Relationship between one-sided and two-sided Green's function representations

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ABSTRACT
The Green's function, defined as the response recorded at the acquisition surface for a source located in the interior of the subsurface, is a combination of the downgoing and upgoing wave fields needed to reconstruct an image of the discontinuities inside the earth. Two-sided Green's function representations require measurement on the full boundary enclosing the domain of interest and allow us to retrieve the Green's function originating from any location inside the medium. Practical constraints usually prevent the placement of receivers at depth inside the earth; hence standard imaging techniques need to apply approximations to two-sided Green's function representations to construct an image of the subsurface. Recently-developed one-sided representations need data acquired on only one side of the boundary enclosing the region of interest. They present a practical advantage because they remove the physical constraints of having receivers at depth; hence one-sided representations do not require the approximations needed by two-sided techniques. In this paper, we show the connection between one-sided and two-sided Green's function representations.

Key words: Greens function representations, Marchenko equations, seismic interferometry, one-sided integral equations, two-sided integral equations

1 INTRODUCTION

Green's function representations requiring measurement on a closed boundary surrounding an inhomogeneous medium allow us to retrieve the Green's function originating from any location inside the medium itself (Aki and Richards, 2002). Techniques based on these representations can be used to create an image of structures present inside the earth. Unfortunately, because of practical constraints, we are unable to place receivers along a closed boundary inside the earth. Since measurements are not available on the full closed boundary, standard imaging techniques need to apply approximations to two-sided Green's function representations to construct an image of the subsurface. Because these equations require measurements on a closed boundary, we refer to these equations as *two-sided* integral equations.

Pioneering work of Rose has shown the connection between one-dimensional autofocusing and the Marchenko equation (Rose, 2001, 2002a; Aktosun and Rose, 2002; Rose, 2002b). We have extended this connection to one-dimensional Green's function retrieval by combining the time-reversed focusing wave field with its reflection response (Broggini and Snieder, 2012; Broggini et al., 2012b). Wapenaar et al. (2012, 2013a,b), and Broggini et al. (2012a) introduce an integral equation that allows one to reconstruct from reflection data at the surface the Green's function propagating from a location inside a medium to receivers located at the acquisition surface. Because reflection data are needed on only one side of the medium, we refer to this equation as *one-sided* integral equation. Following an insight by Lamb (1980), Wapenaar et al. (2013a) show that a *focusing solution* of the source-free wave equation is required to prove the validity of the one-sided integral equation and, additionally, discuss an iterative scheme to solve it.

The techniques introduced by one-side integral equations are elegant, but their derivations, based on the Marchenko equations, restrict their application to a limited number of problems. Here we show that techniques based on one-side representations can be derived from well-known integral equations for Green's function retrieval, known as seismic interferometry in the geophysical community (Bakulin and Calvert, 2006; Wapenaar and Fokkema, 2006; Schuster, 2009). Since these two-sided equations are applicable to a wide range of problems (Lobkis and Weaver, 2001; Larose et al., 2006), one-sided techniques can, in principle, be extended to a large class of problems as well. In this paper, we present a derivation valid for acoustic waves in lossless media.

2 TWO-SIDED INTEGRAL EQUATION FOR GREEN'S FUNCTION RETRIEVAL

We introduce the Green's function $G(\mathbf{x}, \mathbf{x}_B, t)$ as a solution to the wave equation $LG = -\rho\delta(\mathbf{x}-\mathbf{x}_B)\frac{\partial\delta(t)}{\partial t}$, with $L = \rho\nabla\cdot(\rho^{-1}\nabla) - c^{-2}\frac{\partial^2}{\partial t^2}$. The mass density and the velocity are denoted by $\rho(\mathbf{x})$ and $c(\mathbf{x})$, respectively. Following de Hoop (1995), the Green's function $G(\mathbf{x}, \mathbf{x}_B, t)$ corresponds to the acoustic pressure measured at $\mathbf{x}$ due to an impulsive point source of volume injection rate at $\mathbf{x}_B$. We define the spatial coordinate $\mathbf{x}$ as $\mathbf{x} = (\mathbf{x}_H, x_3)$, where $\mathbf{x}_H = (x_1, x_2)$ is the horizontal coordinate vector and x_3 the vertical coordinate. The positive direction of the x_3-axis points downward. The Green's function $G(\mathbf{x}, \mathbf{x}_B, t)$ can be decomposed in its downgoing $G^+(\mathbf{x}, \mathbf{x}_B, t)$ and upgoing $G^-(\mathbf{x}, \mathbf{x}_B, t)$ components. In this paper, these two wave fields are pressure-normalized (Ursin et al., 2012). Using the Fourier convention $f(\mathbf{x}, \omega) = \int_{-\infty}^{+\infty} f(\mathbf{x}, t)\exp(-j\omega t)dt$, the frequency domain Green's function $G(\mathbf{x}, \mathbf{x}_B, \omega)$ obeys the equation $LG = -j\omega\rho\delta(\mathbf{x}-\mathbf{x}_B)$, with $L = \rho\nabla\cdot(\rho^{-1}\nabla) + \omega^2/c^2$. Here, j is the imaginary unit and ω denotes the angular frequency. To keep a simple notation, we use the same symbol for the time-domain version of a function as for its frequency-domain counterpart.

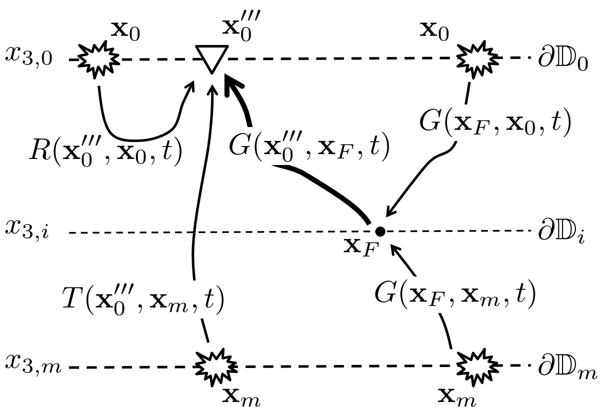

Figure 1. Definition of the variables used in equation (3).

Our starting point is the acoustic representation of Green's functions in the frequency domain given by equation (18) in Wapenaar and Fokkema (2006):

$$\chi_{\mathbb{D}}(\mathbf{x}_A)G(\mathbf{x}_A, \mathbf{x}_B, \omega) + \chi_{\mathbb{D}}(\mathbf{x}_B)G^*(\mathbf{x}_B, \mathbf{x}_A, \omega) = \oint_{\partial\mathbb{D}} \frac{-1}{j\omega\rho(\mathbf{x})}\left\{G^*(\mathbf{x}, \mathbf{x}_A, \omega)\partial_i G(\mathbf{x}, \mathbf{x}_B, \omega) - \partial_i G^*(\mathbf{x}, \mathbf{x}_A, \omega)G(\mathbf{x}, \mathbf{x}_B, \omega)\right\} n_i \mathrm{d}^2\mathbf{x}, \tag{1}$$

where $*$ indicates complex-conjugation and $\chi_{\mathbb{D}}(\mathbf{x})$ is the characteristic function for the domain $\mathbb{D}$, defined as

$$\chi_{\mathbb{D}}(\mathbf{x}) = \begin{cases} 1 & \text{for } \mathbf{x} \in \mathbb{D} \\ \frac{1}{2} & \text{for } \mathbf{x} \in \partial\mathbb{D} \\ 0 & \text{for } \mathbf{x} \in \mathbb{R}^3 \in \setminus \{\mathbb{D} \cup \partial\mathbb{D}\}. \end{cases} \tag{2}$$

In equation (1), ∂_i denotes the partial derivative in the x_i-direction (Einstein's summation convention applies for repeated subscripts) and it acts on $\mathbf{x}$ which corresponds to a receiver location. The term $\frac{-1}{j\omega\rho(\mathbf{x})}\partial_i G(\mathbf{x}, \mathbf{x}_B, \omega)$ in equation (1) corresponds to the particle velocity recorded at $\mathbf{x}$ due to a monopole source located at $\mathbf{x}_B$. We consider an inhomogeneous lossless medium $\mathbb{D}$ enclosed by the boundary $\partial\mathbb{D}$ with outward point normal vector $\mathbf{n} = (n_1, n_2, n_3)$.

By infinitely extending the sides of the domain $\mathbb{D}$, we can replace the closed surface integral over $\partial\mathbb{D}$ by an integral over two horizontal boundaries $\partial\mathbb{D}_0$ and $\partial\mathbb{D}_m$ at depth $x_3 = x_{3,0}$ and $x_3 = x_{3,m}$, respectively, as shown in Figure 1. For this reason, we define equation (1) as a two-sided integral equation. In the following, the upper half-space $x_3 < x_{3,0}$ and the lower half-space $x_3 > x_{3,m}$ are homogeneous and their medium parameters do not have to be the same. The boundaries $\partial\mathbb{D}_0$ and $\partial\mathbb{D}_m$ do not correspond with physical interfaces; hence they do not cause reflection of acoustic waves. With this configuration, the partial derivative ∂_i (acting on $\mathbf{x}$) is along the vertical direction x_3. Next, we take $\mathbf{x}_A$ to be a point $\mathbf{x}_F$ in the interior of $\mathbb{D}$ and take $\mathbf{x}_B$ to be the position $\mathbf{x}_0'''$ of a receiver located just above the surface $\partial\mathbb{D}_0$; see Figure 1, where $\mathbf{x}_0''' = (\mathbf{x}_H, x_3)$. The second term on the left-hand side of equation (1) disappears because $\chi_{\mathbb{D}}(\mathbf{x}_0''') = 0$ when $\mathbf{x}_0'''$ is outside $\mathbb{D}$. These substitutions reduce expression (1) to

$$G(\mathbf{x}_F, \mathbf{x}_0''', \omega) = \int_{\partial\mathbb{D}_0\cup\partial\mathbb{D}_m} \frac{-1}{j\omega\rho(\mathbf{x})}\left\{G^*(\mathbf{x}, \mathbf{x}_F, \omega)\partial_i G(\mathbf{x}, \mathbf{x}_0''', \omega) - \partial_i G^*(\mathbf{x}, \mathbf{x}_F, \omega)G(\mathbf{x}, \mathbf{x}_0''', \omega)\right\} n_i \mathrm{d}^2\mathbf{x} \equiv I_{\partial\mathbb{D}_0} + I_{\partial\mathbb{D}_m}. \tag{3}$$

$I_{\partial\mathbb{D}_0}$ and $I_{\partial\mathbb{D}_m}$ correspond to the integrals over the top $\partial\mathbb{D}_0$ and bottom $\partial\mathbb{D}_m$ boundaries, respectively.

We want to reduce the two terms inside the integral in equation (3) to a single term. To achieve this, we have to analyze separately the integral over the top and bottom boundaries because $\mathbf{x}_0'''$ and $\mathbf{x}_F$ are on different sides with respect to $\partial\mathbb{D}_0$, while they are on the same side with respect to $\partial\mathbb{D}_m$. First, we focus on the integral over the top boundary $\partial\mathbb{D}_0$,

$$I_{\partial\mathbb{D}_0} = \int_{\partial\mathbb{D}_0} \frac{1}{j\omega\rho(\mathbf{x})} \left\{ G^*(\mathbf{x},\mathbf{x}_F,\omega)\partial_3 G(\mathbf{x},\mathbf{x}_0''',\omega) - \partial_3 G^*(\mathbf{x},\mathbf{x}_F,\omega) G(\mathbf{x},\mathbf{x}_0''',\omega) \right\} \mathrm{d}^2\mathbf{x}, \tag{4}$$

where we used that $\partial_i n_i = -\partial_3$ (n_3 points upward at $\partial\mathbb{D}_0$, while x_3 points downward). According to Appendix B.3 of Wapenaar and Berkhout (1989), we can rewrite equation (4) (for the integral over $\partial\mathbb{D}_0$) as

$$I_{\partial\mathbb{D}_0} = \int_{\partial\mathbb{D}_0} \frac{2}{j\omega\rho(\mathbf{x})} \left\{ G^+(\mathbf{x},\mathbf{x}_F,\omega) \right\}^* \partial_3 G^+(\mathbf{x},\mathbf{x}_0''',\omega) \mathrm{d}^2\mathbf{x} + \int_{\partial\mathbb{D}_0} \frac{2}{j\omega\rho(\mathbf{x})} \left\{ G^-(\mathbf{x},\mathbf{x}_F,\omega) \right\}^* \partial_3 G^-(\mathbf{x},\mathbf{x}_0''',\omega) \mathrm{d}^2\mathbf{x}, \tag{5}$$

where $G = G^+ + G^-$. Note that equation (5) is not exact because the evanescent waves are neglected at the boundary $\partial\mathbb{D}_0$. Furthermore, because the medium above $\partial\mathbb{D}_0$ is homogeneous, the downgoing component of $G(\mathbf{x},\mathbf{x}_F,\omega)$ is equal to zero; hence $G(\mathbf{x},\mathbf{x}_F,\omega) = G^-(\mathbf{x},\mathbf{x}_F,\omega)$. This yields

$$I_{\partial\mathbb{D}_0} = \int_{\partial\mathbb{D}_0} \frac{2}{j\omega\rho(\mathbf{x})} \left\{ G^-(\mathbf{x},\mathbf{x}_F,\omega) \right\}^* \partial_3 G^-(\mathbf{x},\mathbf{x}_0''',\omega) \mathrm{d}^2\mathbf{x}. \tag{6}$$

The term $\frac{-1}{j\omega\rho(\mathbf{x})}\partial_3 G^-(\mathbf{x},\mathbf{x}_0''',\omega)$ in equation (6) corresponds to the vertical component of the particle velocity of the upgoing field recorded at $\mathbf{x}$ due to a monopole source located at $\mathbf{x}_0'''$ just above the boundary $\partial\mathbb{D}_0$, or, via reciprocity, to the acoustic pressure recorded at $\mathbf{x}_0'''$ due to a downward radiating dipole source located at $\mathbf{x}$. Since we deal with pressure-normalized wave fields, we use the last definition and define the reflection response as

$$R(\mathbf{x}_0''',\mathbf{x},\omega) = \frac{2}{j\omega\rho(\mathbf{x})}\partial_3 G^-(\mathbf{x},\mathbf{x}_0''',\omega). \tag{7}$$

Wapenaar et al. (2012) use the same definition for $R(\mathbf{x}_0''',\mathbf{x},\omega)$. The reflection response is acquired in a seismic experiment where sources and receivers are both located along the acquisition surface at $x_3 = x_{3,0}$. Using this, we rewrite equation (6) as

$$I_{\partial\mathbb{D}_0} = \int_{\partial\mathbb{D}_0} R(\mathbf{x}_0''',\mathbf{x},\omega) G^*(\mathbf{x},\mathbf{x}_F,\omega) \mathrm{d}^2\mathbf{x}, \tag{8}$$

where $G(\mathbf{x},\mathbf{x}_F,\omega) = G^-(\mathbf{x},\mathbf{x}_F,\omega)$.

Next, we analyze the integral over the bottom boundary $\partial\mathbb{D}_m$ in equation (3):

$$I_{\partial\mathbb{D}_m} = \int_{\partial\mathbb{D}_m} \frac{-1}{j\omega\rho(\mathbf{x})} \left\{ G^*(\mathbf{x},\mathbf{x}_F,\omega)\partial_3 G(\mathbf{x},\mathbf{x}_0''',\omega) - \partial_3 G^*(\mathbf{x},\mathbf{x}_F,\omega) G(\mathbf{x},\mathbf{x}_0''',\omega) \right\} \mathrm{d}^2\mathbf{x}, \tag{9}$$

where we used that $\partial_i n_i = +\partial_3$ (n_3 points downward at $\partial\mathbb{D}_m$, while x_3 also points downward). As previously mentioned, $\mathbf{x}_0'''$ and $\mathbf{x}_F$ are on the same side with respect to $\partial\mathbb{D}_m$; hence, we can use the same arguments given by Wapenaar and Fokkema (2006) to reduce the integral over $\partial\mathbb{D}_m$ to a single term. This yields

$$I_{\partial\mathbb{D}_m} = - \int_{\partial\mathbb{D}_m} \frac{2}{j\omega\rho(\mathbf{x})} \left\{ G^+(\mathbf{x},\mathbf{x}_F,\omega) \right\}^* \partial_3 G^+(\mathbf{x},\mathbf{x}_0''',\omega) \mathrm{d}^2\mathbf{x}. \tag{10}$$

We define the pressure-normalized transmission response as

$$T(\mathbf{x}_0''',\mathbf{x},\omega) = -\frac{2}{j\omega\rho(\mathbf{x})}\partial_3 G^+(\mathbf{x},\mathbf{x}_0''',\omega) \tag{11}$$

and rewrite equation (10) as

$$I_{\partial\mathbb{D}_m} = \int_{\partial\mathbb{D}_m} T(\mathbf{x}_0''',\mathbf{x},\omega) G^*(\mathbf{x},\mathbf{x}_F,\omega) \mathrm{d}^2\mathbf{x}, \tag{12}$$

where $G(\mathbf{x},\mathbf{x}_F,\omega) = G^+(\mathbf{x},\mathbf{x}_F,\omega)$. We finally combine equations (8) and (12) to obtain

$$G(\mathbf{x}_0''',\mathbf{x}_F,\omega) = \int_{\partial\mathbb{D}_0} R(\mathbf{x}_0''',\mathbf{x},\omega) G^*(\mathbf{x}_F,\mathbf{x},\omega) \mathrm{d}^2\mathbf{x} + \int_{\partial\mathbb{D}_m} T(\mathbf{x}_0''',\mathbf{x},\omega) G^*(\mathbf{x}_F,\mathbf{x},\omega) \mathrm{d}^2\mathbf{x}, \tag{13}$$

where we have used the source-receiver reciprocity relation $G(\mathbf{x},\mathbf{x}_F,\omega) = G(\mathbf{x}_F,\mathbf{x},\omega)$. Note that, if R and T are known, we could solve this equation to retrieve the unknown wave field G (but the trivial solution $G = 0$ would also be a valid solution).

Equation (13) is the starting point to derive the relationship with the one-sided integral equation given by Wapenaar et al. (2012, 2013a,b). In the time domain, equation (13) corresponds to

$$G(\mathbf{x}_0''',\mathbf{x}_F,t) = \int_{\partial\mathbb{D}_0} \mathrm{d}^2\mathbf{x} \int_{-\infty}^{+\infty} R(\mathbf{x}_0''',\mathbf{x},t-t') G(\mathbf{x}_F,\mathbf{x},-t') dt' + \int_{\partial\mathbb{D}_m} \mathrm{d}^2\mathbf{x} \int_{-\infty}^{+\infty} T(\mathbf{x}_0''',\mathbf{x},t-t') G(\mathbf{x}_F,\mathbf{x},-t') dt'. \tag{14}$$

3 RELATIONSHIPS BETWEEN GREEN'S FUNCTIONS AND FOCUSING SOLUTION

In the domain $\mathbb{D}$ enclosed by the boundaries $\partial\mathbb{D}_0$ and $\partial\mathbb{D}_m$, we consider three independent acoustic states: the Green's function $G(\mathbf{x},\mathbf{x}_0''',\omega)$ due to an impulsive point source of volume injection rate at $\mathbf{x}_0'''$ (just above $\partial\mathbb{D}_0$), the Green's function due to a similar source at $\mathbf{x}_m$ (just below $\partial\mathbb{D}_m$), and the *focusing solution* $f_1(\mathbf{x},\mathbf{x}_i',\omega)$. This particular wave field $f_1(\mathbf{x},\mathbf{x}_i',\omega)$ is defined anywhere as the wave field that focuses at the location $\mathbf{x}_i'$ in the interior of $\mathbb{D}$. The domain $\mathbb{D}$ is source-free for all states, and the medium parameters are identical between the boundary $\partial\mathbb{D}_0$ and the depth level $\partial\mathbb{D}_i$, where $\partial\mathbb{D}_i$ is an arbitrary depth level between $\partial\mathbb{D}_0$ and $\partial\mathbb{D}_m$. For this situation, the reciprocity theorems of the convolution-type and correlation-type are (Fokkema and van den Berg, 1993), respectively,

$$\int_{\partial\mathbb{D}_0} \frac{2}{j\omega\rho(\mathbf{x}_0)} \left\{(\partial_3 f_1)G - f_1(\partial_3 G)\right\} \mathrm{d}^2\mathbf{x}_0 = \int_{\partial\mathbb{D}_i} \frac{2}{j\omega\rho(\mathbf{x}_i)} \left\{(\partial_3 f_1)G - f_1(\partial_3 G)\right\} \mathrm{d}^2\mathbf{x}_i, \quad (15)$$

$$\int_{\partial\mathbb{D}_0} \frac{2}{j\omega\rho(\mathbf{x}_0)} \left\{(\partial_3 f_1)^* G - f_1^*(\partial_3 G)\right\} \mathrm{d}^2\mathbf{x}_0 = \int_{\partial\mathbb{D}_i} \frac{2}{j\omega\rho(\mathbf{x}_i)} \left\{(\partial_3 f_1)^* G - f_1^*(\partial_3 G)\right\} \mathrm{d}^2\mathbf{x}_i. \quad (16)$$

Substituting $f_1 = f_1^+ + f_1^-$ and $G = G^+ + G^-$ into equations (15) and (16) and following the derivation in Appendix B.3 of Wapenaar and Berkhout (1989), we obtain

$$-\int_{\partial\mathbb{D}_0} \frac{2}{j\omega\rho(\mathbf{x}_0)} \left\{f_1^+(\partial_3 G^-) + f_1^-(\partial_3 G^+)\right\} \mathrm{d}^2\mathbf{x}_0 = \int_{\partial\mathbb{D}_i} \frac{2}{j\omega\rho(\mathbf{x}_i)} \left\{(\partial_3 f_1^+)G^- + (\partial_3 f_1^-)G^+\right\} \mathrm{d}^2\mathbf{x}_i, \quad (17)$$

$$-\int_{\partial\mathbb{D}_0} \frac{2}{j\omega\rho(\mathbf{x}_0)} \left\{(f_1^+)^*(\partial_3 G^+) + (f_1^-)^*(\partial_3 G^-)\right\} \mathrm{d}^2\mathbf{x}_0 = \int_{\partial\mathbb{D}_i} \frac{2}{j\omega\rho(\mathbf{x}_i)} \left\{(\partial_3 f_1^+)^* G^+ + (\partial_3 f_1^-)^* G^-\right\} \mathrm{d}^2\mathbf{x}_i, \quad (18)$$

where in equation (18) the evanescent wave field is neglected.

Figure 2 shows the three different states to which we apply equations (17) and (18). State A1 is defined in the actual medium between the boundaries $\partial\mathbb{D}_0$ and $\partial\mathbb{D}_m$, and a source is located at $\mathbf{x}_0'''$, just above $\partial\mathbb{D}_0$. The downgoing and upgoing Green's function at $\mathbf{x}_0 \in \partial\mathbb{D}_0$ are

$$\frac{2}{j\omega\rho(\mathbf{x}_0)} \partial_3 G^+(\mathbf{x}, \mathbf{x}_0''', \omega)|_{x_3 = x_{3,0}} = -\delta(\mathbf{x}_H - \mathbf{x}_H'''), \quad (19)$$

$$\frac{2}{j\omega\rho(\mathbf{x}_0)} \partial_3 G^-(\mathbf{x}, \mathbf{x}_0''', \omega)|_{x_3 = x_{3,0}} = R(\mathbf{x}_0''', \mathbf{x}_0, \omega). \quad (20)$$

Like state A1, state A2 is also defined in the actual medium, but now we have a source located at $\mathbf{x}_m$, just below $\partial\mathbb{D}_m$. The downgoing and upgoing Green's function at $\mathbf{x}_0 \in \partial\mathbb{D}_0$ are

$$\frac{2}{j\omega\rho(\mathbf{x}_0)} \partial_3 G^+(\mathbf{x}, \mathbf{x}_m, \omega)|_{x_3 = x_{3,0}} = 0, \quad (21)$$

$$\frac{2}{j\omega\rho(\mathbf{x}_0)} \partial_3 G^-(\mathbf{x}, \mathbf{x}_m, \omega)|_{x_3 = x_{3,0}} = T(\mathbf{x}_m, \mathbf{x}_0, \omega). \quad (22)$$

State B1 is defined in a reference medium where the medium parameters are identical to the actual medium between the boundary $\partial\mathbb{D}_0$ and the depth level $\partial\mathbb{D}_i$. Below $\partial\mathbb{D}_i$, state B1 is composed of a reflection-free half-space. The focusing solution f_1 is defined in this reference medium and its dowgoing and upgoing components at $\mathbf{x}_i' \in \partial\mathbb{D}_i$ are

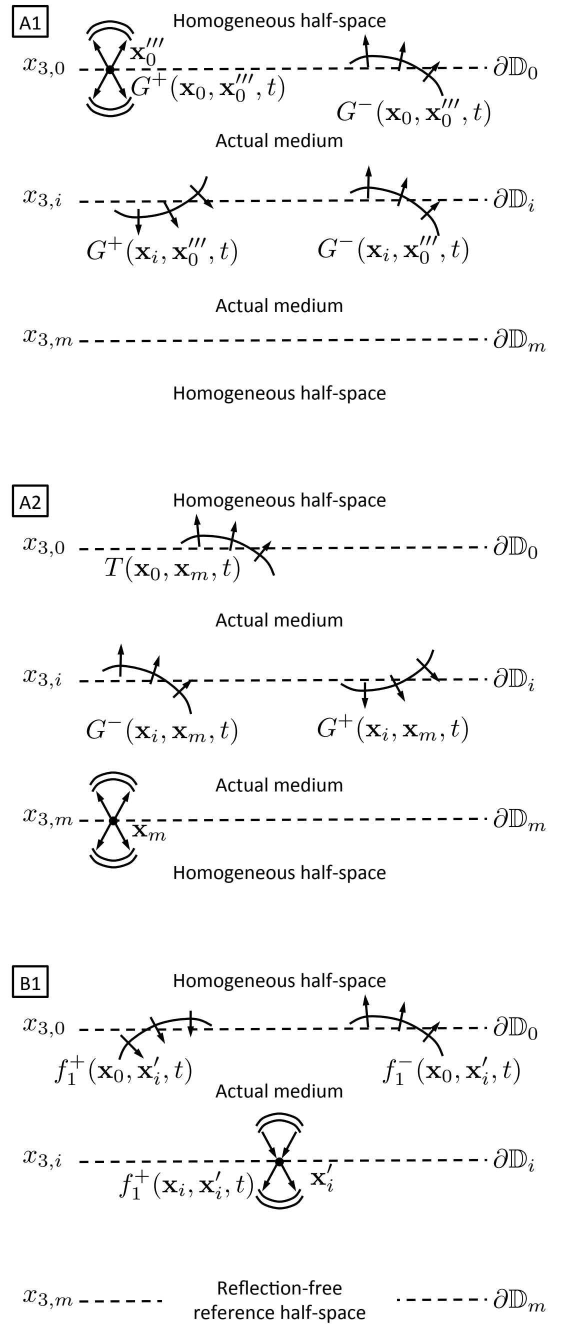

Figure 2. States for reciprocity theorems. Top - State A1: Green's function with a source at $\mathbf{x}_0'''$ just above $\partial\mathbb{D}_0$. Middle - State A2: Green's function with a source at $\mathbf{x}_m$ just below $\partial\mathbb{D}_m$. Bottom - State B1: focusing solution f_1 with a focal point at $\mathbf{x}_i'$ at depth level $\partial\mathbb{D}_i$.

$$\frac{2}{j\omega\rho(\mathbf{x}_i)} \partial_3 f_1^+(\mathbf{x}, \mathbf{x}_i', \omega)|_{x_3 = x_{3,i}} = -\delta(\mathbf{x}_H - \mathbf{x}_H'), \quad (23)$$

$$\frac{2}{j\omega\rho(\mathbf{x}_i)}\partial_3 f_1^-(\mathbf{x},\mathbf{x}_i',\omega)|_{x_3=x_{3,i}}=0. \tag{24}$$

Substituting equations (19-20) and (23-24) into equations (17-18) gives

$$\begin{aligned}&G^-(\mathbf{x}_i',\mathbf{x}_0''',\omega)\\&=\int_{\partial\mathbb{D}_0}R(\mathbf{x}_0''',\mathbf{x}_0,\omega)f_1^+(\mathbf{x}_0,\mathbf{x}_i',\omega)\mathrm{d}^2\mathbf{x}_0\\&-f_1^-(\mathbf{x}_0''',\mathbf{x}_i',\omega),\end{aligned} \tag{25}$$

and

$$\begin{aligned}&G^+(\mathbf{x}_i',\mathbf{x}_0''',\omega)\\&=-\int_{\partial\mathbb{D}_0}R(\mathbf{x}_0''',\mathbf{x}_0,\omega)\left\{f_1^-(\mathbf{x}_0,\mathbf{x}_i',\omega)\right\}^*\mathrm{d}^2\mathbf{x}_0\\&+\left\{f_1^+(\mathbf{x}_0''',\mathbf{x}_i',\omega)\right\}^*.\end{aligned} \tag{26}$$

Adding these expressions yields

$$\begin{aligned}&G(\mathbf{x}_i',\mathbf{x}_0''',\omega)\\&=\int_{\partial\mathbb{D}_0}R(\mathbf{x}_0''',\mathbf{x}_0,\omega)f_2(\mathbf{x}_i',\mathbf{x}_0,\omega)\mathrm{d}^2\mathbf{x}_0\\&+\left\{f_2(\mathbf{x}_i',\mathbf{x}_0''',\omega)\right\}^*,\end{aligned} \tag{27}$$

where we defined a second focusing solution $f_2(\mathbf{x}_i',\mathbf{x}_0''',\omega) = f_1^+(\mathbf{x}_0''',\mathbf{x}_i',\omega) - \left\{f_1^-(\mathbf{x}_0''',\mathbf{x}_i',\omega)\right\}^*$. The focusing solution f_2 has a focal point at $\mathbf{x}_0'''$ (Wapenaar et al., 2013a), as shown in Figure 3. Substituting equations (21-24) into equations (17-18) gives

$$\begin{aligned}&G^-(\mathbf{x}_i',\mathbf{x}_m,\omega)\\&=\int_{\partial\mathbb{D}_0}T(\mathbf{x}_m,\mathbf{x}_0,\omega)f_1^+(\mathbf{x}_0,\mathbf{x}_i',\omega),\mathrm{d}^2\mathbf{x}_0\end{aligned} \tag{28}$$

and

$$\begin{aligned}&G^+(\mathbf{x}_i',\mathbf{x}_m,\omega)\\&=-\int_{\partial\mathbb{D}_0}T(\mathbf{x}_m,\mathbf{x}_0,\omega)\left\{f_1^-(\mathbf{x}_0,\mathbf{x}_i',\omega)\right\}^*\mathrm{d}^2\mathbf{x}_0.\end{aligned} \tag{29}$$

Adding these last two expressions yields

$$\begin{aligned}&G(\mathbf{x}_i',\mathbf{x}_m,\omega)\\&=\int_{\partial\mathbb{D}_0}T(\mathbf{x}_m,\mathbf{x}_0,\omega)f_2(\mathbf{x}_i',\mathbf{x}_0,\omega)\mathrm{d}^2\mathbf{x}_0.\end{aligned} \tag{30}$$

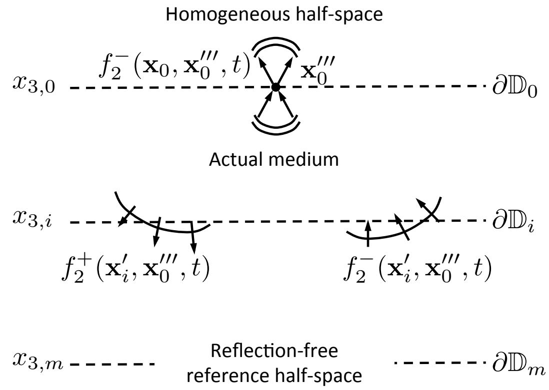

Figure 3. Focusing solution f_2 with a focal point at $\mathbf{x}_0'''$ at the depth level $\partial\mathbb{D}_0$.

4 COMPARISON BETWEEN THE INTEGRAL EQUATIONS

In this section, we show the consistency between two-sided, i.e. equation (13), and one-sided Green's function representations. Substituting equations (27) and (30) into equation (13) gives

$$\begin{aligned}&G(\mathbf{x}_0''',\mathbf{x}_F,\omega)\\&=\int_{\partial\mathbb{D}_0}\mathrm{d}^2\mathbf{x}_0R(\mathbf{x}_0''',\mathbf{x}_0,\omega)\\&\times\int_{\partial\mathbb{D}_0}\left\{R(\mathbf{x}_0,\mathbf{x}_0'',\omega)\right\}^*\left\{f_2(\mathbf{x}_F,\mathbf{x}_0'',\omega)\right\}^*\mathrm{d}^2\mathbf{x}_0''\\&+\int_{\partial\mathbb{D}_0}R(\mathbf{x}_0''',\mathbf{x}_0,\omega)f_2(\mathbf{x}_F,\mathbf{x}_0,\omega)\mathrm{d}^2\mathbf{x}_0\\&+\underbrace{\underbrace{\int_{\partial\mathbb{D}_m}\mathrm{d}^2\mathbf{x}_mT(\mathbf{x}_0''',\mathbf{x}_m,\omega)}_{\mathrm{T1}}\times\int_{\partial\mathbb{D}_0}\left\{T(\mathbf{x}_m,\mathbf{x}_0'',\omega)\right\}^*\left\{f_2(\mathbf{x}_F,\mathbf{x}_0'',\omega)\right\}^*\mathrm{d}^2\mathbf{x}_0''}_{\mathrm{T1}}.\end{aligned} \tag{31}$$

Now, we focus on term T1 of equation (31) and we reorder its terms:

$$\begin{aligned}&\int_{\partial\mathbb{D}_0}\mathrm{d}^2\mathbf{x}_0''\left\{f_2(\mathbf{x}_F,\mathbf{x}_0'',\omega)\right\}^*\\&\times\int_{\partial\mathbb{D}_m}\left\{T(\mathbf{x}_m,\mathbf{x}_0'',\omega)\right\}^*T(\mathbf{x}_0''',\mathbf{x}_m,\omega)\mathrm{d}^2\mathbf{x}_m.\end{aligned} \tag{32}$$

Wapenaar et al. (2004) derives a relationship between flux-normalized reflection and transmission responses for a domain $\mathbb{D}$ enclosed by boundaries $\partial\mathbb{D}_0$ and $\partial\mathbb{D}_m$. In the Appendix, we derive a similar relationship for pressure-normalized reflection and transmission responses, equation (A-14); its expression is:

$$\begin{aligned}&\int_{\partial\mathbb{D}_m}\mathrm{d}^2\mathbf{x}_m\left\{T(\mathbf{x}_m,\mathbf{x}_0'',\omega)\right\}^*T(\mathbf{x}_0''',\mathbf{x}_m,\omega)\\&=\delta(\mathbf{x}_H''-\mathbf{x}_H''')\\&-\int_{\partial\mathbb{D}_0}\mathrm{d}^2\mathbf{x}_0R(\mathbf{x}_0''',\mathbf{x}_0,\omega)\left\{R(\mathbf{x}_0,\mathbf{x}_0'',\omega)\right\}^*.\end{aligned} \tag{33}$$

Note that equation (33) shows the unitarity of the scattering matrix (Rodberg and Thaler, 1967). Then, we use equation (33) to rewrite term T1 as defined in expression (32):

$$\begin{aligned}&\left\{f_2(\mathbf{x}_F,\mathbf{x}_0''',\omega)\right\}^* \\ &\quad -\int_{\partial\mathbb{D}_0} \mathrm{d}^2\mathbf{x}_0'' \left\{f_2(\mathbf{x}_F,\mathbf{x}_0'',\omega)\right\}^* \\ &\quad \times\int_{\partial\mathbb{D}_0} R(\mathbf{x}_0''',\mathbf{x}_0,\omega)\left\{R(\mathbf{x}_0,\mathbf{x}_0'',\omega)\right\}^*\mathrm{d}^2\mathbf{x}_0.\end{aligned} \tag{34}$$

Finally, we insert expression (34) into equation (31) to give

$$\begin{aligned}&G(\mathbf{x}_0''',\mathbf{x}_F,\omega) \\ &\quad = \underbrace{\int_{\partial\mathbb{D}_0} \mathrm{d}^2\mathbf{x}_0 R(\mathbf{x}_0''',\mathbf{x}_0,\omega)}_{\mathrm{T1}} \\ &\quad \underbrace{\times\int_{\partial\mathbb{D}_0}\left\{R(\mathbf{x}_0,\mathbf{x}_0'',\omega)\right\}^*\left\{f_2(\mathbf{x}_F,\mathbf{x}_0'',\omega)\right\}^*\mathrm{d}^2\mathbf{x}_0''}_{\mathrm{T1}} \\ &\quad +\int_{\partial\mathbb{D}_0}\mathrm{d}^2\mathbf{x}_0 R(\mathbf{x}_0''',\mathbf{x}_0,\omega) f_2(\mathbf{x}_F,\mathbf{x}_0,\omega) \\ &\quad +\left\{f_2(\mathbf{x}_F,\mathbf{x}_0''',\omega)\right\}^* \\ &\quad \underbrace{-\int_{\partial\mathbb{D}_0}\mathrm{d}^2\mathbf{x}_0''\left\{f_2(\mathbf{x}_F,\mathbf{x}_0'',\omega)\right\}^*}_{\mathrm{T3}} \\ &\quad \underbrace{\times\int_{\partial\mathbb{D}_0} R(\mathbf{x}_0''',\mathbf{x}_0,\omega)\left\{R^+(\mathbf{x}_0,\mathbf{x}_0'',\omega)\right\}^*\mathrm{d}^2\mathbf{x}_0\,.}_{\mathrm{T3}}\end{aligned} \tag{35}$$

The terms T1 and T3 in equation (35) cancel each other; hence equation (35) becomes

$$\begin{aligned}&G(\mathbf{x}_0''',\mathbf{x}_F,\omega) \\ &\quad =\int_{\partial\mathbb{D}_0} R^+(\mathbf{x}_0''',\mathbf{x}_0,\omega) f_2(\mathbf{x}_F,\mathbf{x}_0,\omega)\mathrm{d}^2\mathbf{x}_0 \\ &\quad +\left\{f_2(\mathbf{x}_F,\mathbf{x}_0''',\omega)\right\}^*.\end{aligned} \tag{36}$$

This one-sided Green's function representation is the key result of our paper. This expression shows that the two-sided and one-sided integral equations for the Green's function retrieval lead to equivalent results. Wapenaar et al. (2013a) use a causality argument to solve equation (36) in the time domain for the focusing solution f_2. Once f_2 is known, it can be used to retrieve $G(\mathbf{x}_0''',\mathbf{x}_F,\omega)$. Note that equation (36) is equivalent to equation (27) which is an intermediate step needed to show the consistency between the two-sided integral equation equation (13) and the one-sided equation (36). Alternative derivations based on the unitarity relation (Rodberg and Thaler, 1967) need further investigation, which is beyond the scope of this paper.

5 CONCLUSION

We have shown the connection between one- and two-sided integral equations for Green's function retrieval. Representations based on one-sided equations are more recent and present a practical advantage because they remove the physical constraints of having receivers and sources at depth; hence, they do not require approximations that are needed because measurements are unavailable at depth. On the other hand, methods requiring data on a closed boundary (two-sided) present well-known applications. For these reasons, their connection can introduce mutual benefits to the different representations.

ACKNOWLEDGMENTS

We thank Ken Larner for his constructive comments. This work was supported by the sponsors of the Consortium Project on Seismic Inverse Methods for Complex Structures at the Center for Wave Phenomena and the Netherlands Research Centre for Integrated Solid Earth Science (ISES).

REFERENCES

Aki, K., and P. G. Richards, 2002, Quantitative seismology, 2nd ed.: University Science Books.

Aktosun, T., and J. H. Rose, 2002, Wave focusing on the line: Journal of Mathematical Physics, **43**, 3717–3745.

Bakulin, A., and R. Calvert, 2006, The virtual source method: Theory and case study: Geophysics, **71**, SI139–SI150.

Broggini, F., and R. Snieder, 2012, Connection of scattering principles: a visual and mathematical tour: European Journal of Physics, **33**, 593–613.

Broggini, F., R. Snieder, and K. Wapenaar, 2012a, Creating a virtual source inside a medium from reflection data: numerical examples for laterally-varying velocity models, spatially-extended virtual sources, and inaccurate direct arrivals: SEG Technical Program Expanded Abstracts, **710**, 1–5.

——, 2012b, Focusing the wavefield inside an unknown 1D medium - Beyond seismic interferometry: Geophysics, **77**, A25–A28.

de Hoop, A. T., 1995, Handbook of radiation and scattering of waves: Academic Press.

Fokkema, J. T., and P. M. van den Berg, 1993, Seismic Applications of Acoustic Reciprocity: Elsevier.

Lamb, G. L., 1980, Elements of soliton theory: Wiley-Interscience.

Larose, E., O. Lobkis, and R. Weaver, 2006, Passive correlation imaging of a buried scatterer (L): Journal of the Acoustical Society of America, **119**, 3549–3552.

Lobkis, O., and R. Weaver, 2001, On the emergence of the Green's function in the correlations of a diffuse field: Journal of the Acoustical Society of America, **110**, 3011–3017.

Rodberg, L. S., and R. M. Thaler, 1967, Introduction to the Quantum Theory of Scattering: Academic Press.

Rose, J. H., 2001, "Single-sided" focusing of the time-dependent Schrödinger equation: Phys. Rev. A, **65**, 012707.

———, 2002a, Single-sided autofocusing of sound in layered materials: Inverse Problems, **18**, 1923–1934.

———, 2002b, Time Reversal, Focusing and Exact Inverse Scattering, *in* Imaging of Complex Media with Acoustic and Seismic Waves: Springer, 97–106.

Schuster, G. T., 2009, Seismic Interferometry: Cambridge University Press.

Ursin, B., O. Pedersen, and B. Arntsen, 2012, Flux-normalized wavefield decomposition and migration of seismic data: Geophysics, **77**, S83–S92.

Wapenaar, K., and A. J. Berkhout, 1989, Elastic wave field extrapolation: Redatuming of single- and multi-component seismic data: Elsevier.

Wapenaar, K., F. Broggini, E. Slob, and R. Snieder, 2013a, Three-Dimensional Single-Sided Marchenko Inverse Scattering, Data-Driven Focusing, Green's Function Retrieval, and their Mutual Relations: Phys. Rev. Lett., **110**, 084301.

Wapenaar, K., F. Broggini, and R. Snieder, 2012, Creating a virtual source inside a medium from reflection data: heuristic derivation and stationary-phase analysis: Geophysical Journal International, **190**, 1020–1024.

Wapenaar, K., and J. Fokkema, 2006, Green's function representations for seismic interferometry: Geophysics, **71**, SI33–SI46.

Wapenaar, K., E. Slob, F. Broggini, R. Snieder, J. Thorbecke, and J. van der Neut, 2013b, Data-driven Green's function retrieval from reflection data: Theory and example: EAGE Expanded Abstracts.

Wapenaar, K., J. Thorbecke, and D. Draganov, 2004, Relations between reflection and transmission responses of three-dimensional inhomogeneous media: Geophysical Journal International, **156**, 179–194.

APPENDIX: RELATIONSHIP BETWEEN PRESSURE-NORMALIZED REFLECTION AND TRANSMISSION RESPONSES

In this appendix, we derive a Green's function representation of the correlation-type for one-way (down- and up-going) fields and a relationship between pressure-normalized reflection and transmission responses measured on the top and bottom boundaries. We start from a two-way reciprocity theorem of the correlation-type:

$$\int_{\partial\mathbb{D}_0} \Big\{ [G_3^{p,f}(\mathbf{x}_0,\mathbf{x}_0'',\omega)]^* G_3^{v,q}(\mathbf{x}_0,\mathbf{x}_0''',\omega) + [G_{3,3}^{v,f}(\mathbf{x}_0,\mathbf{x}_0'',\omega)]^* G^{p,q}(\mathbf{x}_0,\mathbf{x}_0''',\omega) \Big\}\, \mathrm{d}^2\mathbf{x}_0$$
$$= \int_{\partial\mathbb{D}_m} \Big\{ [G_3^{p,f}(\mathbf{x}_m,\mathbf{x}_0'',\omega)]^* G_3^{v,q}(\mathbf{x}_m,\mathbf{x}_0''',\omega) + [G_{3,3}^{v,f}(\mathbf{x}_m,\mathbf{x}_0'',\omega)]^* G^{p,q}(\mathbf{x}_m,\mathbf{x}_0''',\omega) \Big\}\, \mathrm{d}^2\mathbf{x}_m, \tag{A-1}$$

where the superscripts p and v characterize the Green's function as a pressure or velocity, respectively. The superscripts q and f specify if the source is a point source of volume injection rate (monopole) or a point source of force (dipole), respectively. The subscript 3 indicates the vertical component of the velocity or the vertical component of the point source force. We then rewrite $G_{3,3}^{v,f}(\mathbf{x},\mathbf{x}_0'',\omega)$ as

$$G_{3,3}^{v,f}(\mathbf{x},\mathbf{x}_0'',\omega) = -\frac{1}{j\omega\rho(\mathbf{x})}\frac{\partial G_3^{p,f}(\mathbf{x},\mathbf{x}_0'',\omega)}{\partial x_3} + \frac{1}{j\omega\rho(\mathbf{x})}\delta(\mathbf{x}-\mathbf{x}''), \tag{A-2}$$

and insert it into equation (A-1) to obtain

$$\int_{\partial\mathbb{D}_0} \left\{ [G_3^{p,f}(\mathbf{x}_0,\mathbf{x}_0'',\omega)]^* G_3^{v,q}(\mathbf{x}_0,\mathbf{x}_0''',\omega) - \frac{1}{j\omega\rho(\mathbf{x}_0)}[\frac{\partial G_3^{p,f}(\mathbf{x}_0,\mathbf{x}_0'',\omega)}{\partial x_3}]^* G^{p,q}(\mathbf{x}_0,\mathbf{x}_0''',\omega) \right\} \mathrm{d}^2\mathbf{x}_0$$
$$= \int_{\partial\mathbb{D}_m} \left\{ [G_3^{p,f}(\mathbf{x}_m,\mathbf{x}_0'',\omega)]^* G_3^{v,q}(\mathbf{x}_m,\mathbf{x}_0''',\omega) + \frac{1}{j\omega\rho(\mathbf{x}_m)}[\frac{\partial G_3^{p,f}(\mathbf{x}_m,\mathbf{x}_0'',\omega)}{\partial x_3}]^* G^{p,q}(\mathbf{x}_m,\mathbf{x}_0''',\omega) \right\} \mathrm{d}^2\mathbf{x}_m. \tag{A-3}$$

Now, we focus on $\partial\mathbb{D}_0$ and the result for $\partial\mathbb{D}_m$ follows from a similar reasoning. In the following, we assume that the medium parameters are laterally invariant at $\partial\mathbb{D}_0$. We use $G = G^+ + G^-$ and apply Parseval's theorem (Wapenaar and Berkhout, 1989, equation B-27) to equation (A-3), this gives

$$\int_{\partial\mathbb{D}_0} \Big\{ ([G_3^{p,f+}(\chi_0,\mathbf{x}_0'',\omega)]^* + [G_3^{p,f-}(\chi_0,\mathbf{x}_0'',\omega)]^*)[G_3^{v,q+}(\chi_0,\mathbf{x}_0''',\omega) + G_3^{v,q-}(\chi_0,\mathbf{x}_0''',\omega)]$$
$$-\frac{1}{j\omega\rho(x_{3,0})}(jk_3[G_3^{p,f+}(\chi_0,\mathbf{x}_0'',\omega)]^* - jk_3[G_3^{p,f-}(\chi_0,\mathbf{x}_0'',\omega)]^*)[G^{p,q+}(\chi_0,\mathbf{x}_0''',\omega) + G^{p,q-}(\chi_0,\mathbf{x}_0''',\omega)] \Big\}\, \mathrm{d}^2\chi_0, \tag{A-4}$$

where $\chi_0 \equiv (k_1, k_2, x_3)$ and $\left(\frac{\partial G^\pm}{\partial x_3}\right)^* = \pm jk_3(G^\pm)^*$ (Wapenaar and Berkhout, 1989, equation B-29c). We reorder the terms in expression (A-4) and obtain

$$\int_{\partial\mathbb{D}_0} \Big\{ ([G_3^{p,f+}(\chi_0,\mathbf{x}_0'',\omega)]^* + [G_3^{p,f-}(\chi_0,\mathbf{x}_0'',\omega)]^*)[G_3^{v,q+}(\chi_0,\mathbf{x}_0''',\omega) + G_3^{v,q-}(\chi_0,\mathbf{x}_0''',\omega)]$$
$$-([G_3^{p,f+}(\chi_0,\mathbf{x}_0'',\omega)]^* - [G_3^{p,f-}(\chi_0,\mathbf{x}_0'',\omega)]^*)\frac{1}{j\omega\rho(x_{3,0})}[jk_3G^{p,q+}(\chi_0,\mathbf{x}_0''',\omega) + jk_3G^{p,q-}(\chi_0,\mathbf{x}_0''',\omega)] \Big\}\, \mathrm{d}^2\chi_0. \tag{A-5}$$

Then, we use $\mp jk_3(G^\pm)^* = \frac{\partial G^\pm}{\partial x_3}$ (Wapenaar and Berkhout, 1989, equation B-29b) and apply Parsevel's theorem to expression (A-5), yielding

$$\int_{\partial\mathbb{D}_0} \Big\{ ([G_3^{p,f+}(\mathbf{x}_0,\mathbf{x}_0'',\omega)]^* + [G_3^{p,f-}(\mathbf{x}_0,\mathbf{x}_0'',\omega)]^*)[G_3^{v,q+}(\mathbf{x}_0,\mathbf{x}_0''',\omega) + G_3^{v,q-}(\mathbf{x}_0,\mathbf{x}_0''',\omega)]$$
$$-([G_3^{p,f+}(\mathbf{x}_0,\mathbf{x}_0'',\omega)]^* - [G_3^{p,f-}(\mathbf{x}_0,\mathbf{x}_0'',\omega)]^*)\frac{1}{j\omega\rho(\mathbf{x}_0)}[-\frac{\partial G^{p,q+}(\mathbf{x}_0,\mathbf{x}_0''',\omega)}{\partial x_3} + \frac{\partial G^{p,q-}(\mathbf{x}_0,\mathbf{x}_0''',\omega)}{\partial x_3}] \Big\}\, \mathrm{d}^2\mathbf{x}_0, \tag{A-6}$$

where the medium parameters are now laterally varying. Using the relation

$$G_3^{v,q}(\mathbf{x},\mathbf{x}_0''',\omega) = -\frac{1}{j\omega\rho(\mathbf{x})}\frac{\partial G^{p,q}(\mathbf{x},\mathbf{x}_0''',\omega)}{\partial x_3}, \tag{A-7}$$

we rewrite expression (A-6) as

$$\int_{\partial\mathbb{D}_0} \Big\{ ([G_3^{p,f+}(\mathbf{x}_0,\mathbf{x}_0'',\omega)]^* + [G_3^{p,f-}(\mathbf{x}_0,\mathbf{x}_0'',\omega)]^*)[G_3^{v,q+}(\mathbf{x}_0,\mathbf{x}_0''',\omega) + G_3^{v,q-}(\mathbf{x}_0,\mathbf{x}_0''',\omega)]$$
$$-([G_3^{p,f+}(\mathbf{x}_0,\mathbf{x}_0'',\omega)]^* - [G_3^{p,f-}(\mathbf{x}_0,\mathbf{x}_0'',\omega)]^*)[G_3^{v,q+}(\mathbf{x}_0,\mathbf{x}_0''',\omega) - G_3^{v,q-}(\mathbf{x}_0,\mathbf{x}_0''',\omega)] \Big\}\, \mathrm{d}^2\mathbf{x}_0. \tag{A-8}$$

In expression (A-8), the terms containing Green's functions propagating in the same directions (e.g. $[G_3^{p,f+}]^* G_3^{v,q+}$) cancel; hence we obtain

$$2\int_{\partial\mathbb{D}_0} \Big\{[G_3^{p,f+}(\mathbf{x}_0,\mathbf{x}_0'',\omega)]^* G_3^{v,q+}(\mathbf{x}_0,\mathbf{x}_0''',\omega) - [G_3^{p,f-}(\mathbf{x}_0,\mathbf{x}_0'',\omega)]^* G_3^{v,q-}(\mathbf{x}_0,\mathbf{x}_0''',\omega)\Big\}\,\mathrm{d}^2\mathbf{x}_0 \tag{A-9}$$
$$= 2\int_{\partial\mathbb{D}_m} \Big\{[G_3^{p,f+}(\mathbf{x}_m,\mathbf{x}_0'',\omega)]^* G_3^{v,q+}(\mathbf{x}_m,\mathbf{x}_0''',\omega) + [G_3^{p,f-}(\mathbf{x}_m,\mathbf{x}_0'',\omega)]^* G_3^{v,q-}(\mathbf{x}_m,\mathbf{x}_0''',\omega)\Big\}\,\mathrm{d}^2\mathbf{x}_m,$$

where we assumed that the evanescent waves can be neglected. This is a one-way reciprocity theorem of the correlation-type.

We define two acoustic states that will be used in the one-way reciprocity theorem, defined by equation (A-9), to derive a relationship between the pressure-normalized reflection and transmission responses measured on $\partial\mathbb{D}_0$ and $\partial\mathbb{D}_m$. Both states are defined in the actual medium between the boundaries $\partial\mathbb{D}_0$ and $\partial\mathbb{D}_m$, as in state A1 in Figure 2. We choose an impulsive point source of volume injection rate at $\mathbf{x}'''$, just above $\partial\mathbb{D}_0$, and define state C1 as

$$\mathbf{x}\in\partial\mathbb{D}_0 = \begin{cases} G_3^{v,q+}(\mathbf{x}_0,\mathbf{x}_0''',\omega) = -G_3^{p,f+}(\mathbf{x}_0''',\mathbf{x}_0,\omega) = -\delta(\mathbf{x}_H-\mathbf{x}_H'''), \\ G_3^{v,q-}(\mathbf{x}_0,\mathbf{x}_0''',\omega) = -G_3^{p,f-}(\mathbf{x}_0''',\mathbf{x}_0,\omega) = -R(\mathbf{x}_0''',\mathbf{x}_0,\omega), \end{cases} \tag{A-10}$$

$$\mathbf{x}\in\partial\mathbb{D}_m = \begin{cases} G_3^{v,q+}(\mathbf{x}_m,\mathbf{x}_0''',\omega) = -G_3^{p,f+}(\mathbf{x}_0''',\mathbf{x}_m,\omega) = -T(\mathbf{x}_0''',\mathbf{x}_m,\omega), \\ G_3^{v,q-}(\mathbf{x}_m,\mathbf{x}_0''',\omega) = 0. \end{cases} \tag{A-11}$$

For state D1, we choose a point source of force at $\mathbf{x}''$, just above $\partial\mathbb{D}_0$:

$$\mathbf{x}\in\partial\mathbb{D}_0 = \begin{cases} G_3^{p,f+}(\mathbf{x}_0,\mathbf{x}_0'',\omega) = \delta(\mathbf{x}_H-\mathbf{x}_H''), \\ G_3^{p,f-}(\mathbf{x}_0,\mathbf{x}_0'',\omega) = R(\mathbf{x}_0,\mathbf{x}_0'',\omega), \end{cases} \tag{A-12}$$

$$\mathbf{x}\in\partial\mathbb{D}_m = \begin{cases} G_3^{p,f+}(\mathbf{x}_m,\mathbf{x}_0'',\omega) = T(\mathbf{x}_m,\mathbf{x}_0'',\omega), \\ G_3^{p,f-}(\mathbf{x}_0,\mathbf{x}_0'',\omega) = 0. \end{cases} \tag{A-13}$$

Now, we plug these expressions into equation (A-9) and obtain

$$\int_{\partial\mathbb{D}_m} [T(\mathbf{x}_m,\mathbf{x}_0'',\omega)]^* T(\mathbf{x}_0,\mathbf{x}_m,\omega)\mathrm{d}^2\mathbf{x}_m \tag{A-14}$$

$$= \delta(\mathbf{x}_H''-\mathbf{x}_H'') - \int_{\partial\mathbb{D}_0} [R(\mathbf{x}_0,\mathbf{x}_0'',\omega)]^* R(\mathbf{x}_0''',\mathbf{x}_0,\omega)\mathrm{d}^2\mathbf{x}_0. \tag{A-15}$$

This is the pressure-normalized version of equation (20) in Wapenaar et al. (2004).