Time-domain finite-difference modeling for attenuative anisotropic media
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SUMMARY

We present a 2D time-domain finite-difference algorithm for simulating multicomponent data in viscoelastic transversely isotropic media with a vertical symmetry axis (VTI). The generalized standard linear solid (GSLS) model is employed to extend the definitions of the relaxation function and the \( \tau \)-parameter (which quantifies the difference between the stress and strain relaxation times) to anisotropic media. This approach produces nearly constant values of all components of the quality-factor matrix within a specified frequency band. The developed numerical implementation is based on a set of anisotropic viscoelastic wave equations parameterized by memory variables. Application of the spectral-ratio method to synthetic data generated by our algorithm for a layered VTI medium confirms the accuracy of the proposed scheme. The method is also tested on a viscoelastic version of the salt section of the BP TI model.

INTRODUCTION

Viscoelastic properties of subsurface formations have a profound influence on wave propagation and seismic processing. The attenuation-induced amplitude loss and velocity dispersion can cause distortions in amplitude-variation-with-offset (AVO) analysis and imaging. However, attenuation can also provide valuable information about lithology and fluids needed for reservoir characterization.

A prerequisite for accurate attenuation analysis and estimation is efficient viscoelastic modeling (e.g., Shekar and Tsukan, 2014). Time-domain finite-difference (FD) methods have been widely used to model wave propagation in attenuative isotropic media (Day and Minster, 1984; Emmerich and Korn, 1987; Carcione, 1993; Blanch et al., 1995; Bohlen, 2002; Zhu et al., 2013). A nearly constant quality factor \( Q \) over a specified frequency range can be simulated by mechanical models. Memory variables are introduced into the corresponding convolutional stress-strain relationship to facilitate numerical implementation (Robertsson et al., 1994).

The attenuation coefficient is often directionally dependent, and attenuation anisotropy is typically much stronger than velocity anisotropy (Hosten et al., 1987; Zhu and Tsukan, 2006; Behura and Tsukan, 2009a). In one of the few published attempts to include attenuation anisotropy in FD modeling, Mitte and Renline (1996) simulate acoustic full-waveform multipole logging. However, they do not give a clear description of the employed rheological model.

Here, we develop a 2D time-domain FD method designed to simulate P- and SV-waves for models with VTI symmetry for both velocity and attenuation. We present a formalism for generating a nearly constant \( Q \)-factor in a specified frequency band and incorporate it into the viscoelastic wave equation suitable for FD implementation.

METHODOLOGY

Rheology of anisotropic viscoelastic model

The stiffness matrix \( C_{ij} \) in viscoelastic media becomes complex, and attenuation can be described by the quality-factor matrix \( Q_{ij} \) (Carcione, 2007; Zhu and Tsukan, 2006):

\[
Q_{ij} = \frac{\text{Re}(C_{ij})}{\text{Im}(C_{ij})},
\]

In the time domain, attenuation is introduced through the so-called relaxation function \( \Psi \):

\[
\Psi_{ij}(t) = F^{-1}\left\{ \frac{C_{ij}(\omega)}{i\omega} \right\},
\]

where \( F^{-1} \) denotes the inverse Fourier transform, and both \( \Psi_{ij} \) and \( C_{ij} \) are expressed in the two-index Voigt notation. The generalized stress-strain relationship in linear viscoelastic media can be written as:

\[
\sigma_{mn} = \Psi_{mnpq} \epsilon_{pq} = \Psi_{mnpq} \epsilon_{pq},
\]

where the asterisk and dot denote convolution and time derivative, respectively. Equation 3 shows that the stress tensor is determined by the entire history of the strain field, rather than by just its current value, which is the case for purely elastic media.

The relaxation function, which determines the viscoelastic behavior of the material, can be simulated by the so-called generalized standard linear solid (GSLS) model. A single standard linear solid (SLS) consists of two parallel mechanical systems, with one made of a spring and a dashpot in series and the other containing a single spring (Blanch et al., 1995). Several SLS’s in parallel constitute the GSLS, with each individual SLS called a “relaxation mechanism.” Moczo et al. (2007) give an expression for the relaxation function in isotropic media, which we extend to anisotropic models:

\[
\Psi_{ij}(t) = C_{ij} \left[ 1 - \frac{1}{L} \sum_{l=1}^{L} \left( 1 - \frac{\tau_{ij}^{sl}}{\tau_{ij}^{sl}} \right) e^{-t/\tau_{ij}^{sl}} \right] H(t),
\]

where \( C_{ij} \) is called the “relaxed modulus,” \( \tau_{ij}^{sl} \) and \( \tau_{ij}^{sl} \) are the strain and stress relaxation times (respectively) for the \( l \)th mechanism, \( H(t) \) is the Heaviside function, and \( L \) is the number of mechanisms. Generally, the more relaxation mechanisms (or SLS’s) are included, the wider is the frequency range in which it is possible to simulate a nearly constant \( Q_{ij} \). For different components of the anisotropic relaxation tensor
In equation 4, the relaxed modulus $C_{er}$ generally differ (Tromp et al., 2005). The reference frequency is defined at the center of each cell (staggered grid (SSG). In RSG, the particle velocity is assumed, and $\tau_{nj}$ are the memory variables for the $n$th mechanism. The Einstein summation convention over $p$ and $q$ ($p = 1, 3; q = 1, 3$) is assumed, and $mn = 11, 13, 33$.

Viscoelastic VTI wave equation and FD implementation

Using equations 3, 4, and 6, we derive the viscoelastic wave equations for 2D VTI media:

$$\sigma_{mn} = \frac{1}{2} C^R_{mnpq}(v_{p,q} + v_{q,p}) + \sum_{l=1}^{L} r^l_{mn},$$

and

$$r^l_{mn} = -\frac{1}{2} \sum_{i,j} \left( C^U_{mnpq} - C^R_{mnpq} \right) \times (v_{p,q} + v_{q,p}) + r^l_{mn},$$

where $C^U_{mnpq}$ is the relaxation function (equation 4) at zero time and is called the “unrelaxed modulus,” $C^R_{mnpq}$ is the relaxed modulus defined in equation 5, $v_{p,q}$ is the derivative of the $p$th component of the particle velocity with respect to $x_q$, and $r^l_{mn}$ are the memory variables for the $l$th mechanism. The Einstein summation convention over $p$ and $q$ ($p = 1, 3; q = 1, 3$) is assumed, and $mn = 11, 13, 33$.

Equations 8 and 9 allow us to carry out time-domain FD modeling for media with VTI symmetry for both velocity and attenuation. The stress-velocity formulation is adopted here because of its natural connection to staggered grids (Moczo et al., 2007), which generally provide higher numerical accuracy. For anisotropic media, a rotated staggered grid (RSG) (Saenger et al., 2000; Saenger and Bohlen, 2004) is preferable to the standard staggered grid (SSG). In RSG, the particle velocity and density are defined at the center of each cell (staggered grid point), while other parameters including stress, memory variables, stress relaxation time, and $\tau_{nj}$ are assigned to regular grid points. The two sets of parameters are related through FD operators in the auxiliary diagonal directions $\hat{x}$ and $\hat{z}$, as discussed by Saenger et al. (2000).

SYNTHETIC EXAMPLES

Validation test

To check the accuracy of the developed FD algorithm, we apply it to generate the wavefield in a two-layer viscoelastic VTI medium and then estimate the P-wave attenuation coefficient using the spectral-ratio method:

$$\ln \frac{U^{(1)}(\omega)}{U^{(0)}(\omega)} = G - 2\pi Ap ft,$$

where $U^{(1)}(\omega)$ and $U^{(0)}(\omega)$ are the frequency spectra of the vertical component of the P-wave reflection data for the viscoelastic (Figure 2 and Table 1) and reference elastic medium, respectively. The factor $G$, which is assumed to be frequency-independent, accounts for the source radiation pattern, geometric spreading, and reflection/transmission coefficients, $t$ is

Figure 1: Simulated $Q_{ij}$-curve (dashed line) in the frequency range from 2 to 200 Hz. The desired value of $Q_{ij}$ is 30 (solid line). The inverted parameters are: $\tau_j = 0.2124$, $\sigma_{ij} = 22.7$ ms, $\sigma_{ij}^2 = 3.13$ ms, and $\sigma_{ij}^2 = 2 \times 10^{-3}$ ms.

$\Psi$, the stress relaxation times can be identical, while $\tau_{ij}$ generally differ (Tromp et al., 2005).

In equation 4, the relaxed modulus $C^R_{ij}$ is related to the real part of the corresponding complex modulus $C_{ij}$ at the reference frequency $\omega_r$:

$$C^R_{ij} = \text{Re} (C_{ij}) \left[ \sum_{l=1}^{L} \frac{1 + \omega_r^2 \tau_{ij} \tau_{ij}}{1 + (\omega_r \tau_{ij})^2} \right]^{-1}.$$ (5)

The $\tau$-method

Following Blanch et al. (1995), we define the following parameter quantifying the magnitude of attenuation in anisotropic media:

$$\tau_{ij} = \frac{\tau_{ij}}{\tau_{ij}^0} - 1.$$ (6)

The quality-factor element $Q_{ij}$ decreases with increasing $\tau_{ij}$. In the elastic case, the stress and strain relaxation times are equal, and $\tau_{ij}$ vanishes. Because each element $\tau_{ij}$ should remain constant for all relaxation mechanisms, the number of independent parameters ($\tau_{ij}$ and $\tau_{ij}^0$) for the relaxation function $\Psi_{ij}$ reduces from $2L$ to $L + 1$. For P- and SV-waves in a 2D viscoelastic VTI model, the total number of independent parameters is equal to $L + 4$ ($L$ for $\tau_{ij}^0$ and 4 for $\tau_{ij}$).

The expressions for the relaxation function (equation 4) and $\tau_{ij}$ (equation 6) allow us to find the complex modulus $C_{ij}$ from equation 2. Then the inverse of the quality factor is given by

$$Q_{ij}^{-1}(\omega) = \frac{\text{Im}(C_{ij})}{\text{Re}(C_{ij})} \frac{\tau_{ij} \sum_{l=1}^{L} \frac{\omega_r \tau_{ij}}{1 + (\omega_r \tau_{ij})^2}}{L + \tau_{ij} \sum_{l=1}^{L} \frac{\omega_r \tau_{ij}}{1 + (\omega_r \tau_{ij})^2}}.$$ (7)

By applying least-squares inversion to equation 7, we can obtain the corresponding parameters $\tau_{ij}$ and $\tau_{ij}^0$, which produce the desired nearly constant value of $Q_{ij}$ in a specified frequency band (Bohlen, 2002). Figure 1 shows that the simulated $Q_{ij}$-curve using the inverted parameters $\tau_{ij}^0$ and $\tau_{ij}$ is close to the specified constant $Q_{ij}$-value, when three relaxation mechanisms are used. With two mechanisms, $Q_{ij}$ remains almost constant only in the lower frequency band limited by 50 Hz.
\[ V = \text{of } 150 \text{ m}. \text{ In the first layer, } \delta = \text{100 Hz is placed at the origin (white dot). The green line source that excites a Ricker wavelet with a central frequency of 100 Hz is placed at the origin (white dot). The green line marks the receiver locations.} \]

The model size is 900 m × 300 m, with grid spacing \( \Delta x = \Delta z = 3 \text{ m}. \) A horizontal reflector is located at a depth of 150 m. In the first layer, \( V_P = 3.0 \text{ km/s}, V_S = 1.5 \text{ km/s}, \rho = 2.0 \text{ g/m}^3, \varepsilon = 0.2, \) and \( \delta = 0.1; \) in the second layer, \( V_P = 2.0 \text{ km/s}, V_S = 1.0 \text{ km/s}, \rho = 2.0 \text{ g/m}^3, \varepsilon = 0.15, \) and \( \delta = 0.05. \) The attenuation parameters are the same for both layers and are listed in the first row of Table 1. An explosive source that excites a Ricker wavelet with a central frequency of 100 Hz is placed at the origin (white dot). The green line marks the receiver locations.

\[ A_P = \frac{A_{P0}}{Q_{S0}} (1 + \delta_0 \sin^2 \theta \cos^2 \theta + \varepsilon_0 \sin^4 \theta), \quad (11) \]

where \( \theta \) is the phase angle with the symmetry axis, \( A_{P0} \) is the P-wave vertical phase attenuation coefficient [which is close to \( 1/(2Q_{P0}) \)], \( \varepsilon_0 \) is the anisotropy parameter that quantifies the fractional difference between the horizontal and vertical attenuation coefficients, and \( \delta_0 \) controls the curvature of \( A_P(\theta) \) in the vertical direction (Zhu and Tsvankin, 2006).

We process reflections in the offset range from 30 m to 840 m with an increment of 90 m and estimate the corresponding phase angles from the group angles using a linearized relationship (Tsvankin, 2012). The inverted parameters, listed in the second row of Table 1, are close to the actual values. The small errors are likely caused by the linearized approximations for the phase angle and the attenuation coefficient (equation 11), as well as slight deviations of the simulated \( Q_{ij} \) from the desired constant value.

**Example for the salt section of the BP model**

Next, we apply the method to a more complicated model with a salt body (Figure 4). This section is taken from the left part of the 2007 BP TTI model and is resampled with a coarser grid. We remove the tilt of the symmetry axis (i.e., turn the model into VTI) and make the section attenuative, with attenuation anisotropy defined by the parameters \( \varepsilon_0 \) and \( \delta_0 \) (Figure 5).

The reflection energy is significantly damped due to attenuation (compare Figures 6(b) and 6(c) with Figure 6(a)). At large offsets (6-12 km), the diffraction from the left edge of the salt body (Figure 4(a) and 4(b)) interferes with reflections from the thin layers in the overburden (Figures 4(c) and 4(d)). This long-offset interference arrival is significantly influenced by attenuation anisotropy in the shallow layers (0-3 km) (Figure 6(d)). Although attenuation anisotropy is also pronounced at depth, the difference between the amplitudes of the deeper events for the isotropic and VTI models is much smaller because of a more limited range of propagation angles. The spectra of windowed traces (Figure 7) exhibit the attenuation-related amplitude decay and reduction in the dominant frequency.

**CONCLUSIONS**

Using the model of generalized standard linear solid (GSLs), we developed a time-domain FD modeling methodology for anisotropic viscoelastic media based on the convolutional relationship between stress and strain. The modified \( \tau \)-method was employed to obtain the corresponding stress relaxation.
Figure 4: Velocity parameters of the salt section of the BP TI model: (a) $V_p^0$, (b) $V_S^0$, (c) $\varepsilon$, and (d) $\delta$. The modified model size is 11268 m $\times$ 13125 m, with grid spacing $\Delta x = \Delta z = 18.75$ m. An explosive source that excites a Ricker wavelet with a central frequency of 10 Hz is placed at the origin.

Figure 5: Attenuation parameters for the model from Figure 4: (a) $Q_p^0$, (b) $Q_S^0$, (c) $\varepsilon_Q$, and (d) $\delta_Q$.

Figure 6: Vertical component of the reflection data for the model from Figures 4 and 5. The result of (a) elastic VTI modeling; (b) viscoelastic modeling with $\varepsilon_Q = \delta_Q = 0$; and (c) viscoelastic VTI modeling. (d) The difference between (b) and (c).

Figure 7: Spectra of windowed traces (from 6.3 s $\sim$ 8.1 s) at an offset of 10.1 km for the model from Figures 4 and 5. The pink and blue curves correspond to the traces from Figures 6(b) and 6(c), respectively.

time and $\tau_{ij}$ parameters and to simulate nearly-constant $Q_{ij}$ behavior for a certain range of frequencies.

Efficient finite-difference implementation is based on introduction of memory variables and rotated staggered grids (RSG). To validate the algorithm, we reconstructed the attenuation parameters of a VTI layer by applying the spectral-ratio method to the simulated reflection data. The estimated $Q$-factor for the reflected P-wave remains almost independent of frequency in the specified range from 2 to 200 Hz. The method was also applied to a more structurally complicated model with a salt body, in which the offset-varying amplitude decay and reduction in the dominant frequency of the reflection events elucidate the influence of attenuation and attenuation anisotropy.

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