Comparison of seismic dispersion and attenuation models

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ABSTRACT

The frequency-dependent attenuation of seismic waves causes decreased resolution of seismic images with depth, and transmission losses induce amplitude variations with offset. Transmission losses may occur due to friction or fluid movement, or may result from scattering in thin layers. Whatever the physical mechanism, they can often be conveniently described using an empirical formulation wherein the elastic moduli and propagation velocity are complex functions of frequency.

We have compiled and compared algebraically and numerically eight different models involving complex velocity: the Kolsky-Futterman model, the power-law model, Kjartansson’s model, Müller’s model, Azimi’s second and third model, the Cole-Cole model, and the standard linear-solid (SLS) model.

For two different parameter sets, the attenuation and phase velocity are computed in the seismic frequency band, and the plane-wave propagation of a Ricker wavelet for the other models is compared with that for the Kolsky-Futterman model.

By selecting proper parameters, all models, except the standard linear-solid model, show behavior similar to that of the Kolsky-Futterman model. The SLS model behaves differently from the other models as the frequency goes to zero or infinity. Broadband measurement data are needed to select the appropriate model for a given seismic experiment.

Key words: attenuation, dispersion, numerical comparison

Introduction

The frequency-dependent attenuation of seismic waves causes decreased resolution of seismic images with depth, and transmission losses induce variations in amplitude with offset. It is important to understand such wave propagation phenomena so that they can be corrected for in seismic imaging and amplitude-versus-angle analysis.

Transmission losses may occur due to friction or fluid movement (Biot, 1962; Carcione, Gurevich, and Cavallini, 2000), and three-dimensional scattering, and multiple scattering in thin-layers (Backus, 1962; O’Doherty and Austey, 1971; Carcione, 1992; Shapiro, Hubral and Ursin 1996; Ursin and Stovas 2002). Instead of being based on a specific mechanism, attenuation in elastic solids has often been described by an empirical formulation using memory variables. In such a description, the elastic moduli are no longer constants, but rather act as time convolution operators. (Ben-Menahem and Singh, 1981; Fabrizio and Morro, 1992; Gurtin and Herrera, 1965; Carcione, 2001). When applying a Fourier transform with respect to time, the elastic moduli become complex functions of frequency. This results in a complex propagation velocity that gives rise to wave dispersion and attenuation. To preserve causality of a propagating wavelet, these phenomena must be related via the Hilbert transform (Aki and Richards, 1980). Further, to describe the behavior of a medium
that is both viscoelastic and anisotropic, several complex elastic moduli must be used (Carcione, 2001).

We describe eight different mathematical models for the complex elastic modulus or propagation velocity. The model that is most commonly used in seismology and seismic data processing is the Kolosky-Futterman (KF) model (Kolosky, 1956; Futterman 1962) wherein the attenuation coefficient is proportional to frequency. A constant-Q model (Kjartansson, 1979) is also frequently used (Hargreaves, 1992). The standard linear-solid (LS) model is preferred in finite-difference algorithms because it gives additional differential equations that can be approximated by finite differences (Emeridi and Korn, 1987; Causse and Ursin, 2000).

Any specific model that one chooses this must be related to experimental data. Using broadband VSP data, Kan, Batzle and Gaiser (1983) obtained Q values varying with frequency as \( \omega^{-\alpha} \), where \( \alpha \) is in the interval 0-0.4. Prasad and Meissner (1992) conducted laboratory experiments varying the attenuating medium rather than the frequency. They found dry rocks to yield high shear loss while saturated rocks yielded predominantly bulk losses. Leuer (1997) found the effective grain model, which is an extension of the Biot-Stoll model, to give the best fit for his laboratory measurements. Measuring attenuation in the frequency range 10-20 kHz, Klimenkov (1995) found P-wave attenuation to be higher in gas and condensate than in oil and water. For S-waves, he found no dependence on pore fluid. Kang and McMechan (1993) did 2-D modeling with memory variables to obtain better estimates of reflectivity amplitudes in viscoelastic media. Jacobsen (1987) used refraction data to obtain Q values varying with frequency as \( \omega^{-1/2} \) and \( \omega^{1/2} \) for two deep-sea fans in the Indian Ocean. In a series of laboratory measurements on creeps in rocks, Lomnitz (1956) found the creep to have a logarithmic dependence on time, yielding an attenuation coefficient that is almost linear in frequency. Pujol, Luo and Hu (1994) analyzed VSP data from Germany’s continental super deep borehole. They found Q values between 14 and 32 in the frequency range 7.8-46.9 Hz at depths between 3.6 and 4.5 km. Tóverud and Ursin (1999) fitted the parameters of the eight different models to zero-offset VSP data, obtaining equally good fit for all models. It seems therefore that broad-band data are needed in order to choose among the various models.

We compare the eight different attenuation models in order to reveal similarities and differences between them. This could suggest if the models can be used interchangeably or if some models have properties that set them apart from the others. Instead of comparing all models to each of the other models, we used the Kolosky-Futterman model as a reference. Consequently we put more confidence on this model. This is justified by the widespread use of this model in seismic processing, but also in seismology (Aki and Richards, 1980).

Also the standard linear-solid model is used frequently, and could have been applied if this were the only criterion. However, we preferred to use the KF-model since this model shows more similarity with the other models than does the standard linear-solid model.

Unlike most of the others, the standard linear-solid model has finite phase velocity and attenuation coefficient at infinite frequencies. Also unlike most of the others, its phase velocity differs from zero at zero frequency. More elements could have been included in the SLS model to give more freedom in defining its behaviour, but our intention was to compare models with the same number of parameters.

In order to compare the different models we consider plane-wave propagation in a homogeneous viscoelastic medium. First we compute the attenuation and phase velocity in the seismic frequency band for two different parameter sets; then we compare the propagation of a Ricker wavelet, using the KF model as a reference.

**Plane-wave dispersion and attenuation**

We consider a plane wave in a homogeneous viscoelastic medium. With the Fourier-transform defined as in Aki and Richards (1980, Box 5.2) this can be expressed by

\[
P(x, \omega) = P_0(\omega)e^{i[k(\omega)x - \omega t]},
\]

where \( P_0(\omega) \) is the Fourier transform of the propagating pulse, \( x \) is travel distance, \( t \) is time and \( \omega \) is radial frequency;

\[
k(\omega) = \frac{\omega}{c(\omega)} = \frac{\omega}{c_p(\omega)} + i\alpha(\omega)
\]

is a complex wavenumber. The complex propagation velocity is

\[
c(\omega) = \frac{\omega}{k(\omega)} = \sqrt{\frac{M(\omega)}{\rho}},
\]

where

\[
M(\omega) = M_R(\omega) + i M_I(\omega)
\]

is a complex modulus. In equation (2), the phase velocity \( c_p(\omega) \) and the attenuation \( \alpha(\omega) \) are real, positive and even functions of \( \omega \), as is \( M_R(\omega) \) in equation (4). The imaginary part of the modulus, \( M_I(\omega) \), is an odd function of \( \omega \), and is negative for \( \omega > 0 \).

For the transmission response of the medium to be causal, the phase velocity and attenuation must satisfy the Kramers-Kröning relations (Aki and Richards, 1980, Box 5.8)

\[
\frac{\omega}{c_p(\omega)} = \frac{\omega}{c_\infty} + \mathcal{H}\{\alpha(\omega)\},
\]

and

\[
\frac{\alpha(\omega) - \alpha(0)}{\omega} = -\mathcal{H}\left\{\frac{1}{c_p(\omega)} - \frac{1}{c_\infty}\right\},
\]
where $\mathcal{H}\{\}$ denotes the Hilbert transform, and $c_\infty = \lim_{\omega \to \infty} c_p(\omega)$. If these relations are fulfilled, then the transmission response is also minimum delay with respect to the minimum traveltime (first arrival time). The phase velocity is given by

$$
\frac{1}{c_p(\omega)} = \frac{1}{\omega} \text{Re}\{k(\omega)\},
$$

(7)

while the wavepacket, assuming low-loss solid, travels with the group velocity (Casula and Carcione, 1992)

$$
\frac{1}{c_g(\omega)} = \frac{d}{d\omega} \text{Re}\{k(\omega)\}
$$

$$
= \frac{1}{c_p(\omega)} \left[ 1 - \frac{\omega}{c_p(\omega)} \frac{dc_p(\omega)}{d\omega} \right].
$$

(8)

The quality factor $Q$ is the inverse of the dissipation factor

$$
\xi = Q^{-1} = -\frac{M_f}{M_R} \text{sgn}(\omega) = \left| \frac{M_f}{M_R} \right|.
$$

(9)

By squaring the inverse of the complex velocity and using equation (2)-(4) we obtain

$$
\frac{1}{c_p^2} - \frac{\alpha^2}{\omega^2} + 2i \frac{\alpha}{\omega c_p} = \frac{\rho}{M_R + i M_I} = \frac{\rho (M_R - i M_I)}{|M|^2}.
$$

(10)

This gives

$$
\alpha = -\frac{\omega M_f \rho c_p}{2|M|^2},
$$

(11)

and

$$
\frac{1}{c_p^2} = \frac{\rho (M_R + |M|)}{2|M|^2}.
$$

(12)

In terms of the dissipation factor we obtain, approximating for $\xi \ll 1$,

$$
c_p = \sqrt{\frac{M_R}{\rho}} \sqrt{\frac{2(1 + \xi^2)}{1 + \sqrt{1 + \xi^2}}} \approx \sqrt{\frac{M_R}{\rho}} \left(1 + \frac{3}{8} \xi^2\right),
$$

(13)

and

$$
\alpha = \xi |\omega| \sqrt{\frac{\rho}{M_R}} \left(2(1 + \xi^2)(1 + \sqrt{1 + \xi^2})\right)^{-1/2} \approx \frac{\xi |\omega|}{2} \frac{\rho}{M_R} \frac{1}{Q}.
$$

(14)

From equation (9) and (10), we obtain

$$
Q = \frac{|\omega|}{2\alpha c_p} - \frac{\alpha c_p}{2|\omega|}.
$$

(15)

We see that the approximation in equation (14) is valid for $\alpha c_p \ll 2|\omega|$, and then

$$
Q \approx \frac{|\omega|}{2\alpha c_p}.
$$

(16)

**Complex velocity models**

Several expressions have been given in the literature for the complex velocity of a viscoelastic solid. We shall briefly state some of these and later compare them in numerical studies. As a reference model we shall use the Kolsky-Futterman (KF) model (Kolsky, 1956; Futterman, 1962).

**The Kolsky-Futterman model**

The KF model is defined by

$$
\frac{1}{c_p(\omega)} = \frac{1}{c_r} + \frac{1}{\pi \alpha Q} \ln \left| \frac{\omega}{\omega_r} \right|,
$$

(17a)

and

$$
\alpha(\omega) = \frac{|\omega|}{2c_r Q_r}.
$$

(17b)

Here $c_r$ and $Q_r$ are the values of the phase velocity and approximate $Q$ at the reference frequency $\omega_r$. Using equation (16) we obtain

$$
Q = Q_r + \frac{1}{\pi} \ln \left| \frac{\omega}{\omega_r} \right|.
$$

(18)

Even though the KF model does not satisfy causality, it is widely used in seismic data processing and seismology (Aki and Richards, 1980).

**The power law**

The power law proposed by Strick (1967) and Azimi et al. (1968) has attenuation proportional to $|\omega|^\beta$. For $0 < \gamma < 1$ this model satisfies the Kramer-Krönig relations, which gives (with $\beta = 1 - \gamma$)

$$
\frac{1}{c_p(\omega)} = \frac{1}{c_\infty} + a|\omega|^{-\beta} \cot \left( \frac{\pi \beta}{2} \right)
$$

(19a)

for

$$
\alpha(\omega) = a|\omega|^{1-\beta} 0 < \beta < 1.
$$

(19b)

Then, from equation (16),

$$
Q \approx \frac{|\omega|^\beta}{2a c_\infty} + \frac{1}{2} \cot \left( \frac{\pi \beta}{2} \right).
$$

(20)

**Kjartansson’s model**

Kjartansson (1979) proposed a frequently used constant-$Q$ model. It is obtained from the power law by setting $c_\infty = \infty$. This gives

$$
\frac{1}{c_p(\omega)} = a|\omega|^{-\beta} \cot \left( \frac{\pi \beta}{2} \right)
$$

(21a)

and

$$
\alpha(\omega) = a|\omega|^{1-\beta}.
$$

(21b)

Equation (15) now gives

$$
Q = \cot(\pi \beta).
$$

(22)
Müller’s model

Müller (1983) proposed a model with

\[ Q(\omega) = \left( \frac{\omega}{\omega_0} \right)^\beta. \]  

(23)

For \(-1 \leq \beta \leq 1\). The complex modulus is given by

\[ M(\omega) = A(\omega) e^{i \phi(\omega)}, \]  

(24)

where

\[ \phi(\omega) = \arctan(Q^{-1}(\omega)), \]  

(25)

and \(\ln A(\omega)\) can be computed using the Hilbert transform (\(M(\omega)\) is minimum phase). For \(\beta = 0\) this gives Kjartansson’s model. For \(0 < \beta < 1\) and \(Q(\omega)\) large (\(\omega \gg \omega_0\)), the complex velocity is approximated by

\[ c(\omega) = c_\infty \exp \left( -\frac{1}{2} \left| \frac{\omega}{\omega_0} \right|^\beta \left[ \cot \left( \frac{\pi}{2} \beta \right) + \text{isgn} \omega \right] \right). \]  

(26)

When the exponential function is expanded in a first-order Taylor series, this gives exactly the power law with parameters \(c_\infty, \beta,\) and

\[ a = \frac{\left| \omega_0 \right|^\beta}{2c_\infty}. \]  

(27)

For \(-1 \leq \beta < 0\) and \(Q(\omega)\) large (\(\omega \ll \omega_0\), Müller’s model is approximately

\[ c(\omega) = c_0 \exp \left( -\frac{1}{2} \left| \frac{\omega}{\omega_0} \right|^\beta \left[ \cot \left( \frac{\pi}{2} \beta \right) + \text{isgn} \omega \right] \right), \]  

(28)

which is almost of the same form as equation (26), but \(\beta\) is now negative. We set \(\gamma = -\beta\) and obtain

\[ \frac{1}{c(\omega)} = \frac{1}{c_0} \exp \left( \frac{1}{2} \left| \frac{\omega}{\omega_0} \right|^\gamma \left[ -\cot \left( \frac{\pi}{2} \gamma \right) + \text{isgn} \omega \right] \right). \]  

(29)

With \(Q_1 = \left| \omega_0 \right|^\gamma\) and a Taylor expansion, this gives

\[ \frac{1}{c_p(\omega)} \approx \frac{\left| \omega \right|^\gamma}{c_0} \frac{\cot \left( \frac{\pi}{2} \gamma \right)}{2Q_1 \left| \omega_0 \right|}, \]  

(30a)

and

\[ a(\omega) \approx \frac{\left| \omega \right|^{1+\gamma}}{2\alpha_0 Q_1} \quad Q_1 \gg 1. \]  

(30b)

This is roughly a power law with attenuation proportional to \(\left| \omega \right|^{1+\gamma}, 0 < \gamma \leq 1\).

Azimi’s second model

Azimi et al. (1968) proposed three models that satisfy causality and are fairly close to the KF-model. The first model is the power law described previously. The second model is defined by

\[ \frac{1}{c_p(\omega)} = \frac{1}{c_\infty} - 2a \ln(a_2 |\omega|) \]  

\( \pi (1 - a_2^2 |\omega|^2) \)  

(31a)

and

\[ a(\omega) = \frac{a |\omega|}{1 + a_2 |\omega|}. \]  

(31b)

Azimi’s third model

Azimi et al. (1968) obtained their third model, given by

\[ \frac{1}{c_p(\omega)} = \frac{1}{c_\infty} + \frac{a a_3 \sqrt{|\omega|}}{1 + a_3^2 |\omega|} - 2a \ln(a_3^2 |\omega|) \]  

\( \pi (1 - a_3^2 |\omega|^2) \)  

(32a)

and

\[ a(\omega) = \frac{a |\omega|}{1 + a_3 \sqrt{|\omega|}}. \]  

(32b)
The Cole-Cole model

The power law will satisfy causality only for attenuation proportional to frequency to a power less than one. For exponents between one and two we may use the model proposed for dielectrics by Cole and Cole (1941) and extended to viscoelastic media by Jones (1986). The complex modulus is expressed as

\[ M(\omega) = M_0 \left(1 + \left(-i\omega\tau_r\right)^\beta\right) \left(1 + \left(-i\omega\tau_i\right)^\beta\right). \]  

(33)

For \( \tau_r = \tau_i = 0 \), this corresponds to an elastic solid. We shall assume that \( \tau_r < \tau_i \), and that \( 0 < \beta < 1 \). Then

\[ M_R = M_0 \frac{1 + (\omega^2 \tau_r \tau_i)^\beta + [\omega \tau_r]^\beta + [\omega \tau_i]^\beta \cos \left(\frac{\pi \beta}{2}\right)}{1 + 2\cos \left(\frac{\pi \beta}{2}\right) |\omega \tau_r|^\beta + |\omega \tau_i|^\beta}, \]  

(34a)

and

\[ Q^{-1} = \frac{\left(\omega^2 \tau_r \tau_i\right)^\beta - \omega \tau_i |\beta| \sin \left(\frac{\pi \beta}{2}\right)}{1 + (\omega^2 \tau_r \tau_i)^\beta + (\omega \tau_r |\beta| + \omega \tau_i |\beta|) \cos \left(\frac{\pi \beta}{2}\right)}, \]  

(34b)

from which we can compute the phase velocity and the attenuation using equations (13) and (14). Instead, we shall follow Casula and Carcione (1992) and write

\[ \tau_{r,\sigma} = \tau_r \sqrt{1 + \frac{1}{Q_c^2} \pm \frac{1}{Q_c}}, \]  

(35)

where we use \( Q_c \) to distinguish it from \( Q \) in the KE model. Assuming \( Q_c \) large this gives the approximations

\[ |\tau_{r,\sigma}| \approx |\tau_r| \left[1 + \frac{\beta^2}{2Q_c^2} \pm \frac{\beta}{Q_c}\right], \]  

(36)

Then we obtain

\[ M_R \approx M_0 \left[1 + \frac{\frac{2\beta}{Q_c} |\omega \tau_r|^\beta (\cos \left(\frac{\pi \beta}{2}\right) + (\omega \tau_i)^\beta)}{1 + |\omega \tau_r|^\beta (1 - \frac{\beta^2}{4Q_c^2}) + 2|\omega \tau_i|^\beta |\omega \tau_r|^\beta (1 - \frac{\beta^2}{4Q_c^2})}\right], \]  

(37a)

\[ Q^{-1} \approx \frac{\frac{2\beta}{Q_c} |\omega \tau_r|^\beta \sin \left(\frac{\pi \beta}{2}\right)}{1 + |\omega \tau_r|^\beta + 2|\omega \tau_i|^\beta \cos \left(\frac{\pi \beta}{2}\right)}. \]  

(37b)

Using equations (13) and (14) and approximating further gives

\[ \frac{1}{c_p(\omega)} \approx \frac{1}{c_0} \left[1 - \frac{\beta |\omega \tau_r|^\beta (\cos \left(\frac{\pi \beta}{2}\right) + |\omega \tau_i|^\beta)}{1 + |\omega \tau_r|^\beta + 2|\omega \tau_i|^\beta \cos \left(\frac{\pi \beta}{2}\right)}\right], \]  

(38a)

\[ \alpha(\omega) \approx \frac{\beta |\omega \tau_r|^\beta + \sin \left(\frac{\pi \beta}{2}\right)}{c_0 Q_c \tau_r} \]  

(38b)

When \( \omega \to 0 \), \( c_p \to c_0 \) and

\[ \alpha(\omega) \to |\omega \tau_r|^{1 + \beta} \sin \left(\frac{\pi \beta}{2}\right) \frac{\beta}{c_0 Q_c \tau_r}, \]  

(39)

and when \( \omega \to \infty \)

\[ c_p \to c_0 \left(1 + \frac{\beta}{Q_c}\right) \]  

(40a)

and

\[ \alpha(\omega) \to |\omega \tau_r|^{1 - \beta} \frac{\beta \sin \left(\frac{\pi \beta}{2}\right)}{c_0 Q_c \tau_r}. \]  

(40b)

For \( \beta \) very small we may expand the trigonometric functions in equation (36) in Taylor series and neglect second-order and higher terms. This gives, for \( 0 < \beta \ll 1 \),

\[ \frac{1}{c_p(\omega)} \approx \frac{1}{c_0} \left[1 - \frac{\beta |\omega \tau_r|^\beta}{Q_c (1 + |\omega \tau_i|^\beta)}\right]. \]  

(41a)
and

\[ \alpha(\omega) \approx \frac{\pi \beta^2}{2\alpha_0 Q_c \tau_r} \frac{|\omega \tau_r|^{1+\beta}}{(1 + |\omega \tau_r|^2)^2}. \]  \hspace{1cm} (41b)

For \( \beta \) close to unity, we set \( \beta = 1 - \gamma \) and use similar approximations for \( \gamma \) small. This gives

\[ \frac{1}{c_p(\omega)} \approx \frac{1}{\alpha_0} \left[ 1 - \beta \frac{|\omega \tau_r|^\beta}{Q_c} \frac{\pi \beta(1 - \beta) + |\omega \tau_r|^\beta}{1 + |\omega \tau_r|^2 + \pi(1 - \beta)|\omega \tau_r|^2} \right]. \]  \hspace{1cm} (42a)

\[ \alpha(\omega) \approx \frac{\beta |\omega \tau_r|^{1+\beta}}{\alpha_0 Q_c \tau_r [1 + |\omega \tau_r|^2 + \pi(1 - \beta)|\omega \tau_r|^2]}. \]  \hspace{1cm} (42b)

The standard linear-solid

The standard linear-solid (SLS) model (Ben-Menahem and Singh, 1981) is often used in finite-difference modeling because stress and strain are linked by differential operators. This model is obtained by setting \( \beta = 1 \) in the Cole-Cole model, giving the complex modulus

\[ M(\omega) = M_0 \frac{1 - i \omega \tau_r}{1 - i \omega \tau_r}. \]  \hspace{1cm} (43)

Special cases of the SLS model are the Maxwell model obtained when \( \tau_r = 0 \), which gives

\[ M(\omega) = \frac{M_0}{1 - i \omega \tau_r}, \]  \hspace{1cm} (44)

and the Kelvin-Voigt model obtained when \( \tau_r = 0 \) so that

\[ M(\omega) = M_0 (1 - i \omega \tau_v). \]  \hspace{1cm} (45)

Using \( \beta = 1 \) in (38), results in the approximations

\[ \frac{1}{c_p(\omega)} \approx \frac{1}{\alpha_0} \left[ 1 - \frac{(\omega \tau_r)^2}{Q_c [1 + (\omega \tau_r)^2]} \right], \]  \hspace{1cm} (46a)

and

\[ \alpha(\omega) \approx \frac{|\omega \tau_r|^2}{\alpha_0 Q_c \tau_r [1 + (\omega \tau_r)^2]}. \]  \hspace{1cm} (46b)

General linear models

The different models for plane-wave dispersion and attenuation that have been discussed so far may be too simple to describe physical wave propagation over a wide frequency range. It is therefore often suggested to use products and/or sums of these simple functions to obtain a more complicated behavior as a function of frequency (Ben-Menahem and Singh, 1981). For example, Liu et al. (1976) used a sum of standard linear-solids to obtain a model with constant \( Q \) over three decades of frequency.

Hanyga and Seredyńska (1999) proposed an interesting model that they claim is appropriate for porous media. When one replaces the Laplace transform variable \( s \) with \( i \omega \), their equation (13) can be written

\[ \left[ \nabla^2 + \frac{\omega^2}{c(\omega)^2} \right] \tilde{u} = 0 \]  \hspace{1cm} (47)

with

\[ \frac{1}{c(\omega)} = 1 + \frac{2a}{(-i \omega + \beta)^{1/2}} + \frac{b}{-i \omega + \beta} \]

\[ = (1 + \frac{b}{\beta}) \frac{1 - i \omega / (b + \beta)}{1 - i \omega / b} + \frac{2a}{(-i \omega + \beta)^{1/2}}. \]  \hspace{1cm} (48)

This shows that their model is a standard linear-solid in parallel with a non-rational function involving a square root. This model shows some similarity to the Cole-Cole model.
Seismic dispersion & attenuation models

<table>
<thead>
<tr>
<th>Model</th>
<th>Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>KF</td>
<td>$c_r$, $Q_r$, $\omega_r$</td>
</tr>
<tr>
<td>Power law</td>
<td>$c_\infty = c_r$, $a = \frac{1}{\pi Q_r}$, $\beta = \ln</td>
</tr>
<tr>
<td>Kjartanssen</td>
<td>$c_\infty = c_r$, $a = \frac{1}{\pi Q_r}$, $\beta = \ln</td>
</tr>
<tr>
<td>Müller</td>
<td>$c_\infty = c_r$, $\ln</td>
</tr>
<tr>
<td>Azimi's 2nd</td>
<td>$c_\infty = c_r$, $a = \frac{1}{\pi Q_r}$, $a_2 = \omega_r^2$</td>
</tr>
<tr>
<td>Azimi's 3rd</td>
<td>$c_\infty = c_r$, $a = \frac{1}{\pi Q_r}$, $a_3^2 = \omega_r^3$</td>
</tr>
<tr>
<td>Cole-Cole ($\beta$)</td>
<td>$c_0^{-1} = c_r^{-1}(1 + \frac{1}{\pi Q_r\beta})$, $Q_0 = \frac{\pi}{4}Q_r\beta^2$, $\tau_\beta = \omega_r^2$</td>
</tr>
<tr>
<td>SLS</td>
<td>$c_0^{-1} = c_r^{-1}(1 + \frac{1}{2\pi Q_r\beta})$, $Q_\ell = Q_r$, $\tau_\ell = \omega_r^2$</td>
</tr>
</tbody>
</table>

Table 1. Model parameters expressed in terms of the KF model parameters

<table>
<thead>
<tr>
<th>Model</th>
<th>Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>KF</td>
<td>$c_r = 2000$ m/s, $Q_r = 100$, $\omega_r = 2\pi \cdot 100$</td>
</tr>
<tr>
<td>Power law</td>
<td>$c_\infty = 2000$ m/s, $a = 2.5 \cdot 10^{-6}$, $\beta = 0.155$</td>
</tr>
<tr>
<td>Kjartanssen</td>
<td>$c_\infty = 2000$ m/s, $a = 2.5 \cdot 10^{-6}$, $\beta = 0.0031$</td>
</tr>
<tr>
<td>Müller</td>
<td>$c_\infty = 2000$ m/s, $\omega_0 = 1.25 \cdot 10^{-13}$, $\beta = 0.155$</td>
</tr>
<tr>
<td>Azimi's 2nd</td>
<td>$c_\infty = 2000$ m/s, $a = 2.5 \cdot 10^{-6}$, $a_2 = 1.6 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>Azimi's 3rd</td>
<td>$c_\infty = 2000$ m/s, $a = 2.5 \cdot 10^{-6}$, $a_3 = 0.04$</td>
</tr>
<tr>
<td>Cole-Cole</td>
<td>$c_0 = 1872$ m/s, $Q_\ell = \frac{4}{\pi}$, $\tau_\ell^{-1} = 2\pi \cdot 100$</td>
</tr>
<tr>
<td>SLS</td>
<td>$c_0 = 1990$ m/s, $Q_\ell = 100$, $\tau_\ell^{-1} = 2\pi \cdot 100$</td>
</tr>
</tbody>
</table>

Table 2. Model parameters fitted to the KF model as given in Table 1 ($\beta = 0.1$ for the Cole-Cole model).

Numerical comparison of phase velocities and attenuation

For each of the complex velocity models described in the previous section we have computed the phase velocity and attenuation in the frequency band 0-300 Hz. We use the KF model as a reference and compare the other models to it. In the appendix we have derived expressions for the parameters of the different models that make them behave similarly to the KF model. The results are summarized in Table 1 with $c_r = 2000$ m/s, $Q_r = 100$, and $\omega_r = 2\pi \cdot 100$ Hz in the KF model; this gives the parameter set 1 in Table 2. Comparison of the phase velocities and attenuation for the different models with these parameters and the KF model are shown in the upper part of Figures 1 to 7. The results for the KF model are plotted with dashed lines, while the results for the other models are plotted with solid lines. As seen in the figures, the different models are not close to the KF model for this parameter set.

To get a better fit to the KF model, we used parameter set 2 in Table 3 obtained using a trial and error algorithm. The results with these parameters are shown in the lower part of Figures 1 to 7. All models, except the SLS model shown in Figure 7, now seem to behave similarly to the KF model in the selected frequency range. The behavior of the SLS model can be explained by comparing equation (17) and (46). For low frequencies, attenuation in the SLS model is proportional to $|\omega|^2$, while for high frequencies it approaches a constant. The attenuation of the KF model is proportional to $|\omega|$.

<table>
<thead>
<tr>
<th>Model</th>
<th>Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>KF</td>
<td>$c_r = 2000$ m/s, $Q_r = 100$, $\omega_r = 2\pi \cdot 100$</td>
</tr>
<tr>
<td>Power law</td>
<td>$c_\infty = 2065$ m/s, $a = 4.76 \cdot 10^{-6}$, $\beta = 0.1$</td>
</tr>
<tr>
<td>Kjartanssen</td>
<td>$c_\infty = 2070$ m/s, $a = 2.7 \cdot 10^{-6}$, $\beta = 0.0034$</td>
</tr>
<tr>
<td>Müller</td>
<td>$c_\infty = 2035$ m/s, $\omega_0 = 1.0 \cdot 10^{-7}$, $\beta = 0.2$</td>
</tr>
<tr>
<td>Azimi's 2nd</td>
<td>$c_\infty = 2018$ m/s, $a = 2.86 \cdot 10^{-6}$, $a_2 = 1.51 \cdot 10^{-4}$</td>
</tr>
<tr>
<td>Azimi's 3rd</td>
<td>$c_\infty = 2070$ m/s, $a = 2.5 \cdot 10^{-6}$, $a_3 = 0.0002$</td>
</tr>
<tr>
<td>Cole-Cole</td>
<td>$c_0 = 2110$ m/s, $Q_\ell = 0.9348$, $\tau_\ell^{-1} = 3.95 \cdot 100$</td>
</tr>
<tr>
<td>SLS</td>
<td>$c_0 = 1985$ m/s, $Q_\ell = 84.71$, $\tau_\ell^{-1} = 6.75 \cdot 100$</td>
</tr>
</tbody>
</table>

Table 3. Model parameters for an improved fit to the KF model ($\beta = 0.12$ for the Cole-Cole model).
Waveform changes in plane-wave propagation

We compared the performances of the different models in a numerical plane-wave experiment using equation (1), using the Ricker wavelet

\[ P_0(t) = 4ba^2(t^2 - 1/2a) e^{-at^2} \cdot 10^{-7} \]

with parameters \( a = 25^2 \pi^2 s^{-2} \) and \( b = s^2 \), as shown to the left in Figure 8. The Fourier transform, \( P_0(\omega) \), is shown to the right in Figure 8. The wavelet was shifted 80 ms to the right and propagated 4.5 km for the different models. To compensate for the amplitude reduction due to attenuation, the traces were multiplied with \( (x + 0.8)^{1.4} \), where \( x \) is distance in kilometers. The results for the reference KF model, shown in Figure 9, exhibit the expected broadening of the wavelet due to attenuation. The dispersion, while not so apparent, can still be seen as a difference in the amplitudes of the two sidelobes of the wavelet. Plotted in Figure 10 to 16, are the results for the other models: results for parameter set 1 to the left and for parameter set 2 to the right. The top figures show the propagating wavelet and the bottom figures show these results subtracted from those for the KF model shown in Figure 9. As seen in the plots to the left, for parameter set 1, derived from the formulas in Table 1, all models show rather different wave propagation behavior, as compared to the KF model. The modified parameter set 2 shows attenuation that is similar to the KF model, as seen in the plots to the right. In Figure 14 it is seen that Azimi's third model gives almost identical results, which can be predicted from Figure 5. As seen in Figure 16, the SLS model gives results that deviate most from those of the KF model, consistent with the results in Figure 7.
Figure 2. Attenuation and phase velocity for Kjartansson's model (solid lines) compared with the Kolsky-Putterman model (dashed lines). Parameter set 1 is used at the top and parameter set 2 is used at the bottom.
Figure 3. Attenuation and phase velocity for Müller's model (solid lines) compared with the Kolsky-Futterman model (dashed lines). Parameter set 1 is used at the top and parameter set 2 is used at the bottom.
Figure 4. Attenuation and phase velocity for Azimi's second law model (solid lines) compared with the Kolsky-Futterman model (dashed lines). Parameter set 1 is used at the top and parameter set 2 is used at the bottom.
Figure 5. Attenuation and phase velocity for Azimi's third model (solid lines) compared with the Kolsky-Putterman model (dashed lines). Parameter set 1 is used at the top and parameter set 2 is used at the bottom.
Figure 6. Attenuation and phase velocity for the Cole-Cole model (solid lines) compared with the Kolsky-Putterman model (dashed lines). Parameter set 1 is used at the top and parameter set 2 is used at the bottom.
Figure 7. Attenuation and phase velocity for the standard linear-solid model (solid lines) compared with the Košky-Futterman model (dashed lines). Parameter set 1 is used at the top and parameter set 2 is used at the bottom.

Figure 8. Input seismic wavelet (left) and its amplitude spectrum (right).
Figure 9. Plane-wave propagation with the Kolsky-Futterman model.

Figure 10. Plane-wave propagation with the power law model (top) and difference plots using the Kolsky-Futterman model (bottom). Parameter set 1 is used to the left, and parameter set 2 is used to the right.
Figure 11. Plane-wave propagation with Kjartansson’s model (top) and difference plots using the Kolsky-Futterman model (bottom). Parameter set 1 is used to the left, and parameter set 2 is used to the right.
Figure 12. Plane-wave propagation with Müller’s model (top) and difference plots using the Kösky-Futterman model (bottom). Parameter set 1 is used to the left, and parameter set 2 is used to the right.
Figure 13. Plane-wave propagation with Azimi’s second model (top) and difference plots using the Košky-Futterman model (bottom). Parameter set 1 is used to the left, and parameter set 2 is used to the right.
Figure 14. Plane-wave propagation with Azimi’s third model (top) and difference plots using the Kolsky-Futterman model (bottom). Parameter set 1 is used to the left, and parameter set 2 is used to the right.
Figure 15. Plane-wave propagation with the Cole-Cole model (top) and difference plots using the Kolsky-Putterman model (bottom). Parameter set 1 is used to the left, and parameter set 2 is used to the right.
Figure 16. Plane-wave propagation with the standard linear-solid model (top) and difference plots using the Košky-Futterman model (bottom). Parameter set 1 is used to the left, and parameter set 2 is used to the right.
Conclusions

We have compiled and compared algebraically and numerically eight different complex velocity models. The widely used KF model was used as a reference model. By selecting proper parameters, all models except the SLS model behave similarly to the KF model in the seismic frequency range. The SLS model behaves differently from the other models as the frequency goes to zero or infinity. Broadband measurement data are needed to select a specific model for a given seismic experiment.

Acknowledgment

T. Toverud would like to thank VISTA for financial support.

References

Kjartansson, E., 1979, Constant Q-wave propagation and attenuation: Journal of Geophysical Research, 84, 4737–4748.
Prasad, M., and Meisner, R., 1992, Attenuation mecha-
Kjartansson’s model in equation (21) becomes similar to the KF model if we choose $a$ from equation (A-2) and
\[
\beta^{-1} = \pi Q_r + \ln |\omega_r|.
\] (6)
This gives
\[
\frac{1}{c_p(\omega)} = \frac{|\omega|^{-\beta}}{2c_r Q_r} \cot \left( \frac{\pi \beta}{2} \right)
\] (7a)
and
\[
\alpha(\omega) = \frac{|\omega|^{1-\beta}}{2c_r Q_r},
\] (7b)
with $\beta$ given above in equation (A-6).

For Müller’s model we combine equations (26) and (27) with (A-1) to (A-3). This gives the complex velocity $c(\omega) = c_r \exp(-\frac{|\omega|^{-\beta}}{2Q_r} \cot \left( \frac{\pi \beta}{2} \right) + i \text{sgn} \omega)$, (8)
where $\beta$ is given by equation (A-3). We have used
\[
\ln |\omega| = -\ln |\omega_r| \ln Q_r.
\] (9)

Azimi’s second model is similar to the KF model if we use equations (A-1), (A-2) and $a_2 = \omega_r^{-1}$. This gives
\[
\frac{1}{c_p(\omega)} = \frac{1}{c_r} + \frac{1}{\pi c_r Q_r} \ln \frac{|\omega|}{\omega_r} \frac{1}{1 - \left( \frac{\omega_r}{\omega} \right)^2}
\] (10a)
and
\[
\alpha(\omega) = \frac{|\omega|}{2c_r Q_r \left( 1 + \left( \frac{\omega}{\omega_r} \right)^2 \right)}.
\] (10b)

For Azimi’s third model we use equations (A-1), (A-2) and $a_3 = \omega_r^{-1}$. This gives
\[
\frac{1}{c_p(\omega)} = \frac{1}{c_r} + \frac{1}{\pi c_r Q_r} \ln \frac{\omega_r}{\omega} \frac{1}{1 - \left( \frac{\omega_r}{\omega} \right)^2} + \frac{|\omega|^2}{2c_r Q_r \left( 1 + |\frac{\omega_r}{\omega}|^2 \right)^2}
\] (11a)
and
\[
\alpha(\omega) = \frac{|\omega|}{2c_r Q_r \left( 1 + |\frac{\omega_r}{\omega}|^2 \right)^2}.
\] (11b)

In the expression for the inverse of the phase velocity in Azimi’s second and third model there appears the function
\[
F(\omega) = \frac{\ln |\frac{\omega}{\omega_r}|}{1 - \left( \frac{\omega}{\omega_r} \right)^2}.
\] (12)
Using l’Hopital’s rule we obtain that $F(\omega) \to 0$ as $|\omega| \to \infty$. When $\omega$ is close to $\omega_r$, we let $\omega = \omega_r + \Delta \omega$. Expanding the logarithmic function in a Taylor series gives
\[
F(\omega) \approx \frac{1}{2} \left[ 1 - \frac{\Delta \omega}{\omega_r} \right],
\] (13)
which shows that $F(\omega) \to \frac{1}{2}$ as $\omega \to \omega_r$. 

**APPENDIX : COMPARISON WITH THE KOLSKY-FUTTERMAN MODEL**

Here, we show our approach for selecting parameters in the different models for complex propagation velocity so that they become similar to those of the KF model defined by equations (17a) and (17b). To make the power-law model similar to the KF-model we choose
\[
c_\infty = c_r
\] (1)
\[
a = \frac{1}{2c_r Q_r}
\] (2)
\[
\beta^{-1} = \ln |\omega_r|,
\] (3)
which gives
\[
\frac{1}{c_p(\omega)} = \frac{1}{c_r} + \frac{|\omega|^{-\beta}}{2c_r Q_r} \cot \left( \frac{\pi \beta}{2} \right)
\] (4a)
and
\[
\alpha(\omega) = \frac{|\omega|^{1-\beta}}{2c_r Q_r}.
\] (4b)
We shall assume that $\beta$ is small, so that
\[
|\omega|^{-\beta} \approx 1 - \beta \ln |\omega|
\] (5a)
and
\[
\cot \frac{\pi \beta}{2} \approx \frac{2}{\beta r}.
\] (5b)
With $\beta$ given by equation (A-3), the modulus in equation (17a) and (A-4) become fairly close.
For the Cole-Cole model we shall assume that $\beta$ is small and given. Furthermore, we choose

$$\tau_r = \omega_r^{-1}. \quad (14)$$

By using the Taylor series

$$|\omega \tau_r|^\beta = \frac{\omega}{\omega_r} \approx 1 + \beta \ln \frac{\omega}{\omega_r} \quad (15)$$

in equation (38) and comparing with equation (17) we obtain

$$\frac{1}{c_0} = \frac{1}{c_r} \left( 1 + \frac{2}{\pi \beta Q r} \right) \quad (16a)$$

and

$$Q_c = \frac{\pi}{4} Q_r \beta^2 + \frac{1}{2} \beta \approx \frac{\pi}{4} Q_r \beta^2. \quad (16b)$$

Substituting these values into equation (38) gives

$$\frac{1}{c_p(\omega)} \approx \frac{1}{c_r} \left[ 1 + \frac{2}{\pi \beta Q_r} \frac{1 - |\omega_r|^\beta}{1 + |\omega_r|^\beta} \right] \quad (17a)$$

and

$$\alpha(\omega) \approx \frac{2|\omega_r|}{c_r Q_r} \frac{|\omega_r|^{1+\beta}}{(1 + |\omega_r|^\beta)^2}. \quad (17b)$$

It is rather difficult to match the SLS model to the KF model. As an attempt we choose $\tau_r$ from equation (A-14) and then match the phase velocity and attenuation at $\omega_r$. This gives $Q_c = Q_r$ and

$$\frac{1}{c_0} = \frac{1}{c_r} \left( 1 + \frac{1}{2Q_r} \right). \quad (18)$$

Substituting these values into equation (44) results in

$$\frac{1}{c_p(\omega)} \approx \frac{1}{c_r} + \frac{1}{2c_r Q_r} \frac{1 - |\omega_r|^2}{1 + |\omega_r|^2} \quad (19a)$$

and

$$\alpha(\omega) \approx \frac{\omega_r}{c_r Q_r} \frac{|\omega_r|^2}{1 + |\omega_r|^2}. \quad (19b)$$