concept of beams for WEIM is essentially different from that of Gaussian beams used by Hill (1990) for Gaussian-beam migration. In WEIM, each input trace is migrated with an equivalent velocity that depends on the source and receiver position. The beam formation for WEIM is a data operation that transforms the data to be migrated with a translated equivalent velocity field, i.e., prepares each trace of the beam to be migrated with the equivalent velocity of the central position of the beam.

2 THEORY

In this section, we present the basic concepts used to improve the performance of wavefield extrapolation isochron-ray migration method. First, we review the method, then we introduce the concept of redatuming for isochron extrapolation, and explain how to form beams for common-offset isochron-ray migration.

2.1 Wavefield extrapolation isochron-ray migration

This WEIM method, described in detail in Silva & Sava (2008b), is implemented in a trace-by-trace algorithm in three steps: equivalent velocity computation, data conditioning, and migration using a zero-offset wavefield extrapolation (ZOWE) algorithm. Here, we review the concept of equivalent velocity and focus on the data-conditioning step, which is closely related to the adopted accelerating strategies.

Equivalent velocity is the speed of a hypothetical propagating isochron that “moves” into the model as reflection time increases. This speed, which is related to medium velocity, is variable even for homogeneous media. In the absence of caustics, the isochron-field can be reproduced by a hypothetical experiment in which the actual source-receiver pair is replaced by a pseudo seismic source, which has a constant amplitude along a line connecting the source and receiver, and the propagation medium has the equivalent velocity. As the source segment shrinks to a point, the zero-offset case, the speed of propagation becomes unique and equal to the half-velocity.

Data conditioning is the procedure by which a single finite-offset trace is transformed into a gather of traces for zero-offset migration using the equivalent velocity field. For wavefield-extrapolation migration, the input trace is time-shifted by a negative amount that corresponds to the traveltime measured along the raypath connecting the source and receiver. The conditioned data gather is obtained by repeating the shifted trace at every grid position between the source and receiver, while the remaining positions are filled with zeros. Figures 1(a) and 1(b) show a 2D example of data conditioning. In this example, the seismic model is homogeneous so that the applied time-shift is $2h/v$, where $h$ is the half-offset and $v$ is the medium velocity.

In WEIM, each non-null trace of the conditioned data is associated with the isochron ray that starts in the position where the trace is located. During the migration, the field is extrapolated along the isochron rays. Figure 2(a) illustrates the isochron-ray migration process. Observe that horizontal lines in conditioned data are mapped into isochrons in depth.

2.2 Redatuming

In general, redatuming refers to a data transformation that simulates the translation of the source and the receiver from one datum (reference plane) to another. Here, we use the term redatuming to refer to a data transformation associated with isochron extrapolation from one horizontal level to another, without any assumption about the new location of the source-receiver pair.

The isochron redatuming is achieved by applying additional static time-shifts to the input trace during the data-conditioning step. For each position where the input trace is repeated, the time-shift is the difference of the two-way traveltime between this position and the initial position of the corresponding isochron ray. Time-shift calculation involves a mapping procedure along isochron rays, which is analytically determined by doing the redatuming in a homogeneous layer.

Figure 3 illustrates isochron redatuming for an arbitrary trace of the conditioned data. In this example, the source-receiver pair is moved from the surface $z = 0$ to the level $z = datum$. The selected trace is originally located at $A$, which is the initial point of the isochron ray $\Omega$ in the plane $z = 0$. This trace is moved to location $C$ after it is properly shifted in time. $C$ is the vertical projection of $B$, which is the intersection point between the ray $\Omega$ and the plane $z = datum$. The applied time-shift, $t_b - t_a$, is the traveltime computed along the isochron ray $\Omega$ from $A$ to $B$.

Figure 2(b) shows the conditioned data of Figure 1(b) after redatuming. Observe two important modifications in the data-conditioned space: the horizontal event becomes curved, and the trace distribution becomes irregular. In practice, redatuming and data conditioning are performed together, and the trace distribution is regularized, i.e., made uniform by using interpolated isochron rays.

2.3 Beam formation

In common-offset WEIM, each input trace is transformed into a gather (conditioned data), which is used as input for ZOWE migration with equivalent velocity field. The basic idea is to reduce the computational cost by migrating groups of traces instead of individual traces. The input common-offset section is divided in groups with equal numbers of traces and with overlap of adjacent groups. Figure 4 shows an example of beam distribution for a 2D common-offset section. In this example, three groups have central traces located at $A$, $B$, and $C$. In order to keep amplitudes balanced after superposition, traces are
Figure 1. Data-conditioning cartoon: a) input trace, b) conditioned data
Figure 2. Conditioned data for WEIM: a) without redatuming, b) with redatuming.
multiplied by a weight function with an exponential decay; i.e., each group is windowed using Gaussian tapering.

The beam formation for WEIM is performed during the data-conditioning step in such a way that the conditioned data gather contains information from all beam traces. It is equivalent to compute gathers for individual traces and stack them using a proper weight.

The problem of imaging a group of traces by WEIM is that the equivalent velocity depends on the source and receiver positions; thus we have to choose one source-receiver pair to compute this velocity field. The beam center is a natural choice for the equivalent velocity computation. Consequently, traces located away from beam center are migrated using an improper velocity. To overcome this drawback, we create conditioned data for ZOWE migration using a laterally shifted equivalent velocity field.

For a trace located away from the beam center, the conditioning also starts with a time-shift, but it is followed by a dynamic trace operation instead of a simple repetition. In this operation, a point of the input trace is mapped into a curve instead of a horizontal line. We refer to this curve as a spreading line. The spreading line is defined in such a way that it is mapped into an isochron after ZOWE migration with the beam-center equivalent velocity.

Figure 5(b) explains how the spreading line is defined. From the input trace, tr₄, the point N is selected at the time location of M. N is mapped into the spreading line, τ₄, which must be migrated to isochron ζ₄. Thus, for each trace of the conditioned data, the difference between τ₄ and τ₃ is the traveltime between two points of the corresponding isochron ray, which are the points where the isochron ray intersects isochrons ζ₃ and ζ₄. For example, for the trace corresponding to the last isochron ray on the right, the traveltime is measured between points A and B.

For a trace located at the beam center, the conditioning also starts with a time-shift, but it is followed by a dynamic trace operation instead of a simple repetition. In this operation, a point of the input trace is mapped into a curve instead of a horizontal line. We refer to this curve as a spreading line. The spreading line is defined in such a way that it is mapped into an isochron after ZOWE migration with the beam-center equivalent velocity.

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Spreading lines vary in time and depend on the relative position relative to the beam center. Thus, each sample of each trace of the beam has a corresponding spreading line, and each spreading line is mapped into a corresponding isochron.
3 EXAMPLE

In this section, we present the results of application of the WEIM method, using beams and redatuming, in a 2D synthetic common-offset (CO) section. The objective here is not yet to analyze the computational cost, but to illustrate how the method works.

Figure 6(a) shows the CO section and the beam-center locations. The beam width is 200 m and, the distance between beam centers is 100 m. The input section is composed of five linear events, which were artificially inserted, without concern about physical meaning. We assume that: a) the source and receiver are located at the surface $z = 0$, b) the offset is 1 km, c) the velocity is 2 km/s, and d) the distance between traces is 10 m. Figure 6(b) shows the superposition of all beam images using redatuming to $z = 200\, \text{m}$.

Figures 7(a) and 7(b) show the conditioned data for the single trace located at $x = 2\, \text{km}$, with and without redatuming, respectively. Notice that the horizontal events become curved after redatuming. Figures 8(a) and 8(b) are the corresponding conditioned data for the beam, which center is located at $x = 2\, \text{km}$. Notice that events become stretched and asymmetric, being the asymmetry related to the time-slope in common-offset input section. Figures 7(c) and 8(c) show the migration of the single trace and the beam, respectively. In the beam image, notice that the energy is concentrated around the points where the isochrons are tangent to the reflectors.

4 COMPUTATIONAL COST

The computational cost depends on several aspects, including required memory, access to disk and paralleling strategy. Here, we only consider the CPU time for serial programs.

The cost of applying WEIM in common-offset section is proportional to the dataset size and depends on each step involved in WEIM. A reasonable estimate is given by the expression

$$C_{\text{co}}^0 = n_t (C_e + C_d + C_{z0}),$$

where $n_t$ represents the number of traces of the input section and $C$ stands for the computational cost of each involved process, the subscript $e$ pertaining to equivalent-velocity computation, $d$ to data conditioning, and $z_0$ to the zero-offset wavefield extrapolation migration. $C_{\text{co}}^0$ stands for the computational cost of common-offset WEIM without using beams and redatuming.

For a fixed offset, $C_e$ and $C_d$ do not significantly vary from trace to trace, and both are much smaller than $C_{z0}$; thus we can write

$$C_e + C_d = \alpha C_{z0},$$

with $\alpha < 1$.

When inserting expression 2 into equation 1, we obtain

$$C_{\text{co}}^0 = (1 + \alpha)n_tC_{z0},$$

Expression 3 explicitly shows that the cost of common-
offset WEIM can be reduced by either decreasing the number of traces or reducing $C_{zo}$. The use of redatuming and beams influence this cost in different ways. While redatuming reduces the number of steps in the wavefield extrapolation, the use of beams is equivalent of decreasing the number of traces. We can estimate the cost of WEIM using beams and redatuming by the expression

$$C_{rb}^{\text{co}} = (1 + \beta)n_b C_{zo}^{r}$$

(4)
Figure 6. WEIM using redatuming and beams: a) input common-offset section with eleven beam-center locations, b) final image: superposition of all beam contributions.
Figure 7. WEIM of a single trace: a) conditioned data without redatuming, b) conditioned data with redatuming from $z = 0$ to $z = 200$ m, c) image.
Figure 8. WEIM of a 21 traces beam: a) conditioned data without redatuming, b) conditioned data with redatuming from $z = 0$ to $z = 200$ m, c) image.
where \( n_b \) represents the number of beams, and \( \beta \) is the cost of the equivalent-velocity computation plus the cost of the conditioning-data step with redatuming and beam formation. \( C_{zo} \) stands for the cost of zero-offset wave-equation migration with redatuming. Since \( C_{zo} \) is proportional to the size of the wavefield, we can write

\[
C_{zo}^r = \frac{z_{\text{max}} - z_{\text{SR}} - z_{\text{datum}}}{z_{\text{max}} - z_{\text{SR}}} C_{zo}, \tag{5}
\]

with \( z_{\text{max}} \) being the maximum depth, \( z_{\text{SR}} \) is the depth of the plane where the source-receiver pair is located, and \( z_{\text{datum}} \) is the depth to which the source-receiver pair is translated after redatuming.

The cost-reduction factor due from of beams and redatuming is the ratio between expressions 4 and 2. Assuming, with no loss in generality, that \( z_{\text{SR}} = 0 \), the cost-reduction factor can be expressed by

\[
\gamma_{\text{red}} = \frac{C_{zo}^b}{C_{zo}^r} = \frac{1 + \beta n_b}{1 + \alpha n_1} \frac{z_{\text{max}} - z_{\text{datum}}}{z_{\text{max}}}. \tag{6}
\]

For a 2D common-offset section with regular geometry, the ratio \( n_b/n_1 \) is \( 2/(n_{tr,b} - 1) \), where \( n_{tr,b} \) is the number of traces per beam. Typically, \( \alpha \) stays in the range from .01 to .05 in 2D computations, \( \beta \) varies from .03 to .1, so the ratio \((1 + \beta)/(1 + \alpha)\) is close to unity. The cost reduction factor for 2D can be well estimated by the expression

\[
\gamma_{2D} = \frac{2}{(n_{tr,b} - 1)} \frac{z_{\text{max}} - z_{\text{datum}}}{z_{\text{max}}}. \tag{7}
\]

In addition to the gain from use of beams and redatuming, an extra cost reduction can be obtained by using reduced migration aperture, which is equivalent of reducing the conditioned data size.

Although we do not present a formula to estimate the cost reduction for 3D data, we can infer that it is close to the square of \( \gamma_{2D} \) for square grids.

5 APPLICATION TO FIELD-DATA

The WEM using isochron rays was applied to a pseudo 2.5D dataset that consists of 22 common-offset gathers extracted from a 3D dataset following the sequence. (1) the input traces were organized in 22 groups, using as sorting criteria the source-receiver offset, (2) a 3D Kirchhoff time-migration algorithm was applied, and (3), a 2.5D Kirchhoff time-demigration procedure was applied to each image. In the sorting procedure, each input trace was scaled by an areal factor in order to compensate acquisition irregularities. The weight function used in the 3D time-migration algorithm produces a true-amplitude image gather when the medium velocity is constant, i.e., the output amplitudes are proportional to the reflection coefficients. Also, the applied demigration program uses a true-amplitude weight function that produces a 2D common-offset gather in which amplitudes are reduced by a 3D geometrical spreading factor that is correct for homogeneous media.

The minimum offset is 160 m, and the increment between offsets is 200 m. Each common-offset gather has 1351 traces, and the distance between them is 18.75 m. The traces are 5 s long, and the time sampling interval is 4 ms. From this dataset, we selected the section with offset of 1960 m for experiments using beams and redatuming. Figure 9(a) shows the selected common-offset section, and Figure 9(b) shows the image from standard WEIM, i.e., without using beams and redatuming. This image is used as reference in our migration comparisons. The migration aperture and the spatial length of the conditioned data are 7480 m.

The experiments consist of ten migrations with different beam sizes and with redatuming for the same depth. Both redatuming and beam formation were carried out by a constant-velocity algorithm. The number of traces per beam is odd, varying from 3 to 21, while the depth for redatuming is 400 m.

From visual inspection over the ten images (only two of which are shown here), we observe that

- There are no significant kinematic differences between the beam images and the reference image,
- Artifacts are present in all beam images, as well as in the reference image, and
- The greater the beam size, the stronger the artifact amplitudes.

Here, we select for illustration the images with 13 and 17 traces/beam, shown in Figures 10(a) and 10(b), respectively. From the smallest to the largest beam size studied, the 13 traces/beam image is the first where the artifacts become evident, while the 17 traces/beam is the last useful image for migration velocity analysis. Figures 11(a) and 11(b) show a window where the artifact amplitudes become stronger.

Two possible reasons for the presence of artifacts are the use of a constant velocity algorithm to form beams and lack of an appropriated weight function to compensate differential geometrical spreading associated with different trajectories.

The measured CPU time reduction is a factor of 6.5 for 13 traces/beam and 8.4 for 17 traces/beam, while the expected factor according to relation 7 are about seven and nine respectively.

6 DISCUSSION

These cost reductions from use of beams and redatuming can make WEIM feasible for MVA. Let us compare an estimate of the MVA-WEIM cost with the cost of conventional zero-offset wavefield-extrapolation (ZOWE) migration.

The computational cost reduction from beams depends on offset because the number of traces per beam increases with offset. To simplify, we consider a linear relationship for this dependency and assume that the average reduction is approximated well by the cost reduction for the average offset.

Consider an MVA algorithm that consists of analyzing image strips from several common-offset sections. The average computational cost of imaging one common-offset section is given by

\[
C_{mva} = f_{tr} f_a n_b C_{zo}, \tag{8}
\]
Figure 9. Field-data experiments: a) input common-offset section (offset=1960 m), b) image from standard WEIM (without using beams and redatuming).
Figure 10. WEIM images using beam and redatuming: a) 13 traces/beam, b) 17 traces/beam.
Figure 11. Zoomed WEIM images using beam and redatuming: a) 13 traces/beam, b) 17 traces/beam.
where \( n_t \) is the number of traces in the common-offset section, \( f_{br} \) is the factor of cost reduction from beams and redatuming, \( f_s \) is the reduction from the use of strips, \( f_a \) is the ratio between migration aperture and spatial size of the input section, and \( C_{zo} \) is the ZOWE-migration cost.

For the presented field-data example, \( f_{br} \) is close to 1/8, \( f_s \) is roughly 1/4, and \( n_t \) is 1351. Suppose that the use of strips reduces the number of input traces to 20%, i.e., \( C_{mva} = 8C_{zo} \). This result suggests that the computational cost of individual WEIM experiments for MVA has the same order of magnitude of ZOWE migration, indicating that MVA using WEIM is feasible.

7 CONCLUSION

The objective of improving the computational performance of WEIM was achieved by using beams and redatuming. The effective reduction of the computational cost observed in a 2D field-data example is in accordance with the reduction predicted by a simplified formula. In this example, the maximum speedup from beams is around 7.5 times, while the gain from redatuming is close to 10%. The obtained results indicate that MVA using WEIM is feasible.

Numeric experiments indicate that the higher the density of traces, the better the relative performance expected for WEIM using beams. This fact is particularly useful for the application of MVA based on WEIM to 3D surveys with dense acquisition grid. We expect more significant gains for deep-water data, in which case the redatuming can efficiently extrapolate isochrons from the surface to a deep level in one step, perhaps providing a gain of 50% instead of the 10% observe here.

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Asymptotically true-amplitude one-way wave equations in $t$: Modeling, migration and inversion

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ABSTRACT
Currently used true-amplitude one-way wave equations in depth yield high quality solutions for forward modeling problems and inversion. A shortcoming of these equations is that they yield poor results near and fail at horizontal propagation, making the application to turned waves problematic, at the very least. One-way wave equations in time do not have this shortcoming; they are omni-directional in space. We introduce fairly simple forward and reverse time first order wave equations; they are essentially adjoints of one another. Spatial derivatives appear in these equations through a pseudo-differential operator—the square root of the Laplacian. With an appropriate definition of this operator, we have proved via ray theory that the solutions of one-way wave equations in time asymptotically approximate the solutions to the two-way wave equation to leading order for forward or reverse time propagation. For us “true-amplitude” is meant in this ray-theoretic (asymptotic) sense. The inverse series in powers of $i\omega$ in the frequency domain becomes a series in progressing waves in the time domain. The propagation of the leading order progressing wave is governed by the eikonal equation for the two-way wave equation and the slowly varying amplitude of this leading order progressing wave satisfies the same transport equation as for the two-way wave equation. This theory provides a solid theoretical base for the Explicit Marching algorithm for solving reverse time migration and anticipates an inversion—a true-amplitude reverse time migration. We present the equivalent initial value problem for Green’s functions. For homogeneous media, the 3D Green’s function derived by integral transform methods is complex-valued. Its real part is the propagating delta function that we expect and its imaginary part is the Hilbert transform of the real part, completing the so-called analytic Green’s function. We confirm that the Kirchhoff approximation of asymptotic ray theory in frequency domain applies to progressing waves in time domain. A Green’s identity relating solutions of the one-way wave equations and their adjoint is derived. This allows us to develop Kirchhoff integral representations from propagation of surface data into the Earth and the propagation of reflection data to the upper surface. Those identities plus identification of the adjoint operator for the forward modeling operator lead to migration and inversion formulas using our analytic Green’s functions. Observed data at the upper surface must be extended to analytic data in order to apply the inversion theory.

Key words: true-amplitude, one-way in time, modeling, migration, inversion, pseudo-differential operators
1 INTRODUCTION

Asymptotically true-amplitude one-way wave equations in depth yield high quality solutions for forward modeling problems and inversion. Zhang [1993] introduced two new one-way wave equations in depth and verified that the leading order their asymptotic solutions agreed with the asymptotic solutions of the two-way wave equation. This original work was for waves in two spatial dimensions and frequency. Zhang et al [2003] extended the theory to 3D, and based on the generalization, they proposed the theory of true-amplitude one-way wave equation migrations.

In theory, this separation into two one-way wave equations for up-going and down-going waves has a pathology in the neighborhood of horizontal propagation where the terminology, up-going and down-going, loses meaning.

Zhang et al [2007] derived a new one-way wave equation in time. All spatial directions are treated the same way. Thus, it can handle horizontal and turning waves routinely. Furthermore, this method provides a new way of doing reverse-time migration, that those authors call the “Explicit Marching” (EM) method. Unlike the conventional finite-difference methods, EM does not suffer from stability and numerical dispersion problems. In contrast to the one-way equations in depth, the pseudo-differential operator involved in EM is non-singular; the new equation can be numerically solved efficiently.

Here, we adapt the methodology of Zhang [1993] and Zhang et al [2003] to analyze the one-way wave equations in time. Though the equations we use look similar to those proposed by Gazdag [1981], and later were used by Baysal, et al [1983] for reverse-time migration, ours can deal with steep deeps (near horizontal propagation) and overhangs (turned waves). That is, these equations avoid the pathology of the spatial one-way wave equations in z at and near horizontal propagation: they have no favored direction in space.

A crucial difference between the method of Baysal, et al [1983] and the method proposed in Zhang et al [2007] and used here is that the real-valued space/time Green’s function of the two-way wave equation used in Baysal, et al [1983] is necessarily replaced in Zhang et al [2007] and this paper by the analytic Green’s function. The analytic Green’s function has the real Green’s function of the two-way wave equation as its real part and its Hilbert transform as the imaginary part.

1.1 Our methodology

As in the development of reverse time migration, we present the theory here in time rather than in frequency. Recall that an inverse power of $i\omega$ as a multiplier of a function in the frequency domain corresponds to integration of that function in the time domain. Thus, the successive terms in an inverse power series in $i\omega$, used in asymptotic ray theory—terms of the form, $f(\omega)\exp{i\omega\tau(x)}/(i\omega)^n$—correspond to multiple integrals of the basic function, say $F(\tau(x) - t)$, in the time domain. (Here, $\tau$ is the travel time function.) Hence the sequence of functions in the frequency domain transforms into increasingly smoother functions in the time domain.

We assume a relatively compact domain for the function $F$, such as a bandlimited delta function, so that with each term of the sequence of integrals of $F$ in time a relatively smoother progressing wave. Hence, ray theory in the time domain is called the “progressing wave formalism”; see Lewis [1964]. Each progressing wave is multiplied by a more “slowly varying” spatial amplitude that needs to be determined. As in the frequency domain, the formalism determines the traveltime $\tau$ and these slowly varying amplitudes through a hierarchy of equations just as in ray theory in the frequency domain.

The spatial derivative in the one-way equations is the square root of the Laplace operator. We use the method of Zhang [1993] to write down an integral representation of this operator that avoids the square root. This is achieved through a formal application of pseudo-differential operator theory.

It is then possible to derive the hierarchy of equations for traveltime and amplitude familiar from ray theory. The leading order equation in the hierarchy leads to the correct eikonal equation. The next order equation in the hierarchy yields the transport equation for the leading order amplitude. We usually go no further than this leading order amplitude in solving problems asymptotically.

That first transport equation yields an amplitude that differs from the corresponding transport equation for the full wave equation only by a power of $v$, the wave speed. Thus, it is straightforward to write down the asymptotic solutions of the two-way wave equation in terms of the solutions of these one-way wave equations and vice versa. The calculations are tedious, but the method is straightforward.

One-way wave equations need initial data to generate well-posed problems for solution. This is a challenge for the Green’s function where the initial data is prescribed by a balance between a second derivative with respect to time and a delta function in time as a factor of the source. We show that the “right” initial data for the Green’s function for this one-way wave equation is actually a pseudo-differential operator in spacial coordinates acting on the spatial delta function. This echoes the corresponding result for the one-way wave equation in space; see Zhang et al [2003], where we consider “initial value” problems in z and prescribe “initial data” in z.
In the next section, we introduce the idea of a progressing wave formalism and present the asymptotic analysis for the two-way wave equation. This discussion provides a basis of comparison for the asymptotic analysis of the one-way wave equations to follow and also demonstrates the methodology of the progressing wave formalism for the reader who might be unfamiliar with this approach.

The following section introduces the one-way wave equations for forward and reverse time propagation. Here is where we use Zhang’s [1993] method to express the square root operator as the integral of a quotient of second order operators.

The derived eikonal equation matches the eikonal equation for the two-way wave equation. The derived transport equation agrees with each of the transport equations derived previously for the two-way wave equation within a power of the wavespeed $v$, different for each of those two two-way wave equations. The latter equations arise from conservation laws for displacement or pressure, for example. The one-way equation is a mathematical device yielding analytic extensions of the solutions of the two-way wave equation. This is a device for approximating solutions of two-way wave equations for which there are no apparent physical conservation laws; we accept these solutions for what they are, absent a physical interpretation.

Next, we define the initial value problem for the Green’s function and show that the solution for homogeneous media is the analytic Green’s function, with the “right” real part for the two-way wave equation and imaginary part being the Hilbert transform of the real part.

We also develop a Kirchhoff approximation to generate data at a reflector in terms of the incident data and the reflection coefficient. We need this for modeling of observed data in the derivation of a Kirchhoff-type inversion formula.

Next, we introduce the adjoint operator for each of our one-way wave operators. With those in place, we can derive a Green’s theorem from which we can derive representations of the downward continuation of observed surface data and for the upward propagation of our Kirchhoff-approximate data from a reflector to the upper surface.

A general feature of modeling with one-way wave equations is that (i) source information for the two-way wave equation provides initial data or final data for the one-way wave equations and (ii) boundary data for the two-way wave equation provides sources for the one-way wave equations. The latter echoes the exploding reflector model for wave propagation.

We expect modeling and inversion outputs to exhibit the same level of quality and smoothness as is achieved by using the one-way wave equations in spatial coordinates. However this inversion is expected to have the additional advantage that there is no pathology for horizontal propagation or turning waves. Thus, this inversion should routinely image vertical flanks and the underside of salt domes just as reverse time migration does.

Finally, we derive common-shot inversion formulas in the frequency domain and in the time domain. The simplicity of form of our Green’s functions in the frequency domain makes the derivation of an inversion formula in that domain a simple imitation of previous derivations by pseudo-inversion, as presented, for example, in Bleistein et al [2005]. Then return to the time domain and provide an inversion formula there. Both of these inversion formulas form an image by correlation of the downward continued data with the downward continued source.

The data used in this inversion must be consistent with forward modeling for this one-way wave equation. Acquired data is the solution of a two-way wave equation. As with the Green’s function, we need to take those acquired data and extend them to analytic data by adding $i$ times their Hilbert transform in time.

## 2 PROGRESSING WAVE FORMALISM FOR THE TWO-WAY WAVE EQUATION

Here, we present the derivation of the progressing wave formalism for the two-way wave equation

$$
\mathcal{L} U = \frac{1}{v^2} \left\{ \frac{\partial^2 U}{\partial t^2} - (v \nabla)^2 U \right\} = 0, \quad U = U(x, t), \quad (v \nabla)^2 \equiv v \nabla \cdot (v \nabla).
$$

(1)

Here, we artificially introduce $v$ as the bulk modulus for a medium of density equal to one or the inverse of density with a bulk modulus equal to one. The former would be a wave equation for displacement; the latter would be a wave equation for pressure. See, for example, Chapman [2004]. We are not as concerned about the physics here as we are with the versatility of the progressing wave formalism for dealing with something more complex than the standard wave equation. However, we also state without proof the corresponding eikonal and transport equation for the simpler wave equation

$$
\mathcal{L} \hat{U} = \frac{1}{v^2} \frac{\partial^2 \hat{U}}{\partial t^2} - \nabla^2 \hat{U} = 0.
$$

(2)

Note that the previous wave equation (1) reduces to this one in piecewise constant media.
We develop a progressing wave formalism for asymptotic solution of these wave equations in a manner that is completely analogous to asymptotic ray theory.

To begin, we introduce a sequence of progressing wave functions $F_0[\tau(x) - t], F_1[\tau(x) - t], \ldots$, with the property that

$$F_{n+1}' = F_n', \quad n = 0, 1, 2, \ldots . \quad (3)$$

Here the prime \{\}' means derivative with respect to total argument of the function. As defined, each wave function is therefore smoother (in the sense of having one more nonsingular derivative) than its predecessor as was the case for the example in the Introduction. We then assume that the solution $U$ can be written as a series in these functions as follows.

$$U(x,t) = A_0(x)F_0[\tau(x) \mp t] + A_1(x)F_1[\tau(x) \mp t] + \ldots . \quad (4)$$

We need save only these terms to determine the governing equations for $\tau$ and $A_0$, as we are performing a leading-order asymptotic analysis.

The representation of $U$ in equation (4) is substituted into the wave equation (1). The series for the derivatives are as follows.

$$\frac{\partial U}{\partial t} = \mp[A_0F_0' + A_1F_1'] + \ldots \quad (5)$$

$$\frac{\partial^2 U}{\partial t^2} = A_0F_0'' + A_1F_1'' + \ldots . \quad (6)$$

We need nothing smoother than these two orders to derive the eikonal equation and the transport equation for $A_0$.

The slownesses,

$$p = \nabla \tau = \begin{cases} (p_x, p_y), & 2D, \\ (p_x, p_y, p_z), & 3D, \end{cases} \quad (7)$$

are used below in calculating the spatial derivatives.

$$(v\nabla)^2 = v^2\nabla^2 + v\nabla v \cdot \nabla. \quad (8)$$

That is, $(v\nabla)^2$ can be written as a sum of a second order differential operator plus a first order differential operator. Explicitly, these terms are

$$v\nabla v \cdot \nabla U = F_0'[\nabla v \cdot p + \ldots , \quad (9)$$

and

$$\nabla^2 U = [A_0F_0'' + A_1F_1'']p^2 + F_0''[2p \cdot \nabla A_0 + A_0\nabla \cdot p] + \ldots . \quad (10)$$

In terms of these two operators,

$$(v\nabla)^2 U = v \left\{ v[A_0F_0'' + A_1F_1'']p^2 + F_0''[2vp \cdot \nabla A_0 + 2A_0p \cdot \nabla v + A_0\nabla \cdot p] + \ldots \right\}. \quad (11)$$

The series representation of the second time derivative of $U$ in equation (6) and that of $(v\nabla)^2 U$ in this last equation (11) are substituted into the wave equation (1). Collecting the coefficients of $F_0''$ and $F_0'$ and setting them separately equal to zero leads to the following pair of equations.

$$A_0 \left[ (vp)^2 - 1 \right] = 0 \quad (12)$$

and

$$A_1 \left[ (vp)^2 - 1 \right] + v[2vp \cdot \nabla A_0 + A_0p \cdot \nabla v + A_0\nabla \cdot p] = 0. \quad (13)$$

For $A_0 \neq 0$, the first equation here leads to the eikonal equation, more familiarly written as

$$p^2 = (\nabla \tau)^2 = \frac{1}{v^2}. \quad (14)$$

In the second order equation (13), the multiplier of $A_1$ is now zero. Thus, this equation becomes
\[ v[2vp \cdot \nabla A_0 + A_0p \cdot \nabla v + A_0 \nabla \cdot p] = \frac{1}{A_0} \nabla \cdot [v^2A_0^2p] = 0, \quad (15) \]

leading to
\[ \nabla \cdot [vA_0^2p] = 0. \quad (16) \]

This is the transport equation for the amplitude \( A_0 \) written in divergence form. This form, which is a statement of the conservation of energy flowing through the end caps of tube made of rays, has standard solutions which form the basis of ray amplitude theory.

The eikonal equation for the simpler wave equation (2) is the same as above. We call the new amplitude \( \tilde{A}_0 \). The transport equation for \( \tilde{A}_0 \) is the same as equation (15), except for lacking the term containing \( \nabla v \) in that equation.

As a consequence, the transport equation for \( \tilde{A}_0 \) is
\[ \nabla \cdot [\tilde{A}_0^2p] = 0. \quad (17) \]

The fact that the two transport equations preserve quantities in ray tubes that differ only by a power of \( v \) suggests that it is not necessary to deal with the more difficult wave equation (1) when addressing the one-way wave equation; that is confirmed in the next section.

3 ONE-WAY WAVE EQUATION IN TIME

We carry out the same asymptotic analysis as in the previous section for the one-way wave equations
\[ L_\pm W = \frac{1}{v} \frac{\partial W}{\partial t} \pm \sqrt{\nabla^2 W} = 0. \quad (18) \]

We will see below that the upper sign (+) corresponds to waves in which the traveltime function \( \tau(x) \) increases on the wavefront with increasing time and the lower sign (-) corresponds to waves for which \( \tau(x) \) increases on the wavefront with decreasing time; that is, forward and backwards propagation in time, respectively.

We need a formalism for defining the square root of the Laplacian here. Let us first introduce the straightforward correspondence,
\[ \nabla \leftrightarrow ik, \quad k = \left\{ \begin{array}{ll} (k_x, k_z), & 2D, \\ (k_x, k_y, k_z), & 3D. \end{array} \right. \quad (19) \]

By using the correspondence in this equation we obtain the symbolic correspondence
\[ \sqrt{(\nabla)^2} \leftrightarrow \sqrt{(ik)^2} = ik, \quad k = \sqrt{k^2}. \quad (20) \]

Then, by imitating the methodology of Zhang et al [2003], we show in Appendix A—equation (A-6) that\(^1\)
\[ ik = i |k_z| [I(k) + 1], \quad (21) \]
with
\[ I(k) = \frac{1}{\pi} \int_{-1}^{1} \sqrt{1 - s^2k_z^2} k_x^2 ds, \quad k_T = \left\{ \begin{array}{ll} k_x, & 2D, \\ (k_x, k_y), & 3D. \end{array} \right. \quad (22) \]

Both the factor \( i |k_z| \) in equation (21) and the integral \( I(k) \) defined just above require interpretation as pseudo-differential operators. We interpret \( i |k_z| \) as follows.

\[ k_z > 0, \quad i|k_z| = ik_z \leftrightarrow \frac{\partial}{\partial z}; \quad k_z < 0, \quad i|k_z| = -ik_z \leftrightarrow -\frac{\partial}{\partial z}. \quad (23) \]

We remark that we could have as easily distinguished the \( x \)- or \( y \)-direction in defining these pseudo-differential operators. We would then have had an analogous interpretation for \( i |k_x| \) or \( i |k_y| \). We remind the reader that our method is merely a device for carrying out the necessary asymptotic analysis here. We will see below that the distinction of the \( z \)-direction plays no role in our final results of this section or beyond.

Now the differential equation (18) can be symbolically written as

\(^1\)Although we define the gradient operator and the square root of the Laplacian operator below in 2D and 3D, only the analysis in 3D is be presented here.
\[
\mathcal{L}_\pm W = \frac{1}{v} \frac{\partial W}{\partial t} \pm i|k_s| \left\{ W + \frac{1}{\pi} \int_{-1}^{1} \frac{\sqrt{1 - s^2 k_s^2}}{k_s^2 + s^2 k_T^2} \, ds \right\} = 0,
\]  
(24)

with the integral operator here subject to interpretation. If multiplication by functions of \(ik\) correspond to differentiation, then division by functions of \(ik\) should correspond to integration, or equivalently, solving a differential equation or convolution with a Green’s function for the differential operator in the denominator. The Green’s function would be convolved with the function in the numerator. Equivalently, we solve the differential equation for an auxiliary function \(W_T(x, t, s)\) as follows

\[
W_T = \frac{k_s^2 W}{k_T^2 + s^2 k_T^2} \quad \Leftrightarrow \quad \{k_s^2 + s^2 k_T^2\} W_T = k_T^2 W
\]

\[\Leftrightarrow\]  
(25)

\[
\frac{\partial^2 W}{\partial z^2} + s^2 \nabla_T^2 W_T = \nabla_T^2 W, \quad \nabla_T = \begin{cases} \frac{\partial}{\partial x}, & 2D, \\ \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right), & 3D. \end{cases}
\]

This is exactly what is needed to interpret \(I(k)W\) with \(I\) defined in equation (22) and \(ik\) in equation (21). By this device, we have interpreted the pseudo-differential operator \(ikW \iff \sqrt{\nabla^2} W\) in terms of standard differential operators and an auxiliary function: determine \(ikW\) by solving for \(W_T(x, t, s)\) in equation (25) as a function of \(W\) and \(s\) and then carry out the necessary integration in \(s\) to determine \(ikW\).

We now have a compound of \(\pm\) signs due to the signs in the differential equation (18) and the signs of \(i|k_s|\) in equation (23). Therefore, below we separate \(\mathcal{L}_\pm\) of equation (18) and allow the \(\pm\) to correspond to the signs of \(i|k_s|\) in equation (23). Then, the correspondence between this last expression and the square root of the Laplacian in equation (20) allows us to rewrite the one-way wave operators in equation (18) as

\[
\mathcal{L}_+ W = \frac{1}{v} \frac{\partial W}{\partial t} + \frac{\partial W}{\partial z} \pm \frac{1}{\pi} \int_{-1}^{1} \sqrt{1 - s^2} W_T \, ds = 0,
\]

\[\mathcal{L}_- W = \frac{1}{v} \frac{\partial W}{\partial t} + \frac{\partial W}{\partial z} \pm \frac{1}{\pi} \int_{-1}^{1} \sqrt{1 - s^2} W_T \, ds = 0
\]

(26)

Here, whether the waves propagate forward in time (\(\mathcal{L}_+\)) or backward in time (\(\mathcal{L}_-\)), the upper sign corresponds to downgoing waves and the lower sign corresponds to upgoing waves. Notice, however, that those signs are opposite in this pair of equations.

The procedure for asymptotic analysis of the one-way wave equations (18) is as follows.

(i) First write down progressing wave series for both \(W\) and \(W_T\).
(ii) Use the spatial differential equation (25) to determine the coefficients of the series for \(W_T\) in terms of the coefficients of the series for \(W\) and functions of \(s\). Substitute the series for \(W_T\) into the integral in the one-way wave equations (26) just above, and carry out the integrals with respect to \(s\).
(iii) What results is a progressing wave series totally in terms of the traveltime and amplitudes of the progressing wave representation for \(W\). The most singular part of this equation is the coefficient of \(F_0^{\prime}\) and the next order will be the coefficient of \(F_0\), itself. Setting those two coefficients equal to zero yields the eikonal equation and transport equation that we seek.

We step through this list in the following subsections.

3.1 Relationship between \(W\) and \(W_T\)

Here, we introduce a progressing wave series of the form of \(U\) in equation (4) and then use the partial differential equation (25) to relate the coefficients of the progressing wave expansion of \(W_T\) to the coefficients in the progressing wave formalism of \(W\).
The two series that we use for \( W \) and \( W_T \) are

\[
W(x,t) = B_0(x)F_0[\tau(x) \mp t] + B_1(x)F_1[\tau(x) \mp t] + \ldots ,
\]

and

\[
W_T(x,t,s) = B_{0T}(x)F_0[\tau(x) \mp t] + B_{1T}(x)F_1[\tau(x) \mp t] + \ldots ,
\]

The relationships among the coefficients are derived in Appendix B. They are

\[
B_{0T} = \frac{p_T^2}{p_z^2 + s^2 p_T^2} B_0,
\]

which is equation (B-7) of Appendix B, and

\[
B_{1T} = \frac{p_T^2}{p_z^2 + s^2 p_T^2} B_1
\]

\[
+ \frac{1}{B_0} \left\{ \frac{1}{p_T^2} \nabla_T \cdot \left( \frac{p_T^2 B_0^2 p_T}{(p_z^2 + s^2 p_T^2)^2} \right) - \frac{1}{p_T^2} \frac{\partial}{\partial z} \left( \frac{B_0^2 p_T^2 p_z}{(p_z^2 + s^2 p_T^2)^2} \right) \right\},
\]

which is equation (B-10) of Appendix B. Note that equation(29) is an algebraic equation in slownesses that matches the pseudo-differential operator equation \( ? ? \) relating \( W_T \) to \( W \). This should be expected: to leading order the differentiation of the traveltime function dominates the differentiation process just as it does in asymptotic ray theory in the frequency domain.

### 3.2 Progressing Wave Expansion for the One-Way Wave Operator

The equations (29) and (30) relating the coefficients in the progressing wave series for \( W \) and \( W_T \) are what we need to analyze the progressing wave series for the one-way differential equations (18).

Equation (5) provides the necessary time derivative with \( A \)'s replaced by \( B \)'s. In this manner The differential equation (26) leads to the following series to two orders in progressing wave functions.

\[
\nu \nabla_1 W = -[B_0 F_0' + B_1 F_0] \pm \nu[B_0 F_0' + B_1 F_0] p_z \pm \nu \frac{\partial B_0}{\partial z} F_0
\]

\[
\pm \frac{\nu}{\pi} \left\{ p_z F_0' \int_{-1}^{1} \sqrt{1 - s^2 B_{0T}} ds + p_z F_0 \int_{-1}^{1} \sqrt{1 - s^2 B_{1T}} ds \right. \\
\left. + F_0 \frac{\partial}{\partial z} \int_{-1}^{1} \sqrt{1 - s^2 B_{0T}} ds \right\} = 0,
\]

and

\[
\nu \nabla_1 W = -[B_0 F_0' + B_1 F_0] \mp \nu[B_0 F_0' + B_1 F_0] p_z \mp \nu \frac{\partial B_0}{\partial z} F_0
\]

\[
\pm \frac{\nu}{\pi} \left\{ p_z F_0' \int_{-1}^{1} \sqrt{1 - s^2 B_{0T}} ds + p_z F_0 \int_{-1}^{1} \sqrt{1 - s^2 B_{1T}} ds \right. \\
\left. + F_0 \frac{\partial}{\partial z} \int_{-1}^{1} \sqrt{1 - s^2 B_{0T}} ds \right\} = 0.
\]

As expected, the most singular terms here are of order \( F_0' \). Concentrating on those terms,
for $L_+ :  F_0' \left\{ B_0[-1 \pm v p_z] + \frac{v p_z}{\pi} \int_{-1}^{1} \sqrt{1 - s^2 B_{oT}} \, ds \right\} = 0.$ 

(33)

for $L_- :  F_0' \left\{ B_0[-1 \mp v p_z] + \frac{v p_z}{\pi} \int_{-1}^{1} \sqrt{1 - s^2 B_{oT}} \, ds \right\} = 0.$

We use equation (29) to rewrite this last equation totally in terms of $B_0$; that is,

for $L_+ :  F_0' B_0 \left\{ -1 \pm v p_z \pm \frac{v p_z}{\pi} \int_{-1}^{1} \sqrt{1 - s^2 p_z^2} \, ds \right\} = 0.$

(34)

for $L_- :  F_0' B_0 \left\{ -1 \mp v p_z \pm \frac{v p_z}{\pi} \int_{-1}^{1} \sqrt{1 - s^2 p_z^2} \, ds \right\} = 0.$

The integral here is expressible in terms of the integral $I(p)$, with $I(k)$ defined in equations (21) and (22). That is,

$$\frac{1}{\pi} \int_{-1}^{1} \frac{\sqrt{1 - s^2 p_z^2}}{p_z^2 + s^2 p_T^2} \, ds = \frac{\sqrt{p_T^2}}{|p_z|} - 1.$$  

(35)

For waves propagating forward in time ($L_+$), $p_z$ is positive for downgoing waves and negative for upgoing waves; For waves propagating backward in time ($L_-$), $p_z$ is negative for downgoing waves and positive for upgoing waves. Thus, we interpret $|p_z|$ as follows.

For $L_+ :  |p_z| = \pm p_z$

(36)

for $L_- :  |p_z| = \mp p_z$

By using this values for $|p_z|$ in the integral in the equation (35) and then substituting into the two leading order equations for $L_\pm$ in equation (34) we conclude that in all cases

$$F_0' B_0 \left\{ -1 \pm v \sqrt{p_T^2} \right\} = 0.$$  

(37)

Now, for $B_0 \neq 0$, setting the coefficient of $F_0'$ equal to zero leads to

$$\sqrt{p_T^2} = 1/v.$$  

(38)

Because of the sign conventions we chose in the one-way wave equations (18) and in the progressing wave series (27) and (28), the single eikonal equation with positive sign arises here. However, in the arguments $\tau \mp t$, wavefronts move in the direction of $\nabla \tau$ for the upper sign and in the direction of $-\nabla \tau$ for the lower sign; this is forward or backward propagation in time. This eikonal equation is thus equivalent to the eikonal equation (14) for the two-way wave equation.

Let us now return to the operator equations (31) and (32). Because we have eliminated the term of order $F_0'$ in those equations, the operator is now of order $F_0$ with error of order $F_1$. Collecting the terms of order $F_0$ in equation (31) leads to

$$L_+ :  F_0 \left\{ B_1[-1 \pm v p_z] \pm \frac{\partial B_0}{\partial z} \pm \frac{v p_z}{\pi} \int_{-1}^{1} \sqrt{1 - s^2 B_{1T}} \, ds \pm \frac{\partial}{\partial z} \left[ \frac{1}{\pi} \int_{-1}^{1} \sqrt{1 - s^2 B_{oT}} \, ds \right] \right\},$$

(39)

$$L_- :  F_0 \left\{ B_1[-1 \mp v p_z] \mp \frac{\partial B_0}{\partial z} \mp \frac{v p_z}{\pi} \int_{-1}^{1} \sqrt{1 - s^2 B_{1T}} \, ds \mp \frac{\partial}{\partial z} \left[ \frac{1}{\pi} \int_{-1}^{1} \sqrt{1 - s^2 B_{oT}} \, ds \right] \right\}.$$

Note that the dependence of $B_{1T}$ on $B_1$ in equation (30) is exactly the same as the dependence of $B_{oT}$ on
for $L_+ : W = \pm F_0 \left\{ \frac{\partial B_0}{\partial z} + \frac{1}{p_z B_0} \nabla_T \cdot \left( p_z^2 B_0^2 \frac{1}{\pi} \int_{-1}^{1} \frac{\sqrt{1-s^2}}{p_z^2 + s^2 p_T^2} ds \right) \right. $} \tag{40}

for $L_- : W = \mp F_0 \left\{ \frac{\partial B_0}{\partial z} + \frac{1}{p_z B_0} \nabla_T \cdot \left( p_z^2 B_0^2 \frac{1}{\pi} \int_{-1}^{1} \frac{\sqrt{1-s^2}}{p_z^2 + s^2 p_T^2} ds \right) \right. $} \tag{41}

The error in this approximation is of order $F_1$.

The integral in the last term here is given in equation (35), except that we now know that $\sqrt{p^2} = 1/v$ from equation (38). The integral in the other two terms here is discussed in Appendix A. Its evaluation is

$$I_1(p) = \frac{1}{\pi} \int_{-1}^{1} \frac{\sqrt{1-s^2}}{p_z^2 + s^2 p_T^2} ds = \frac{1}{2 |p_T|^2} \sqrt{\frac{1}{p_T}} = \frac{v}{2 |p_T|^2}. $$} \tag{42}

By substituting this expression for the middle integral in the previous equation and substituting the value of the last integral as given in equation (35), we find that

$$\frac{\partial B_0}{\partial z} + \frac{1}{p_z B_0} \nabla_T \cdot \left( p_z^2 B_0^2 \frac{1}{\pi} \int_{-1}^{1} \frac{\sqrt{1-s^2}}{p_z^2 + s^2 p_T^2} ds \right) = 0.$$} \tag{43}

Here, we have used the same logic of distinguishing the signs of $|p_z|^2$ as above.

Expanding out the terms here and setting the sum of terms in the braces equal to zero leads to

$$\nabla B_0 + \frac{B_0}{2} \nabla \cdot (\nabla B_0) + \frac{B_0}{2 p_z} \left[ \nabla \frac{1}{v} \right] = 0.$$} \tag{44}

The term in square brackets here is zero. To see this, consider the solution of the eikonal equation (38). The eikonal equation is solved by the method of characteristics, with the characteristic equations determining the rays. Those equations are [Bleistein, et al, 2001]

$$\frac{dx}{ds} = \nabla B_0, \quad \frac{dp}{ds} = \nabla \left( \frac{1}{v} \right) = - \frac{1}{v^2} \nabla v. $$} \tag{45}

By examining the third component of the second equation here more closely,

$$\frac{\partial p_z}{\partial s} = \frac{dx}{ds} \cdot \nabla p_z = v \nabla \cdot v = - \frac{1}{v^2} \nabla v.$$} \tag{46}

Here, we have used the chain rule and the first line of the previous equation to write the $s$ derivative of $p_z$ in terms of the gradient. Substitution of the last identity here into the term in square brackets in equation (44) confirms that this term is zero as claimed.

The first two terms in equation (44) for $\pm L_\pm W$ combine into a single expression as follows.
\[ \nabla \cdot [v_B^2 p] = 0. \]  
(47)

The expression in square brackets here is slightly different from the ones obtained for the two two-way wave equations. See the previous transport equations, (16) and (17). Thus,

\[ A_0 = B_0, \quad \tilde{A}_0 = \sqrt{v} B_0. \]  
(48)

That is, the solution \( W \) of the one-way wave equation is

\[ \mathcal{L}_\pm W = \frac{\partial W}{\partial t} \pm v \sqrt{\nabla^2} W = 0, \]  
(49)

is asymptotically equal to the solution \( U \)

\[ U \sim W/\sqrt{v}, \]  
(50)

of the two-wave equation (1) and

\[ \tilde{U} \sim \sqrt{v} W \]  
(51)

provides an asymptotic solution of the wave equation (2).

4 AN APPROACH TO SOLVING INITIAL/BOUNDARY VALUE PROBLEMS FOR THE ONE-WAY WAVE EQUATIONS IN TIME

The analytical scheme used in the previous section to derive the asymptotic solutions of the one-way wave equation (18) is not a viable solution technique in practice; solving for the auxiliary function \( W \), in order to find \( W \), is not efficient. We propose, instead, to use the pseudo-spectral method of solution as introduced by Kosloff and Baysal [1982]. This approach was suggested by Y. Zhang.

To apply this method, we introduce the forward spatial Fourier transform

\[ \mathcal{F}[W(x, t)] = \tilde{W}(k, t) = \int_{-\infty}^{\infty} W(x, t) \exp\{-ik \cdot x\} d^3x, \]  
(52)

and its inverse,

\[ W(x, t) = \mathcal{F}^{-1}[\tilde{W}(k, t)] = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \tilde{W}(k, t) \exp\{ik \cdot x\}. \]  
(53)

We now rewrite the one-way wave equations (18) as

\[ \frac{1}{v} \frac{\partial W}{\partial t} \pm \mathcal{F}^{-1}[ik \mathcal{F}[W(x, t)]] = 0. \]  
(54)

That is, we apply the square root operator indirectly by computing the forward Fourier transform of \( W \), multiplying by \( ik \), and computing the inverse transform.

We are by no means suggesting that this is the only way to solve the one-way differential equation (18), but it is a classical approach to the solution. For a computationally more efficient approach, see Zhang et al [2007].

5 THE GREEN’S FUNCTIONS FOR THE ONE-WAY WAVE EQUATIONS

The Green’s function for the two-way wave equation satisfies equation (1) with right side \(-\delta(t)\delta(x)\) and initial data that \( U \equiv 0 \) for \( t < 0 \). In homogeneous media, \( v = \text{constant} \), the known solution of this equation in 3D is

\[ G(x, t) = \frac{\delta(r/v - t)}{4\pi r}, \]  
(55)

with \( r \) being radial distance from the source point. Note that this Green’s function is a one-term progressing wave series.

We propose to derive the Green’s function for the one-way forward and reverse time wave equations by considering the following problem.
\[
L \pm G_\pm(x, x_0, t, t_0) = \frac{1}{v(x)} \frac{\partial G_\pm}{\partial t} \pm \sqrt{v^2} G_\pm = \frac{1}{v(x)} \frac{\partial G_\pm}{\partial t} \pm ik G_\pm = 0,
\]

(56)

\[
G_\pm(x, x_0, t_0, t_0) = \mp v(x_0) \frac{\delta(x - x_0)}{ik}, \quad G_\pm \equiv 0, \quad \pm (t - t_0) < 0.
\]

Here, the operator \(1/ik\) applied to the \(\delta\)-function should be interpreted in a completely analogous manner as the operator \(ik\) in equation (54), above. The explicit calculation is carried out in Appendix 22 with the final result given by equation (C-4). We also observe from the form of the equation and initial conditions that the Green’s functions are shift invariant and depend only on the temporal difference, \(t - t_0\) since there is no temporal dependence in the differential equation, itself. Therefore, we always write

\[
G_\pm(x, x_0, t, t_0) \equiv G_\pm(x, x_0, t - t_0).
\]

(57)

We can simplify this problem by shifting the source point to the origin and considering instead,

\[
L_\pm G_\pm = \frac{1}{v} \frac{\partial G_\pm}{\partial t} \pm \sqrt{v^2} G_\pm = 0, \quad G_\pm(x, 0, 0) = \mp \frac{\nu(0)}{ik} \delta(x).
\]

(58)

The initial data \(G_\pm(x, 0, 0)\) has point support at \(x = 0\). Thus, for the leading order asymptotic solution, we should expect that the solution for constant \(v\) will have the proper “initial weight” to match the solution for variable \(v\). So consider the problem of equation (56) for constant \(v\) and proceed to solve by employing Fourier transforms where the dual of the operator \(\sqrt{v^2}\) is just \(ik\), as suggested in the previous section.

We define

\[
\gamma_\pm(k, \omega) = \pm \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx \, dy \, dz \, G_\pm(x, 0, t) \exp\{-ik \cdot x + i\omega t\}.
\]

(59)

The choice of lower limit in the temporal transform provides a causal solution (upper sign) or anti-causal solution (lower sign) in different half-planes of complex frequency. That is, the frequency domain transform is defined initially only for \(\pm \Re[\omega] \) being above or below all singularities of \(\gamma_\pm\) in the complex \(\omega\)-domain, respectively, for the upper or lower signs. Extension of the functions \(\gamma_\pm\) beyond these initial domains of definition are achieved by analytic continuation.

To proceed, compute the temporal Fourier transform of the time derivative of \(G_\pm\) as follows.

\[
\int_{-\infty}^{\infty} dt \frac{\partial G_\pm(x, 0, t)}{\partial t} \exp\{i\omega t\} = -G_\pm(x, 0, 0) - i\omega \int_{0}^{\infty} dt G_\pm(x, 0, t) \exp\{i\omega t\}
\]

\[
= \pm \frac{\nu(0)}{ik} \delta(x) - i\omega \int_{0}^{\infty} dt G_\pm(x, 0, t) \exp\{i\omega t\}.
\]

(60)

With this representation of the transform of the temporal derivative we can complete the Fourier transform of the differential equation (56) for \(G_\pm\):

\[
\pm [-i\omega \pm ivk] \gamma_\pm = -\frac{\nu}{ik},
\]

(61)

for which the solution is

\[
\gamma_\pm = \mp \frac{\nu}{ik \pm \omega v k}.
\]

(62)

We calculate the inverse Fourier transforms of \(\gamma_\pm\) in Appendix D, stating the result for initial time zero and source location at \(0\) in equation (D-6). By restoring the temporal and spatial shifts to \((t_0, x_0)\), we obtain

\[
G_\pm(x, x_0, t - t_0) \sim \frac{1}{4\pi^2} \left\{ \delta(r/v \mp (t - t_0)) \pm \frac{i}{\pi(r/v \mp (t - t_0))} \right\}, \quad r = |x - x_0|.
\]

(63)

Here we have restored the temporal and spatial shifts of the original differential equations (56) for the Green’s functions. The real parts of \(G_\pm\) are the causal/anti-causal Green’s functions of the two-way wave equation. The imaginary parts here yield the analytic solutions that we claimed as being typical of the solution of one-way wave equations in time. The complex values arise from the implementation of the operator \(ik\). This operator yields analytic solutions whenever it is applied.

These Green’s functions have the structure of the progressing wave \(W(x, t)\) of equation (27), modulo the additional shifts, \((x_0, t_0)\). We conclude that for variable wave speed \(v(x)\), the asymptotic Green’s functions are
The use of the square root of the Laplacian complicates the derivation of an adjoint operator and its consequences. Here, we propose an adjoint and show how it leads to integral representations of upward and downward propagation of wave fields. We use the symbol $L^\dagger_\pm$ for adjoint and reserve $*$ for complex conjugate. Below, complex conjugate will be applied to progressively larger expressions and the over-bar notation would become intractable for the reader. The operators $L_\pm W$ were defined in equation (18). Formally, we know that the adjoint of a first order operator such as $\partial/\partial t$ is just its negative, modulo initial and final data. Surprisingly, the adjoint of the operator $ikW$ will prove to be just itself. Boundary data for the two-way wave equation are sources for the one-way wave equation, so that no boundary terms arise from the domain integration of this operator.

We propose, then, that

$$L^\dagger_\pm W = -L_\mp W = -\left\{ W \frac{\partial W}{\partial t} \mp ikW \right\}. \quad (65)$$

Here and below, products in which one of the terms is $ikW$ arise. It is important to note to which function in the product the multiplication by $ik$ applies: it requires applying Fourier transform to that function, multiplying the function by $ik$ and applying inverse Fourier transform. Hence, we use the careful notation where that term is placed in square brackets.

Let us now consider

$$I = \int d^3x' \int_{t_-}^{t_+} dt' \left\{ U L_\pm W - W L^\dagger_\pm U \right\} (x', t')$$

$$I = I_1 + I_2,$$  \quad (66)

with

$$I_1 = \int d^3x' \int_{t_-}^{t_+} dt' \left\{ U \frac{\partial W}{\partial t} + W \frac{\partial U}{\partial t} \right\}$$

$$I_2 = \pm \int d^3x' \int_{t_-}^{t_+} dt' \left\{ W[ikW] - W[ikU] \right\}.$$  \quad (67)

The integral $I_1$ is an exact differential in time. Therefore, we simplify this integral by carrying out the integration in $t$ as follows.

$$I_1 = \int d^3x' \left. \frac{U(x', t')W(x', t')}{{v(x')}} \right|_{t_-}^{t_+}.$$  \quad (68)

By choosing one of these functions to have initial or final data equal to a Dirac delta function in space, this integral yields the other function evaluated at $(x, t_\pm)$. This is one ingredient in integral representations of solutions in terms of Green’s functions.

Now let us consider the integral $I_2$ in equation (67). In particular, let us consider the first term and only its spatial integral.

$$I_3 = \int d^3x' U(x') [ikW(x')].$$  \quad (69)
For our purposes here, the dependence on \( t' \) is unimportant and we have dropped it for brevity. The difficulty we face is that the operator \( ikW \) is defined in terms of forward and inverse Fourier transforms in equation (54).

Thus,

\[
I_3 = \frac{1}{(2\pi)^3} \int d^3x' U(x') \int d^3k d^3\bar{x}' W(\bar{x}') \exp\{ik \cdot (x' - \bar{x}')\}
\]

\[
= \frac{1}{(2\pi)^3} \int d^3x' U(x') \int d^3k d^3\bar{x}' W(\bar{x}') \exp\{ik \cdot (x' - \bar{x}')\} \tag{70}
\]

In each step here, we have expended the braces around complex conjugation further to the left while correspondingly conjugating the appropriate functions in the interior to leave the value unchanged.

Given what we started out with for \( I_3 \) in equation (69) and what we ended up with in equation (70), we conclude that the integral \( I_2 \) in equation (67) is equal to zero!

Now, we equate the expression for \( I_1 \) in equation (68) and the definition of \( I \) in equation (66) to find that

\[
\int_{t_-}^{t_+} d^3x' \int d^t \{U \mathcal{L}_\pm W - W \mathcal{L}_\pm U\} (x', t') = \int_{t_-}^{t_+} d^3x' W(x') U(x', t') \tag{71}
\]

This is the basic identity that we use to downward continue data from the upper surface and to upward continue data from a reflector at depth.

### 6.1 Symmetries of the Green’s functions

There are certain symmetries of the Green’s functions that is needed in the discussion below. We present those here.

#### 6.1.1 Symmetry in temporal variables

From the differential equations (56) for \( G_\pm(x, x_0, t - t_0) \), it should be clear that, in fact,

\[
G_\pm(x, x_0, t, t_0) \equiv G_\pm(x, x_0, t - t_0). \tag{72}
\]

Thus, let us introduce

\[
\tilde{G}_\pm(x, x_0, t - t_0) = G_\pm(x, x_0, t_0 - t), \tag{73}
\]

and observe that

\[
\frac{\partial}{\partial t} \tilde{G}_\pm(x, x_0, t - t_0) = \frac{\partial}{\partial t} G_\pm(x, x_0, t_0 - t). \tag{74}
\]

Returning to the defining equation (56) for the Green’s functions, we can now see that

\[
0 = \mathcal{L}_\pm G_\pm(x, x_0, t - t_0) = \frac{1}{v} \frac{\partial G_\pm(x, x_0, t - t_0)}{\partial t} \pm \sqrt{\nu^2} G_\pm(x, x_0, t - t_0)
\]

\[
= \frac{1}{v} \frac{\partial G_\pm(x, x_0, t_0, t)}{\partial t} \pm \sqrt{\nu^2} G_\pm \tag{75}
\]
Thus, \( \tilde{G}_\pm(x, x_0, t - t_0) \) satisfy the differential equations (56) of \( G_\pm(x, x_0, t - t_0) \) with the data for the former at \( t_0 \) of opposite sign to the data of the latter at \( t_0 \). We conclude then that

\[
\tilde{G}_\pm(x, x_0, t - t_0) = G_\pm(x, x_0, t) = -G_\mp(x, x_0, t).
\]  

(76)

This symmetry is used below.

### 6.1.2 Symmetry in spatial variables

Let us suppose that \( U \) and \( W \) satisfy the following equation.

\[
\mathcal{L}_\pm(x', x) U_\pm(x', x, t, t_\pm) = 0, \quad U_\pm(x', x, t_-, t_+) = \pm v(x) \delta(x' - x),
\]

\[
\mathcal{L}_\pm^1(x', x) W_\pm(x', y, t_+, t_+) = -\mathcal{L}_\pm(x', x) W_\pm(x', y, t_+, t_+) = 0,
\]

\[
W_\pm(x', y, t_+, t_+) = \pm v(y) \delta(x' - y)
\]

Apply our Green’s identity, equation (71) to these two functions to obtain

\[
0 = \int \frac{d^2 x'}{\sqrt{v(x')}} \left[U_\pm(x', x, t, t_\pm) v(y) \delta(x' - y) - W_\pm(x', y, t_+, t_+) v(x) \delta(x' - x)\right].
\]

(78)

Now use the \( \delta \)-functions to evaluate the integral and conclude that

\[
U_\pm(y, x, t_+, t_-) = W_\pm(x, y, t_-, t_+).
\]

(79)

By comparing the problems for \( U_\pm \) and \( W_\pm \) here with the differential equations (56), we conclude that

\[
U_\pm(y, x, t_+, t_-) = -ik G_\pm(y, x, t_+ - t_-), \quad W_\pm(x, y, t_-, t_+) = ik G_\mp(x, y, t_+ - t_-).
\]

(80)

Now we apply the temporal symmetry of equation (76) to conclude that

\[
\text{ik} G_\pm(y, x, t_+ - t_-) = \text{ik} G_\pm(x, y, t_+ - t_-).
\]

(81)

We see here that the spatial symmetry applies to \( \text{ik} G_\pm \) and not to the Green’s functions themselves.

However, let us now use the differential equations (56) for \( G_\pm \) to rewrite this last symmetry as follows

\[
\frac{1}{\sqrt{v(y)}} \frac{\partial G_\pm(y, x, t - t_-)}{\partial t} = \frac{1}{\sqrt{v(x)}} \frac{\partial G_\pm(x, y, t - t_-)}{\partial t}.
\]

(82)

We have now generated a list of symmetries among the Green’s functions that will allow us to proceed below to derive modeling formulas for downward continuation of data, upward propagation of reflection data and a pseudo inverse of the modeling operator for observed data at the upper surface. That pseudo-inverse will provide a true-amplitude migration of the observed data for a reflectivity function—a reflector map and an estimate of a specular reflection coefficient at each point of the reflector.

### 6.1.3 Complex conjugates

Let us first rewrite the differential equation and problem for \( G_\pm \) in equation (58) as follows.

\[
\frac{1}{\sqrt{v}} \frac{\partial G_\pm}{\partial t} \pm i k G_\pm = 0, \quad G_\pm(x, 0, 0) = \mp \frac{v(0)}{i k} \delta(x).
\]

(83)

Next, we take complex conjugates in this equation:

\[
\frac{1}{\sqrt{v}} \frac{\partial G_\mp^*}{\partial t} \mp i k G_\mp^* = 0, \quad G_\mp^*(x, 0, 0) = \mp \frac{v(0)}{i k} \delta(x).
\]

(84)

By interchanging signs here,

\[
\frac{1}{\sqrt{v}} \frac{\partial G_\mp^*}{\partial t} \pm i k G_\mp^* = 0, \quad G_\mp^*(x, 0, 0) = \mp \frac{v(0)}{i k} \delta(x).
\]

(85)
Thus, we see that $G^*_\pm$ is the same solution of the anti-causal problem here as is $G_-$ and, similarly, $G^*_\pm$ is the same solution of the causal problem as is $G_\pm$. That is,

$$G^*_\pm(x, x_0, t) = G_{\pm}(x, x_0, t), \quad \mp t > 0. \quad (86)$$

### 6.2 Symmetries in the frequency domain

We consider here the temporal Fourier transforms of the differential equations and initial data for the Green’s functions $G_{\pm}$ in equation (58).

$$-\frac{i\omega}{v}g_{\pm} \pm ikg_{\pm} = \pm \frac{v(0)}{ik} \delta(x). \quad (87)$$

Here, the inhomogeneous right hand side arises exactly as it did in equation (60) from integration by parts applied to the temporal Fourier transform of the time derivative of $G_{\pm}$.

Let us now consider the complex conjugate of this equation.

$$\frac{i\omega}{v}g_{\pm}^* \mp ikg_{\pm}^* = \pm \frac{v(0)}{ik} \delta(x). \quad (88)$$

Note that conjugation of the Green’s functions defines a new Green’s function that is analytic in the opposite half-$\omega$-plane as the original function. Thus, $g^*_\pm$ is analytic in a lower half-plane and $g^*_\mp$ is analytic in an upper half-plane. Just as in the time domain above, conjugation switches the causal and anti-causal property of these functions.

Furthermore, if we replace $\omega$ by $-\omega$; in the last equation, we have

$$-\frac{i\omega}{v}g_{\pm}^*(x, 0, -\omega) = \pm ikg_{\pm}^*(x, 0, -\omega) = \pm \frac{v(0)}{ik} \delta(x) \quad \text{or}$$

$$\frac{i\omega}{v}g_{\pm}^*(x, 0, -\omega) = \mp ikg_{\pm}^*(x, 0, -\omega) = \mp \frac{v(0)}{ik} \delta(x) \quad (89)$$

We conclude then that

$$g^*_\pm(x, 0, -\omega) = g_{\mp}(x, 0, \omega). \quad (90)$$

### 6.3 Downward continuation

Suppose now that we are given observed data for $U$ at $z = 0$. For the one-way wave equation, these data propagated forward in time to get to $z = 0$. Thus we want to propagate the data backwards in time to observe where it was located in depth at earlier times.

$$L_-(x', t')W(x', t') = \delta(z')D(x', y', t'), \quad W(x', \infty) = 0, \quad (91)$$

$$L^+_\pm(x', t')U(x', x, t', t) = -L_+U(x', x, t', t) = 0, \quad U(x', x, t, t) = v(x) \delta(x' - x).$$

We see here $W$ is the response to an “exploding reflector” at $z' = 0$ and

$$U(x', x, t', t) = ikG_+(x', x, t' - t) = -\frac{1}{v(x')} \frac{\partial G_+}{\partial t'}(x', x, t' - t), \quad (92)$$

with $G_-$ being the Green’s function as defined in equation (56). Furthermore, the propagation is backwards in time because $t' > t$. By using this information in the Green’s identity of equation (71), we find that

$$W(x, t) = \int_{z' = 0}^{\infty} dx' dy' \int_{t}^{\infty} dt' D(x', y', t') \frac{1}{v(x')} \frac{\partial G_+(x', x, t' - t)}{\partial t'} \quad (93)$$

$$= -\int_{z' = 0}^{\infty} dx' dy' \int_{t}^{\infty} dt' D(x', y', t') \frac{1}{v(x')} \frac{\partial G_-(x', x, t' - t)}{\partial t'}.$$

In the second line here, we used the temporal symmetry of the Green’s functions in equation (76). The first line emphasizes that the solution at time $t$ depends on the data for all times greater than $t$. The second form reminds us that this is really a back propagation in time from times $t'$ to earlier time $t$.\[93\]
Although we do not normally think of the downward propagation of observed data as conforming to the exploding reflector model, it really does. Here, it becomes explicit in that the problem for $W$ in equation (91) uses the observed data as source at $z' = 0$.

### 6.4 Upward propagation of reflection data

Now, suppose that we have reflection data on a surface $S$, such as we derived in Section 7.7. The surface would be described by a function of two parameters, $x = x_0(\sigma_1, \sigma_2)$.

We introduce $\gamma(x)$, the singular function of the surface. This is a Dirac delta function of signed normal distance from the surface at each point. As with any other distribution, we define this one by its “action” on nice functions:

$$\int_D dV f(x) \gamma(x) = \int_S dS f(x),$$

as long as the domain $D$ includes the surface $S$.

Let us now set up the appropriate two problems for the Green’s identity in equation (71).

$$\mathcal{L}^+(x', t') W(x', t') = \gamma(x') D(x', t'), \quad W(x', 0) = 0,$$

$$\mathcal{L}^+(x', t') U(x', x, t', t) = -\mathcal{L} U(x', x, t', t) = 0, \quad U(x', x, t, t) = v(x) \delta(x' - x).$$

As above,

$$U(x', x, t', t) = -ik G_-(x', x, t' - t) = -\frac{1}{v(x')} \frac{\partial G_-(x', x, t' - t)}{\partial v}. \quad (96)$$

In analogy with the previous section, we apply the Green’s identity of equation (71), to conclude that

$$W(x, t) = -\int_S dS' \int_0^t dt' D(x', y', t') \frac{1}{v(x')} \frac{\partial G_-(x', x, t' - t)}{\partial v} \quad (97)$$

$$= \int_S dS' \int_0^t dt' D(x', y', t') \frac{1}{v(x')} \frac{\partial G_+(x', x, t - t')}{\partial v}.$$

In the last line here, we used the temporal symmetry of the Green’s functions in equation 49.

The solution propagates forward in time from the data values on the surface $S$. As in the previous example, we use the differential equation (56) to replace $ik G_+$ by the time derivative of $G_+$ divided by $v$ in the last line.

In summary, we have derived a Green’s identity in equation (71). We used it to derive propagator integrals of Kirchhoff-type to downward continue surface data backwards in time and downwards into $z > 0$ and then to upward continue reflection data prescribed on a surface at depth.

### 7 REFLECTION AND THE KIRCHHOFF APPROXIMATION

When an acoustic wave is incident on an interface, it gives rise to reflected and transmitted waves. For elastic waves, there are additional mode-converted waves initiated at an interface. Here, we confirm that the standard procedure used in asymptotic ray theory for acoustic waves does not change when we do the same analysis in the time domain. Implicit in this example is the fact that the same will be true for elastic waves with their initiation of mode-converted waves.

For the two-way wave equation, the derived data for the reflected and transmitted waves provide input to a Green’s function integral representation of the propagation of the reflected and transmitted waves away from the interface, by providing leading order approximations of these waves themselves and their normal derivatives, as well. Theory tells us that this is an over-determination of prescribed data, but the leading order wave field produced by this method is sufficiently accurate to be useful in modeling and inversion.

The original Kirchhoff approximation was introduced for the specific boundary conditions $u = 0$ or $\partial u / \partial n = 0$ at the interface. In Bleistein [1984], we simply used these asymptotic ray theory data to extend the Kirchhoff approximation to interfaces. We have not been able to find an earlier reference to this extension although we believe it was in the “folklore” of users of the Kirchhoff approximation well before that.
Let us suppose that there is an interface $S$ defined by
\[ S : \quad x = x_0(\sigma), \quad \sigma = (\sigma_1, \sigma_2) \] (98)
and that a wave,
\[ U_i(x, t) = B_i F_i(\tau_i(x) \mp t), \] (99)
is incident on $S$. This wave gives rise to reflected and transmitted waves near $S$, $U_R$ and $U_T$, respectively:
\[ U_R(x, t) = B_R F_R(\tau_R(x) \mp t), \quad U_T(x, t) = B_T F_T(\tau_T(x) \mp t). \] (100)
The total solution on the two sides of $S$ are
\[ U = U_i + U_R, \quad \text{and} \quad U = U_T. \] (101)

Our objective is to derive initial data on $S$ for the propagation of these waves away from $S$. Standardly, we impose two continuity conditions on $S$ that the total solutions and their normal derivatives be continuous on $S$; that is
\[ U_i + U_R = U_T, \quad \partial U_i / \partial n + \partial U_R / \partial n = -\partial U_T / \partial n \quad \text{on } S. \] (102)

We have no hope of satisfying these equations unless we require that
\[ F_i(\tau_i(x_0) \mp t) = F_R(\tau_R(x_0) \mp t) = F_T(\tau_T(x_0) \mp t), \quad \text{on } S, \] (103)
and further that
\[ \tau_i(x_0) = \tau_R(x_0) = \tau_T(x_0) \quad \text{on } S. \] (104)

This is exactly equivalent to matching of the phases of the complex wave forms in the frequency domain. Here, it is necessary because of the localization of the progressing waves in space at a given time.

This last equation provides initial data for the travel times $\tau_R$ and $\tau_T$ on $S$. Furthermore, we can differentiate these last equations with respect to $\sigma_1$ and $\sigma_2$. Those equations tell us that the tangential component of the travel time gradients must also be equal; that is Snell’s law:
\[ \nabla_{\tau_i} \cdot \frac{\partial x_0(\sigma)}{\partial \sigma_i} = \nabla_{\tau_R} \cdot \frac{\partial x_0(\sigma)}{\partial \sigma_i} = \nabla_{\tau_T} \cdot \frac{\partial x_0(\sigma)}{\partial \sigma_i}, \quad i = 1, 2. \] (105)

More succinctly, we write this equation as
\[ \nabla_{\text{tang}} \tau_i = \nabla_{\text{tang}} \tau_R = \nabla_{\text{tang}} \tau_T, \] (106)
with the subscript “tang” denoting the tangential part of the gradient. We find the normal component of the gradient by using the eikonal equation, (37). However, we have to distinguish $v$ on the two sides of the surface $S$; denote them by $v_-$ as the wave speed on the same side of $S$ as the incident wave and $v_+$ on the transmitted side. Then we conclude that
\[ \frac{\partial \tau_R}{\partial n} = -\text{sgn} \left[ \frac{\partial \tau_T}{\partial n} \right] \sqrt{\frac{1}{v_-^2} - \left[ \nabla_{\text{tang}} \tau_T \right]^2}, \] (107)
\[ \frac{\partial \tau_T}{\partial n} = \text{sgn} \left[ \frac{\partial \tau_T}{\partial n} \right] \sqrt{\frac{1}{v_+^2} - \left[ \nabla_{\text{tang}} \tau_T \right]^2}, \quad \text{on } S. \] (108)

Now we return to the continuity conditions in equation (102) to determine how $B_R$ and $B_T$ are related to $B_i$. Using the representations of the wave functions themselves in equations (99) and (100), we can now strip away the progressing waves in the first equation and its derivative with respect to argument in the second equation, leaving the normal derivatives of the travel times in the resulting equation. This leads to the standard equations,
\[ B_i + B_R = B_T, \quad \frac{\partial \tau_i}{\partial n} [B_i - B_R] = \frac{\partial \tau_T}{\partial n} B_T, \] (108)
and solutions,
\[ B_R = R B_i, \quad B_T = T B_i, \quad \text{on } S, \] (109)
238  N. Bleistein, Y. Zhang & G. Zhang

\[ R = \frac{\partial \tau_i}{\partial n} - \frac{\partial \tau_T}{\partial n}, \quad T = \frac{2\partial \tau_T}{\partial n} + \frac{\partial \tau_T}{\partial n}. \]

These are the standard reflection and transmission coefficients derived in asymptotic ray theory in the frequency domain.

In summary, of the reflected and transmitted waves on the surface \( S \) are given by

\[ U_R(x, t) = RB_1(x)F_1(\tau_i(x) \mp t), \quad U_T(x, t) = TB_1(x)F_1(\tau_T(x) \mp t), \quad x \text{ on } S, \]

with the reflection coefficient \( R \) and the transmission coefficient \( T \) defined in equation (109) above.

8 A PSEUDO-INVERSE OF MODELING FOR COMMON-SHOT INVERSION

In this section, we consider the modeling identity for reflected data in equation (97). We assume that we have a common-shot data set at \( x \equiv (x_r, y_r, 0) \). In that wave field representation, we use the Kirchhoff approximation for \( U_R \) in equation (110) with \( F_1 \) being the Green’s function \( G_1(x, x_s, t) \). In this manner, we arrive at

\[ W(x_r, t) = \int d^3 x' \int_0^t dt' \frac{R_0(x', x_s)}{v(x')} G_1(x', x_s, t') \frac{\partial G_1(x', x_r, t - t')}{\partial t'}, \]

\[ R_0(x', x_s) = R(x', x_s) \gamma(x') \]  

Here, \( R_0 \) is the reflectivity function that we seek in the inverse problem and \( R(x', x_s) \) is the ray-theoretic reflection coefficient \( R \) as given in equation (109). The function \( R_0 \) corresponds to the reflectivity \( \beta \) of Bleistein et al [2001]. This notation is consistent with the notation in Bleistein et al [2005]. When we divide the integrand by \( |\nabla \tau(x, x_s) + \tau(x, x_s)| \) we obtain the reflectivity \( \beta(x, x_s) \) denoted by \( R \) in Bleistein et al [2005].

We apply the temporal Fourier transform to both sides of this equation, using lower case letters in the frequency domain to correspond to the capital letters for the functions here in the time domain. We provide explanations of each line of this multi-line equation directly below.

\[ w(x_r, \omega) = \int_0^\infty dt \int_0^t dt' \int d^3 x' \frac{R_0(x', x_s)}{v(x')} G_1(x', x_s, t') \frac{\partial G_1(x', x_r, t - t')}{\partial t'} \exp(i\omega t) \]

\[ = -\int_0^\infty \exp(i\omega t') dt' \int_0^\infty \exp(i\omega(t - t')) dt \int d^3 x' \frac{R_0(x', x_s)}{v(x')} G_1(x', x_s, t') \frac{\partial G_1(x', x_r, t - t')}{\partial t}, \]

\[ = -\int_0^\infty \exp(i\omega t') dt' \int_0^\infty \exp(i\omega \sigma) d\sigma \int d^3 x' \frac{R_0(x', x_s)}{v(x')} G_1(x', x_s, t') \frac{\partial G_1(x', x_r, \sigma)}{\partial \sigma}, \]

\[ = \int d^3 x' \frac{R_0(x', x_s)}{v(x')} i\omega g_1(x', x_s, \omega) g_1(x', x_r, \omega). \] (112)

Here, the right sides are obtained as follows.

(i) The first equality just introduces the definition of the Fourier transform on the right hand side.

(ii) In the second equality, the orders of integration in \( t \) and \( t' \) are interchanged, a differentiation with respect to \( t' \) has been rewritten as a differentiation with respect to \( t \) and the single exponential has been rewritten as a product of two exponentials.

(iii) In the third equality, the origin in the \( t \)-integral has been shifted from \( t' \) to zero with the change of variable of integration, \( t - t' = \sigma \).

(iv) In the fourth equality, we recognize the temporal integrals as a product of the Fourier transforms of the two Green’s functions of the previous line with the differentiation with respect to \( \sigma \) leading to a multiplier of \(-i\omega\).

The final result here has the form of a linear integral operator on the reflectivity \( R_0 \); that is,
Formally,
\[
K^{-1} = \frac{K^\dag}{\|K^\dag K\|}
\]

Here, the adjoint \(K^\dag\) is an integral over the variables \((x, y, \omega)\) and the norm \(\|K^\dag K\|\) is an integral over \((x, y, \omega)\) and the three elements of \(x\). Thus, our next task is to identify \(K^\dag\).

We find the operator \(K^\dag\) by considering the inner product between two functions, one \(w(x, \omega)\) and the other \(u(x')\) and then seek the operator for which,
\[
\langle w, K[u] \rangle = \langle K^\dag[w], u \rangle,
\]
with
\[
\langle w, K[u] \rangle = \int d^2x, d\omega \int d^3x' w^*(x, \omega) u(x', x, \omega) \frac{i\omega}{\sqrt{v(x')}} g_i(x', x, \omega).
\]

It is fairly straightforward to recast this integral with the kernel of the operator attached to the function \(w\) as follows
\[
\langle w, K[u] \rangle = \int d^2x, d\omega \int d^3x' \left[w(x, \omega) g_i^*(x', x, \omega) \frac{i\omega}{\sqrt{v(x')}} g_i(x', x, \omega)\right] u(x')
\]
\[
= \langle K^\dag[w], u \rangle,
\]
with
\[
K^\dag[w](x) = \int d^2x, \frac{i\omega d\omega}{\sqrt{v(x')}} w(x', \omega) g_i^*(x, x, \omega) g_i(x, x, \omega)
\]
\[
= -\int d^2x, \frac{i\omega d\omega}{\sqrt{v(x')}} w(x', \omega) g_-(x, x, \omega) g_+(x, x, \omega)
\]
\[
= \int d^2x, \frac{i\omega d\omega}{\sqrt{v(x')}} w(x', \omega) g_-(x, x, \omega) g_-(x, x, \omega).
\]

Here, the second equality comes from the relation between \(g\) and \(g^*\) in equation (90) and the last equality arises from replacing \(\omega\) by \(-\omega\) in the previous line. We claim that this last equality provides a correlation-type migration operator in the frequency domain. We interrupt our development of an inversion operator to transform this last equality into a migration operator in the time domain. Note that the \(\omega\) integration in the last line is restored to an integral below all singularities in the frequency domain.

### 8.1 Migration operator in time deduced from \(K^\dag[w]\) in the frequency domain.

We remark that if \(w(x, \omega)\) is the observed data on the upper surface, then this last integral is a correlation-type migration. This formula can be expressed in the time domain, as well. To do that, we need a notation for the temporal Fourier transform. We have already used \(\mathcal{F}\) to denote the spatial Fourier transform, so let us use \(\mathcal{F}\) to denote the temporal Fourier transform of an anti-causal function. That is,
\[
\mathcal{F}[U(\ldots,t)] = u(\ldots, \omega)
\]
\[
= \int_{-\infty}^{0} U(\ldots, t) \exp\{i\omega t\} \, dt,
\]
\[
\mathcal{F}^{-1}[u(\ldots, \omega)] = U(\ldots, t) = \frac{1}{2\pi} \int_{\Gamma} u(\ldots, \omega) \exp\{-i\omega t\} \, d\omega.
\]
Here, \( \Gamma \) is a line parallel to the \( \Re[\omega] \)-axis, below all singularities of the function \( u(\ldots, \omega) \).

We begin our analysis here with the last line of equation (119) for \( K^\dagger[w](x) \) as a frequency domain integral and being introducing the definitions of various functions in terms of their temporal transforms as defined in equation (240). N. Bleistein, Y. Zhang & G. Zhang

The substitutions in each line here proceed as follow.

(i) In line one, \( g_-(x, x_s, \omega) \) has been replaced by its definition in terms of \( G_-(x, x_s, t) \)

(ii) In line two, the same was done with \( g_-(x, x_r, \omega) \), except that we also absorbed the factor of \( i\omega \) into this transform by introducing a \(-\partial / \partial t'\).

(iii) In line three the inverse transform of \( w \) was introduced.

(iv) In line four, the change of variable of integration from \( t' \) to \( u \) was introduced, with \( t + t' = -u \).

(v) In line five, the variable \( t \) was replaced by the variable \(-t\) with appropriate changes in the limits of integration.

(vi) In line six, we identified the downward propagation of the data \( v(x_r)W(x_r, u) \) to \( v(x)W(x, t) \) by using the derived formula in equation (93). In this form, we see the migration formed by a correlation of the downward propagated data with the backward propagate point source.

(vii) Here, the same correlation is presented as an integral with \( G_+ \) by using the symmetry of these Green's functions stated in equation (76).

8.2 Asymptotic analysis of \( ||K^\dagger K|| \)

To transform our migrations into inversions, we need to calculate \( ||K^\dagger K|| \). Actually, “true-amplitude” processing only makes sense when an image is formed by a single arrival at the image point. Furthermore, the interpretation of the output amplitude at the image point in terms of reflectivity is based on leading order asymptotic theory—asymptotic ray theory—for single arrivals. Hence, for normalization purposes, we may use leading order asymptotic analysis to estimate the norm \( ||K^\dagger K|| \) that we seek.

The operator \( K \) is implicit in the representation of \( w(x_r, \omega) \) in equation (112). Similarly, the operator \( K^\dagger \) is implicit in the representation of \( K^\dagger[w] \) in equation (119). The cascade of these two operators is

\[
||K^\dagger K|| = \int d^3x_r d\omega \frac{\omega}{v(x_r)} g_-(x, x_s, \omega) g_-(x, x_r, \omega) \int d^3x_r' \frac{\omega}{v(x_r')} g_+(x', x_s, \omega) g_+(x', x_r, \omega).
\]
As noted above, we can use leading order asymptotic approximations here for the Green’s functions. Those leading order approximations are given in equation (E-7). When we use those values here with appropriate substitution of arguments, we obtain the following integral representation for \( |K^\dagger K| \).

\[
|K^\dagger K| = 16 \int d^2x \, d\omega d^3x' \omega^2 \cdot B(x, x_s)B(x', x_r)B(x', x_s)B(x', x_r) \exp\{\Phi(x, x', x_s, x_r)\},
\]

(123)

\[
\Phi(x, x', x_s, x_r) = \tau(x', x_s) + \tau(x', x_r) - [\tau(x, x_s) + \tau(x, x_r)].
\]

In fact, this same calculation was carried out in Appendix E of Bleistein et al [2005], and stated in equation (28) of that reference as

With these changes, in equation (18) of the reference, we obtain

\[
\|K^\dagger K\|^{-1} = \frac{v^2(x)}{8\pi^2} |h(x, x_r)| \frac{|A_0(x, x_s)|}{|A_0(x, x_r)|^2}.
\]

(124)

The Beylkin determinant for common-shot is know to be

\[
h(x, x_r) = \frac{16\pi^2 |\tilde A_0^2(x, x_r)| \cos \beta_r}{v^2(x)} \cos \beta_r,
\]

with \( \beta_r \) being the emergence angle of the ray from \( x \) to \( x_r \) with respect to the vertical. With this value substitute into the previous equation, we find that

\[
\|K^\dagger K\|^{-1} = \frac{1}{2\pi |A_0(x, x_s)|^2} \frac{\cos \beta_r}{v(x_r)} = \frac{1}{2\pi |A_0(x, x_s)|^2 v(x_r)} \cos \beta_r.
\]

(126)

Here, the last result arises from the relationship between \( A_0 \) and \( B_0 \) in equation (48).

### 8.2.1 Common shot inversion

Note that the inversions and migrations that we have written above are integrals over \( x_r \) and \( \omega \) or \( t \). It is easy now to slip this expression now into the migration in frequency domain in equation (119) or migration in the time domain in equation (121). We claim then that the following formulas provide inversions of frequency and time domain data.

\[
\mathcal{R}_0(x, x_s) = \frac{1}{2\pi v^2(x)|B_0(x, x_s)|^2} \int d^2x_r \frac{\cos \beta_r}{v(x_r)} \int i\omega \, d\omega (x_r, \omega)g_-(x, x_s, \omega)g_-(x, x_r, \omega),
\]

(127)

and

\[
\mathcal{R}_0(x, x_s) = \frac{1}{v^2(x)|B_0(x, x_s)|^2} \int d^2x_r \frac{\cos \beta_r}{v(x_r)}
\]

\[
\times \int_0^\infty dt G_-(x, x_s, -t) \int_0^\infty \frac{\partial G_-(x, x_r, t - u)}{\partial u} W(x_r, u).
\]

\[
= \frac{1}{v^2(x)|B_0(x, x_s)|^2} \int_0^\infty dt G_-(x, x_s, -t)W(x, t).
\]

(128)

In the frequency domain formula, equation (127), the integration over \( x_r \) provides the downward continuation of the observed data and the integral over frequency forms the image. In the time domain, the integral over \( x_r \) and the integral over \( u \) provide the downward continuation of the data and the integral over \( t' \) forms the image.

The data \( W \) must be consistent with forward modeling for this one way wave equation. Acquired data is the solution of a two-way wave equation. As with the Green’s function, we need to take those acquired data and extend them to analytic data by adding \( t \) times their Hilbert transform in time.

The scaling factor \( v^2(x)|B_0(x, x_s)|^2 \) is an asymptotic expansion of \( |K^\dagger K| \) under the assumption that the incident wave does not have a caustic near the image point \( x \). The interpretation of the reflectivity \( \mathcal{R}_0 \) in terms of a reflection coefficient as in equation (111), has not been shown to be valid near a caustic. Thus, we can content ourselves with simply regularizing this denominator so that it does not become infinite at the caustic.
Here is one recipe for such regularization. Using ray theory in ray-centered coordinates, one can show that
\[
\frac{1}{|B_0(x, x_s)|^2} = (4\pi)^2 v(x_s) |\det[Q]|, \tag{129}
\]
with \(Q\) the standard notation of dynamic ray theory in ray-centered coordinates. It is this determinant that goes to zero at caustics. On the other hand, in homogeneous media, \(|\det[Q]| = |x - x_s|^2\). Thus we propose the replacement
\[
\frac{1}{|B_0(x, x_s)|^2} \Rightarrow (4\pi)^2 v(x_s) \left[ |\det[Q]| - i\epsilon |x - x_s|^2 \right], \tag{130}
\]
for some appropriately chosen “small” \(\epsilon\). Now, away from caustics, this factor has phase near zero and we simply ignore the imaginary part to estimate the reflection coefficient. Near a caustic, the phase of this new term moves away from zero and the estimate of a reflection coefficient is unreliable. However, the output does not have a zero at the caustic and we are not evaluating \(1/|B_0(x, x_s)|^2\) which becomes infinite at the caustic.

A KMAH index can also be incorporated into the frequency domain formula to account for phase shifts at caustics. The phase shifts would introduce multipliers of powers of \(i\) in the time domain, progressively interchanging the real and imaginary parts of the Green’s functions, with \(-1\) factors introduced with each pair of phase shifts.

9 SUMMARY AND CONCLUSION

We have shown that a simple pair of one-way first order wave equations have the right asymptotic properties to serve as replacements for the usual second order two-way wave equation. These equations use \(t\) as the distinguished variable in contrast to the usual use of \(z\) in this role in the literature. Where the one-way wave equations in \(z\) fail at horizontal propagation, the one-way equations in time do not; all directions of propagation are treated alike by these equations. An analytic implementation of the method was presented to calculate the Green’s function for homogeneous media. This method produces an analytic Green’s function with its real part providing the Green’s function for the two-way wave equation and its imaginary part being the Hilbert transform of the real part. This also happens for the one-way wave equations in \(z\). Downward propagation of data prescribed at \(z = 0\) was also demonstrated by using the complex conjugate of the anti-causal Green’s function as propagator.

We derived a Green’s identity to connect the solution of a one-way equation to the solution of the adjoint equation. That allowed us to derive integral formulas to describe downward propagation of observed data from \(z = 0\) and upward propagation of reflection data from a surface at depth. The latter would use Kirchhoff-approximate data that was also derived here.

Finally, we derived the pseudo-inverse of the forward modeling formula for Kirchhoff-approximate reflected data. That pseudo-inversion operator provides an inversion of the data in the same sense as Kirchhoff inversion [Bleistein et al, 2001].

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Figure A-1. The complex s-plane with the contours, Γ₁ and Γ₂. The integrand has branch points at s = ±1 and poles at the points ±s₀ with s₀ = i |kz|/√kt₂. When the vertical pieces of contour come together, the integrals along them cancel, leaving only the integrals around the poles and the integral on the large circle whose radius becomes infinite.

APPENDIX A: ANALYSIS OF THE INTEGRALS I(k) AND I₁(p)

The purpose of this appendix is to calculate the two integrals I(k) of equation (22) and I₁(p) of equation (42).

We discuss I first, repeated here:

\[ I(k) = \frac{1}{\pi} \int_{-1}^{1} \frac{\sqrt{1-s^2 k_x^2}}{k_x^2 + s^2 k_T^2} ds, \quad k_T = \begin{cases} k_x, & 2D, \\ (k_x, k_y), & 3D. \end{cases} \]  

(A-1)

The integrand here has branch points at s = ±1 and poles at ±s₀, with

\[ s₀ = i \frac{|k_x|}{\sqrt{k_T^2}}. \]  

(A-2)

Furthermore, \( \sqrt{1-s^2} \) is positive and real on the initial interval (-1, 1).

The interval of integration can be replaced by the contour Γ₁ of Figure A-1 by standard contour integration techniques; this introduces a factor of 1/2; that is,

\[ I(k) = \frac{1}{2\pi} \int_{Γ₁} \frac{\sqrt{1-s^2 k_x^2}}{k_x^2 + s^2 k_T^2} ds. \]  

(A-3)

The contour Γ₁ can be replaced by the contour Γ₂. The vertical segments of the contour are shown as separate to make their orientation clearer. However, in practice we bring them together. Since the integrand is single valued around the poles, these integrals on the vertical pieces cancel, both above the real axis and below. We are then left with integrals around poles ±s₀ which are evaluated as a sum of residues at those poles. In addition, we have the integral on the large circle whose radius is allowed to approach infinity.

It is fairly straightforward to calculate the sum of the residues, so that

\[ I(k) = \frac{k}{|k_x|} - \frac{1}{2\pi} \int_{|s|=R} \frac{\sqrt{1-s^2 k_x^2}}{k_x^2 + s^2 k_T^2} ds. \]  

(A-4)

Here, we have changed the sign on the integral so that its orientation is positive; that is, counterclockwise. As \( R \to \infty \), we need only retain the leading powers of s in the numerator and the denominator; that is,

\[ \frac{1}{2\pi} \int_{|s|=R} \frac{\sqrt{1-s^2 k_x^2}}{k_x^2 + s^2 k_T^2} ds = \frac{1}{2\pi} \int_{|s|=R \to \infty} \frac{-is}{s^2} ds = -\frac{1}{2\pi} \int_{|s|=R \to \infty} \frac{ids}{s} = 1. \]  

(A-5)
Here, the first equality just effects the large-\(|s|\) assumption. That is, it extends to \(s > 1\) as \(-i\sqrt{s^2-1}\) by standard complex analysis, which we have approximated as \(-is\) for large \(|s|\). The second equality simplifies the quotient. The final equality applies Cauchy’s residue formula, in which case we see that this integral on a circle of increasingly large radius has a finite limit, namely, one.

When this evaluation of the integral appearing in equation (A-4) for \(I(k)\), the final evaluation of \(I\) is

\[
I(k) = \frac{k}{|k_z|} - 1. \quad (A-6)
\]

This is equivalent to equation (21) for \(I\).

Now consider \(I_1\) as defined in equation (42) and repeated here.

\[
I_1(p) = \frac{1}{\pi} \int_{-1}^{1} \frac{\sqrt{1-x^2}}{(p_x^2 + s^2 p_T^2)^{\frac{3}{2}}} \, ds. \quad (A-7)
\]

The analysis of this integral uses the same contour deformations as above. However, now, the poles are second order and the integral on the circle of radius \(R\) approaches zero as \(R \to \infty\). The reason is that now the denominator is \(O(|s|^4)\) for large \(|s|\) while the numerator is again of order \(|s|\). The calculation of the sum of residues in this case leads to the stated result in equation (42).

**APPENDIX B: DETERMINING \(B_{0T}, B_{1T}\) IN TERMS OF \(B_0, B_1\)**

In this appendix, we present the details of the determination of the coefficients \(B_{0T}\) and \(B_{1T}\) in terms of \(B_0\) and \(B_1\). To this end, the series of equations (27) and (28) need to be substituted into the partial differential equation (25) that relates the two functions \(W_T\) and \(W_x\). The series for the derivatives are as follows. For the transverse Laplacians,

\[
\nabla_T^2 W = [B_0 F_0' + B_1 F_0']p_T^2 + F_0'[2p_T \cdot \nabla_T B_0 + +B_0 \nabla_T \cdot p_T] + \ldots, \quad (B-1)
\]

and

\[
\nabla_T^2 W_x = [B_{0T} F_0' + B_{1T} F_0']p_T^2 + F_0'[2p_T \cdot \nabla_T B_{0T} + B_{0T} \nabla_T \cdot p_T] + \ldots. \quad (B-2)
\]

Similarly, for the second derivative with respect to \(z\),

\[
\frac{\partial^2 W_T}{\partial z^2} = [B_0 F_0'' + B_1 F_0'']p_z^2 + F_0'[2p_z \frac{\partial B_0}{\partial z} + B_0 \frac{\partial p_z}{\partial z}] + \ldots, \quad (B-3)
\]

and

\[
\frac{\partial^2 W_x}{\partial z^2} = [B_{0T} F_0'' + B_{1T} F_0'']p_z^2 + F_0'[2p_z \frac{\partial B_{0T}}{\partial z} + B_{0T} \frac{\partial p_z}{\partial z}] + \ldots. \quad (B-4)
\]

Here,

\[
p_T = \begin{cases} 
  p_x, & \text{2D}, \\
  (p_x, p_y), & \text{3D}. 
\end{cases} \quad (B-5)
\]

The series in equations (B-1), (B-2) and (B-4) are substituted into equation (25) and the coefficients of \(F_0''\) and \(F_0'\) on the two sides of the equation are set equal. This leads to the following pair of equations.

\[
[p_z^2 + s^2 p_T^2] B_{0T} = p_T^2 B_0, \quad (B-6)
\]

\[
[p_z^2 + s^2 p_T^2] B_{1T} = p_T^2 B_1 + 2p_T \cdot \nabla_T B_0 + B_0 \nabla_T \cdot p_T
\]

\[
- \left[ 2p_z \frac{\partial B_{0T}}{\partial z} + B_{0T} \frac{\partial p_z}{\partial z} + s^2 \left[ 2p_T \cdot \nabla_T B_{0T} + B_{0T} \nabla_T \cdot p_T \right] \right].
\]

The first equation here has solution

\[
B_{0T} = \frac{p_T^2}{p_T^2 + s^2 p_T^2} B_0. \quad (B-7)
\]
We observe that the second equation has transverse derivatives multiplying the combination
\[ B_0 - s^2 B_{\omega T} = \frac{p_z^2}{p_z^2 + s^2 p_T^2} B_0. \]
(B-8)

By using this identity and the solution for \( B_{\omega T} \) of equation (29) in the second equation in (B-6), we can rewrite that equation as
\[
[p_z^2 + s^2 p_T^2] B_{\omega T} = \left( p_z^2 B_0 + 2 p_T \cdot \nabla_T \left( \frac{p_z^2 B_0}{p_z^2 + s^2 p_T^2} \right) \right) + \frac{p_z^2 B_0}{p_z^2 + s^2 p_T^2} \nabla_T \cdot p_T
\]
(B-9)

\[
-2p_z \frac{\partial}{\partial s} \left( \frac{p_T B_0}{p_z^2 + s^2 p_T^2} \right) - \frac{p_z^2 B_0}{p_z^2 + s^2 p_T^2} \frac{\partial p_z}{\partial z}.
\]

The solution of this equation is
\[
B_{\omega T} = \frac{p_T^2 B_0}{p_z^2 + s^2 p_T^2} B_1 + \frac{1}{(p_z^2 + s^2 p_T^2)} \left\{ 2 p_T \cdot \nabla_T \left( \frac{p_z^2 B_0}{p_z^2 + s^2 p_T^2} \right) + \frac{p_z^2 B_0}{p_z^2 + s^2 p_T^2} \nabla_T \cdot p_T \\
-2p_z \frac{\partial}{\partial s} \left( \frac{B_0 p_T^2}{p_z^2 + s^2 p_T^2} \right) - \frac{p_z^2 B_0}{p_z^2 + s^2 p_T^2} \frac{\partial p_z}{\partial z} \right\}.
\]
(B-10)

In the last line here, we have isolated the \( s \)-dependence in a single factor under a 2D divergence and a similar single factor under the \( z \)-derivative—very nearly a divergence.

**APPENDIX C: THE DISTRIBUTION \( \delta(x - \xi)/IK \)**

In equation (56) we introduced the distribution \( \delta(x - \xi)/ik \) as the final data for an anti-causal Green’s function. Earlier, in equation (56), we introduced the same distribution with \( \xi = 0 \) in defining the Green’s function for the causal one-way Green’s function.

Here, we reinterpret this distribution by applying the same spectral method as we used to define multiplication by \( ik \) in equations (52) and (53).

Thus, we write
\[
\frac{\delta(x - \xi)}{ik} = \frac{1}{(2\pi)^3} \int d^3k \frac{1}{ik} \int d^3x' \delta(x' - \xi) \exp\{ik \cdot (x - x')\}
\]
\[
= \frac{1}{(2\pi)^3} \int d^3k \frac{1}{ik} \exp\{ik \cdot (x - \xi)\}
\]
(C-1)
\[
= -\frac{i}{(2\pi)^3} \int_0^{2\pi} dk \int_0^\pi \sin \theta \ d\theta \int_0^{2\pi} d\phi \exp\{ikr \cos \theta\}, \ r = |x - \xi|.
\]

Here, in the second line, we used the \( \delta \) function to evaluate the integral over \( x' \). In the third line, we transformed to polar coordinates in \( k \) with the polar axis parallel to the vector \( x - \xi \). In this last line, we see that the integrand is independent of the angle \( \phi \), so that the \( \phi \)-integration only introduces a factor of \( 2\pi \). Further, the integral in \( \theta \) is exact, leading to
\[
\frac{\delta(x - \xi)}{ik} = \frac{1}{(2\pi)^3} \int_0^{2\pi} dk \left[ \exp\{-ikr\} - \exp\{ikr\} \right].
\] (C-2)
FIGURE D-1. The contours, $\Gamma_{\pm}$, for the frequency domain integration of $\gamma_{\pm}$. Also shown are the poles on the $\Re\{\omega\}$-axis at $\pm v k$ for $\gamma_{\pm}$, respectively.

These two integrals are known distributions. They can be derived from the identities in equation (A.7) of Bleistein et al [2001], namely that

$$\frac{1}{2\pi} \int_{0}^{\infty} dk \exp\{\pm ikr\} = \frac{1}{2} \left[ \delta(r) \pm \frac{i}{\pi r} \right].$$  \hspace{1cm} (C-3)

We add the two together with opposite signs as indicated in the previous equation to find that

$$\frac{\delta(x - \xi)}{ik} = \frac{1}{2\pi^2 ir}.$$  \hspace{1cm} (C-4)

APPENDIX D: INVERTING THE FOURIER TRANSFORM, $\gamma_{\pm}$, FOR THE GREEN'S FUNCTIONS IN HOMOGENEOUS MEDIA

Here we derive the inverse Fourier transforms of the functions $\gamma_{\pm}$ in equation (62). To begin, we write the inverse transform as an integral in $\omega$ and $k$ as follows.

$$G_{\pm}(x, 0, t) = -\frac{v}{(2\pi)^{3}} \int_{r_{\pm}} d\omega \int_{-\infty}^{\infty} dk_{1} dk_{2} dk_{3} \frac{\exp\{ik \cdot x - i\omega t\}}{k[\omega \mp v k]}$$  \hspace{1cm} (D-1)

In this equation, the contours $\Gamma_{\pm}$ are infinite lines parallel to the $\Re\{\omega\}$-axis above (+) or below (-) all singularities of the integrand, namely, any line above (+) or below (-) the real $\omega$-axis, respectively, for the causal and anti-causal solutions. See Figure ??, where the contours $\Gamma_{\pm}$ are depicted as well as the poles $\pm v k$ of the functions $\gamma_{pm}$, respectively.

In the next sequence, we rewrite the $k$-domain integrals in equation (D-1) in polar coordinates, then cancel a common factor of $k$ in numerator. As a last step in this sequence, we integrate in $\phi$, thereby cancelling a factor of $2\pi$ since the integrand is independent of $\phi$.

$$G_{\pm}(x, 0, t) = -\frac{v}{(2\pi)^{3}} \int_{r_{\pm}} d\omega \int_{0}^{\infty} k^{2} dk \int_{0}^{\pi} \sin \theta d\theta \int_{0}^{2\pi} d\phi \frac{\exp\{ikr \cos \theta - i\omega t\}}{k[\omega \mp v k]}$$  \hspace{1cm} (D-2)

$$= -\frac{v}{(2\pi)^{3}} \int_{r_{\pm}} d\omega \int_{0}^{\infty} kdk \int_{0}^{\pi} \sin \theta d\theta \int_{0}^{2\pi} d\phi \frac{\exp\{ikr \cos \theta - i\omega t\}}{[\omega \mp v k]}$$

$$= -\frac{v}{(2\pi)^{3}} \int_{r_{\pm}} d\omega \int_{0}^{\infty} kdk \int_{0}^{\pi} \sin \theta d\theta \int_{0}^{2\pi} d\phi \frac{\exp\{ikr \cos \theta - i\omega t\}}{[\omega \mp v k]}.$$  \hspace{1cm} (D-3)

Next, the integration in $\theta$ can readily be calculated because of the factor of $\sin \theta$ in the amplitude. Therefore,

$$G_{\pm}(x, 0, t) = -\frac{v}{i(2\pi)^{3}r} \int_{r_{\pm}} d\omega \int_{0}^{\infty} dk \frac{\exp\{ikr - i\omega t\} - \exp\{-ikr - i\omega t\}}{\omega \mp v k}.$$  \hspace{1cm} (D-3)
The contour \( \Gamma_+ \) lies above the pole of the integrand at \( \omega = \nu k \); similarly, the contour \( \Gamma_- \) lies below the pole of the integrand at \( \omega = -k \). Let us consider \( G_+ \) for the moment. For \( t < 0 \) we can close the contour of integration \( \Gamma_+ \) by a semicircle in the upper half plane whose radius is allowed to approach infinity. The integral around the closed path is equal to the integral along \( \Gamma_+ \), but the integral around the closed path encloses no singularities of the integrand. Therefore, by Cauchy’s theorem, \( G_+ = 0 \) for \( t < 0 \).

We characterize this below by a multiplier of the Heaviside function, \( H(t) \). Similarly, for \( t > 0 \) we can apply the same arguments to the integral on the contour \( \Gamma_- \) and conclude that \( G_- = 0 \) for \( t > 0 \). We can handle both of these observations at the same time with the multiplier \( H(\pm t) \).

For the complimentary temporal domains, we close the contours along the opposite semicircles, now enclosing a pole in each case. Thus, we compute the causal and anti-causal Green’s functions as residues of the integrands, as follows.

\[
G_\pm(x, 0, t) = \pm H(\pm t) \frac{\nu}{(2\pi)^2} \int_0^\infty dk \left[ \exp\{i k (r \mp \nu t)\} - \exp\{-i k (r \pm \nu t)\} \right].
\]

This integral is a difference of well-known distributions. They can be determined using the basic discussion of the Fourier transform of the Heaviside function in Section A.7 of Bleistein et al. [2001], among other places. They are the Fourier transform of the Heaviside function in Section A.7 of Bleistein et al. [2001], among other places. They are

\[
\delta(\omega - \nu t) \pm \delta(\omega + \nu t) \quad \text{and the support of the source delta function shifted from } 0 \text{ to } x_0.
\]

**APPENDIX E: THE TEMPORAL FOURIER TRANSFORMS OF THE ASYMPTOTIC GREEN’S FUNCTIONS.**

In this appendix, we derive the temporal Fourier transforms \( g_\pm \), equation (??), of the asymptotic Green’s functions \( G_\pm \) in equation (64). That is, we consider the pair of integrals

\[
g_\pm(x, x_0, \omega) = B \int_0^{\pm \infty} \left\{ \delta(\tau \mp t) \pm \frac{i}{\pi} \cdot \frac{1}{\tau \mp t} \right\} \exp\{i \omega t\} dt. \quad \pm \Im(\omega) > 0.
\]

Here, the spatial dependencies are unimportant, so we have suppressed them. Furthermore, the Fourier transform of the imaginary parts in curly brackets are to be interpreted as Cauchy principal value integrals.

In the representation of \( g_- \) in equation (E-1), let us replace \( t \) by \( -t \), leading to the following:

\[
g_\pm(x, x_0, \omega) = \pm B \int_0^{\infty} \left\{ \delta(\tau - t) \pm \frac{i}{\pi} \cdot \frac{1}{\tau - t} \right\} \exp\{\pm i \omega t\} dt. \quad \pm \Im(\omega) > 0.
\]

First, let us consider the two integrals with the delta functions:

\[
\pm B \int_0^{\infty} \delta(\tau - t) \exp\{\pm i \omega t\} dt = \pm B \exp\{\pm i \omega t\}, \quad \pm \Im(\omega) > 0.
\]

Of course, the analytic continuation of these two functions to the entire complex \( \omega \)-plane are just the explicit exponentials.

Next, we consider the principal value integrals of the second terms here. We introduce the integral on the counterclockwise contour of Figure E-1 in the complex \( t \)-plane. Call the whole contour \( C \). On that contour, \( C_R \) and \( C_r \) are
semi-circles of radius $R$ and $r$, respectively. Let us consider the following limit on the integral on the contour $C$.

$$\lim_{R \to \infty} \int_C \frac{i}{\pi} \frac{1}{\tau - t} \exp\{\pm i\omega t\} dt = \frac{i}{\pi} \int_{-\infty}^{0} \frac{1}{\tau - t} \exp\{\pm i\omega t\} dt + \frac{i}{\pi} \int_{0}^{\infty} \frac{1}{\tau - t} \exp\{\pm i\omega t\} dt$$

$$+ \lim_{R \to \infty} \frac{i}{\pi} \int_{C_R} \frac{1}{\tau - t} \exp\{\pm i\omega t\} dt$$

$$+ \text{Res}\left\{ \frac{1}{\tau - t} \exp\{\pm i\omega t\} \right\}, \quad \pm \Im(\omega) > 0.$$  \hspace{1cm} (E-4)

Here, $\int$ denotes the Cauchy principal value that we seek and “Res” denotes the residue of the integrand at the pole at $t = \tau$. We remark that, in fact, this integral over the contour $C$ for any choice of $R$ and $r$ is zero by Cauchy's theorem, because the integrand is analytic inside the contour.

We first have to deal with the integral on the path $C_R$. Note that $\exp\{\pm i\omega t\}$ has a negative real part for $\pm \Im(\omega) > 0$. Thus, by standard analysis, the integral over the contour $C_R$ approaches zero as $R \to \infty$.

Next, we consider

$$\frac{i}{\pi} \int_{-\infty}^{0} \frac{1}{\tau - t} \exp\{\pm i\omega t\} dt = \mp \frac{1}{\pi\omega} + O\left(\frac{1}{\omega^2}\right)$$  \hspace{1cm} (E-5)

by using integration by parts. Even to leading order, this term is smaller than order one in $\omega$ and hence does not contribute to the leading order asymptotic expansion of this Fourier transform.

Finally, we observe that

$$\text{Res}\left\{ \frac{1}{\tau - t} \exp\{\pm i\omega t\} \right\} = - \exp\{i\omega \tau\}.$$  \hspace{1cm} (E-6)

We combine all of our observations here below equation (E-4) and solve for the principal value integral there to conclude that

$$\frac{i}{\pi} \int_{0}^{\infty} \frac{1}{\tau - t} \exp\{\pm i\omega t\} dt = \exp\{i\omega \tau\}.$$  \hspace{1cm} (E-7)

That is, the Cauchy principal value integrals have the same value as the Fourier transforms of the delta functions, but only to leading order asymptotically.

We now combine the transforms in equation (E-6) with the transforms of the delta functions in equation (E-3) and substitute into equation (E-2) for $g_{\pm}$ to conclude that

$$g_{\pm}(x, x_0, \omega) = \pm 2B(x, x_0) \exp\{\pm i\omega \tau(x, x_0)\}.$$  \hspace{1cm} (E-8)

We remark here that if we consider the effect of caustics, then we would introduce a phase shift,

$$g_{\pm}(x, x_0, \omega) = \pm 2B(x, x_0) \exp\{\pm i\omega \tau(x, x_0) + i \pm K\pi/2\}.$$  \hspace{1cm} (E-9)

Here, $K$ is the KMAH index, counting the number of caustics passed on the ray from $x_0$ to $x$. 

---

Figure E-1. The oriented contour of integration in the complex $t$-plane for determination of the Cauchy principal value integrals. $R$ and $r$ are the radii of the semi-circles.
Source distribution in interferometry for wave and diffusion

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ABSTRACT
For the wave equation, the Green’s function that describes the wave propagating between two receivers can be reconstructed by cross-correlation if the receivers are enclosed by sources on a closed surface. This technique is normally called interferometry. The ordinary operator used in this technique is cross-correlation. The same technique for Green’s function extraction can be applied to the solution of the diffusion equation if there are sources throughout in the volume. In practice, we only have a finite number of active sources. We address the question what minimum source density is needed for the accurate extraction of the Green’s function, and how these sources should be located on the source in the wave problem and if it is possible to reconstruct the Green’s function of the diffusion equation by using a limited number of sources within a finite volume. We study these questions for homogeneous and isotropic media for both wave propagation and diffusion using numerical simulations. These simulations show that for the used model, the angular distribution of sources is critical in wave problems. For diffusion, the sensitivity of the sources decays away from the center of the two receivers. The required width of the source distribution decreases with frequency, and therefore the required source distribution for early time and late time reconstruction is different.

Key words: interferometry, waves, diffusion, virtual source

1 INTRODUCTION
The term interferometry generally refers to the study of the interference of two signals to obtain a measure of the difference between them (Curtis et al., 2006). It also refers to the technique used to extract the response which describes the wave propagating between two receivers as if one of the receivers were an active source (Lobkis & Weaver, 2001; Derode et al., 2003; Weaver & Lobkis, 2004; Wapenaar, 2004; Snieder, 2004; Snieder, 2007; Wapenaar et al., 2005). This technique has also been applied in many fields: in ultrasound (Weaver & Lobkis, 2001; Malcolm et al., 2004; van Wijk, 2006; Larose et al., 2006), crustal seismology (Campillo & Paul, 2003; Shapiro et al., 2005; Roux et al., 2005; Sabra et al., 2005a; Sabra et al., 2005b), exploration seismology (Bakulin & Calvert, 2004; Calvert et al., 2004; Bakulin & Calvert, 2006), helioseismology (Rickett & Claerbout, 1999), structural engineering (Snieder & Safak, 2006; Snieder et al., 2006), and numerical modeling (van Manen et al., 2005).

In seismic imaging, the studies of interferometry and its applications have grown rapidly in recent years. The seismic interferometry technique was first applied to wave propagation in non-attenuating and time-reversal invariance media (Lobkis & Weaver, 2001; Derode et al., 2003; Weaver & Lobkis, 2004; Wapenaar, 2004; Snieder, 2004). Later, it was proved that Interferometry can not only be applied to wave fields, but also to diffusive fields (Snieder, 2006b). Recent proofs have been given that the Green’s function can be extracted for a wide class of linear systems that may be attenuating, and that may not be invariant for time reversal because of flow (Godin, 2006; Wapenaar, 2006b; Wapenaar et al., 2006; Snieder et al., 2007; Weaver, 2008).

Seismic interferometry is in the exploration community referred to as the virtual source method (Bakulin & Calvert, 2004; Calvert et al., 2004; Bakulin & Calvert,
to the surface dS, sources are located, r

functions that describe waves received by receiver A and becomes S

c density and functions that describe wave propagation from receiver at position r

Although the extraction of the Green’s function is usually based on cross-correlation, deconvolution can also be used (Snieder & Safak, 2006; Vasconcelos & Snieder, 2006). The term interferometry in this paper refers to the cross-correlation based interferometry.

Interferometry applied to fields governed by the wave equation can be expressed, in the frequency domain, as (Snieder et al., 2007):

\[ G(r_A, r_B, \omega) - G^*(r_A, r_B, \omega) = \]

\[ \frac{1}{\rho c^2} \int_S \left( G^*(r_A, r, \omega) \nabla G(r_B, r, \omega) \right) dS, \]  

\[ - \left( \nabla G^*(r_A, r, \omega) \right) G(r_B, r) \hat{n} dS, \]  

where \( G(r_A, r_B, \omega) \) is the displacement Green’s function that describes wave propagation from receiver at \( r_B \) to the receiver at \( r_A \) respectively, * indicates complex conjugation, \( G(r_A, r, \omega) \) and \( G(r_B, r, \omega) \) are the Green’s functions that describe waves received by receiver A and B from a source at position \( r \), \( S \) is the surface where sources are located, \( \hat{n} \) is the unit vector perpendicular to the surface \( dS \), \( \omega \) is the angular frequency, \( \rho \) is the density and \( c \) is the medium velocity. When the waves satisfy a radiation boundary condition on the surface \( S \), \( \nabla G(r_A, r, \omega) \approx i(\omega/c)G(r_A, r, \omega) \hat{r} \), and equation (1) becomes

\[ G(r_A, r_B, \omega) - G^*(r_A, r_B, \omega) \approx \]

\[ \frac{2i\omega}{\rho c^2} \int_S G(r_A, r, \omega)G^*(r_B, r, \omega)\hat{r} \cdot \hat{n} dS, \]  

\[ 2i\omega \int_V G(r_A, r, \omega)G^*(r_B, r, \omega) dV, \]  

in which \( V \) is the volume containing the sources. The meaning of other terms are the same as those in equation (3).

Equations (3) and (4) show that the main difference between wave and diffusion interferometry is the required source distribution. For waves, equation (3) shows that if two receivers are surrounded by active sources on a closed surface, the response which describes the wave propagating between the two receivers can be reconstructed as if one of the receivers were an active source. For diffusion, equation (4) states that the sources are required to be everywhere in the volume (Snieder, 2006b). In practice, there are only a finite number of sources. Therefore, we can never have a closed source surface for waves or sources throughout the volume for diffusion. This raises the question: what is the required source density and how should we locate these sources in order to reconstruct the Green’s function accurately?

The importance of cross-correlation-based interferometry for waves has been addressed by numerous authors. On the other hand, cross-correlation-based interferometry for diffusion is still at the theory stage and there are no field applications yet. In exploration geophysics, there are at least two important diffusive fields: pore pressure and low-frequency inductive electromagnetic fields. From the pore pressure we can infer the fluid conductivity between wells (Voyiadjis & Song, 2003; Nakken & Hooton, 2007). Electromagnetic fields carry information about the resistivity of the pore fluid and may thus help distinguish between hydrocarbons and water. For the offshore oil exploration, controlled-
source electromagnetic (CSEM) is one of the most important techniques used to detect hydrocarbon (Hoversten et al., 2006; Constable & Srnka, 2007; Darnet et al., 2007; Scholl & Edwards, 2007).

In this paper, we study the required source distribution in a simple homogeneous model for both waves and diffusion with a finite number of sources.

MODEL AND RESULTS

Waves

For simplicity, we first use a 2-D homogeneous model, with constant velocity 1 km/s, in the numerical tests. For defining the source position, we use two parameters: source angle and source radius as shown in Figure 2. A and B are two receivers with a separation \( d \). The position vectors of the two receivers are denoted by \( \mathbf{r}_{SA} \) and \( \mathbf{r}_{SB} \), respectively. The source function for waves we use in all the examples in this paper is a Ricker wavelet with a central frequency of 0.5 Hz. The source amplitude is the same for all sources.

**Experiment 1: uniformly distributed source angle.**

We study the effect of the source angle distribution. Sources are uniformly distributed on the circle with a radius of 40 km. The distance between the two receivers is 6 km. Figure 3 shows the reconstructed response between the two receivers for a homogeneous distribution of sources with increasing source number. The response has two parts, the causal and anti-causal parts as represented by equation (3). The causal part of the signal represents the signal propagating from receiver A to B and the anti-causal part is the time-reverse of this, i.e. the signal propagating from receiver B to A. If we replace one of the receivers with an active source, the received signal arrives after a propagation time of 6 s. To make the shape of the received signal the same as the reconstructed signal, we correlate the received signal with the source-time function. This new signal is represented by the dashed line in the bottom panel, it is virtually the same as the causal part of the reconstructed response with 50 sources (the amplitudes of both reconstructed and active signals are normalized). The main point in figure 3 is that the fluctuation energy in the middle part of the reconstructed signal decreases with increasing source number \( N \), and a minimum source density needs to be exceeded in order to extract the response successfully. This required source density is derived in the discussion part of this section.

We quantify the spurious fluctuations that arrive between the anti-causal and the causal response by defining the fluctuation energy

\[
E_m = \frac{1}{N_m} \sum_{i=1}^{N_m} A[i]^2,
\]

in which \( N_m \) is the number of discrete sample points in the middle part of the signal, i.e. the part between the two main pulses. Figure 4 shows this fluctuation energy decay as a function of source number \( N \). Weaver and Lobkis (Weaver & Lobkis, 2005) showed that these fluctuations decay as \( N^{-1} \) if the sources are randomly distributed. Figure 4 shows that when the sources are uniformly distributed in angle, the decay rate is much faster than \( N^{-1} \). The reason of this is shown in the discussion part of the wave problem.
Figure 4. Fluctuation energy decay as a function of source number $N$ for the uniform angle distribution. The dashed and solid line represent two different power-laws in the log-log coordinate system.

Figure 5. Reconstructed responses (solid lines) for random angle distribution with different number of sources $N$ (the dashed line in the bottom panel is the exact response between the two receivers).

Experiment 2: randomly distributed source angle.

In this experiment, the source angles are randomly distributed while the source radius is constant. Figure 5 shows the reconstructed response as a function of source number $N$ for a random distribution of sources along the circle. Compared with figure 3, the random distribution gives a much poorer reconstruction than does the uniform distribution with the same source number.

Figure 6 shows this fluctuation energy decay, as defined in equation (5), as a function of $N$ for randomly distributed sources. The fluctuation decay behavior is consistent with the prediction of Weaver and Lobkis (2005): the decay is proportional to $N^{-1}$. Experiment 2 suggests that not only the number of the source is important, but also their angular distribution. The difference of this decay rate of uniformly and randomly distributed sources is explained in the discussion part of this section.

Experiment 3: smooth source angle distribution.

From experiments 1 and 2, we might conclude that the source angle needs to be uniformly distributed to apply this technique successfully with a small number of sources. This experiment shows that the angle is not necessary to be uniformly distributed but is smoothly varying. Next we show examples with non-uniform but smoothly varying angle distribution where the response is accurately reconstructed.

In the example shown in figure 7, sources are uniformly distributed on the circle with the center of the two receivers moved away from the center of the circle (from $(0,0)$ to $(-5,6)$). This makes the source angle distribution no longer uniform, but it’s still smooth. The numerical simulation shows accurate reconstruction of the response from 50 sources. The amplitude difference of the causal and anti-causal parts are due to the different energy from the two stationary-phase zones on the left and right side of the receivers as illustrated by the two dashed curves in the upper panel of figure 7. Only the sources within these two stationary zones contribute to the extraction of the direct wave (Snieder, 2004; Roux et al., 2005). In this case, the stationary zone on the right side corresponds to the causal pulse and the left stationary zone corresponds to the anti-causal pulse. We notice that in the right stationary zone there are more sources than on the left side. This explains why the causal pulse is stronger than the anti-causal one. Notice that the distance of the sources to the midpoint of the receiver locations is not constant in this experi-
Interferometry for waves and diffusion

Figure 7. Reconstructed response (solid line in the lower panel, dashed line is the exact response) for a smoothly varying source angle distribution (upper panel)

Figure 8. Reconstructed response (solid line in the lower panel, dashed line is the exact response) for source spaced equidistantly on a triangle (upper panel)

Figure 9. Two source distributions with the same angle distribution but different radii (top) and the reconstructed responses: red solid (same radius), black dashed line (different radius)

Note that in this example the source radii are much larger than the distance between the two receivers. In this case, the influence of varying source radii is negligible. This is shown in detail in the next experiment.

**Experiment 4: source radius importance.**

In the previous three experiments we learned how the angle distribution influences the response extraction. The best angle distribution is the uniform distribution and the decay rate of the non-physical fluctuations in the middle part is faster than $N^{-10}$. The slowest fluctuation decay is from the random angle distribution: and varies with the number of sources as $N^{-1}$. The number of source required for a smoothly varying angle distribution is close to the uniform distribution and the value of the number depends how smooth the angular distribution is. There is, however, still another parameter: the radius $r$ as defined in figure 2. In the next example we compare the result from two distributions with the same angle distribution but different source radii. The first one is the example we showed in experiment 1, when 50 sources are uniformly distributed on a circle (stars in the upper panel of figure 9). The second one is for sources with the same angle distribution but the radius is randomly varying in a range in which all radii are much larger than the distance between the two receivers (dots in the upper panel of figure 9). The reconstructed responses in figure 9 suggest that varying the source radii does not degrade the accuracy of the Green’s function extraction. This is only true when source radii are much larger than the distance between the two receivers. The reason for this is explained in the discussion part.

The geometrical spreading also affects the amplitude of the reconstructed signal. One might think that the geometrical spreading also affects the amplitude of the reconstructed signal.

In the example shown in figure 8, sources are uniformly distributed on the sides of a triangle. The source angle is not uniform but smoothly varying. The lower panel in figure 8 shows that for the employed number of sources, the reconstruction of the response is still accurate. In this case, the required source number is a little bit larger than that for the uniform distribution but much smaller than for the accurate reconstruction of the Green’s function with the random source distribution. The value of this number depends how smooth the angle is varying. This geometry is closer to the practical cases in which controlled sources are distributed on a line (Bakulin & Calvert, 2006; Mehta et al., 2007a).
Figure 10. 1-D source distribution and the definitions of geometric parameters.

Figure 11. 1-D source distribution (upper panel) with $W_s = 14$ km, $\rho_s = 1.143$ $km^{-1}$ and the extracted Green’s function (lower panel).

Figure 12. 1-D source distribution (upper panel) with $W_s = 34$ km, $\rho_s = 0.47$ $km^{-1}$ and the extracted Green’s function (lower panel).

Figure 13. 1-D source distribution (upper panel) with $W_s = 34$ km, $\rho_s = 1.147$ $km^{-1}$ and the extracted Green’s function (lower panel).

Diffusion

Equation (4) shows that sources in the whole volume are needed to extract the Green’s function for diffusion. To simplify the problem, we start with a 1-D medium with constant diffusion coefficient. Then we extend this to 3-D.

Experiment 1: diffusion Green’s function recovery in 1-D.

We choose the origin of the coordinate system at the center of the two receivers. The distance between the two receivers is 2 km. The diffusion coefficient used in this model is $D = 1$ $km^2/s$. We distribute sources uniformly on the 1-D line with the center of the distribution at origin. Figure 10 shows the geometry of 1-D source distribution.

We define two parameters to characterize this source distribution. As shown in figure 10, $W_s$ is the width of the distribution and $\rho_s = N/W_s$ is the source density. Next we test three different distributions. The first one is a distribution with narrow width $W_s$ and high source density $\rho_s$ (figure 11). The second one is a distribution with the same number of sources, but with a wide width $W_s$ and low density (figure 12). The third distribution has more sources and has wide $W_s$ and high density $\rho_s$ (figure 13).

Figures 11 to 13, show that for the reconstruction of the Green’s function, different source distributions are needed for the accurate reconstruction of the early-time and the late-time response. The early time response is defined as the response before the peak in the Green’s function of the diffusion equation, the late time part is defined as the response after the main peak. The early-time reconstruction is controlled by the source density $\rho_s$ (figure 11 and 13) and late-time reconstruction is more affected by the distribution width $W_s$ (figure 12 and 13).
Interferometry for waves and diffusion

Experiment 2: Green’s function recover for diffusion in 3-D.

Following the same strategy we extend the diffusion experiment to 3-D. Instead of putting the sources on a line, we uniformly distributed them in a cube as shown in figure 14. We define \( W_s \) as the side length of the cube, and the source density is defined as \( \rho_s = \frac{N}{(W_s)^3} \).

In figure 15a, a source distribution with small \( W_s \) is used. As in 1-D, the early-time of the Green’s function is reconstructed well, but the late-time behavior is not. When the width of the distribution increased, with sufficiently high source density \( \rho_s \), both early and late time can be extracted well (figure 15b). In the 3-D diffusion problem, higher source density is not always helpful for an accurate reconstruction. When the sources are getting too close to the receiver position, the drawback of the spatial singularity becomes severe. Consequently, the reconstruction is less accurate.

DISCUSSION

Waves in homogeneous media

The Green’s function of the wave equation in 2D is represented in the frequency domain by the first Hankel function of degree zero (Snieder, 2006a):

\[
G(r) = \frac{i}{4} H_0^{(1)}(kr).
\]  

In all the numerical simulations in this paper, we use the far field approximation of equation (6), which is

\[
G(r) = \sqrt{\frac{1}{8\pi kr}} e^{i(kr + \pi/4)}. \tag{7}
\]

Inserting this into equation (3), we obtain

\[
G(r_A, r_B, \omega) - G^*(r_A, r_B, \omega) \approx \frac{i}{4\pi \rho} \int_{S'} \frac{1}{r_{SA} r_{SB}} e^{ik(r_{SA} - r_{SB})} dS'. \tag{8}
\]

When the source radius is much larger than the distance between the two receivers, the distance in the geometrical spreading can be approximated as \( r_{SA} \approx r_{SB} \approx r \), while for the phase the approximation \( r_{SA} - r_{SB} \approx dcos\theta \) is accurate to first order in \( d/r \). (These parameters are defined in figure 2.) Using these approximations and the relationship \( dS' = rd\theta \), equation (8) becomes

\[
G(r_A, r_B, \omega) - G^*(r_A, r_B, \omega) \approx \frac{i}{4\pi \rho} \int_0^{2\pi} e^{ikdcos\theta} d\theta. \tag{9}
\]

Note that the right hand side does not depend on the source radius \( r \). Experiment 4 in the wave part supports this conclusion: in that experiment, variations in the source radius do not influence the Green’s function extraction.

The source radius enters this interferometry problem in three ways. The first one is the geometrical spreading term \( 1/r \), the second one is the relation between the surface element and the increment in source angle \( dS' = rd\theta \), and the third one is the width of the stationary-phase zones as illustrated in the upper panel of figure 7. Equation (9) confirms that the first two factors compensate each other. Consequently, only the width of the stationary-phase zones contribute to the amplitude of the reconstructed signal. The different source number in the left and right stationary-phase
zones cause the asymmetry in the amplitude of causal and anti-causal response as shown in the lower panel of figure 7.

Another interesting observation is that the right hand side of equation (9) is the integral representation of the Bessel function (Snieder, 2006a), which is related to the exact Green’s function:

$$\frac{1}{2\pi} \int_0^{2\pi} e^{ikd\cos \theta} d\theta = J_0(kd) = \frac{1}{2} (H_0^{(1)}(kd) - H_0^{(1)}(-kd))$$

(10)

This shows that by using only far field of the waves in the interferometry, both far field and near field response are reconstructed. This was shown for elastic waves by (Sánchez-Sesma et al., 2006; Sánchez-Sesma & Campillo, 2006).

For the dependence on the angle $\theta$, we need to study the character of the integrand in equation (9). The real part of this integrand is the oscillatory function shown in figure 16. The extraction of the Green’s function depends on the sampling of this integral over source angle $\theta$, and reduces to the numerical integration of a continuous oscillatory function. In the experiment we sum over all the sources to represent this integral. This actually assumes that the sampling in $\theta$ is uniform. Therefore if the sources are uniformly sampled in angle, this integral can be represented well by summation. Provided the sampling is sufficiently dense, with the random distribution of source angles, the angle separations are different everywhere. This would not give an accurate estimation of the integral just using summation. If the angle separation for each source is known, we can use this $d\theta$ as a weight in the summation as it does in the numerical integral. But if there is no information on $d\theta$, the average over repeated experiments would converge to the accurate reconstruction. Figure 17 shows a histogram of 100 repeated estimations of the integral in figure 16. Each time, 1000 randomly distributed sources are used to estimate this integral. The estimation value at a specific realization can be far from the exact value while the average over all realizations (dashed line) is close to the accurate value (solid line). For the smoothly varying source angle, $d\theta$ is still fairly constant locally. Therefore oscillations cancel and the result is still accurate. For most of the applications of interferometry using control-shots, the source angle is actually smoothly changing (Bakulin & Calvert, 2006; Mehta et al., 2007a). Here we explain why those smooth source angle distributions from experiment 3 for waves give accurate Green’s function reconstruction.

Waves in heterogeneous media

We next investigate whether these observations for source distribution from a homogeneous medium also hold in heterogeneous media. To answer this question, we need to consider which parts of the problem are changing from a homogeneous medium to a heterogeneous one. Equation (3) is valid for any medium, regardless of its complexity. Therefore, we can still study the integrand in equation (3) as we show for the homogeneous case. For a homogeneous medium, this integrand becomes a simple form as shown in equation (9). For a heterogeneous medium the integrand can be in a very complicated form for the heterogeneous media. But the point is that we are still estimating the integral of this complicated function using summation. Consequently, the uniform angle distribution should still give the most accurate estimation. In order to support this statement, we show a comparison of two source distributions in a heterogeneous medium with 200 isotropic point scatters around the two receivers in figure 18. The wavefield was modeled using the theory of (Groenenboom & Snieder, 1995) that takes all multiple scattering events into ac-
Green’s function between the scatter points $A$ in which $\psi$ as summation of the primary field and the scattered field are used. (1) and (2) show the same comparison in different cases. 

$$\psi(r) = \psi_0(r) + \sum_{j=1, j \neq i}^n G^{(0)}(r, r_j) A_j \psi(r_j), \quad (11)$$

in which $\psi_0$ denotes the primary field, $G^{(0)}(r, r_j)$ is the Green’s function between the scatter points $r_j$ and $r_i$, $A_j$ is the scattering amplitude of scatterer $j$, and $\psi(r_j)$ is the field incident on that scatterer. Then the system of linear equations are solved to calculate the total wave field. In figure 18, the red curve is the signal received by one receiver when the other one becomes a real source. The black curve is the signal reconstructed by interferometry. The letter $a$ denotes the case when 300 randomly distributed sources are used while $b$ shows the case when 300 uniformly distributed sources are used. (1) and (2) show the same comparison in different scales.

Note in some cases, the source distribution becomes less important. Malcolm (2004) showed that the ensemble averaged Green’s function in a rock can be retrieved from a single source by moving the receiver pair around the source with uniform angle step. For a layered model with enough horizontal layers to give strong scattering, the full Green’s function can be reconstructed by one-sided illumination with sources uniformly distributed on the free surface (Wapenaar, 2006a).

What is the minimum required source density if the sources are uniformly distributed in a homogeneous medium? As shown in figure 16, the oscillations have a variable period. In order to make the highest frequency oscillations cancel we need to have enough sampling points for the highest frequency. This oscillation depends on the phase term $\Phi = k d \cos \theta$ of equation (9). The change in the phase for an angular increment $\Delta \theta$ is $\Delta \Phi = k d \sin \theta \Delta \theta$. The most rapid oscillation happens at $\sin \theta = 1$. In order to have $N_r$ number of sources within the period of the most rapid oscillation, the required source density becomes

$$\rho_{\text{source}} = \frac{N_r k d}{2 \pi} \text{(radian}^{-1}), \quad (12)$$

From experience, when $N_r > 2.5$, the fluctuation energy between the two main pulses in the reconstruction vanishes, this gives the sampling criterion

$$\rho_{\text{source}} = 0.4 k d \text{ (radian}^{-1}), \quad (13)$$

In conclusion, for wave interferometry in a homogeneous model, the most important parameter is the source angle distribution. If we know the source distribution, and hence the source angle, different weight function can be used to estimate the oscillation function better. If randomly distributed sources are used and there is no information on the source angle distribution, the average over large amount of extracted signals is more accurate to describe the real response. Otherwise, we need to choose the source angle smoothly varying to apply this technique accurately.

This conclusion holds when all sources have the same amplitude. If the amplitude of the sources fluctuates randomly, a uniform angle distribution gives similar reconstruction of the Green’s function as the random angle distribution for a constant source strength. For a heterogeneous medium, the importance of the source distribution depends on how strong the heterogeneities are.

**Diffusion**

The frequency domain Green’s function of the diffusion equation in a 1-D homogeneous medium is given by

$$G^{1D}(x, \omega) = \frac{1}{(1+i)(2\pi x)} e^{-i(1-i)x \sqrt{2/\omega}} \quad (14)$$

Inserting this expression into equation (4), gives

$$G(r_A, r_B, \omega) - G^*(r_A, r_B, \omega) = \frac{i}{2D} \int_x e^{-i(r_A+r_B) \sqrt{2/\omega}} e^{-i(r_A-r_B) \sqrt{2/\omega}} \sqrt{2/\omega} \ dx \quad (15)$$

Similar as for the analysis of the wave, we study the real part of the integrand of equation (15) as a function of space variable $x$. Notice that the integrand is also a function of frequency $\omega$. Therefore, for different frequencies the real part of the integrand behaves differently. Figure 19 shows this integrand for two different frequencies. The width of the distribution decreases with fre-
Inserting this into equation (4) we obtain

\[ x_{\text{behavior}} \text{ in the } \text{behavior } \text{ is reconstructed well. With increasing } W_s \text{, the early time behavior is reconstructed well. With increasing } W_s \text{, more lower frequency components are recovered. Since the tail of the Green’s function mostly contains low frequencies, Green’s function of late time is recovered accurately with a large width } W_s. \text{ Consequently, the source density } \rho_s \text{ controls the retrieval of the high frequency components of the Green’s function (e.g. early time of the Green’s function), and the width of the distribution } W_s \text{ controls lower frequency components (e.g. the late time of the Green’s function).} \]

The frequency domain Green’s function for diffusion in 3-D is

\[ G^{3D}(r, \omega) = \frac{1}{4 \pi D r} e^{(-1-i)\sqrt{\omega/2D}} \]  

(16)

Inserting this into equation (4) we obtain

\[ G(r_A, r_B, \omega) - G^*(r_A, r_B, \omega) = \frac{1}{2i\omega \sqrt{d}} \int_V e^{-(r_{SA}+r_{SB})/\sqrt{\omega/2D}} e^{-i(r_{SA}-r_{SB})/\sqrt{\omega/2D}} dV \]  

(17)

The real part of the integrand is not easily displayed as a function of three volume variables. Since it is invariant for rotation along the axis joining the receivers (the x-axis in the used coordinate system), we display its behavior in the x-y plane. Figure 20 shows the real part of the integrand of equation (17) at a frequency of 0.15 Hz. There are two isolated peaks at the location of the two receivers. These peaks at the locations of the two receivers correspond to the two singularities in equation (17). It seems that the largest contribution to the integral comes from the sources at the positions of the receivers. These singularities are, however, integrable. In the numerical integration, if there are no sampling points at the singularities, the integration is still accurate as in the example of figure 15. In contrast, if there are sources very close the singularities, the integration is inaccurate because of the contributions from the singularities. But if there is a source very close to one of the receivers, there in no need for the interferometric extraction of the Green’s function because this function can then be directly recorded by the other receiver.

This explains that why we can reconstruct the 3-D Green’s function for diffusion in the experiment 2 of the diffusion part even though there are two singularities in equation (17). Because of these singularities, we should not have any sampling point very close to or at the singularities when we do numerical simulations or reality. This suggests that high source density in 3-D diffusion problem does not always give accurate Green’s function extraction since some sources might be too close to the singularities.

We next address the question how to quantify the required source distribution width \( W_s \) and source density \( \rho_s \). As we learned from the examples in the diffusion part, \( W_s \) determines the late time reconstruction of the Green’s function. Suppose that the diffusion Green’s function needs to be reconstructed accurately up to time \( \tau_a \) (as shown in figure 21a), sources contribute up to a source-receiver distance: \( r_{\text{max}}^2/(4Dt) = 1 \). This is based on the decay term \( e^{-r^2/(4Dt)} \) in the time domain Green’s function of diffusion equation. Therefore, the required \( W_s \) should be

\[ W_s = 4\sqrt{D\tau_a} + d \]  

(18)

for an accurate reconstruction up to time \( \tau_a \), in which \( d \) is the distance between the two receivers. Figure 21b shows the error of the reconstruction at \( \tau_a \) and \( 2\tau_a \) by using different \( W_s \). The error is defined as the ratio of
the difference between the exact and extracted signals to the exact signal. The time \( \tau \) for a certain \( Ws \) is determined by equation (18). The model parameters used in this test are the same as those in the experiment 1 of the diffusion part. For this sampling criterion, the error in the Green’s function up to \( \tau \) is less than 5%.

The other parameter \( \rho \) controls the early time reconstruction. In other words, it determines the accuracy of the reconstruction for the high frequency components. For the maximum frequency \( f_{\text{max}} \) in the problem – either the highest frequency component of the Green’s function itself or the maximum frequency of the source function – there is a sensitivity function of source position as shown in figure 19 for the 1-D problem or figure 20 for the 3-D problem. This sensitivity is controlled by the decay factor is \( e^{-r_{\text{SA}}+r_{\text{SB}}}/\sqrt{2D} \) as shown in equation (15) and (17). The 1/e width of this sensitivity function is:

\[
\sigma = \sqrt{2D/\omega} \tag{19}
\]

Then if \( N_r \) is the number of source needed in this range \( \sigma \) to estimate the integral accurately, the required source density is:

\[
\rho_{sa} = N_r \sqrt{\omega/2D} \tag{20}
\]

Based on the numerical examples, when \( N_r \) is larger than 2, it is adequate to reconstruct the early time response. Then equation (20) becomes:

\[
\rho_{sa} = 2\sqrt{\omega/2D} \tag{21}
\]

We can estimate the maximum frequency component in the Green’s function as \( 1/(4t_p) \), in which \( t_p \) is the arrival time of the amplitude peak in the Green’s function. Figure 22 shows the error in the reconstructed Green’s function at time \( t_p/2 \) using different source densities. Green’s functions with different \( t_p \) are tested. The adequate source density \( \rho_{sa} \) is calculated using equation (21).

In conclusion, for cross-correlation-based diffusion interferometry, instead of having sources everywhere in the volume, it suffices to have sources in only a small volume surrounding the receivers as shown in figures 19 and 20. For the 1-D problem, the source distribution width controls the late-time (low-frequency components) reconstruction of the Green’s function and source density controls the early-time (high-frequency components) reconstruction. For the 3-D problem, sources should not be located too close to the receivers position because of the singularities. As we focus on the possibility of diffusion Green’s function reconstruction and the uniform source distribution is the best case to understand this problem, we do not include non-uniform source distribution in this paper.

Let us consider next what happens if we apply deconvolution rather than cross-correlation. For the 3-D Green’s function of the diffusion in equation (16), the deconvolution of \( G_A \) and \( G_B \) is given by

\[
\frac{G_A}{G_B} = \frac{r_{SB}}{r_{SA}} e^{(-1-\alpha)(r_{SA}-r_{SB})/\sqrt{2D}} \tag{22}
\]

So if we have a single source on the \( x \) axis and outside of the two receivers, equation (22) becomes

\[
\frac{G_A}{G_B} = \frac{r_{SB}}{r_{SA}} e^{(-1-\alpha)\sqrt{2D}} = \frac{r_{SA}}{4\pi D \sigma_{SB}} \frac{G_{AB}}{r_{SA}} \tag{23}
\]

From this equation we can see that, apart from a scaling factor, deconvolution gives us an accurate reconstruction for a single source. This shows that cross correlation and deconvolution behave different for diffusion.
problem. Deconvolution-based interferometry might be a better approach to diffusion problem, but this may be a peculiarity of the homogeneous medium. For waves, researchers have started to use deconvolution-based interferometry (Trampert et al., 1993; Snieder & Safak, 2006; Vasconcelos & Snieder, 2006; Mehta et al., 2007b).

**Conclusion**

The cross-correlation-based interferometry used to extract the Green’s function which describes the field propagation between two receivers can be applied to the solution of both the wave equation and the diffusion equation. The main difference is the required source distribution.

For interferometry for the wave equation in a homogeneous medium, the source angle distribution is the most important parameter. With the assumption that the source radii are much larger than the distance between the two receivers, the variation in the source radius has a negligible effect and the interferometry problem can be represented by a numerical integral of an oscillatory function of source angle. If cross-correlations from different sources are simply added together in the calculation, the uniform source angle distribution gives the fastest decay rate of the non-physical fluctuation as a function of source number (faster than $N^{-10}$). With the same source number, the random distribution gives much poorer Green’s function reconstruction. The rate of the non-physical fluctuation decay is approximately $N^{-1}$. The distribution with the source angle smoothly varying behaves closer to the uniform distribution than random distribution.

For interferometry for the diffusion equation in a homogeneous medium, a finite number of source suffices to reconstruct the Green’s function. For a 1-D model, the sensitivity of the sources decays away from the center of the two receivers. The width of the distribution controls the late-time of the reconstructed Green’s function while the source density controls the early-time of the reconstructed Green’s function. For a 3-D model, the main properties are the same as the 1-D problem. What special in 3-D problem is that the two receiver positions are the two singularities in the calculation. Consequently, the source should not be too close to the receiver positions. Otherwise, the contribution from that source is over weighted and the reconstruction becomes not accurate. Different from the cross-correlation-based interferometry, the deconvolution-based interferometry shows special properties for the diffusion equation. For 3-D diffusion problem in a homogeneous medium, one source is enough to reconstruct accurate Green’s function if deconvolution is used in the calculation.

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Seismic and electromagnetic controlled-source interferometry in dissipative media

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ABSTRACT
Seismic interferometry deals with the generation of new seismic responses by crosscorrelating existing ones. One of the main assumptions underlying most interferometry methods is that the medium is lossless. We develop an ‘interferometry-by-deconvolution’ approach which circumvents this assumption. The proposed method applies not only to seismic waves, but to any type of diffusion and/or wave field in a dissipative medium. This opens the way to applying interferometry to controlled-source electromagnetic (CSEM) data. Interferometry-by-deconvolution replaces the overburden by a homogeneous half space, thereby solving the shallow sea problem for CSEM applications. We demonstrate this at the hand of numerically modeled CSEM data.

Key words: interferometry, CSEM, overburden removal

INTRODUCTION
Seismic interferometry is the branch of science that deals with the creation of new seismic responses by crosscorrelating seismic observations at different receiver locations. Since its introduction around the turn of the century, the literature on seismic interferometry has grown spectacularly. Interferometric methods have been developed for random fields (Larose et al., 2006; Gerstoft et al., 2006; Draganov et al., 2007) as well as for controlled-source data (Schuster & Zhou, 2006; Bakulin & Calvert, 2006). The underlying theories range from diffusion theory for enclosures (Weaver & Lobkis, 2001), stationary phase theory (Schuster et al., 2005; Snieder, 2004) to reciprocity theory (Wapenaar et al., 2004; Weaver & Lobkis, 2004; van Manen et al., 2005). All these theories have in common the underlying assumption that the medium is lossless and non-moving. The main reason for this assumption is that the wave equation in lossless non-moving media is invariant for time-reversal, which facilitates the derivation.

Until 2005 it was commonly thought that time-reversal invariance was a necessary condition for interferometry, but recent research shows that this assumption can be relaxed. Slob et al. (2006) analyzed the interferometric method for ground penetrating radar data (GPR), in which losses play a prominent role. They showed that losses lead to amplitude errors as well as to the occurrence of spurious events. By choosing the recording locations in a specific way, the spurious events arrive before the first desired arrival and can thus be identified. Snieder (2006, 2007) followed a different approach. He showed that a volume distribution of uncorrelated noise sources, with source strengths proportional to the dissipation parameters of the medium, precisely compensates for the energy losses. As a consequence, the responses obtained by interferometry for this situation are free of spurious events and their amplitudes decay the way they should in a dissipative medium. This approach does not only hold for waves in dissipative media, but also for pure diffusion processes.

Time-reversal invariance as well as source-receiver reciprocity break down in flowing or rotating media, but with some minor modifications interferometry also appears to work for these situations (Wapenaar, 2006; Godin, 2006; Ruigrok et al., 2007). Recently we showed that interferometry, including its extensions for waves and diffusion in dissipative and/or moving media, can be represented in a unified form (Wapenaar et al., 2006; Snieder et al., 2007). In turn, from this unified formulation it follows that the interferometric method can also
be used for more exotic applications like electroseismic prospecting and quantum mechanics.

Interferometry in the strict sense makes use of crosscorrelations, but in the following we will extend the definition of interferometry so that it also includes crossconvolution and deconvolution methods. Slob et al. (2007) introduce interferometry by crossconvolution and show that it is valid for arbitrary dissipative media. The crossconvolution method does not require a volume distribution of sources, but a restriction is that it only works for transient signals in specific configurations with receivers at opposite sides of the source array. The latter restriction does not apply to ‘interferometry-by-deconvolution’, which is the method discussed in this paper.

INTERFEROMETRY-BY-DECONVOLUTION: 1-D VERSION

‘Interferometry-by-deconvolution’ is a generalization of a 1-D deconvolution method introduced by Riley & Claerbout (1976). Here we briefly review this 1-D method. The 3-D extension is introduced in the next section.

Consider a plane wave experiment in a horizontally layered medium. At a particular depth level the total wave field is decomposed into down going and up going waves. Assuming the actual source is situated above this depth level, the total down going wave field can be seen as the illuminating wave field and the total up going wave field as its response. Subsequently, the up going wave field is deconvolved by the down going wave field. The deconvolution result is the reflection impulse response of the 1-D medium below the chosen depth level. In the frequency domain, where deconvolution is replaced by division, this can be formulated as

\[
\hat{R}^0_1(x_{3,1}, \omega) = \hat{p}^+ (x_{3,1}, \omega) / \hat{p}^- (x_{3,1}, \omega),
\]

where \(x_{3,1}\) is the \(x_3\)-coordinate of the depth level at which the decomposition and division take place (in this paper the \(x_3\)-axis points downwards), and \(\hat{p}^+\) and \(\hat{p}^-\) are the down going and up going wave fields, respectively (the Fourier transform of a time-dependent function \(f(t)\) is defined as \(\hat{f} (\omega) = \int_{-\infty}^{\infty} f(t) \exp(-j\omega t) \, dt\), where \(j\) is the imaginary unit and \(\omega\) denotes the angular frequency, which is taken non-negative throughout this paper). The reflection response \(\hat{R}^0_1(x_{3,1}, \omega)\) is the response that would be measured with source and receiver at \(x_{3,1}\) and a homogeneous half-space above \(x_{3,1}\). This is independent of the actual configuration above \(x_{3,1}\). For example, if \(x_{3,1}\) is chosen just below the sea-bottom, \(\hat{R}^0_1(x_{3,1}, \omega)\) is the response of the medium below the sea-bottom, free of multiples related to the sea-bottom as well as to the water surface. Hence, \(\hat{R}^0_1(x_{3,1}, \omega)\) obeys different boundary conditions than \(\hat{p}^+\) and \(\hat{p}^-\).

Throughout this paper we will loosely use the term ‘deconvolution’ for division in the frequency domain (as in equation (1)). When the division is carried out for a sufficient range of frequencies, the result can be inverse Fourier transformed, yielding the time domain deconvolution result (e.g. \(\hat{R}^0_1(x_{3,1}, t)\)).

The analogy of equation (1) with interferometry is as follows (see also Snieder et al., 2006): the right-hand side is a ‘deconvolution’ of two received wave fields (instead of a correlation of two wave fields), whereas the left-hand side is the response of a virtual source at the position of a receiver (just as in interferometry). Moreover, independent of the actual source signature (transient or noise), the time domain deconvolution result \(\hat{R}^0_1(x_{3,1}, t)\) is an impulse response. Of course in practice the division in equation (1) should be carried out in a stabilized sense, meaning that the result becomes a band-limited impulse response. An important difference with most versions of interferometry is that equation (1) remains valid even when the medium is dissipative. Another difference is that the application of equation (1) changes the boundary conditions, as explained above.

Bakulin & Calvert (2006) proposed a similar 1-D deconvolution to improve their virtual source method. Snieder et al. (2007) employed a variant of this method (with source and receiver at different depth levels) to derive the impulse response of a building from earthquake data, and Mehta et al. (2007) used a similar approach to estimate the near-surface properties of a dissipative medium.

INTERFEROMETRY-BY-DECONVOLUTION: 3-D SCALAR VERSION

The 1-D deconvolution approach formulated by equation (1) has been extended by various authors to a multi-dimensional deconvolution method as a means for surface related and sea-bottom related multiple elimination (Wapenaar & Verschuur, 1996; Ziolkowski et al., 1998; Amundsen, 1999; Wapenaar et al., 2000; Holvik & Amundsen, 2005). In the following we derive this multi-dimensional deconvolution method along the same lines as our derivation for seismic interferometry by crosscorrelation (Wapenaar et al., 2004). First we consider the situation for scalar fields; in the next section we generalize the derivation for vector fields. Note that when we speak of ‘fields’ we mean wave and/or diffusion fields.

The starting point for our derivation is a reciprocity theorem of the convolution type for one-way scalar fields, which reads in the space-frequency domain

\[
\int \delta_\mathbb{D}_1 \left(\hat{p}^+_A \hat{p}^-_B - \hat{p}^-_A \hat{p}^+_B\right) d^2 x = \int \delta_\mathbb{D}_m \left(\hat{p}^+_A \hat{p}^-_B - \hat{p}^-_A \hat{p}^+_B\right) d^2 x, \tag{2}
\]

where \(x = (x_1, x_2, x_3)\) is the Cartesian coordinate vector, \(\delta_\mathbb{D}_1\) and \(\delta_\mathbb{D}_m\) are two horizontal boundaries of infinite extent (with \(\partial \mathbb{D}_m\) below \(\partial \mathbb{D}_1\)), and \(\hat{p}^+ = \hat{p}^+(x, \omega)\)
and $\hat{p}^- = \hat{p}^-(x, \omega)$ are flux-normalized down going and up going fields, respectively (see Appendix A for the derivation). The terms ‘down going’ and ‘up going’ should be interpreted in a broad sense: for down going fields these terms mean ‘decaying in the positive or negative $x_3$-direction, respectively’. The subscripts $A$ and $B$ refer to two independent states. Equation (2) holds for lossless as well as dissipative 3-D inhomogeneous media. The underlying assumptions for equation (2) are that there are no sources between $\partial \mathcal{D}_1$ and $\partial \mathcal{D}_m$ and that in the region enclosed by these boundaries the medium parameters in states $A$ and $B$ are identical. Above $\partial \mathcal{D}_1$ and below $\partial \mathcal{D}_m$ the medium parameters and boundary conditions in states $A$ and $B$ need not be the same. The condition that $\partial \mathcal{D}_1$ and $\partial \mathcal{D}_m$ are horizontal boundaries can be relaxed. Frijlink & Vlapan (2007) show that under certain conditions equation (2) also holds when $\partial \mathcal{D}_1$ and $\partial \mathcal{D}_m$ are smoothly curved boundaries.

Note that other variants of equation (2) exist, containing vertical derivatives of the down going and up going fields in one of the two states. This is the case, for example, when $\hat{p}^+$ and $\hat{p}^-$ represent down going and up going acoustic pressure fields. Since we consider flux-normalized fields these derivatives are absent in equation (2).

In the following, state $B$ will represent the measured response of the real earth, whereas state $A$ will represent the new response of a redatumed source in an earth with different boundary conditions, obtained by interferometry. Hence, state $B$ is the actual state whereas state $A$ is the desired state. First we discuss state $B$. Consider a dissipative 3-D inhomogeneous earth bounded by a free surface $\partial \mathcal{D}_0$, see Figure 1b. The source of the actual field at $x_S$, with source spectrum $\hat{s}(\omega)$, is situated below $\partial \mathcal{D}_0$ and above the receivers. The receivers are located, for example, at the sea-bottom or in a horizontal borehole. The boundary $\partial \mathcal{D}_1$ is chosen an $\epsilon$-distance below the receivers (e.g. just below the sea-bottom) and $\partial \mathcal{D}_m$ is chosen below all inhomogeneities. The measured field at the receivers is represented by a $2 \times 1$-vector $\mathbf{Q}(x, x_S, \omega)$, containing for example the acoustic pressure and vertical component of the particle velocity, or the inline electric field and crossline magnetic field components. The quantities in this vector are continuous in the depth direction, hence, at $\partial \mathcal{D}_1$ (i.e., just below the receivers) we have the same $\mathbf{Q}(x, x_S, \omega)$. This field vector is decomposed at $\partial \mathcal{D}_1$ into flux-normalized down going and up going fields, according to

$\mathbf{P}(x, x_S, \omega) = \hat{L}^{-1} \mathbf{Q}(x, x_S, \omega), \quad (3)$

where $\hat{L}^{-1}$ is a decomposition operator containing the medium parameters at $\partial \mathcal{D}_1$, and

$\mathbf{P}(x, x_S, \omega) = \left( \begin{array}{c} \hat{p}^+(x, x_S, \omega) \\ \hat{p}^-(x, x_S, \omega) \end{array} \right), \quad (4)$

see Appendix B for details. Hence, in state $B$ we have

$x \in \partial \mathcal{D}_1 : \begin{cases} \hat{p}^+_B(x, \omega) = \hat{p}^+_1(x, x_S, \omega), \\ \hat{p}^-_B(x, \omega) = \hat{p}^-(x, x_S, \omega). \end{cases} \quad (5)$

Since we chose $\partial \mathcal{D}_m$ below all inhomogeneities, there are only down going fields at $\partial \mathcal{D}_m$, hence

$x \in \partial \mathcal{D}_m : \begin{cases} \hat{p}^+_B(x, \omega) = \hat{p}^+_1(x, x_S, \omega), \\ \hat{p}^-_B(x, \omega) = 0. \end{cases} \quad (6)$

Note that the decomposition at $\partial \mathcal{D}_1$, as formulated by equation (3), requires that the field components in $\mathbf{Q}(x, x_S, \omega)$ are properly sampled and that the (laterally varying) medium parameters at $\partial \mathcal{D}_1$ are known. When, in case of sea-bottom measurements, there is a very thin layer of soft sediment on top of a hard rock sea floor, then the parameters of the hard rock should be used in $\hat{L}^{-1}$. Schalikwijk et al. (2003) and Mujs et al. (2004) discuss adaptive decomposition schemes for sea-bottom seismic data, which estimate the sea-bottom parameters directly from the data by optimizing the decomposition result.

For the desired state $A$ we replace the medium above $\partial \mathcal{D}_1$ by a non-reflecting half-space, see Figure 1a, which is accomplished by choosing the medium parameters continuous across $\partial \mathcal{D}_1$ and independent of $x_3$ above $\partial \mathcal{D}_1$. We choose a point source for a down going field at $x_A$ just above $\partial \mathcal{D}_1$; the receivers are chosen at $x \in \partial \mathcal{D}_1$. We define $\hat{R}^+_A(x, x_A, \omega)$ as the reflection response of the medium below $\partial \mathcal{D}_1$ with a source for a down going field at $x_A$ and a receiver for an up going field at $x \in \partial \mathcal{D}_1$. The subscript ‘0’ denotes that no multiples related to reflectors above $\partial \mathcal{D}_1$ are included; the superscript ‘+’ denotes that this is the response of a down going source field. For the down going and up going fields in state $A$ we thus have

$x \in \partial \mathcal{D}_1 : \begin{cases} \hat{p}^+_A(x, \omega) = \delta(x_H - x_H, A) \hat{s}_A(\omega), \\ \hat{p}^-_A(x, \omega) = \hat{R}^+_A(x, x_A, \omega) \hat{s}_A(\omega), \end{cases} \quad (7)$

where $\hat{s}_A(\omega)$ denotes the spectrum of the source at $x_A$. 

![Figure 1](attachment:image1.png) 

**Figure 1.** State A: the desired reflection response of the medium below $\partial \mathcal{D}_1$, for the situation of a non-reflecting half-space above $\partial \mathcal{D}_1$. State B: the actual response of the real earth, bounded by a free surface at $\partial \mathcal{D}_0$. The medium parameters exhibit dissipation, are 3-D inhomogeneous functions of position, and below $\partial \mathcal{D}_1$ they are the same in both states.
We used the subscript \(H\) to denote the horizontal coordinates, hence \(x_H = (x_1, x_2)\) and \(x_{H,A} = (x_{1,A}, x_{2,A})\) (the latter denoting the horizontal coordinates of \(x_A\)). At \(\partial D_m\) we have again only down going fields, hence
\[
x \in \partial D_m : \begin{cases} \hat{p}_A^+(x, \omega) = \hat{T}_0^+(x, x_A, \omega) \hat{s}_A(\omega), \\
\hat{q}_A(x, \omega) = 0, \end{cases}
\tag{8}
\]
where \(\hat{T}_0^+(x, x_A, \omega)\) is the transmission response of the medium between \(\partial D_1\) and \(\partial D_m\) with a source at \(x_A\) and a receiver at \(x \in \partial D_m\). Substitution of equations (5) – (8) into equation (2), using source-receiver reciprocity [i.e., \(\hat{R}_0^+(x, x_A, \omega) = \hat{R}_0^+(x_A, x, \omega)\)] and dividing the result by \(\hat{s}_A(\omega)\) gives
\[
\hat{p}^-(x_A, x_S, \omega) = \int_{\partial D_1} \hat{R}_0^+(x, x_A, \omega) \hat{p}^+(x, x_S, \omega) d^2 x.
\tag{9}
\]
This is an integral equation of the first kind for \(\hat{R}_0^+(x_A, x, \omega)\). Note that \(\hat{R}_0^+\) is the Fourier transform of an impulse response, whereas \(\hat{p}^+\) and \(\hat{p}^-\) are proportional to the source spectrum \(\hat{s}(\omega)\) of the source at \(x_S\). For laterally invariant media equation (9) can easily be solved via a scalar division in the wavenumber-frequency domain. For 3-D inhomogeneous media it can only be solved when the down going and up going fields \(\hat{p}^+(x, x_S, \omega)\) and \(\hat{p}^-(x_A, x_S, \omega)\) are available for a sufficient range of source positions \(x_S\). In matrix notation (Berkhout, 1982), equation (9) can be written as
\[
\hat{p}^- = \hat{R}_0^+ \hat{p}^+.
\tag{10}
\]
For example, the columns of matrix \(\hat{p}^+\) contain \(\hat{p}^+(x, x_S, \omega)\) for fixed \(x_S\) and variable \(x\) at \(\partial D_1\), whereas the rows of this matrix contain \(\hat{p}^+(x, x_S, \omega)\) for fixed \(x\) and variable \(x_S\) at \(\partial D_S\), where \(\partial D_S\) represents the depth level of the sources. Inversion of equation (10) involves matrix inversion, according to
\[
\hat{R}_0^+ = \hat{p}^- (\hat{p}^+)^{-1}.
\tag{11}
\]
We conclude this section by comparing 3-D interferometry-by-deconvolution with the virtual source method of Bakulin & Calvert (2006). We start by ignoring the inverse matrix in equation (12), according to
\[
\hat{R}_0^+ \approx \hat{p}^- (\hat{p}^+)^{-1}.
\tag{13}
\]
If we rewrite this equation again in integral form we obtain
\[
\hat{R}_0^+(x_A, x, \omega) \approx \int_{\partial D_S} \hat{p}^-(x_A, x_S, \omega) \{\hat{p}^+(x, x_S, \omega)\}^* d^2 x_S,
\tag{14}
\]
where the superscript * denotes complex conjugation. Note that \(\hat{R}_0^+\) is now proportional to the power spectrum \(|\hat{s}(\omega)|^2\) of the sources at \(\partial D_S\). Transforming equation (14) to the time domain yields
\[
R_0^+(x_A, x, t) \approx \int_{\partial D_S} \hat{p}^-(x_A, x_S, t) \star p^+(x, x_S, -t) d^2 x_S,
\tag{15}
\]
where \(\star\) denotes temporal convolution. The latter equation corresponds to the virtual source method of Bakulin & Calvert (2006). The integrand on the right-hand side represents the convolution of the up going field at \(x_A\) due to a source at \(x_S\) and the time-reversed down going field at \(x\) due to the same source. The integral is carried out along all sources at \(x_S \in \partial D_S\). The left-hand side is the response at \(x_A\) of a virtual source at \(x\). Bakulin & Calvert (2006) actually use a time-windowed version of \(p^+(x, x_S, t)\), containing the first arrival (which is possible for wave fields but not for diffusion fields). The main effect of their method is the suppression of propagation distortions of the overburden. For comparison, inversion of equation (9) not only removes the propagation distortions of the overburden, but also eliminates all multiple reflections related to all reflectors above \(\partial D_1\), including the free surface \(\partial D_0\).

INTERFEROMETRY-BY-DECONVOLUTION: 3-D VECTOR VERSION

We generalize the approach discussed in the previous section. We replace the one-way scalar fields by general one-way vector fields (waves and/or diffusion fields) and we derive an ‘interferometry-by-deconvolution’ approach for these vector fields.

The vectorial extension of equation (2) for a dissipative 3-D inhomogeneous medium reads
\[
\int_{\partial D_1} \{(\hat{p}_{\lambda}^+)^t \hat{p}_B - (\hat{p}_{\lambda}^-)^t \hat{p}_B^+\} d^2 x
\]
\[
= \int_{\partial D_m} \{(\hat{p}_{\lambda}^+)^t \hat{p}_B - (\hat{p}_{\lambda}^-)^t \hat{p}_B^+\} d^2 x,
\tag{16}
\]
where the superscript \(t\) denotes transposition and where
 Controlled-source interferometry in dissipate media

by decomposition, see Appendix using the parameters denoted the reflection response of the medium below ten as down going fields at \( x \) is a matrix containing the reflection responses of the receivers for up going fields at \( x \) and a reflected \( P \) it can only be solved when the down going and up going fields by Amundsen & Holvik (2004, Processing electromagnetic data, Patent GB2415511) and for elastodynamic wave fields by Holvik & Amundsen (2005). In equation (17) \( \hat{p}^+(x, x', \omega) \) and \( \hat{p}^-(x, x', \omega) \) are obtained from a field (Holvik & Amundsen, 2005), vector \( \hat{Q}(x, x', \omega) \) at \( \partial D_1 \) by decomposition, see Appendix C for details. Vector \( \hat{Q}(x, x', \omega) \) contains, for example, an elastodynamic or electromagnetic field, measured by multicomponent receivers. The multicomponent sources for these fields are located at \( x_S \in \partial D_S \). \( \hat{R}_0^+(x, x', \omega) \) is a matrix containing the reflection responses of the medium below \( \partial D_1 \) with multicomponent sources for down going fields at \( x \in \partial D_1 \) and multicomponent receivers for up going fields at \( x_A \). For example, for the situation of elastodynamic waves, vectors \( \hat{p}^+ \) and \( \hat{p}^- \) in equation (17) are defined as

\[
\hat{p}^+ = \begin{pmatrix}
\hat{\Phi}^+ \\
\hat{\Psi}^+
\end{pmatrix}
\quad \text{and} \quad
\hat{p}^- = \begin{pmatrix}
\hat{\Phi}^- \\
\hat{\Psi}^-
\end{pmatrix},
\]

where \( \hat{\Phi}^\pm, \hat{\Psi}^\pm \) and \( \hat{T}^\pm \) represent flux-normalized down going and up going \( P \), \( S_1 \) and \( S_2 \) waves, respectively. Moreover, for this situation matrix \( \hat{R}_0^+(x, x', \omega) \) is written as

\[
\hat{R}_0^+(x, x', \omega) = \begin{pmatrix}
\hat{R}_0^{+\phi\phi} & \hat{R}_0^{+\phi\psi} & \hat{R}_0^{+\phi\upsilon} \\
\hat{R}_0^{+\psi\phi} & \hat{R}_0^{+\psi\psi} & \hat{R}_0^{+\psi\upsilon} \\
\hat{R}_0^{+\upsilon\phi} & \hat{R}_0^{+\upsilon\psi} & \hat{R}_0^{+\upsilon\upsilon}
\end{pmatrix}
\]

(no summation convention) where \( \hat{R}_0^{+q,q}(x, x', \omega) \) denotes the reflection response of the medium below \( \partial D_1 \) in terms of a down going \( q \)-type wave field at \( x \) and a reflected \( p \)-type wave field at \( x_A \).

Equation (17) is an integral equation of the first kind for \( \hat{R}_0^+(x, x', \omega) \). For 3-D inhomogeneous media it can only be solved when the down going and up going fields \( \hat{p}^+(x, x_S, \omega) \) and \( \hat{p}^-(x, x_S, \omega) \) are available for a sufficient range of source positions \( x_S \) and for a sufficient number of independent source components at each source position. To be more specific, since \( \hat{p}^+ \) and \( \hat{p}^- \) are \( K \times 1 \) vectors (see Appendix A), \( K \) independent source components are needed to solve equation (17) uniquely. For example, when for the elastodynamic situation three orthogonal forces are employed at each source position, equation (17) can be solved (Holvik & Amundsen, 2005). The least-squares solution procedure for the general \( \frac{K}{2} \times \frac{K}{2} \) reflection response matrix is similar to that described in the previous section, in particular by equations (10) – (12).

**NUMERICAL EXAMPLE OF CSEM INTERFEROMETRY**

We illustrate interferometry-by-deconvolution with a numerical example. We choose to apply it to simulated controlled-source electromagnetic (CSEM) data because this best demonstrates the ability of the proposed method to deal with dissipation. Although the spatial resolution of CSEM data is much lower than that of seismic data, the main advantage of CSEM prospecting is its power to detect a hydrocarbon accumulation in a reservoir due to its high conductivity contrast (Ellingsrud et al., 2002; Moser et al., 2006). Amundsen et al. (2006) showed that decomposition of CSEM data into down going and up going fields improves the detectability of hydrocarbon reservoirs. Below we show that the combination of decomposition and interferometry-by-deconvolution not only improves the detectability but also results in improved quantitative information about the reservoir parameters.

The model consists of a plane layered Earth. The 2-D TM-mode (see Table B1) is modeled as a two-dimensional approximation of the CSEM method as applied in Seabed Logging applications. The model is shown in Figure 2, where the seawater layer contains an inline electric current source at 25 m above the sea bottom. The receivers are located at the sea bottom with a total extent of 40 km. The water layer is modeled with variable thickness and we take the values of 50 m as a model for a shallow sea and of 500 m as a model of a deep sea. The seawater has a conductivity of \( \sigma_w = 3 \) S/m. Below the sea bottom there is a layer with a conductivity of \( \sigma_1 = 1 \) S/m with a thickness of 250 m. This is followed by a half-space with \( \sigma_2 = 0.5 \) S/m, which is intersected after 900 m by a reservoir-type layer with a thickness of 50 m and a conductivity of \( \sigma_3 = 20 \) mS/m. Note that the top of this reservoir layer is located at 1150 m below the sea bottom.

For the modeling, a unit strength AC current with an oscillation frequency of 0.25 Hz, a receiver separation of 40 m and a total offset range of 40 km have been used. The array measures both the horizontal electric and magnetic field components (\( E_1 \) and \( H_1 \) as depicted in Figure 3 for the two water depths of 50 m and 500 m for the situation with and without the reservoir layer. It can be observed that for small offsets, the in-line electric field component shows more decay as a function of offset than the cross-line magnetic field component. For large offset the situation is reversed. For the deep sea the difference between the presence and absence of the reservoir layer is more pronounced than in the shallow sea situation. We can decompose the measured fields into flux-normalized down going and up going fields (equation 3). The decomposition is carried out using the parameters.
It should be noted that the electromagnetic field, at \( f=0.25 \) Hz, at the sea bottom.

The electromagnetic field, at \( f=0.25 \) Hz, at the sea bottom. (a) The electric and (b) the magnetic field amplitudes for a water depth \( h_w=50 \) m. (c) The electric and (d) magnetic field amplitudes for a water depth \( h_w=500 \) m. The red and blue curves represent the situation with and without the reservoir layer. The result is shown in Figure 5 for the situation with and without the reservoir layer. After this step the effect of the water layer has been completely removed and the reservoir response. The model is the same as in Figure 2, but now the water depth is maintained constant at 50 m, while the first two Earth layers have in the first example thicknesses of \( h_1=250 \) m and \( h_2=900 \) m, and in the second example \( h_1=900 \) m and \( h_2=250 \) m (keeping the total thickness fixed). The receiver array is located in a horizontal borehole at 950 m below the sea bottom, 200 m above the top of the reservoir, as indicated in Figure 3.

Figure 2. The configuration with a reservoir-type layer at 1150 m below the sea bottom and a water layer with variable thickness: \( h_w=50 \) m or \( h_w=500 \) m. The source is 25 m above the sea bottom and the receiver array is located at the sea bottom.

Figure 3. The electromagnetic field, at \( f=0.25 \) Hz, at the sea bottom. (a) The electric and (b) the magnetic field amplitudes for a water depth \( h_w=50 \) m. (c) The electric and (d) magnetic field amplitudes for a water depth \( h_w=500 \) m. The red and blue curves represent the situation with and without the reservoir layer.

of the first layer below the sea bottom. Hence, the resulting down going and up going fields correspond to the fields just below the sea bottom. Both are shown in Figure 4 for the two water depths of 50 m and 500 m for the situation with and without the reservoir layer. The effect of the reservoir response is clearly visible in the up going fields for offsets larger than 2 km in the shallow sea and for offsets larger than 1 km in the deep sea. It is almost not visible in the down going field (except at offsets larger than 4 km, indicating that its multiple interaction with the water layer is very small). The sharp minima that occur in the up going fields correspond to sign changes in the real and imaginary parts. By comparing Figure 4b with 4d it can be seen that the shallow water layer has a strong effect on the amplitude and shape of the up going field, which makes quantitative analysis of the reservoir parameters very difficult. This effect can be eliminated by performing interferometry-by-deconvolution, that is, by solving equation (9) for \( \hat{F}_R(x_1,x_2,\omega) \). Since in this example the medium is horizontally layered, we solve equation (9) by applying a division in the wavenumber-frequency domain and transforming the result back to the space-frequency domain. The result is shown in Figure 5 for the situation with and without the reservoir layer. After this step the effect of the water layer has been completely removed and therefore this result is independent of the water depth in the original model. It is as if the upper most Earth layer now extends upward to infinity. Moreover, note that in \( \hat{F}_R(x_1,x_2,\omega) \) the original source has been replaced by a source at the receiver level, while only the reflection response of the medium below the sea bottom is retained. Hence, the strong direct field has also been eliminated. In theory, this procedure solves the shallow sea problem of the seabed logging method. It should be noted that we considered an ideal situation of well sampled data, measured with high precision and no noise added.

When a horizontal borehole is available at depth we can perform these steps again, thereby removing the direct field and all overburden effects, leaving only the reservoir response. The model is the same as in Figure 2, but now the water depth is maintained constant at 50 m, while the first two Earth layers have in the first example thicknesses of \( h_1=250 \) m and \( h_2=900 \) m, and in the second example \( h_1=900 \) m and \( h_2=250 \) m (keeping the total thickness fixed). The receiver array is located in a horizontal borehole at 950 m below the sea bottom, 200 m above the top of the reservoir, as indicated in Figure 4.
Figure 5. The reflection response $\tilde{R}_\omega^1(x_A, x, \omega)$ at $f=0.25$ Hz just below the sea bottom, obtained by interferometry-by-deconvolution. The result is independent of the water depth. The red and blue curves represent the situation with and without the reservoir layer, respectively.

Figure 6. The configuration with a reservoir-type layer at 1150 m below the sea bottom and a water layer with a fixed thickness of 50 m. The source is 25 m above the sea bottom and the receiver array is located in a horizontal borehole 950 m below the sea bottom and 200 m above the top of the reservoir layer. The Earth layers have variable thicknesses: $h_1=250$ m and $h_2=900$ m, or $h_1=900$ m and $h_2=250$ m.

6. Again we assume that $\tilde{E}_1$ and $\tilde{H}_2$ are available so that decomposition is possible. The fields are modeled with a receiver separation of 160 m and the results are shown in Figure 7 for the situation with a water depth of 50 m, with and without reservoir layer and with a first layer of 250 m and a second layer of 900 m as well as the reversed situation. The decomposition result is shown in Figure 8, where the up going fields in 8b and 8d clearly demonstrate the detectability of the reservoir layer. Note that these up going fields have similar shapes, but their field amplitudes differ by a factor of 4. This can be understood from the fact that the interaction between the reservoir and the interface above the reservoir plays a more prominent role in the up going field in the second example (Figure 8d) where the interface is located 250 m above the top of the reservoir and only 50 m above the receivers. Figure 9 shows $\tilde{R}_\omega^1(x_A, x, \omega)$, obtained by interferometry-by-deconvolution. Sources as well as receivers are now in the horizontal borehole, 200 m above the top of the reservoir layer. This result is independent of the overburden in the original model. In theory it is the exact reflection response of the reservoir layer as if it were embedded in a homogeneous medium with a conductivity of $\sigma_2=0.5$ S/m.

CONCLUSIONS

One of the main assumptions in most seismic interferometry schemes is that the medium is lossless. We have shown that this assumption can be circumvented when the crosscorrelation procedure (the central step in seismic interferometry) is replaced by a multi-dimensional deconvolution procedure. We derived an algorithm for ‘interferometry-by-deconvolution’ for the situation of sources at or below the Earth’s surface and multicomponent receivers at depth, for example at the sea-bottom or in a horizontal borehole. The proposed algorithm not only moves the source to the receiver depth level (‘source redatuming’), but also changes the boundary conditions in such a way that the overburden becomes non-reflecting. The result is a reflection response observed relatively close to the target, without the disturbing effects of the overburden. As in all interferometry approaches, no knowledge of the medium is required, except at the depth level of the receivers, where a decomposition into down going and up going fields takes place.

An important application of ‘interferometry-by-
solving the shallow sea.

processing CSEM data at f=0.25 Hz in a horizontal borehole 200 m above the top of the reservoir layer. (a) Down going field and (b) up going field for $h_1=250$ m and $h_2=900$ m. (c) Down going field and (d) up going field for $h_1=900$ m and $h_2=250$ m. The red and blue curves represent the situation with and without the reservoir layer, respectively. In (a) and (c) the blue curves are hidden by the red curves. In (b) and (d) there are only red curves since in the situation without reservoir layer there are no up going fields.

Figure 8. Decomposition of 2-D TM field at f=0.25 Hz in a horizontal borehole 200 m above the top of the reservoir layer. (a) Down going field and (b) up going field for $h_1=250$ m and $h_2=900$ m. (c) Down going field and (d) up going field for $h_1=900$ m and $h_2=250$ m. The red and blue curves represent the situation with and without the reservoir layer, respectively. In (a) and (c) the blue curves are hidden by the red curves. In (b) and (d) there are only red curves since in the situation without reservoir layer there are no up going fields.

Deconvolution’ is removing the air/sea interface and the direct field in CSEM data. The two main factors that complicate standard CSEM data processing are the presence of the direct field and, for shallow seas, the presence of the field reflected at the sea surface (effect of the air wave). Both effects interfere with the subsurface response in the measurements. Due to the diffuse character of the EM fields the direct field and the effect of the air wave cannot be separated in the measurement by time windowing. The method we propose here removes both effects, thereby in theory solving the shallow sea problem of CSEM applications. For deep receivers, the method removes the direct field and all overburden effects.

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APPENDIX A: ONE-WAY RECIPROCITY THEOREMS FOR 3-D INHOMOGENEOUS DISSIPATIVE MEDIA

We derive the convolution-type reciprocity theorems for one-way scalar and vector fields (equations 2 and 16). There are two approaches for deriving these theorems. The first approach starts with decomposing the equation for the total field into one-way equations, followed by deriving reciprocity theorems for the one-way fields. The advantage of this approach is that it leads to relatively simple expressions, even when the medium parameters in both states are different and sources are present in the considered domain (e.g., Wapenaar et al. [2001]). The disadvantage is that for the situation of a 3-D inhomogeneous medium (dissipative or lossless), exact derivations exist only for scalar fields, whereas in the derivations for vector fields approximations are made throughout the domain of application.

In the second approach the order of steps is reversed, hence, it starts with deriving a reciprocity theorem for the total field, followed by decomposition of the fields in this reciprocity theorem into one-way fields. The first step in this approach is exact for scalar as well as vector fields in 3-D inhomogeneous media (dissipative or lossless). For the special situation that the medium parameters in both states are identical and the considered domain is source-free, this reciprocity theorem reduces to an integral over the boundary of the domain. Hence, for the subsequent decomposition step, approximations only need to be made at this boundary. When the medium is laterally invariant at the boundary of the domain, no approximations need to be made at all. In this paper, we consider the situation of a source-free domain with identical medium parameters in both states (i.e., the domain between $\partial \Omega_1$ and $\partial \Omega_2$ in Figure 1), we follow the second approach.

Our starting point is the following equation

$$\frac{\partial \mathbf{Q}}{\partial x_3} = \hat{\mathbf{A}} \mathbf{Q},$$

(A1)

where $\mathbf{Q} = \mathbf{Q}(\mathbf{x}, \omega)$ is a $K \times 1$ field vector and $\hat{\mathbf{A}} = \hat{\mathbf{A}}(\mathbf{x}, \omega)$ a $K \times K$ operator matrix containing a particular combination of the medium parameters and the horizontal differentiation operators $\partial/\partial x_\alpha$ for $\alpha = 1, 2, 3$. This equation holds for acoustic wave fields in fluids ($K = 2$), electromagnetic wave and/or diffusion fields in matter ($K = 4$, Reid [1972]), elastodynamic fields in solids ($K = 6$, Woodhouse [1974]), porous waves in porous solids ($K = 8$) and seismic waves in porous solids ($K = 12$, Pride & Haartsen, 1996; Haartsen & Pride, 1997). Vector $\mathbf{Q}$ and operator matrix $\hat{\mathbf{A}}$ are specified for some of these cases in Appendices B and C.

We consider a dissipative 3-D inhomogeneous medium in a domain $\Omega$ enclosed by two horizontal boundaries $\partial \Omega_1$ and $\partial \Omega_2$, with outward pointing normal vector $\mathbf{n} = (0, 0, -1)$ on $\partial \Omega_1$ and $\mathbf{n} = (0, 0, +1)$ on $\partial \Omega_2$, and a cylindrical boundary $\partial \Omega_{cyl}$ with a vertical axis and normal vector $\mathbf{n} = (n_1, n_2, 0)$, see Figure A1. In the following we assume that the horizontal boundaries $\partial \Omega_1$ and $\partial \Omega_2$ are of infinite extent (which implies that the radius of the cylindrical boundary $\partial \Omega_{cyl}$ is also infinite), the domain $\Omega$ between boundaries $\partial \Omega_1$ and $\partial \Omega_2$ is source-free and the medium parameters in states $A$ and $B$ are identical in this domain.

For an arbitrary operator matrix $\hat{\mathbf{u}}$ containing $\partial/\partial x_\alpha$ for $\alpha = 1, 2$, we introduce the transposed $\hat{\mathbf{u}}^\top$ via

$$\int_{\Omega^2} (\hat{\mathbf{u}} f)^\top \mathbf{g} \, d^2 \mathbf{x}_H = \int_{\Omega^2} \hat{\mathbf{r}}^\top (\hat{\mathbf{u}}^\top \mathbf{g}) \, d^2 \mathbf{x}_H,$$

(A2)

where $f = f(\mathbf{x}_H)$ and $\mathbf{g} = \mathbf{g}(\mathbf{x}_H)$ are arbitrary square-integrable $K \times 1$ vector functions. According to this equation, $\hat{\mathbf{u}}^\top$ is a transposed matrix, containing transposed operators (with $(\partial/\partial x_\alpha)^\top = -\partial/\partial x_\alpha$, which follows from the rules for partial integration). The operator matrix $\hat{\mathbf{A}}$ is organized such that it obeys the following symmetry relation

$$\hat{\mathbf{A}}^\top \mathbf{N} = -\mathbf{N} \hat{\mathbf{A}},$$

(A3)

with

$$\mathbf{N} = \begin{pmatrix} \mathbf{O} & \mathbf{I} \\ -\mathbf{I} & \mathbf{O} \end{pmatrix},$$

(A4)

where $\mathbf{I}$ and $\mathbf{O}$ are $K \times K$ identity and null matrices (note that $K = 2$ for scalar fields, hence $\mathbf{I} = 1$ and $\mathbf{O} = 0$).

We define an interaction quantity $\frac{\partial}{\partial x_3} \{\hat{\mathbf{Q}}^\top \mathbf{N} \hat{\mathbf{Q}}\}$, where subscripts $A$ and $B$ denote two independent states. Applying the product rule for differentiation and substituting equation (A1) we obtain

$$\frac{\partial}{\partial x_3} \{\hat{\mathbf{Q}}^\top \mathbf{N} \hat{\mathbf{Q}}\} = (\hat{\mathbf{A}} \hat{\mathbf{Q}})^\top \mathbf{N} \hat{\mathbf{Q}} + \hat{\mathbf{Q}}^\top \mathbf{N} \hat{\mathbf{A}} \hat{\mathbf{Q}}.$$ (A5)

Integrating over $\mathbf{x}_H$ and using equation (A2) gives

$$\int_{\Omega^2} \frac{\partial}{\partial x_3} \{\hat{\mathbf{Q}}^\top \mathbf{N} \hat{\mathbf{Q}}\} \, d^2 \mathbf{x}_H = \int_{\Omega^2} \hat{\mathbf{Q}}^\top (\hat{\mathbf{A}}^\top \mathbf{N} + \mathbf{N} \hat{\mathbf{A}}) \hat{\mathbf{Q}} \, d^2 \mathbf{x}_H.$$ (A6)

From equation (A3) it follows that the right-hand side is equal to zero. Integrating the left-hand side over $x_3$ from $x_{31}$ to $x_{3m}$ (which are the depth levels of boundaries $\partial \Omega_1$ and $\partial \Omega_2$) we obtain

$$\int_{\partial \Omega_1} \hat{\mathbf{Q}}^\top \mathbf{N} \hat{\mathbf{Q}} \, d^2 \mathbf{x} = \int_{\partial \Omega_2} \hat{\mathbf{Q}}^\top \mathbf{N} \hat{\mathbf{Q}} \, d^2 \mathbf{x}. $$ (A7)

At the boundaries $\partial \Omega_1$ and $\partial \Omega_2$ we decompose operator matrix $\hat{\mathbf{A}}$ as follows

$$\hat{\mathbf{A}} = \hat{\mathbf{L}} \hat{\mathbf{N}} \hat{\mathbf{L}}^{-1}.$$ (A8)

(Corones et al., 1983, 1992; Fishman et al., 1987; de Hoop, 1992, 1996; Fishman, 1993; Haines & de Hoop, 1996; Wapenaar et al., 2001). Examples of this decomposition are given in Appendices B and C. We scale the operator $\hat{\mathbf{L}}$ in such a way that

$$\hat{\mathbf{L}}^\top \mathbf{N} \hat{\mathbf{L}} = -\mathbf{N} \quad \text{or} \quad \hat{\mathbf{L}}^{-1} = -\mathbf{N}^{-1} \hat{\mathbf{L}}^\top \mathbf{N}$$ (A9)
(de Hoop, 1992; Wapenaar et al., 2001).

Using this specific scaling, we introduce the $K \times 1$
flux-normalized decomposed field vector $\hat{P} = \hat{P}(\mathbf{x}, \omega)$
via

$$\hat{Q} = \hat{L}\hat{P} \quad \text{and} \quad \hat{P} = \hat{L}^{-1}\hat{Q}, \quad (A10)$$

with

$$\hat{P} = \begin{pmatrix} \hat{p}^+ \\ \hat{v}_3 \end{pmatrix}, \quad (A11)$$

where $\frac{\partial}{\partial x} \times 1$ vectors $\hat{p}^+ = \hat{p}^+(\mathbf{x}, \omega)$ and $\hat{p}^- = \hat{p}^-(\mathbf{x}, \omega)$
represent down going and up going fields, respectively.
Here ‘down going’ and ‘up going’ should be interpreted
in a broad sense: for diffusion fields these terms mean
decaying in the positive or negative $x_3$-direction, respectively’.
Substitution of $\hat{Q} = \hat{L}\hat{P}$ into equation (A7) gives, using equation (A2),

$$\int_{\partial \Omega_1} \hat{P}^A_i \hat{L}^i \hat{C}^B \hat{P}^B_i \, d^2x = \int_{\partial \Omega_m} \hat{P}^A_i \hat{L}^i \hat{C}^B \hat{P}^B_i \, d^2x, \quad (A12)$$

or, using $\hat{L}^i \hat{C}^B = -N$,

$$\int_{\partial \Omega_1} \hat{P}^A_i \hat{N} \hat{P}^B_i \, d^2x = \int_{\partial \Omega_m} \hat{P}^A_i \hat{N} \hat{P}^B_i \, d^2x. \quad (A13)$$

Substitution of equations (A4) and (A11) into equation (A13) yields equation (2) (for $K = 2$) or equation (16)
(for all other cases).

**APPENDIX B: DECOMPOSITION OPERATORS FOR SCALAR FIELDS**

We discuss the field vectors and operators introduced in
Appendix A for scalar fields ($K = 2$). For an acoustic
wave field in a dissipative 3-D inhomogeneous fluid we have
(de Hoop, 1992; Wapenaar et al., 2001)

$$\hat{Q} = \begin{pmatrix} \hat{p} \\ \hat{v}_3 \end{pmatrix}, \quad \hat{A} = \begin{pmatrix} 0 & -j\omega \hat{p} \\ \hat{v}_3 & \frac{\partial}{\partial x_3} (\hat{H}\hat{p} - \frac{j}{2}) \end{pmatrix}, \quad (B1)$$

where $\hat{p} = \hat{p}(\mathbf{x}, \omega)$ is the acoustic pressure,
$\hat{v}_3 = \hat{v}_3(\mathbf{x}, \omega)$
the vertical component of the particle velocity and
$\hat{\rho} = \rho(\mathbf{x}, \omega)$ the complex-valued mass density of the dissipative medium.
$\hat{H}_2$ is the Helmholtz operator, defined as

$$\hat{H}_2 = \omega^2/c^2 + \frac{\partial}{\partial x_3} \frac{\partial}{\partial x_3}. \quad (B2)$$

The summation convention applies for repeated subscripts;
Greek subscripts take on the values 1 and 2.
In equation (B2), $\hat{c} = \hat{c}(\mathbf{x}, \omega)$ is the complex-valued
propagation velocity, obeying the Klein-Gordon dispersion
relation known from relativistic quantum mechanics (Messiah, 1962; Anno et al., 1992), according to

$$\frac{\omega^2}{c^2} = \omega^2 \hat{\kappa} \hat{\rho} - \frac{3}{4} \frac{\partial \hat{p}}{\partial x_3} \frac{\partial \hat{p}}{\partial x_3} + \frac{1}{2} \frac{\partial}{\partial x_3} \frac{\partial}{\partial x_3}, \quad (B3)$$

with $\hat{\kappa} = \kappa(\mathbf{x}, \omega)$ the complex-valued compressibility of the dissipative medium.
Note that $\hat{H}_2 = \hat{H}_2^\dagger$, hence, symmetry relation (A3) is fulfilled.

The decomposition of $\hat{A}$ is given by equation (A8), where

$$\hat{H} = \begin{pmatrix} \hat{H}_1 & 0 \\ 0 & \hat{H}_1 \end{pmatrix}, \quad \hat{H}_1 = \hat{H}_2^\dagger, \quad (B4)$$

with $\hat{H}_1 = \hat{H}_1^\dagger$ (Wapenaar et al., 2001), and

$$\hat{L} = \begin{pmatrix} \hat{L}_1 & \hat{L}_2 \\ \hat{L}_2 & -\hat{L}_2 \end{pmatrix}, \quad \hat{L}^{-1} = \frac{1}{2} \begin{pmatrix} \hat{L}_1^{-1} & -\hat{L}_2^{-1} \\ \hat{L}_2^{-1} & \hat{L}_1^{-1} \end{pmatrix}, \quad (B5)$$

with

$$\hat{L}_1 = \left( \hat{\rho} \hat{L}_1 \right)^{\frac{1}{2}}, \quad \frac{1}{2} \hat{L}_1^{-1} = \hat{L}_2 = \hat{H}_1^\dagger \left( \frac{1}{2} \hat{\rho} \right)^{\frac{1}{2}}, \quad (B6)$$

$$\hat{L}_2 = \left( \frac{1}{2} \hat{\rho} \right)^{\frac{1}{2}} \hat{H}_1^\dagger, \quad \frac{1}{2} \hat{L}_2^{-1} = \hat{L}_1 = \hat{H}_1^\dagger \left( \frac{\hat{\rho}}{2} \right)^{\frac{1}{2}}. \quad (B7)$$

with $\hat{H}_1^\dagger = (\hat{H}_1^\dagger)^\dagger$ and $\hat{H}_1^\dagger = (\hat{H}_1^\dagger)^\dagger$. Note that symmetry relation (A9) is fulfilled as well.

The field quantities and medium parameters involved in acoustic decomposition are summarized in the first row of Table B1. In a dissipative fluid the imaginary parts of $\hat{\rho}$ and $\hat{\kappa}$ are negative (for positive $\omega$).
The imaginary part of the eigenvalue spectrum of the square-root operator $\hat{H}_1$ is chosen negative as well (Wapenaar et al., 2001). The same applies for the spectrum of $\hat{H}_1^\dagger$, i.e., the square-root of the square-root operator.

For mass diffusion of a species through a mixture, $\hat{Q}$ is given by

$$\hat{Q} = \begin{pmatrix} \hat{Y} \\ \hat{J}_3 \end{pmatrix}, \quad (B8)$$

where $\hat{Y} = \hat{Y}(\mathbf{x}, \omega)$ is the mass fraction of the species
and $\hat{J}_3 = \hat{J}_3(\mathbf{x}, \omega)$ the vertical component of the mass
flux relative to the mixture. In equations (B1) – (B7) we replace the quantities in the first row of Table B1 by those in the second row (where $D$ is the diffusion coefficient). We thus obtain the decomposition operators for mass diffusion. Note that the term $1/j\omega D$ is

![Diagram](image-url)
purely negative imaginary (for positive \( \omega \)). The imaginary parts of the spectra of \( \mathcal{H}_1 \) and \( \mathcal{H}_1^T \) are again chosen negative as well.

The last three rows in Table B1 show the field quantities and medium parameters for three other applications of equations (B1) – (B7), but this time for the 2-D situation (i.e., assuming that the field quantities and medium parameters are independent of the \( x_2 \)-coordinate). ‘SH’ stands for horizontally polarized shear waves. In this row \( \tau_{23} \) is the shear stress and \( \mu^t \) the shear modulus of the medium. ‘TE’ and ‘TM’ stand for transverse electric and transverse magnetic fields, respectively. \( \mathbf{E}_{1,2}, \mathbf{H}_{1,2} \) are the electric and magnetic field components, \( \varepsilon \) is the permittivity, \( \mu \) the permeability, \( \sigma \) the conductivity and \( \Gamma \) the magnetic hysteresis loss term. Note that depending on the choices of the medium parameters, the TE and TM fields can be wave or diffusion fields, or a combination of the two. In all cases the imaginary parts of the spectra of \( \mathcal{H}_1 \) and \( \mathcal{H}_1^T \) are chosen negative.

### APPENDIX C: DECOMPOSITION OPERATORS FOR VECTOR FIELDS

For 3-D electromagnetic diffusion and/or wave propagation in a dissipative 3-D inhomogeneous medium we have (Reid, 1972)

\[
\mathbf{Q} = \begin{pmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{H}_2 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} \mathbf{0} & \mathbf{A}_{12} \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} \mathbf{0} & \mathbf{C}_{11} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{pmatrix}, \quad \mathbf{L} = \begin{pmatrix} \mathbf{L}_1 & \mathbf{L}_2 \end{pmatrix}, \quad \mathbf{L}^{-1} = \begin{pmatrix} \mathbf{L}_1^{-1} & -\mathbf{L}_2^{-1} \end{pmatrix}
\]

where

\[
\mathbf{A}_{12} = \begin{pmatrix} -j\omega \mathbf{M} + \frac{1}{j\omega} \frac{\partial}{\partial x_1} \left( \frac{\partial}{\partial x_1} \right) & -j\omega \mathbf{M} + \frac{1}{j\omega} \frac{\partial}{\partial x_2} \left( \frac{\partial}{\partial x_2} \right) \\ \frac{1}{j\omega} \frac{\partial}{\partial x_2} \left( \frac{\partial}{\partial x_1} \right) & \frac{1}{j\omega} \frac{\partial}{\partial x_1} \left( \frac{\partial}{\partial x_2} \right) \end{pmatrix}, \quad (C2)
\]

\[
\mathbf{A}_{21} = \begin{pmatrix} -j\omega \mathbf{E} + \frac{1}{j\omega} \frac{\partial}{\partial x_2} \left( \frac{\partial}{\partial x_1} \right) & -j\omega \mathbf{E} + \frac{1}{j\omega} \frac{\partial}{\partial x_1} \left( \frac{\partial}{\partial x_2} \right) \\ -\frac{1}{j\omega} \frac{\partial}{\partial x_1} \left( \frac{\partial}{\partial x_2} \right) & -\frac{1}{j\omega} \frac{\partial}{\partial x_2} \left( \frac{\partial}{\partial x_1} \right) \end{pmatrix}, \quad (C3)
\]

with

\[
\dot{\mathbf{E}} = \dot{\varepsilon} + \sigma/j\omega, \quad \dot{\mathbf{M}} = \dot{\mu} + \dot{\Gamma}/j\omega. \quad (C4)
\]

Note that \( \mathbf{A} \) obeys symmetry relation (A3) in an arbitrary 3-D inhomogeneous dissipative medium, which validates reciprocity theorem (A7). Next, for the decomposition of \( \mathbf{A} \) at \( \partial \mathcal{D}_1 \) and \( \partial \mathcal{D}_m \) we assume that there are no lateral variations of the medium parameters at these boundaries. We define the two-dimensional spatial Fourier transform from the space-frequency domain to the wavenumber-frequency domain as follows

\[
\hat{f}(k_H, x_3, \omega) = \int_{\mathbb{R}^2} f(x_H, x_3, \omega) \exp(jk_H \cdot x_H) \, d^2 x_H, \quad (C5)
\]

with \( k_H = (k_1, k_2) \). Applying this transform to equation (A8) gives

\[
\hat{A} = \hat{\mathbf{C}} \hat{\mathcal{H}} \hat{\mathbf{C}}^{-1}, \quad (C6)
\]

where \( \hat{A} \) is obtained from \( \mathbf{A} \) defined in equations (C1) – (C3) by replacing \( \partial/\partial x_3 \) by \(-j k_3 \). Note that equation (A3) transforms to

\[
\hat{\mathbf{A}}'(\mathbf{k}_H, x_3, \omega) \mathbf{N} = -Ne \hat{\mathbf{A}}(\mathbf{k}_H, x_3, \omega). \quad (C7)
\]

The sign-change of \( \mathbf{k}_H \) in the argument of the transposed matrix is due to the relation \( (\partial/\partial x_3)' = -\partial/\partial x_3 \). Similarly, equation (A9) transforms to

\[
\{\mathbf{C}(\mathbf{k}_H, x_3, \omega)\}^T \mathbf{N} \hat{\mathbf{C}}(\mathbf{k}_H, x_3, \omega) = \mathbf{N}, \quad (C8)
\]

or

\[
\{\hat{\mathbf{C}}(\mathbf{k}_H, x_3, \omega)\}^{-1} \mathbf{N} = -\mathbf{N}^{-1}\{\hat{\mathbf{C}}(\mathbf{k}_H, x_3, \omega)\}^T \mathbf{N}. \quad (C9)
\]

With this scaling we obtain (Ursin, 1983)

\[
\hat{\mathcal{H}} = \begin{pmatrix} -j\mathcal{H}_1 & 0 \\ 0 & j\mathcal{H}_1 \end{pmatrix}, \quad \hat{\mathcal{H}}_1 = \begin{pmatrix} \mathcal{H}_1 & 0 \\ 0 & \mathcal{H}_1 \end{pmatrix}, \quad (C10)
\]

\[
\hat{\mathbf{C}} = \begin{pmatrix} \hat{\mathbf{L}}_1 & \hat{\mathbf{L}}_2 \\ \hat{\mathbf{L}}_2 & -\hat{\mathbf{L}}_2 \end{pmatrix}, \quad \hat{\mathbf{L}}^{-1} = \begin{pmatrix} \hat{\mathbf{L}}_1^{-1} & -\hat{\mathbf{L}}_2^{-1} \end{pmatrix}, \quad (C11)
\]
where
\[
\hat{\mathcal{L}}_1(k_H, x_3, \omega) = \frac{1}{\sqrt{2}} \begin{pmatrix}
\frac{\zeta}{\omega} \hat{H}_1^{1/2} & 0 \\
-\frac{k_1 k_2}{\omega \zeta} \hat{H}_1^{1/2} & \hat{H}_1^{1/2}
\end{pmatrix},
\]
for an elastodynamic wave field in a dissipative inhomogeneous anisotropic solid we have (Woodhouse, 1974)
\[
\hat{\mathcal{L}}_2(k_H, x_3, \omega) = \frac{1}{\sqrt{2}} \begin{pmatrix}
\frac{\zeta}{\omega} \hat{H}_1^{1/2} & 0 \\
-\frac{k_2 k_3}{\omega \zeta} \hat{H}_1^{1/2} & \hat{H}_1^{1/2}
\end{pmatrix},
\]
with
\[
\hat{H}_1 = \sqrt{k_2^2 - k_3 k_3}, \quad \Im(\hat{H}_1) < 0,
\]
and
\[
k^2 = \frac{\omega^2}{c^2} = \omega^2 \hat{\varepsilon} \hat{\mu},
\]
\[
\hat{\zeta} = \left(\omega \hat{\varepsilon} - \frac{k^2}{\omega \hat{\mu}}\right)^{1/2}, \quad \hat{\vartheta} = \left(\omega^{1/2} \hat{\varepsilon} \right)^{1/2}
\]
For the decomposition of \(\hat{\mathcal{A}}\) at \(\partial \Omega_1\) and \(\partial \Omega_m\) we assume that the medium is laterally invariant and isotropic at these boundaries, hence
\[
\hat{\mathcal{A}}_1 = \begin{pmatrix}
-\frac{\partial}{\partial x_\alpha} (C_{\alpha \beta} \hat{C}_{\beta \beta}^{-1}) \\
\end{pmatrix},
\]
\[
\hat{\mathcal{A}}_2 = \begin{pmatrix}
-j \omega \hat{\rho} I + \frac{1}{j \omega} \frac{\partial}{\partial x_\alpha} (U_{\alpha \beta} \frac{\partial}{\partial x_\beta}) \\
\end{pmatrix},
\]
\[
\hat{\mathcal{A}}_2 = \begin{pmatrix}
-j \omega \hat{C}_{\alpha \alpha}^{-1} \\
\end{pmatrix},
\]
\[
\hat{\mathcal{A}}_2 = \begin{pmatrix}
-j \hat{C}_{\alpha \beta} \frac{\partial}{\partial x_\beta} \\
\end{pmatrix},
\]
\[
U_{\alpha \beta} = \begin{pmatrix}
C_{\alpha \beta} - C_{\alpha \alpha} C_{\beta \beta}^{-1} C_{\beta \beta}
\end{pmatrix},
\]
with \((\hat{C}_{\beta \beta})_{\beta} = \hat{C}_{\beta \beta}(k, \omega)\) is the complex-valued stiffness tensor and \(\hat{\rho} = \hat{\rho}(x, \omega)\) the complex-valued mass density of the dissipative medium. Since \(\hat{C}_{\beta \beta} = \hat{C}_{\beta \beta}\), \(\hat{\mathcal{A}}\) obeys symmetry relation (A3) in an arbitrary 3-D inhomogeneous anisotropic dissipative medium. This validates reciprocity theorem (A7). Next, for the decomposition of \(\hat{\mathcal{A}}\) at \(\partial \Omega_1\) and \(\partial \Omega_m\) we assume that the medium is laterally invariant and isotropic at these boundaries, hence
\[
\hat{\mathcal{L}}_1 = \begin{pmatrix}
-\frac{\partial}{\partial x_\alpha} (C_{\alpha \beta} \hat{C}_{\beta \beta}^{-1}) \\
\end{pmatrix},
\]
\[
\hat{\mathcal{L}}_2 = \begin{pmatrix}
-j \omega \hat{\rho} I + \frac{1}{j \omega} \frac{\partial}{\partial x_\alpha} (U_{\alpha \beta} \frac{\partial}{\partial x_\beta}) \\
\end{pmatrix},
\]
\[
\hat{\mathcal{L}}_2 = \begin{pmatrix}
-j \omega \hat{C}_{\alpha \alpha}^{-1} \\
\end{pmatrix},
\]
\[
\hat{\mathcal{L}}_2 = \begin{pmatrix}
-j \hat{C}_{\alpha \beta} \frac{\partial}{\partial x_\beta} \\
\end{pmatrix},
\]
\[
U_{\alpha \beta} = \begin{pmatrix}
C_{\alpha \beta} - C_{\alpha \alpha} C_{\beta \beta}^{-1} C_{\beta \beta}
\end{pmatrix},
\]
with \((\hat{C}_{\beta \beta})_{\beta} = \hat{C}_{\beta \beta}(k, \omega)\) is the complex-valued stiffness tensor and \(\hat{\rho} = \hat{\rho}(x, \omega)\) the complex-valued mass density of the dissipative medium. Since \(\hat{C}_{\beta \beta} = \hat{C}_{\beta \beta}\), \(\hat{\mathcal{A}}\) obeys symmetry relation (A3) in an arbitrary 3-D inhomogeneous anisotropic dissipative medium. This validates reciprocity theorem (A7). Next, for the decomposition of \(\hat{\mathcal{A}}\) at \(\partial \Omega_1\) and \(\partial \Omega_m\) we assume that the medium is laterally invariant and isotropic at these boundaries, hence
\[
\hat{\mathcal{L}}_1 = \begin{pmatrix}
-\frac{\partial}{\partial x_\alpha} (C_{\alpha \beta} \hat{C}_{\beta \beta}^{-1}) \\
\end{pmatrix},
\]
\[
\hat{\mathcal{L}}_2 = \begin{pmatrix}
-j \omega \hat{\rho} I + \frac{1}{j \omega} \frac{\partial}{\partial x_\alpha} (U_{\alpha \beta} \frac{\partial}{\partial x_\beta}) \\
\end{pmatrix},
\]
\[
\hat{\mathcal{L}}_2 = \begin{pmatrix}
-j \omega \hat{C}_{\alpha \alpha}^{-1} \\
\end{pmatrix},
\]
\[
\hat{\mathcal{L}}_2 = \begin{pmatrix}
-j \hat{C}_{\alpha \beta} \frac{\partial}{\partial x_\beta} \\
\end{pmatrix},
\]
\[
U_{\alpha \beta} = \begin{pmatrix}
C_{\alpha \beta} - C_{\alpha \alpha} C_{\beta \beta}^{-1} C_{\beta \beta}
\end{pmatrix},
\]
with \((\hat{C}_{\beta \beta})_{\beta} = \hat{C}_{\beta \beta}(k, \omega)\) is the complex-valued stiffness tensor and \(\hat{\rho} = \hat{\rho}(x, \omega)\) the complex-valued mass density of the dissipative medium. Since \(\hat{C}_{\beta \beta} = \hat{C}_{\beta \beta}\), \(\hat{\mathcal{A}}\) obeys symmetry relation (A3) in an arbitrary 3-D inhomogeneous anisotropic dissipative medium. This validates reciprocity theorem (A7). Next, for the decomposition of \(\hat{\mathcal{A}}\) at \(\partial \Omega_1\) and \(\partial \Omega_m\) we assume that the medium is laterally invariant and isotropic at these boundaries, hence
\[
\hat{\mathcal{L}}_1 = \begin{pmatrix}
-\frac{\partial}{\partial x_\alpha} (C_{\alpha \beta} \hat{C}_{\beta \beta}^{-1}) \\
\end{pmatrix},
\]
\[
\hat{\mathcal{L}}_2 = \begin{pmatrix}
-j \omega \hat{\rho} I + \frac{1}{j \omega} \frac{\partial}{\partial x_\alpha} (U_{\alpha \beta} \frac{\partial}{\partial x_\beta}) \\
\end{pmatrix},
\]
\[
\hat{\mathcal{L}}_2 = \begin{pmatrix}
-j \omega \hat{C}_{\alpha \alpha}^{-1} \\
\end{pmatrix},
\]
\[
\hat{\mathcal{L}}_2 = \begin{pmatrix}
-j \hat{C}_{\alpha \beta} \frac{\partial}{\partial x_\beta} \\
\end{pmatrix},
\]
\[
U_{\alpha \beta} = \begin{pmatrix}
C_{\alpha \beta} - C_{\alpha \alpha} C_{\beta \beta}^{-1} C_{\beta \beta}
\end{pmatrix},
\]
For the 2-D situation (i.e., assuming that the field quantities and medium parameters are independent of the \(x_2\)-coordinate), the elastodynamic wave field decouples into horizontally polarized shear waves (SH-waves) and waves polarized in the vertical plane (P and SV waves). The matrices for SH-waves have been discussed in Appendix A. The matrices for P and SV waves can be obtained from equations (C28) and (C29) by setting \(k_2\) to zero and deleting all matrix elements that are equal to zero. The disadvantage of this approach is that the resulting matrices will contain terms proportional to the discontinuous functions \(k_r = |k_1|\) and \(k_1/k_1 = \text{sign}(k_1)\). As an alternative, we may define \(\mathcal{L}^\pm_1(k_1, x_3, \omega)\) and \(\mathcal{L}^\pm_2(k_1, x_3, \omega)\) as follows
\[
\mathcal{L}^\pm_1(k_1, x_3, \omega) = \frac{\mu}{\omega^{3/2}(2\rho)^{1/2}} \begin{pmatrix}
\pm k_1 \hat{H}_1^{1/2} \\
\end{pmatrix},
\]
\[
\mathcal{L}^\pm_2(k_1, x_3, \omega) = \frac{1}{\omega^{3/2}(2\rho)^{1/2}} \begin{pmatrix}
\pm k_1 \hat{H}_1^{1/2} \\
\end{pmatrix},
\]
With these choices the above mentioned discontinuities are avoided and equations (C8) and (C9) are again obeyed.

With the matrices discussed in this appendix, wave field decomposition at $\partial D_1$ and $\partial D_m$ is accomplished through

$$\tilde{P} = \tilde{L}^{-1} \tilde{Q},$$

or, applying an inverse spatial Fourier transform,

$$\tilde{P}(x, \omega) = \left( \frac{1}{2\pi} \right)^2 \int_{\mathbb{R}^2} \tilde{L}^{-1} \tilde{Q} \exp\{-j k_H \cdot x_H\} d^2 k_H.$$

This decomposition is exact when the medium parameters at $\partial D_1$ and $\partial D_m$ are laterally invariant (everywhere else the medium can be arbitrarily inhomogeneous). When the medium parameters at $\partial D_1$ and $\partial D_m$ are smoothly varying, this equation can still be used in an approximate sense. To this end $\tilde{L}^{-1}(k_H, x_3, \omega)$ should be replaced by $\tilde{L}^{-1}(k_H, x, \omega)$, based on the local medium parameters at $x$ on $\partial D_1$ and $\partial D_m$. 


The extraction of the Green’s function and the generalized optical theorem

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ABSTRACT
The relationship between the Green’s function and the average cross-correlation of a diffuse field is shown to be equivalent to the generalized optical theorem. The correspondence to the usual optical theorem holds for the extraction of the imaginary component of the Green’s function at the source for general linear scalar systems from the autocorrelation of field fluctuations. The imaginary component of the Green’s function at the source is finite and shown to be related to the loss in energy (acoustic waves) and probability (quantum mechanics) due to radiation and attenuation.

Key words: interferometry, autocorrelation of noise

1 INTRODUCTION
The extraction of the Green’s function from ambient fluctuations within a diffuse field has received considerable attention for different research areas (Weaver, 2005; Larose et al., 2006b). It has been shown that the extraction of the Green’s function can be carried out for a wide variety of systems (Snieder et al., 2007; Wapenaar et al., 2006). For scalar systems that are invariant to time-reversal, theory relates the imaginary part of the Green’s function to a bilinear combination of the Green’s function (Snieder et al., 2007):

\[ \text{Im}(G_{\mathbf{r}_A, \mathbf{r}_B}) = \int G(\mathbf{r}_A, \mathbf{r})O(\mathbf{r})G^*(\mathbf{r}_B, \mathbf{r})dS, \quad (1) \]

where \( O(\mathbf{r}) \) is an operator that follows from theory, and \( \text{Im} \) denotes the imaginary part. A similar expression holds for systems that are not invariant for time reversal, but for such systems a volume integral with the same functional form as the surface integral should be added to the right hand side of this expression (Snieder et al., 2007). Equation (1) forms the basis for extraction of the Green’s function from ambient fluctuations. When spatially uncorrelated noise with power spectrum \(|S(\omega)|^2\) excites a field \( u(\mathbf{r}) \), then the Green’s function follows from (Larose et al., 2006b; Snieder et al., 2007; Sánchez-Sesma & Campillo, 2006)

\[ \text{Im}(G_{\mathbf{r}_A, \mathbf{r}_B}) \propto \frac{\langle u(\mathbf{r}_A)u^*(\mathbf{r}_B) \rangle}{|S(\omega)|^2}, \quad (2) \]

where \( \langle \cdots \rangle \) denotes a source-average. In practice this source average is replaced by an average over a set of non-overlapping windows (Larose et al., 2006b). The right-hand side corresponds, in the frequency domain, to the cross correlation of the field measured at locations \( \mathbf{r}_A \) and \( \mathbf{r}_B \), respectively. By setting \( \mathbf{r}_A = \mathbf{r}_B \) the right-hand side reduces to the autocorrelation of the field fluctuations, while the left-hand side gives the imaginary component of the Green’s function at the source. This quantity measures the return of waves to the source location and can be used to study localization (Larose et al., 2004; Larose et al., 2006a).

This relationship between the imaginary part of the Green’s function at the source and radiated energy has been shown earlier for the special case of elastic waves in a homogeneous space (Weaver, 1985; Sánchez-Sesma et al., 2007). The presence of a free surface and of scatterers has been explored for both scalar and elastic sys-
2 THE IMAGINARY PART OF GREEN’S FUNCTION AT THE SOURCE

Consider, first, the damped oscillator: \( mx + \gamma \dot{x} + sx = f \), with \( m, \gamma \) and \( s \) the mass, damping and stiffness parameters, respectively. Using the Fourier convention \( f(t) = (2\pi)^{-1} \int f(\omega) \exp(-i\omega t) d\omega \), the Green’s function satisfies the relation \( G = 1/(s - i\gamma - m\omega^2) \) in the frequency domain, which implies that \( \text{Im}(G) = \gamma |G|^2 \).

The imaginary part of the Green’s function thus is proportional to the rate of dissipation by this one-degree-of-freedom system. We next derive a general expression that relates the imaginary part of the Green’s function at the source to both the radiated and dissipated power lost by a unit harmonic source.

Consider the following scalar partial differential equation for a field \( u \) that is of \( N \)th order in the time derivatives

\[
\left( a_N(r, t) \frac{\partial^N}{\partial t^N} + \cdots + a_2(r, t) \frac{\partial^2}{\partial t^2} + a_1(r, t) \frac{\partial}{\partial t} \right) u(r, t) = H(r, t) \star u(r, t) + q(r, t),
\]

where \( \star \) denotes convolution in time, while the operator \( H \) contains the space derivatives of the field equation. The field is excited by the forcing \( q(r, t) \). Using the above Fourier convention, this equation corresponds in the frequency domain to

\[
\sum_{n=1}^{N} a_n(r, \omega) (-i\omega)^n u(r, \omega) = H(r, \omega) u(r, \omega) + q(r, \omega).
\]

Henceforth the derivation is in the frequency domain, and we suppress the frequency-dependence. The operator \( H \) is not necessarily self-adjoint over the volume \( V \) under consideration, and, following ref. \((\text{Snieder et al., 2007})\), we define the bilinear form \( L \) by \( \int_V (f(Hg) - L(f)g) \, dV \equiv \int_{\partial V} L(f)g \, dS \), where the surface integral is over the surface \( \partial V \) that bounds the volume \( V \) with respect to.

Now assume that the \( q(r) \) are spatial delta functions \( \delta(r - r_{A,B}) \) at locations \( r_A \) and \( r_B \), respectively, with harmonic time variation. The fields \( u_A \) and \( u_B \) are then the Green’s functions \( G(r, r_{A,B}) \). Using reciprocity (\( G(r_A, r_B) = G(r_B, r_A) \)) and setting \( r_A = r_B = r_0 \) it follows from expression (16) of ref. \((\text{Snieder et al., 2007})\) that, for the general system (3):

\[
\text{Im}(G(r_0, r_0)) = -\sum_{n \text{ odd}} (-1)^{(n+1)/2} \omega^n \int_V \text{Re}(a_n) |G(r_0, r)|^2 dV
\]

\[
-\sum_{n \text{ even}} (-1)^n \omega^n \int_V \text{Im}(a_n) |G(r_0, r)|^2 dV
\]

\[
-\frac{i}{2} \int_{\partial V} L(G^*(r_0, r), G(r_0, r)) dS
\]

\[
+ \int_V G(r_0, r) \text{Im}(H) G^*(r_0, r) dV,
\]

where \( \text{Re} \) denotes the real part. In order to see its meaning we next present three examples.

3 DAMPED ACOUSTIC WAVES

These waves satisfy the following partial differential equation

\[
\kappa \frac{\partial^2 p}{\partial t^2} = \nabla \cdot \left( \frac{1}{\rho} \nabla p \right) + q,
\]

now \( p \) is pressure, \( \rho \) mass density, and \( \kappa \) compressibility. For attenuating media, \( \kappa \) is complex and frequency-dependent. In the notation of the general equation (3), \( a_2 = \kappa \), and \( H = \nabla \cdot (\rho^{-1} \nabla) \); all other \( a_n \) vanish. Using Green’s theorem, the bilinear form \( L \) for this example is given by \( L(f, g) = \rho^{-1} (f(\partial g/\partial n) - (\partial f/\partial n)g) \).

We assume that the boundary \( \partial V \) is a sphere in the far field and that radiation boundary conditions hold on this boundary; that is, \( \nabla p = ikp \). In this case \((-i/2) \int_{\partial V} L(p^*, p) dS = \omega \int (pe)^{-1} |p|^2 dS \), where \( c = \omega/k \) is the speed of sound, and the general equation (5) reduces to

\[
\omega^{-1} \text{Im}(G(r_0, r_0)) = \omega \int_V \text{Im}(\kappa) |G(r_0, r)|^2 dV
\]

\[
+ \int_{\partial V} \frac{\kappa}{\rho} |G(r_0, r)|^2 dS.
\]

The first term in the right-hand side gives the energy dissipated within the volume \( V \), while the last term accounts for the energy flux through the boundary \( \partial V \). The imaginary part of the Green’s function at the source thus accounts for the two ways in which energy injected into the medium by the source leaves the volume \( V \).

Note that the volume \( V \) need not be the volume of all space; it can be an arbitrary sub-volume, and its boundary \( \partial V \) need not be a physical boundary. When \( V \) is a volume with radius much less than the extinction length \( l_{ext} = [\text{Im}(\sqrt{\kappa})/\omega]^{-1} \), most of the energy is lost through radiation through the boundary \( \partial V \), and the second term in the right-hand side dominates. In contrast, where the volume defines a region much larger than the extinction length, the first term in the right-hand side of expression (7) dominates.
4 QUANTUM MECHANICS

For this case, the field is governed by Schrödinger’s equation:

\[ i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi. \quad (8) \]

In the notation of equation (3), \( H = -(\hbar^2/2m)\nabla^2 + V \), \( a_1 = i\hbar \), and all other \( a_n \) are equal to zero. Using Green’s theorem it follows that \( L(\psi^*, \psi) = -(\hbar^2/2m) (\psi^*(\partial\psi/\partial n) - (\partial\psi^*/\partial n)\psi) \). This quantity is related to the probability density current \( J = (\hbar/2m) (\psi^*\nabla\psi - \psi\nabla\psi^*) \) (Merzbacher, 1970). Inserting these results in expression (5) gives

\[ -\frac{2}{\hbar} \text{Im} (G(r_0, r_0)) = \int_{\partial V} \mathbf{J}_G \cdot d\mathbf{S}, \quad (9) \]

where \( \mathbf{J}_G \) is the probability density current associated with the Green’s function \( G \). The right-hand side gives the probability that the particle leaves the volume \( V \). Here, the imaginary part of the Green’s function is not equal to the energy loss; but instead corresponds to a loss in probability.

When waves with power spectrum \( |S(\omega)|^2 \) strike the scatterer, the imaginary component of the Green’s function follows from the autocorrelation of the field fluctuations (Snieder et al., 2007):

\[ -\frac{2}{\hbar^2} \text{Im} (G(r_0, r_0)) = \frac{k}{m|S(\omega)|^2} \langle |\psi(r_0, \omega)|^2 \rangle. \quad (10) \]

The combination of expressions (9) and (10) shows that the probability that particles leave the volume \( V \) is related to the fluctuations in the probability density \( \langle |\psi(r_0, \omega)|^2 \rangle \).

5 A SIMPLE ACOUSTIC MODEL

As a prototype of the potential applications, consider the problem of acoustic waves propagating within a layer of thickness \( h \). At the upper and lower edge, the pressure gradient is assumed to vanish \( (\partial p/\partial n) = 0 \); hence the reflection coefficient is equal +1 at the boundaries. It follows from the method of images that the wave-field at the source location for a source placed at the top of the layer is given by

\[ G(0, 0) = \frac{2\rho}{4\pi} \left( \lim_{r \to 0} \frac{e^{ikr}}{r} + 2\frac{e^{i2kh}}{2h} + 2\frac{e^{i4kh}}{4h} + \cdots \right). \quad (11) \]

The overall factor of 2 results because the source radiates only into the plate rather than in all directions, while the factor 2 for the backscattered waves arises from the interaction of the waves with the surface (where the receiver is located). The first term in the right-hand side gives the direct wave at the source in infinite space, corrected with a factor 2 that accounts for the boundary condition. Since this wave is singular, it is written in the form of a limit. The subsequent terms in expression (11) come from the successive waves that bounce back and forth within the layer.

The first term is twice the Green’s function in a homogeneous space \( G_0(r) = \rho \exp(ikr)/4\pi r \). While at the source \( (r = 0) \) this Green’s function is singular, the imaginary part of this Green’s function, given by \( \text{Im} (G_0(r)) = \rho \sin(kr)/4\pi r \), tends to the finite value \( \rho k/4\pi \) as \( r \to 0 \).

For the layer model, the imaginary component of the Green’s function is given by

\[ \text{Im} (G(0, 0)) = \frac{2\rho k}{4\pi} \left( 1 + \frac{2\sin(2kh)}{2kh} + \frac{2\sin(4kh)}{4kh} + \cdots \right). \quad (12) \]

This function, normalized to the corresponding value \( \text{Im} (G_0(0, 0)) = \rho k/4\pi \) for the free-space Green’s function, is shown in figure 1 as a function of normalized frequency \( f \tau \), where \( \tau = 2h/c \) is the two-way vertical travel time in the layer. The peaks in the response curve correspond to the resonances of the layer at frequencies \( f_n = n\tau = nc/2h \), with \( n \) an integer. Since energy is radiated sideways in the layer, the peaks have nonzero width. According to expression (2), the imaginary part of the Green’s function can, in general, be extracted from field fluctuations using the relation

\[ \text{Im} (G(r_0, r_0)) \propto \frac{\langle |u(r_0)|^2 \rangle}{|S(\omega)|^2}. \quad (13) \]

The right-hand side depends on the autocorrelation \( \langle |u(r_0)|^2 \rangle \) of recorded field fluctuations. This means that properties of the medium, in this case the travel-time \( h/c \) through the layer, can be inferred from the autocorrelation of field fluctuations.
6 DISCUSSION

This theorem, which concerns the scattering amplitude \( A_k(n, n') \) of scattered waves with wave number \( k \), and \( n \) and \( n' \) unit vectors representing the directions of the incoming and outgoing waves, respectively, states that (Heisenberg, 1943)

\[
\text{Im} (A_k(n_A, n_B)) = \frac{k}{4\pi} \int A_k(n_A, n) A_k^*(n_B, n) d^2n. \tag{14}
\]

This expression has the same form as equation (1) for the extraction of the Green’s function. The only difference is that equation (1) holds in the space domain, while the generalized optical theorem (14) is formulated in the wave-number domain. The optical theorem follows from expression (14) by setting \( n_A = n_B = n_0 \):

\[
\sigma_e = \int |A_k(n_0, n)|^2 d^2n = \frac{4\pi}{k} \text{Im} (A_k(n_0, n_0)). \tag{15}
\]

where \( \sigma_e \) is the extinction cross section, and \( A_k(n_0, n_0) \) is the scattering amplitude in the forward direction. This theorem, which relates the radiation loss by scattering to the properties of the forward-scattered wave (Newton, 1976), is equivalent to the general expression (5) that relates the imaginary component of the Green’s function at the source to the loss of generalized energy.

For uncorrelated noisy sources with power spectrum \(|S(\omega)|^2\) on a large sphere surrounding the scattering object (Wapenaar et al., 2005), equation (14) can be written analogously to equation (2) as

\[
\text{Im} (A_k(n_A, n_B)) = \frac{k}{4\pi|S(\omega)|^2} \langle \psi^s(n_A) \psi^{s*}(n_B) \rangle, \tag{16}
\]

where \( \psi^s(n) \) is the scattered field in the direction \( n \) excited by the noise. By setting \( n_A = n_N = n_0 \), the extinction cross section is given by

\[
\sigma_e = \frac{4\pi}{k} \text{Im} (A_k(n_0, n_0)) = \frac{|\langle \psi^s(n_0) \rangle|^2}{|S(\omega)|^2}. \tag{17}
\]

The scattering amplitude and scattering cross section thus can be inferred from the auto-correlation of field fluctuations. Note the resemblance between expressions (10) and (17).

Discussion. – The generalized optical theorem and the equations for the extraction of the Green’s function have the same functional form. Both equations express the same physical principle, but in different spaces. The generalized optical theorem is given in wave-number space while the imaginary part of Green’s function is expressed in position space. There is an even deeper connection with scattering theory, because these expressions are equivalent to relations used in the Marchenko equation for inverse scattering (e.g., ref. (Budreck & Rose, 1992)).

The optical theorem and the general expression for the imaginary part of Green’s function at the source account for generalized energy loss at the source. We use the term general energy, because the imaginary component of the Green’s function is related to integrals that are quadratic in the field representing energy for acoustic waves and probability in quantum mechanics.

In general, one cannot measure the Green’s function at the source because it is singular. The imaginary part of the Green’s function, however, is regular. As shown in expression (13), this quantity can be inferred from the autocorrelation of measured field fluctuations. The autocorrelation of observed field fluctuations has been used for monitoring a volcano and a fault zone during an earthquake (Sens-Schönfelder & Wegler, 2006) and to study the coherent backscattering effect for ultrasound (Larose et al., 2006a). The example in the previous section illustrates that the formal relationship, presented here, between the autocorrelation of the field fluctuations and the imaginary part of the Green’s function can be used to infer properties of the medium using field fluctuations recorded by a single sensor.

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The cancellation of spurious arrivals in Green’s function extraction and the generalized optical theorem

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ABSTRACT

The extraction of the Green’s function by cross-correlation of waves recorded at two receivers nowadays finds much application. We show that for an arbitrary isolated scatterer, the cross-terms of scattered waves give an unphysical wave with an arrival time that is independent of the source position. This constitutes a paradox because theory predicts that such spurious arrivals do not arise, after integration over a complete source aperture. The paradox can be resolved by integrating the contribution of all sources in the stationary phase approximation to show that the stationary phase contributions to the source integral cancel the spurious arrival by virtue of the generalized optical theorem. This work constitutes an alternative derivation of this theorem. When the source aperture is not complete, the spurious arrival is not canceled and could be misinterpreted to be part of the Green’s function.

Key words: interferometry, Green’s function extraction

1 INTRODUCTION

In recent years the extraction of the Green’s function from field fluctuations has received considerable attention. This technique is described in recent tutorials [Curtis et al., 2006; Larose et al., 2006], and is in the seismic community known as seismic interferometry. The Green’s function can be retrieved by cross-correlating the fields recorded at two receivers. This approach can be used to extract the impulse response from field fluctuations from thermal noise exciting elastic waves [Weaver & Lobkis, 2001], from oceanic noise exciting surface waves [Shapiro et al., 2005; Sabra et al., 2005], from turbulent flow over an airfoil [Sabra et al., 2008], from chaotic earthquake signals [Snieder & Şafak, 2006; Mehta et al., 2007], or from skeletal muscle noise [Sabra et al., 2007]. Alternatively, one can use controlled sources. In this case the advantage of extracting the impulse response from cross-correlation lies in the removal of the imprint of medium complexity between the sources and the receivers [Bakulin & Calvert, 2006], or in a more optimal illumination of the target [van Wijk, 2006; Hornby & Yu, 2007].

Even though the theory of Green’s function extraction is well-developed and numerous applications have been implemented, there are puzzling open questions; this work presents one of those questions. The Green’s function $G$ can be extracted by cross-correlating field fluctuations in two locations $\mathbf{r}_A$ and $\mathbf{r}_B$. In the frequency domain the expression (21) of ref. [Wapenaar
& Fokkema, 2006] is:
\[
\frac{1}{\rho(r)} \int G^*(r_B, r) \nabla G(r_A, r) - G(r_A, r) \nabla G^*(r_B, r) \cdot \hat{n} dS = 2i \text{Im}(G(r_A, r_B)),
\]
where \(\rho(r)\) is the density, \(\hat{n}\) is the normal outward on the integration surface. For sources far from the receivers \((r \gg r_{A,B})\) the Green’s function satisfies a radiation boundary condition, so that for a spherical surface with a normal vector in the radial direction \(\nabla G(r_{A,B}, r) = ikG(r_{A,B}, r)\hat{n}\). Using this radiation boundary condition gives, for a constant wave-number and density on the surface:
\[
\oint G(r_A, r)G^*(r_B, r) dS = -\frac{\rho}{k} \text{Im}(G(r_A, r_B)),
\]
where \(k\) is the wavenumber and \(\text{Im}\) denotes the imaginary part. Expression (2) is applicable for the case of acoustic waves treated here. By replacing \(\rho/k\) in the right hand side by \(kh^2/m\), the analysis is equally applicable to quantum mechanics [Snieder et al., 2007].

The imaginary part of the Green’s function can be written as the sum of the Green’s function and its complex conjugate, which corresponds to the time domain to the superposition of the causal and a-causal Green’s function. This reflects the well-known fact that the cross-correlation leads to the superposition of the causal Green’s function and its a-causal counterpart (e.g., [Wapenaar et al., 2005; Lobkis & Weaver, 2001]). Expression (2) forms the basis for the Green’s function retrieval from the cross-correlation of waves excited by uncorrelated sources on a closed surface surrounding the observation points [Wapenaar et al., 2005]. A similar relation is valid for open systems where the surface integration needs be replaced by an integration over all angles of incidence [Weaver & Lobkis, 2004]. Expression (2) contains a surface integral. The counterpart of this expression for general linear systems contains a volume integral as well [Wapenaar et al., 2006; Snieder et al., 2007; Weaver, 2008].

According to equation (2), the Green’s function can be found by cross-correlating the waves excited by sources on a closed surface. We present in section 2 a paradox that suggests that unphysical arrivals arise from expression (2). We resolve this paradox in section 3, and illustrate this with a numerical example in the subsequent section. This work is not only of academic interest, we discuss the implications for practical applications of the Green’s function retrieval in the conclusion. Details of the employed stationary phase approximations are shown in an appendix.

2 A PARADOX

We consider the special case of an isolated scatterer as shown in figure 1 with scattering amplitude \(f_k(\hat{n}, \hat{n}'\) for incident waves with wavenumber \(k\) traveling in the \(\hat{n}'\)-direction that are scattered in the \(\hat{n}\)-direction. The scatterer may have a finite extent and does not need to be either isotropic or weak. The subsequent analysis is in the frequency domain, and, because \(k\) is constant at a fixed frequency, we suppress the subscript \(k\) in the following.

The scattered waves travel over a time \(t_A\) from the scatterer to the receiver at \(r_A\) and in a time \(t_B\) from the scatterer to the receiver at \(r_B\). The Green’s function contains a scattered wave that propagates from \(r_A\) via the scatterer to \(r_B\). The arrival time of this scattered waves is given by the sum \(t_A + t_B\). The cross-correlation of these waves gives, in the time domain, a wave arriving at the difference of these arrivals times, hence it gives after cross-correlation a wave arriving at time \(t_A - t_B\). Note that this time difference is the same for any source location \(r\). Following this argument, the correlation of the scattered waves in expression (2) gives a wave arriving at a time \(t_A - t_B\) at which no physical wave arrives. We call such a wave a spurious arrival. Since this arrival has the same travel time for all source positions \(r\) it appears that there is no reason why this arrival vanishes by averaging over all source positions. In the following section we investigate this paradox by a detailed evaluation of the integral in expression (2).

3 STATIONARY PHASE EVALUATION OF THE INTEGRAL

In our analysis we follow the treatment of van der Hulst [van der Hulst, 1949] and assume sources far away \((r \gg r_{A,B})\), consistent with our assumption in equation (2). This makes it possible to evaluate the surface integral with the stationary phase analysis. The stationary phase approximation is exact in the limit \(r \to \infty\). The waves
excited by a point source at \( \mathbf{r} \) recorded at locations \( \mathbf{r}_{A,B} \) is given by

\[
G(\mathbf{r}_A, \mathbf{r}) = -\frac{\rho}{4\pi} e^{ik|\mathbf{r} - \mathbf{r}_A|} - \frac{\rho}{4\pi} \frac{e^{ikr}}{r} f(\mathbf{r}_A, -\hat{\mathbf{r}}) e^{ikr_A} \frac{\mathbf{G}(\hat{\mathbf{r}}_A)}{r_A}, \tag{3}
\]

\[
G(\mathbf{r}_B, \mathbf{r}) = -\frac{\rho}{4\pi} e^{ik|\mathbf{r} - \mathbf{r}_B|} - \frac{\rho}{4\pi} \frac{e^{ikr}}{r} f(\mathbf{r}_B, -\hat{\mathbf{r}}) e^{ikr_B} \frac{\mathbf{G}(\hat{\mathbf{r}}_B)}{r_B}. \tag{4}
\]

The cross correlation of these fields corresponds, in the frequency domain, to

\[
\int G(\mathbf{r}_A, \mathbf{r}) G^*(\mathbf{r}_B, \mathbf{r}) dS
\]

\[
= \frac{\rho^2}{(4\pi)^2} \int \frac{\exp(ik(|\mathbf{r} - \mathbf{r}_A| - |\mathbf{r} - \mathbf{r}_B|))}{|\mathbf{r} - \mathbf{r}_A| |\mathbf{r} - \mathbf{r}_B|} dS
\]

\[
+ \frac{\rho^2}{(4\pi)^2} \int \frac{\exp(ik(|\mathbf{r} - \mathbf{r}_A| - t - \mathbf{r}_B))}{|\mathbf{r} - \mathbf{r}_A| t r_B} f(\mathbf{r}_A, -\hat{\mathbf{r}}) dS
\]

\[
+ \frac{\rho^2}{(4\pi)^2} \int \frac{\exp(-ik(|\mathbf{r} - \mathbf{r}_B| - t - \mathbf{r}_A))}{|\mathbf{r} - \mathbf{r}_B| t r_A} f(\mathbf{r}_A, -\hat{\mathbf{r}}) dS
\]

\[
+ \frac{\rho^2}{(4\pi)^2} \int \frac{\exp(ik(|\mathbf{r} - \mathbf{r}_A - \mathbf{r}_B|))}{t^2 r_{AB}} f(\mathbf{r}_A, -\hat{\mathbf{r}}) f(\mathbf{r}_B, -\hat{\mathbf{r}}) dS. \tag{5}
\]

The term \( T_1 \) represents the cross-terms of the direct waves at the two receivers, the terms \( T_2 \) and \( T_3 \) represent cross-terms of the direct wave and a scattered wave, while the term \( T_4 \) accounts for the cross-term of the scattered waves. Note that for the latter term the phase is for every integration point \( \mathbf{r} \) given by \( \exp(ik(r_A - r_B)) \), hence it is this term that corresponds, in the time domain, to the spurious arrival at time \( t_A - t_B \).

We carry out the surface integrals using a system of spherical coordinates as shown in figure 2. Without loss of generality we use a coordinate system with the scatterer centered on the origin and where the points \( \mathbf{r}_A \) and \( \mathbf{r}_B \) are located in the \((x, z)\)-plane with \( x_A > x_B \) and \( z_B > z_A \). In this coordinate system:

\[
\mathbf{r}_A = r_A \begin{pmatrix} \sin \theta_A \\ 0 \\ \cos \theta_A \end{pmatrix},
\]

\[
\mathbf{r}_B = r_B \begin{pmatrix} \sin \theta_B \\ 0 \\ \cos \theta_B \end{pmatrix}, \tag{6}
\]

\[
\mathbf{r} = \begin{pmatrix} \sin \theta \cos \varphi \\ \sin \theta \sin \varphi \\ \cos \theta \end{pmatrix}.
\]

![Figure 2. Definition of geometric variables.](image)

For sources located far away \( (r \gg r_{A,B}) \), this gives to first order in \( r_{A,B}/r \):

\[
|r - r_{A,B}| = r - r_{A,B} (\cos \varphi \sin \theta \sin \theta_{A,B} + \cos \theta \cos \theta_{A,B}) \tag{7}
\]

We use this approximation in the exponents of expression (5) while we replace \( |r - r_{A,B}| \) in the denominators by \( r \). With these replacements the incoming waves effectively are plane waves, which make our analysis applicable for the treatment of the extraction of the Green’s function from incoming plane waves [Weaver & Lobkis, 2004] as well. The surface integral is related to an integration over solid angle by the relation \( dS = r^2 d\Omega \). Making these simplifications, expression (5) reduces to

\[
\int G(\mathbf{r}_A, \mathbf{r}) G^*(\mathbf{r}_B, \mathbf{r}) dS
\]

\[
= \frac{\rho^2}{(4\pi)^2} \int \frac{\exp(ikL_1)}{T_1} d\Omega
\]

\[
+ \frac{\rho^2}{(4\pi)^2} \int \frac{\exp(ikL_2)}{r_B} f(\mathbf{r}_B, -\hat{\mathbf{r}}) d\Omega
\]

\[
+ \frac{\rho^2}{(4\pi)^2} \int \frac{\exp(-ikL_3)}{r_A} f(\mathbf{r}_A, -\hat{\mathbf{r}}) d\Omega
\]

\[
+ \frac{\rho^2}{(4\pi)^2} \int \frac{\exp(ik(r_A - r_B))}{r_{AB}} f(\mathbf{r}_A, -\hat{\mathbf{r}}) f(\mathbf{r}_B, -\hat{\mathbf{r}}) d\Omega. \tag{8}
\]

with

\[
L_1 = |r - r_A| - |r - r_B|, \tag{9}
\]

\[
L_2 = |r - r_A| - r - r_B. \tag{10}
\]
the complex conjugate of the contribution from point B.

This term accounts for the complex conjugate of the scattered wave \(G_S\) that propagates between the points \((r_B, r_A)\).

The stationary phase point 2B of term T2 gives the correlation between the direct wave arriving from that point at location \(r_A\) and a scattered wave that propagates from the stationary phase point 2B through the scatterer to \(r_B\). As shown in the appendix, the contribution of this stationary phase point is given by

\[
T_{2B} = -\frac{i\rho^2}{8\pi k} \frac{\exp(ik(r_A - r_B))}{r_A r_B} f^*(\hat{r}_B, \hat{r}_A).
\]  

This term is not a physical arrival because there is no wave that arrives with a phase given by the length difference \(r_A - r_B\). Note that the phase of this term is identical to the phase of the spurious arrival \(T4\) in equation (8).

The contribution of term T3 is due to the stationary points 3A and 3B in figure 3 that have the same physical interpretation as the stationary phase contribution of the points 2A and 2B, respectively for term T2. Term T3 follows most simply by interchanging A and B in term T2 and by taking the complex conjugate. It thus follows from expressions (14) and (15) that

\[
T_3 = \frac{i\rho}{2k} G_S(r_A, r_B) + \frac{\rho^2}{8\pi k} \frac{\exp(ik(r_A - r_B))}{r_A r_B} f(\hat{r}_A, \hat{r}_B).
\]  

Taking all contributions into account by summing expressions (12) through (16) with term T4 from equation (8), using reciprocity \((G(r_1, r_2) = G(r_2, r_1))\) and \(f(\hat{r}_1, \hat{r}_2) = f(\hat{r}_2, \hat{r}_1))\), and replacing the integration variable \(\hat{r}\) by \(-\hat{r}\) gives

\[
\hat{f} G(r_A, \hat{r}) G^*(r_B, \hat{r}) d\hat{S} = -\frac{\rho}{k} Im (G(r_A, r_B)) + \rho^2 \frac{\exp(ik(r_A - r_B))}{4\pi k r_A r_B}
\]

\[
\times \left[ -Im (f(\hat{r}_A, \hat{r}_B)) + \frac{k}{4\pi} \hat{f} f(\hat{r}_A, \hat{r}) f^*(\hat{r}_B, \hat{r}) d\hat{\Omega} \right],
\]  

where \(G = G_0 + G_S\) is the sum of the direct and scattered waves.

The last term in expression (17) contains the spurious event discussed in section 2 that, having in the time domain a travel time \(t_A - t_B\), does not correspond to a physical arrival. Perhaps surprisingly, there are two terms within the square brackets that arise from different stationary-phase arrivals from the different terms in the cross-correlation. For this spurious arrival to cancel, and to make expression (17) equal to the general relation (2), the terms within the square brackets must vanish, hence:

\[
Im (f(\hat{r}_A, \hat{r}_B)) = \frac{k}{4\pi} \hat{f} f(\hat{r}_A, \hat{r}) f^*(\hat{r}_B, \hat{r}) d\hat{\Omega}.
\]  

This relation is known as the generalized optical theorem [Heisenberg, 1943; Glauber & Schomaker, 1953; Schiff,
1968] which guarantees that the spurious arrivals in expression (17) cancel.

4 NUMERICAL EXAMPLE

We illustrate the cancellation of the spurious arrival with a numerical example based on the Spectral-Element Method. This is a high-order variational numerical technique [Priolo et al., 1994; Faccioli et al., 1997] that combines the flexibility of the finite-element method with the accuracy of global pseudo-spectral techniques. Widely used in seismology [Komatitsch & Tromp, 2002; Komatitsch et al., 2002], we use the Spectral Element Method to simulate wave propagation in an acoustic model that contains an isolated scatterer. The numerical example is in two dimensions, but it shows the same behavior as the theory derived above for three dimensions. The model parameters are given the following. Background velocity and density of the 1500 m by 3 m model is 1000 kg/m$^3$ and 1500 m/s, respectively, and a single square scatterer (10 m by 10 m) is located in the origin. This scatterer has a velocity of 3000 m/s and a density of 2000 kg/m$^3$. 720 sources are distributed evenly on a circle with a radius of 500 m from the center of the scatterer. The source wavelet is a Ricker wavelet with a central frequency of 50 Hz. In the Spectral Element Method, we use Lagrange polynomials of degree $N=4$ to interpolate the wave field in each quadrangular cell; the total number of spectral-elements is 90601. The time step used in the explicit integration scheme is $\Delta t = 0.1$ ms and we propagate the signal for 0.53 s. The wave field is recorded at two receivers $\mathbf{r}_A = (900, 750)$ and $\mathbf{r}_B = (750, 950)$. The geometry of the experiment is drawn in Figure 4, indicating that $G(\mathbf{r}_A, \mathbf{r}_B)$ contains a direct arrival at $t \approx 0.17$ s, and a scattered event at $t \approx 0.23$ s.

The top panel of Figure 5 contains the cross-correlations of the waves recorded at $\mathbf{r}_B$ with $\mathbf{r}_A$, for each source. The bottom panel displays the sum of the cross-correlations over all sources. The sinusoidal features in the top panel are the causal and a-causal direct and scattered events with stationary points around $t \approx \pm 0.17$ s, and $t \approx \pm 0.23$ s, respectively. For example, the stationary point 1A gives the causal direct wave arriving at $t \approx 0.17$ s, while the stationary point 1B gives the a-causal direct wave at $t \approx -0.17$ s. Similarly, the stationary points 2A and 3A gives the causal and a-causal scattered waves arriving at $t \approx \pm 0.23$ s. Of special interest is the arrival, marked with the label “4” at a constant travel time of about 0.03 s. This is the spurious arrival described in the previous sections. This spurious arrival is canceled by the contribution of the stationary points 2B and 3B. Indeed, there is no spurious arrival in the bottom panel at arrival time 0.03 s.

This numerical example thus confirms that the spurious arrival vanishes because of the destructive interference of the wave with constant arrival time (marked with the label “4”) with two stationary phase contributions 2B and 3B. Note that in this example we did not specify the scattering amplitude $f_0(\mathbf{n}, \mathbf{n}')$. Its properties are implicitly accounted for by the spectral element code.

In Figure 6 we show the sum (in red) over the sources located along the lower half of the circle, and the sum (in blue) over the sources in the upper half circle. The sources along the lower half circle show the causal direct wave at $t \approx 0.17$ s and the causal scattered wave at $t \approx 0.23$ s, but not their a-causal counterparts. The

![Figure 4. Geometry of the numerical example. The 720 sources are located on a circle with radius 500 m. The scatterer in the origin has dimensions 10 m by 10 m and is not shown to scale. All dimensions are in meters.](image)

![Figure 5. Top panel: cross-correlation of the waves recorded at the receivers as a function of time (horizontal axis) and source number (vertical axis) where the sources are numbered counterclockwise from east. The labels at the stationary points are the same as in figure 3. The color bar on the left side in the upper panel indicates the subsets of sources used in figure 6. Bottom panel: the cross-correlations after summation over all sources.](image)

![Figure 6.](image)
reason is that the sources on the lower half circle only launch direct and scattered waves that propagate from receiver A to receiver B. The sources along the upper half circle launch waves in the reverse direction, and give the a-causal direct and scattered waves shown in blue.

The spurious arrival is marked with the label “S”. Both subsets of sources give a nonzero spurious arrival. The contribution from the sources at the lower half mostly contribute to the constant-time arrival marked with “4” in figure 5, while the sources from the lower half mostly contribute to the stationary phase points 2B and 3B in figure 5. As shown in figure 6, each of these contributions is nonzero, but they do cancel when summed. The abrupt truncation of the sum over sources leads to additional truncation phases marked with “T”, that individually are non-zero, but whose sum vanishes as well. Truncation phases result from the dominant end-point contributions of oscillatory integrals [Bender & Orszag, 1978], and are a known complication in modeling waveforms with the reflectivity method [Bardick & Orcutt, 1979]. Truncation phases also occurred in field application of the extraction of the Green’s function from ocean-bottom seismic data [Mehta et al., 2008]. Note that the arrival time of the truncation phases depends on the employed sources, while the arrival time of the spurious arrival is the same for every truncated source distribution. Also, the truncated phases can be suppressed by suitable tapering of the source strength, but this does not suppress the spurious arrival.

5 CONCLUSION
This work has two implication. First, the comparison of expressions (2) and (17) shows that the generalized optical theorem holds. This work thus constitutes an alternative derivation of the generalized optical theorem, although Heisenberg’s original derivation [Heisenberg, 1943], which was based on the unitary of the scattering matrix, is much simpler.

The second implication of this work is that, as shown in figure 6, for a limited source aperture the spurious arrivals do not vanish. This is the same phenomena as the spurious arrivals that arise through cross-terms of single reflected waves from different layers in the subsurface [Snieder et al., 2006]. In seismic applications, those cross-terms could be misinterpreted as reflectors in the subsurface.

From a practical point of view the spurious arrivals, as those shown in figure 6, are undesirable because these waves could be mistaken for true physical waves that are part of the Green’s function. One should be aware of these spurious arrivals whenever the source distribution used for the extraction of the Green’s function does not fulfill theoretical requirements.

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REFERENCES
Spurious arrivals and Green’s function extraction


APPENDIX A: EVALUATION OF THE STATIONARY PHASE INTEGRALS

This section features details of the various stationary phase integrations starting with term T1 in equation (8). Using the vectors in expression (6), the length L1 of equation (9) is given by

$$L_1 = \cos \varphi \sin \theta (r_B \sin \theta_B - r_A \sin \theta_A)$$

$$+ \cos \theta (r_B \cos \theta_B - r_A \cos \theta_A)$$

$$= (x_B - x_A) \cos \varphi \sin \theta + (z_B - z_A) \cos \theta .$$

(A1)

The integration is over the angles $\theta$ and $\varphi$ and the stationary points are determined by the conditions

$$\frac{\partial L_1}{\partial \varphi} = -(x_B - x_A) \sin \varphi \sin \theta = 0 ,$$

(A2)

$$\frac{\partial L_1}{\partial \theta} = (x_B - x_A) \cos \varphi \cos \theta - (z_B - z_A) \sin \theta = 0 .$$

(A3)

These expressions determine the angles $\theta_0$ and $\varphi_0$ for which the phase of term T1 is stationary. The first condition gives $\sin \varphi_0 = 0$, hence the stationary points are located in the $(x, z)$-plane.

In the following we treat the stationary point $\varphi_0 = 0$, the contribution of the stationary point $\varphi_0 = \pi$ follows by complex conjugation. For $\varphi_0 = 0$, expression (A3) gives

$$\tan \theta_0 = \frac{x_B - x_A}{z_B - z_A} .$$

(A4)

This angle is depicted in figure A1 where it can be seen that the stationary phase point is aligned with the points $r_A$ and $r_B$.

Differentiation of expression (A3), and using the geometry of figure A1, gives the second derivative at the stationary phase point

$$\frac{\partial^2 L_1}{\partial \theta^2} = -(x_B - x_A) \sin \theta_0 - (z_B - z_A) \cos \theta_0$$

$$= -(x_B - x_A) z_B - z_A \left| \frac{r_B - r_A}{r_B - r_A} \right| - (z_B - z_A) \left| \frac{z_B - z_A}{r_B - r_A} \right|$$

$$= -|r_B - r_A| .$$

(A5)

Differentiation of expression (A2) gives at the stationary
The stationary phase approximation of the derivative vanishes at the stationary phase point:
\[ \theta = \theta_A. \]  
It follows from the geometry of figure A1 that at the stationary phase point
\[ L_1 = |r - r_A| - |r - r_B| = |r_A - r_B|. \]  
The stationary phase approximation of the \( \theta \) - and \( \varphi \)-integration applied to the term \( T_1 \) of expression (8) gives [Bleistein & Handelsman, 1975; Bender & Orszag, 1978]
\[ T_{1A} = \frac{\rho^2}{(4\pi)^2} \exp(ik|r_A - r_B|) \times \left( e^{-i\pi/4} \right)^2 \sqrt{\frac{2\pi}{k|\partial^2 L_1/\partial \varphi^2|}} \sqrt{\frac{2\pi}{k|\partial^2 L_1/\partial \theta^2|}} \sin \theta_A. \]  
The factors \( e^{-i\pi/4} \) arise because the second derivatives both are negative. The \( \sin \theta \) term comes from the Jacobian in the angular integration. With expressions (A5) and (A6) term \( T_{1A} \) reduces to
\[ T_{1A} = -\frac{i\rho^2}{8\pi k} \exp(-ik(r_A - r_B)) \frac{\sin \theta_A}{\sqrt{|r_A - r_B|}} \sqrt{\frac{\sin \theta_A|x_A - x_B|}{z_A}}. \]  
As shown in figure A1, \( \sin \theta_A = (x_B - x_A)/|r_A - r_B|. \) Inserting this in expression (A10) gives equation (12). Term \( T_{1B} \) in equation (13) follows by complex conjugation.

Next we treat the contribution of point 2A to term \( T_2 \) of expression (8). Using expression (6), the length \( L_2 \) is given by
\[ L_2 = -r_A \cos \varphi \sin \theta \sin \theta_A - r_A \cos \theta \sin \theta_A - r_B \]  
\[ = -x_A \cos \varphi \sin \theta - z_A \cos \theta - r_B. \]  
The stationary phase condition for the angle \( \varphi \) gives \( \partial L_2/\partial \varphi = x_A \sin \varphi \sin \theta = 0 \), which implies that the stationary point lies in the \( (x, z) \)-plane: \( \sin \varphi = 0 \). We first analyze the point \( \varphi = 0 \). For this point, the stationary phase condition for the variable \( \theta \) is:
\[ \partial L_2/\partial \theta = -x_A \cos \theta + z_A \sin \theta = 0. \]  
This gives the stationary point
\[ \tan \theta_A = \frac{x_A}{z_A}. \]  
This stationary phase point is sketched in figure A2. An analysis similar as for term \( T_1 \) shows that at this stationary point: \( L_2 = -r_A - r_B \), \( \hat{r} = -r_A \), \( \partial^2 L_2/\partial \varphi^2 = r_A \), and \( \partial^2 L_2/\partial \theta^2 = x_A \sin \theta_A \). The stationary phase contribution from point 2A to term \( T_2 \) of expression (8) thus is given by
\[ T_{2A} = \frac{i\rho^2}{8\pi k} \frac{\exp(-ik(r_A + r_B))}{r_B\sqrt{r_A}} \frac{\sin \theta_A}{\sqrt{\sin \theta_A x_A}}. \]  
According to figure A2, \( x_A = r_A \sin \theta_A \), which leads to expression (14).

The stationary phase point \( 2B \) corresponds to \( \varphi = \pi \). The stationary phase condition for \( \theta \) is in this case \( \partial L_2/\partial \theta = x_A \cos \theta + z_A \sin \theta = 0 \). This gives the stationary point
\[ \tan \theta_A = -\frac{x_A}{z_A}. \]  
This point is sketched in figure A3. Using that \( \theta_A = \pi - \theta_A \), it follows that at this stationary point:
\[ L_2 = r_A - r_B, \]  
\[ \hat{r} = r_A, \ \partial^2 L_2/\partial \varphi^2 = -r_A, \]  
and \( \partial L_2/\partial \theta^2 = -x_A \sin \theta_A \).
The stationary phase contribution of this point is thus given by

\[
T_{2A} = -\frac{i\rho^2}{8\pi k} \frac{\exp\left(ik(r_A - r_B)\right)}{r_B\sqrt{r_A}} f^*(\hat{r}_B, \hat{r}_A) \frac{\sin \theta_A}{\sqrt{\sin \theta_s x_A}}.
\]  
(A15)

Using the geometric relation \(\sin \theta_A = x_A/r_A\) reduces expression (A15) to equation (15).