Scale factor for ray theoretic Green's function amplitudes

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ABSTRACT
Dynamic ray theory is a method for obtaining the asymptotic solution of wave equations in the limit of high frequency. The method reduces the description to a system of ordinary differential equations that describe the rays and then two further equations to describe how the travel time and amplitude of a simple wave-like solution propagates along the rays. This system of equations requires initial values in order for the solution to be carried out. For the asymptotic Green's function, the amplitude is singular right at the source point and a simple evaluation is not possible. Instead, a comparison technique is used, with the additional criterion that the amplitude have the proper order singularity at the source point. That allows the final determination of a scale factor in the amplitude. This scale factor depends on the parameters used to describe the rays. Here, for the acoustic wave equation, we derive formulas for those scale factors in general terms for arbitrary choice of the ray parameters. We do this in 2D, 2.5D and 3D.

Introduction

This is a short note on the scaling factors for the amplitude of the ray-theoretic Green's function for the scalar wave equation or Helmholtz equation. Dynamic ray tracing is a method for determining the rays and the propagation of travel time and amplitude along those rays. In addition to the system of ordinary differential equations, one needs initial values of all of the variables at the staring point of each ray. For Green's functions, the amplitude is infinite at the source point which is also the starting point of the rays. Thus, for the ray-theoretic amplitude in a heterogeneous medium it is necessary to characterize the amplitude by the nature of its singular behavior and thereby determine an appropriate scaling constant for the amplitude solution. The objective of this note is to derive a formula for that scale factor for any choice of running parameter along the ray; examples of such running parameter are travel time, arc length or σ, the variable that measures the rate of out-of-plane geometrical spreading in 2.5D propagation. We use Appendix E of Bleistein, et al. [2001] (referenced as MMSIMI below) as a point of departure. Indeed, we view this discussion as a follow-on to Sections E.4.1, E.5.2 and E.4.2, where the initial data along the rays for all of the constituents of the ray theoretic solution are discussed in 3D, 2.5D and 2D, respectively.

The new feature here is that we discuss the initial data for the amplitude for generic ray coordinates. We find that the scale factor is independent of the choice of the running parameter along the ray, thus, for example, being the same if we use any of the above mentioned variables.

The Ray Equations for the 3D Green's Function.

The starting point for this discussion is the equations that govern the propagation of the rays, phase and leading order amplitude of ray theory. The leading-order term of the ray theoretic solution in 3D has the form

$$u(\boldsymbol{x},\omega) = A_{3D}(\boldsymbol{x})e^{i\omega\tau(\boldsymbol{x})}. \tag{1}$$

In this equation and for the remainder of this section, we emphasize that this is the amplitude in 3D by that subscript. For this solution, the rays emanate from a single point, say $\boldsymbol{x} = \boldsymbol{x}_0$, where we can take the travel time, τ to be zero. Furthermore, we know the exact solution for the amplitude in this case when the medium is homogeneous. We will require that the solution for heterogeneous media have the same local behavior near $\boldsymbol{x}_0$, that is,

$$A_{3D}(\boldsymbol{x}) \approx \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{x}_0|}, \quad \boldsymbol{x} \text{ “near” } \boldsymbol{x}_0. \tag{2}$$

In this equation, the right side is the exact solution for homogeneous media and therefor is certainly the leading-order asymptotic solution, as well.

The plan now is to study the solution of the ray equations for a heterogeneous medium in the neighborhood of $\boldsymbol{x}_0$ to obtain an approximate solution there in terms of ray parameters, modulo the scale we seek. We then write that solution in terms of the Cartesian coordinates $\boldsymbol{x}$ and carry out the comparison in (2). That will determine the constant we seek.

We begin by quoting the governing differential equations of ray theory from MMSIMI: equations (E.2.10) and (E.3.2). One change we make here is that we use γ_3 for the independent variable along the rays in this discussion. Thus, we write

$$\begin{aligned} \frac{d\boldsymbol{x}}{d\gamma_3} &= 2\lambda\boldsymbol{p}, \quad \boldsymbol{x}(\gamma_1,\gamma_2,0)=\boldsymbol{x}_0; \\ \frac{d\boldsymbol{p}}{d\gamma_3} &= 2\lambda(\boldsymbol{x})\nabla\left[\frac{1}{c^2(\boldsymbol{x})}\right] = 2\lambda p(\boldsymbol{x})\nabla p(\boldsymbol{x}) \\ &= -\frac{2\lambda}{c^3(\boldsymbol{x})}\nabla c(\boldsymbol{x}), \quad \boldsymbol{p}(\gamma_1,\gamma_2,0)=\boldsymbol{p}_0(\gamma_1,\gamma_2). \\ \frac{d\tau}{d\gamma_3} &= \frac{2\lambda}{c^2(\boldsymbol{x})} = 2\lambda p^2(\boldsymbol{x}), \quad \tau(\gamma_1,\gamma_2,0)=0. \end{aligned} \tag{3}$$

In these equations, we have used

$$\nabla\tau=\boldsymbol{p}, \quad \boldsymbol{p}\cdot\boldsymbol{p}=\frac{1}{c^2(\boldsymbol{x})}=p^2(\boldsymbol{x}), \tag{4}$$

and we have taken the initial value of $\boldsymbol{p}$ on the ray to be a function of the two variables, (γ_1,γ_2), denoted by $\boldsymbol{p}_0(\gamma_1,\gamma_2)$. Each choice of (γ_1,γ_2) labels a ray and defines its initial direction through the first of the ray equations where the direction of the tangent to the ray at any point is seen to be just the direction of $\boldsymbol{p}$.

The parameter λ is a mathematical device that allows us to include all choices of running parameter along the ray in one system of equations. For example, if we set $2\lambda = c^2(\boldsymbol{x})$, then the right side of the equation for $\boldsymbol{x}$ are seen to have the dimensions of velocity. Thus, the same must be true on the left side, which means that γ_3 must have the dimension of time. This can be further confirmed by checking that the equation for τ now becomes $d\tau/d\gamma_3 = 1$. On the other hand, if we set $2\lambda = c(\boldsymbol{x})$, then the right side of the differential equation for $\boldsymbol{x}$ is dimensionless and has magnitude of one. Hence the same must be true on the left side and γ_3 must therefore be arc length. Similarly, the reader might check that setting $2\lambda = 1/p_3$ recasts γ_3 as z.

These changes are also achievable if we did not include λ in the equations. For example, dividing all equations except the differential equation for τ by that equation for τ leads to a system of equations with τ as the independent variable along the ray. Dividing the equations by $\sqrt{(d\boldsymbol{x}/d\gamma_3)^2}$ leads to equations in which γ_3 is arc length, and so on. However, proceeding in that manner will not make the point that we want to make here, namely, that the results derived here are true for any of these parameter choices—easily seen as meaning any choice of λ in the formulation of the ray equations as presented.

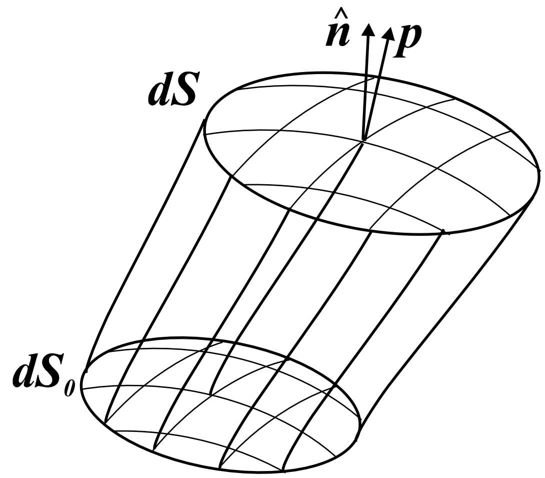

Figure 1. Mapping by rays of a differential surface area element. An element of surface dS_0 across a family of rays at level γ_{30} is extended by the rays to a new differential surface element dS at level γ_3. The lower surface is described by two parameters, γ_1 and γ_2 and the rays that map the lower surface to the upper surface are therefore also distinguished by the labels γ_1 and γ_2.

The equation governing the propagation of the amplitude, (E.3.2), is

$$\begin{aligned} &2A_{3D}\nabla\tau(\boldsymbol{x})\cdot\nabla A_{3D}(\boldsymbol{x}) + A_{3D}^2\nabla^2\tau(\boldsymbol{x}) \\ &\qquad = \nabla\cdot(A_{3D}^2\nabla\tau(\boldsymbol{x})) = 0. \end{aligned} \tag{5}$$

Without going into the details here that can be found in MMSIMI and many other places, we write down the main conclusion of this last equation, namely that

$$A_{3D}^2\boldsymbol{p}\cdot\hat{\boldsymbol{n}}dS\Big|_{\gamma_{30}}^{\gamma_3} = 0. \tag{6}$$

Here, dS is a differential surface area element mapped by the rays from a level, γ_{30} to another level γ_3. See Figure (1. In more detail, we label the rays by two parameters, γ_1 and γ_2. We take a differential cross section dS_0 of the rays at γ_3, a fixed level of the running parameter γ. We use the rays through dS_0 to create another differential surface dS for an arbitrary value of γ. The range of values of γ_1 and γ_2 on each surface is the same. Thus, the differential domain of surface is the same if expressed in terms of these variables. Thus, if we choose as that element of surface $d\gamma_1 d\gamma_2$, then as a standard result of the calculus of surface integration

$$dS = \left|\frac{\partial\boldsymbol{x}}{\partial\gamma_1}\times\frac{\partial\boldsymbol{x}}{\partial\gamma_2}\right| d\gamma_1 d\gamma_2,$$

for any choice of γ_3. Therefore, the entire change in the surface area element in the physical space is characterized by the cross product appearing in this equation. That is,

$$A_{3D}^2 \boldsymbol{p}\cdot\hat{\boldsymbol{n}}\left|\frac{\partial \boldsymbol{x}}{\partial\gamma_1}\times\frac{\partial \boldsymbol{x}}{\partial\gamma_2}\right|\Bigg|_{\gamma_{30}}^{\gamma_3} = 0. \tag{7}$$

Now, let us use the first line of the ray equations in (3) to set

$$\boldsymbol{p} = \frac{1}{2\lambda}\frac{d\boldsymbol{x}}{d\gamma_3}$$

and further observe that

$$\hat{\boldsymbol{n}}\left|\frac{\partial \boldsymbol{x}}{\partial\gamma_1}\times\frac{\partial \boldsymbol{x}}{\partial\gamma_2}\right| = \frac{\partial \boldsymbol{x}}{\partial\gamma_1}\times\frac{\partial \boldsymbol{x}}{\partial\gamma_2},$$

in order to rewrite (7) in terms of the triple scalar product, which is also a determinant. That is,

$$\begin{aligned}
A_{3D}^2 \boldsymbol{p}\cdot\hat{\boldsymbol{n}}\left|\frac{\partial \boldsymbol{x}}{\partial\gamma_1}\times\frac{\partial \boldsymbol{x}}{\partial\gamma_2}\right|\Bigg|_{\gamma_3} &= \\
& A_{3D}^2 \boldsymbol{p}\cdot\hat{\boldsymbol{n}}\left|\frac{\partial \boldsymbol{x}}{\partial\gamma_1}\times\frac{\partial \boldsymbol{x}}{\partial\gamma_2}\right|\Bigg|_{\gamma_{30}}, \quad\Rightarrow \\
\frac{A_{3D}^2}{2\lambda}\left|\frac{d\boldsymbol{x}}{d\gamma_3}\cdot\left(\frac{\partial \boldsymbol{x}}{\partial\gamma_1}\times\frac{\partial \boldsymbol{x}}{\partial\gamma_2}\right)\right|\Bigg|_{\gamma_3} &= \\
& \frac{A_{3D}^2}{2\lambda}\left|\frac{d\boldsymbol{x}}{d\gamma_3}\cdot\left(\frac{\partial \boldsymbol{x}}{\partial\gamma_1}\times\frac{\partial \boldsymbol{x}}{\partial\gamma_2}\right)\right|\Bigg|_{\gamma_{30}}, \\
\Rightarrow \quad \frac{A_{3D}^2}{2\lambda}\left|\frac{\partial(\boldsymbol{x})}{\partial(\boldsymbol{\gamma})}\right|\Bigg|_{\gamma_3} &= K^2(\gamma_1,\gamma_2).
\end{aligned} \tag{8}$$

In this equation, we have evaluated the expression in (7) at the fixed point, γ_{30} as a constant in γ_3, $K^2(\gamma_1,\gamma_2)$. The second line uses the observations above to rewrite all the factors multiplying the amplitude as a triple scalar product. The third line reinterprets that triple scalar product as the Jacobian of transformation from the variables $\boldsymbol{x}$ to $\boldsymbol{\gamma}$ induced by the ray equations.

Now, the solution for $A_{3D}(\boldsymbol{x}(\boldsymbol{\gamma}))$ can be written as

$$\begin{aligned}
A_{3D}(\boldsymbol{x}(\boldsymbol{\gamma})) &= \frac{K_{3D}(\gamma_1,\gamma_2)\sqrt{2\lambda(\boldsymbol{x}(\boldsymbol{\gamma}))}}{\sqrt{\left|\dfrac{\partial(\boldsymbol{x})}{\partial(\boldsymbol{\gamma})}\right|}} \\
&= \frac{K_{3D}(\gamma_1,\gamma_2)\sqrt{2\lambda(\boldsymbol{x}(\boldsymbol{\gamma}))}}{|\sqrt{J_{3D}(\boldsymbol{\gamma}|)}}.
\end{aligned} \tag{9}$$

In the next subsection, I will verify that the Jacobian has the singular behavior characterized in (2) by examining the solution near the singular point. Thus, we need only pick the "constant" $K_{3D}(\gamma_1,\gamma_2)$ correctly in order to obtain exactly the behavior indicated by that equation.

Determining $K_{3D}(\gamma_1,\gamma_2)$ for the 3D Green's Function.

In order to determine $K_{3D}(\gamma_1,\gamma_2)$, we need to examine the solution of the ray equations in the neighborhood of $\boldsymbol{x}_0$. We return to (3) with $c(\boldsymbol{x}) = c(\boldsymbol{x}_0)$ and $\lambda(\boldsymbol{x}) = \lambda(\boldsymbol{x}_0)$. This implies that the derivatives of $\boldsymbol{p}$ are all zero. In this case, $\boldsymbol{p}$ is given by its initial value, $\boldsymbol{p}_0(\gamma_1,\gamma_2)$. Then, the equation for $\boldsymbol{x}$ can be solved, since both λ and $\boldsymbol{p}$ are now constants. That solution is

$$\begin{aligned}
\boldsymbol{x} &\approx \boldsymbol{x}_0 + 2\lambda(\boldsymbol{x}_0)\boldsymbol{p}_0\gamma_3, \quad\Rightarrow \\
& \quad |\boldsymbol{x}-\boldsymbol{x}_0| \approx 2\lambda(\boldsymbol{x}_0)|\boldsymbol{p}_0|\gamma_3, \\
\frac{\partial\boldsymbol{x}}{\partial\gamma_j} &\approx 2\lambda(\boldsymbol{x}_0)\frac{\partial\boldsymbol{p}_0}{\partial\gamma_j}\gamma_3, \quad j=1,2.
\end{aligned} \tag{10}$$

The third line here comes from differentiating the solution for $\boldsymbol{x}$ in the first line since only $\boldsymbol{p}_0$ in that equation depends on γ_1, and γ_2. Now, one can verify that

$$\begin{aligned}
J_{3D}(\boldsymbol{\gamma}) = \frac{\partial(\boldsymbol{x})}{\partial(\boldsymbol{\gamma})} &\approx 8\lambda^3(\boldsymbol{x}_0)\gamma_3^2 \det\begin{bmatrix}\boldsymbol{p}_0 \\ \dfrac{\partial\boldsymbol{p}_0}{\partial\gamma_1} \\ \dfrac{\partial\boldsymbol{p}_0}{\partial\gamma_2}\end{bmatrix} \\
&= 8\lambda^3(\boldsymbol{x}_0)\gamma_3^2 p_0\left|\frac{\partial\boldsymbol{p}_0}{\partial\gamma_1}\times\frac{\partial\boldsymbol{p}_0}{\partial\gamma_2}\right|.
\end{aligned} \tag{11}$$

To determine $K_{3D}(\gamma_1,\gamma_2)$, we need only compare the approximate solution in (2) with the approximate solution in terms of the parameters $\boldsymbol{\gamma}$ that is deduced by using the results derived just above. Thus, we will use the Jacobian in the previous equation and the distance function in (10) in the solution representation (9):

$$\begin{aligned}
A_{3D} &\approx \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{x}_0|} \approx \frac{1}{8\pi\lambda(\boldsymbol{x}_0)|\boldsymbol{p}_0|\gamma_3} \\
&\approx \frac{K_{3D}(\gamma_1,\gamma_2)\sqrt{2\lambda(\boldsymbol{x}_0)}}{2\lambda(\boldsymbol{x}_0)\gamma_3\sqrt{2\lambda(\boldsymbol{x}_0)p_0\left|\dfrac{\partial\boldsymbol{p}_0}{\partial\gamma_1}\times\dfrac{\partial\boldsymbol{p}_0}{\partial\gamma_2}\right|}} \\
&\approx \frac{K_{3D}(\gamma_1,\gamma_2)}{2\lambda(\boldsymbol{x}_0)\gamma_3\sqrt{p_0\left|\dfrac{\partial\boldsymbol{p}_0}{\partial\gamma_1}\times\dfrac{\partial\boldsymbol{p}_0}{\partial\gamma_2}\right|}}.
\end{aligned} \tag{12}$$

In this equation, the rightmost equality in the first line simply uses the last equality in the first line of (10). The next line uses the solution representation (9) with the Jacobian evaluated from (11). Finally, then, the last line is just a simplification of the previous result. Next we compare the second and last expressions here to determine $K_{3D}(\gamma_1, \gamma_2)$. The result is

$$K_{3D}(\gamma_1,\gamma_2) = \frac{1}{4\pi}\frac{\sqrt{\left|\dfrac{\partial \boldsymbol{p}_0}{\partial\gamma_1}\times\dfrac{\partial \boldsymbol{p}_0}{\partial\gamma_2}\right|}}{\sqrt{p_0}} = \frac{1}{4\pi}\sqrt{c(\boldsymbol{x}_0)\left|\frac{\partial \boldsymbol{p}_0}{\partial\gamma_1}\times\frac{\partial \boldsymbol{p}_0}{\partial\gamma_2}\right|}. \tag{13}$$

It can be seen, here, that $K_{3D}(\gamma_1, \gamma_2)$ is independent of $\lambda(\boldsymbol{x})$. Thus, whether τ (travel time), σ (the 2.5D out-of-plane spreading factor) or s (arc length) is used as the independent variable along the rays, the constant is the same, and the solution for the amplitude is

$$A_{3D}(\boldsymbol{x}(\boldsymbol{\gamma})) = \frac{1}{4\pi}\sqrt{c(\boldsymbol{x}_0)\left|\frac{\partial \boldsymbol{p}_0}{\partial\gamma_1}\times\frac{\partial \boldsymbol{p}_0}{\partial\gamma_2}\right|}\sqrt{\frac{2\lambda(\boldsymbol{x}(\boldsymbol{\gamma}))}{J_{3D}(\boldsymbol{\gamma})}}. \tag{14}$$

We refrain from simplifying the powers of 2 here because the choice of ray parameter will lead to a "natural" choice of 2λ rather than λ. Note, from (11), that in our approximate solution $J_{3D} = O((2\lambda)^3)$ so that the quotient appearing under the second square root here is $O((2\lambda)^{-2})$. Then the square root of this quantity, which is the dependence of the amplitude on λ is $O((2\lambda)^{-1})$. This estimate correctly matches the power of λ in $1/|\boldsymbol{x} - \boldsymbol{x}_0|$ in the second expression in (12). This is just another way of seeing that K_{3D} is appropriately independent of λ.

Examples of K_{3D}.

Here, I present two examples of the determination of K_{3D}. The first one uses the polar coordinates of $\boldsymbol{p}_0$ and the second uses the first two components of $\boldsymbol{p}_0$ for (γ_1, γ_2).

K_{3D} for initial polar angles as ray-labeling parameters.

Here, we set

$$\boldsymbol{p}_0 = p_0(\boldsymbol{x}_0)(\sin\gamma_1\sin\gamma_2, \cos\gamma_1\sin\gamma_2, \cos\gamma_2) \tag{15}$$

and find that

$$\left|\frac{\partial \boldsymbol{p}_0}{\partial\gamma_1}\times\frac{\partial \boldsymbol{p}_0}{\partial\gamma_2}\right| = p^2(\boldsymbol{x}_0)\sin\gamma_2. \tag{16}$$

Then, by substituting into (13) and (14), we find that

$$K_{3D}(\gamma_1,\gamma_2) = \frac{1}{4\pi}\sqrt{p(\boldsymbol{x}_0)\sin\gamma_2} = \frac{1}{4\pi}\sqrt{\frac{\sin\gamma_2}{c(\boldsymbol{x}_0)}}, \tag{17}$$

so that

$$A_{3D}(\boldsymbol{x}(\boldsymbol{\gamma})) = \frac{1}{4\pi}\sqrt{\frac{2\lambda(\boldsymbol{x}(\boldsymbol{\gamma}))\sin\gamma_2}{c(\boldsymbol{x}_0)J_{3D}(\boldsymbol{\gamma})}}. \tag{18}$$

K_{3D} for initial values, (p_{01}, p_{02}), as ray-labeling parameters.

In this case, we set

$$\boldsymbol{p}_0 = \left(p_{01},\ p_{02},\ p_{03} = \sqrt{p_0^2 - p_{01}^2 - p_{02}^2}\right) \tag{19}$$

and find that

$$\left|\frac{\partial \boldsymbol{p}_0}{\partial\gamma_1}\times\frac{\partial \boldsymbol{p}_0}{\partial\gamma_2}\right| = \frac{p_0}{p_{03}}. \tag{20}$$

Consequently, in this case,

$$A_{3D}(\boldsymbol{x}(\boldsymbol{\gamma})) = \frac{1}{4\pi}\sqrt{\frac{2\lambda(\boldsymbol{x}(\boldsymbol{\gamma}))}{p_{03}J_{3D}(\boldsymbol{\gamma})}}. \tag{21}$$

Two-and-one-half Dimensions (2.5D).

Two-and-one-half dimensions (2.5D) is the name we use for 3D propagation in the (x, z)-plane for a medium in which the wave speed is independent of y; that is, $c = c(x, z)$. Results for 2.5D can be deduced from the 3D wave propagation results of the previous section when we specialize to this type of medium. We describe that process here and then derive the corresponding constant for the Green's function, $K_{2.5D}$.

The major consequence of a y-independent wave speed arises in the ray equation for p_2 in (3):

$$\frac{dp_2}{d\gamma_3} = 0, \tag{22}$$

so that p_2 is given by its initial value, p_{20}. It then makes sense to take this as one of the ray-labeling parameters; that is, set $\gamma_2 = p_{20}$. Now, from the differential equation for $y = x_2$ in (3), we find that

$$x_2 = x_{20} + 2\lambda\gamma_2\gamma_3. \tag{23}$$

We remark that the solution we seek is still the 3D Green's function, but specialized to in-plane propagation in the (x_1, x_3) plane. From this equation, we conclude that such in-plane propagation arises when we set $p_{20} = 0$, which we will do, below. For the moment, note that the ray equations in (3) reduce now to the 2D equations in $(x_1, x_3, p_1, p_3, \tau)$. Thus, the in-plane propagation is a solution of the same system of equations as that which arises in 2D; however, the amplitude must still honor 3D geometrical spreading.

We can now complete the story of the initial values, $\boldsymbol{p}_0$, as

$$\begin{aligned}\boldsymbol{p}_0 &= (p_T\sin\gamma_1, \gamma_2, p_T\cos\gamma_1),\\ p_T &= \sqrt{p_0^2 - \gamma_2^2} = \sqrt{c(\boldsymbol{x}_0)^{-2} - \gamma_2^2}.\end{aligned} \tag{24}$$

Next, let us consider the 3D Jacobian for this case, when we evaluate it at $p_{20} = 0$. First, note from (23) that for any choice of the constant p_{20}

$$\frac{\partial x_2}{\partial \gamma_1} = 0.$$

Then,

$$J_{3D}(\gamma_1, 0, \gamma_3) = \left|\frac{\partial(\boldsymbol{x})}{\partial(\boldsymbol{\gamma})}\right|\Bigg|_{\gamma_2=0} = \left|\det\begin{bmatrix} \cdot & 0 & \cdot \\ \cdot & 2\lambda\gamma_3 & \cdot \\ \cdot & 0 & \cdot \end{bmatrix}\right| \tag{25}$$

$$= 2\lambda\gamma_3 \frac{\partial(x_1, x_3)}{\partial(\gamma_1, \gamma_3)} = 2\lambda\gamma_3 J_{2D}(\gamma_1, \gamma_3).$$

We also need to calculate the cross product in (13) by using (24) and setting $\gamma_2 = 0$. That result is

$$\left|\frac{\partial \boldsymbol{p}_0}{\partial \gamma_1} \times \frac{\partial \boldsymbol{p}_0}{\partial \gamma_2}\right|\Bigg|_{\gamma_2=0} = p_0\,|(\cos\gamma_1, 0, -\sin\ \gamma_1) \times (0,1,0)| \tag{26}$$

$$= p_0 = 1/c(\boldsymbol{x}_0).$$

By using these last two results in the equations for K and A in (13) and (14), we find the corresponding 2.5D results,

$$K_{2.5D}(\gamma_1) = \frac{1}{4\pi}; \quad A_{2.5D}(\boldsymbol{x}(\boldsymbol{\gamma})) = \frac{1}{4\pi\sqrt{\gamma_3 J_{2D}(\boldsymbol{\gamma})}}. \tag{27}$$

Here, $\boldsymbol{\gamma} = (\gamma_1, \gamma_3)$ and the result is specific to the choice of γ_1 being the angle that the in-plane ray makes with the vertical direction. This is a convenient choice that leads to the simplest representation.

We remark that this last solution is not completely independent of λ because $J_{2D} = O(\lambda)$. Usually, we set $2\lambda = 1$ and denote the special value of γ_3 by σ. This ray parameter has unusual dimensions. If we return to (3) with $\lambda = 1$, we can write the dimensional equation

$$\text{dimension}[\sigma] = \frac{\text{dimension}[d\boldsymbol{x}]}{\text{dimension}[\boldsymbol{p}]}$$

$$= \text{dimension}[d\boldsymbol{x}]\text{dimension}[c(\boldsymbol{x})]$$

$$= \frac{\text{length}^2}{\text{time}}.$$

On the other hand, (27) reveals that there is nothing sacred about σ. For example, we could as well characterize the out-of-plane geometrical spreading by τ (for which $2\lambda = c^2(\boldsymbol{x})$) and J_{2D} would compensate for this new out-of-plane spreading factor through the corresponding new choice of λ.

Two Dimensions (2D).

In two dimensions, the Green's function has a slightly different form compared to (1); namely,

$$u(\boldsymbol{x}, \omega) = \frac{A_{2D}(\boldsymbol{x})}{\sqrt{|\omega|}} e^{i\omega\tau(\boldsymbol{x}) + i\pi/4\text{sgn}(\omega)} \tag{28}$$

$$= \frac{A_{2D}(\boldsymbol{x})}{\sqrt{-i\omega}} e^{i\omega\tau(\boldsymbol{x}))}.$$

For this solution, again the rays emanate from a single point, say $\boldsymbol{x} = \boldsymbol{x}_0$, where, again, we can take the travel time, τ to be zero. determination of the proper behavior for the amplitude is more subtle. The exact solution for the 2D Green's function has a logarithmic singularity in $|\boldsymbol{x} - \boldsymbol{x}_0|$ at $\boldsymbol{x}_0$. However, it is not the exact solution that we want to match, but the asymptotic solution. That asymptotic solution is valid for "large" values of the dimensionless parameter, $\omega L/c$, with L any one of a number of length scales of the problem and c a "typical" value of the wave speed. One of the candidates for L is the range from the source, $|\boldsymbol{x} - \boldsymbol{x}_0|$. At the very least, this distance cannot be nearly zero, which precludes matching the asymptotic solution to the log singularity of the exact solution. Quite the contrary, we want our ray theoretic solution to characterize propagation "many" units of inverse wave number c/ω from the source. Luckily, the asymptotic expansion of the Green's function for homogeneous media is well known and leads to the requirement that

$$A_{2D}(\boldsymbol{x}) \approx \frac{1}{2\sqrt{2\pi|\boldsymbol{x} - \boldsymbol{x}_0|/c(\boldsymbol{x}_0)}}. \tag{29}$$

The right side here is the leading-order amplitude of the asymptotic Green's function for constant wave speed $c(\boldsymbol{x}_0)$.

We need to modify the ray equations (3) by interpreting $\boldsymbol{x} = (x_1, x_3)$ and $\boldsymbol{p} = (p_1, p_3)$ being two dimensional vectors with only two ray parameters, $\boldsymbol{\gamma} = (\gamma_1, \gamma_3)$. That is,

$$\frac{d\boldsymbol{x}}{d\gamma_3} = 2\lambda\boldsymbol{p}, \quad \boldsymbol{x}(\gamma_1, 0) = \boldsymbol{x}_0;$$

$$\frac{d\boldsymbol{p}}{d\gamma_3} = 2\lambda(\boldsymbol{x})\nabla\left[\frac{1}{c^2(\boldsymbol{x})}\right]$$

$$= 2\lambda p(\boldsymbol{x})\nabla p(\boldsymbol{x}) \tag{30}$$

$$= -\frac{2\lambda}{c^3(\boldsymbol{x})}\nabla c(\boldsymbol{x}), \quad \boldsymbol{p}(\gamma_1, 0) = \boldsymbol{p}_0(\gamma_1).$$

$$\frac{d\tau}{d\gamma_3} = \frac{2\lambda}{c^2(\boldsymbol{x})} = 2\lambda p^2(\boldsymbol{x}), \quad \tau(\gamma_1, 0) = 0.$$

Similarly, (6) is replaced by

$$A_{2D}^2 \boldsymbol{p}\cdot\hat{\boldsymbol{n}}ds\Big|_{\gamma_{30}}^{\gamma_3} = 0. \tag{31}$$

In this case, ds is differential arc length mapped by rays between the level, γ_{30}, and the level γ_3. Furthermore, in place of the equation for dS, we write,

$$ds = \left|\frac{\partial \boldsymbol{x}}{\partial \gamma_1}\right| d\gamma_1.$$

Again, the differential $d\gamma_1$ is a constant along the rays, so that (31) leads to the following counterpart of (7):

$$A_{2D}^2 \boldsymbol{p}\cdot\hat{\boldsymbol{n}}\left|\frac{\partial \boldsymbol{x}}{\partial\gamma_1}\right|\Bigg|_{\gamma_{30}}^{\gamma_3} = 0. \tag{32}$$

As in the 3D case, let us use the first line of the ray equations, this time in (30), to set

$$\boldsymbol{p} = \frac{1}{2\lambda}\frac{d\boldsymbol{x}}{d\gamma_3}$$

and further observe that

$$\begin{aligned} |\boldsymbol{p}\cdot\hat{\boldsymbol{n}}|\left|\frac{\partial \boldsymbol{x}}{\partial\gamma_1}\right| &= \left|\boldsymbol{p}\times\hat{\boldsymbol{t}}\right|\left|\frac{\partial \boldsymbol{x}}{\partial\gamma_1}\right| = \frac{1}{2\lambda}\left|\frac{\partial \boldsymbol{x}}{\partial\gamma_1}\times\frac{d\boldsymbol{x}}{d\gamma_3}\right| \\ &= \frac{1}{2\lambda}\left|\frac{\partial(\boldsymbol{x})}{\partial(\boldsymbol{\gamma})}\right| = \frac{1}{2\lambda}\left|J_{2D}\right|. \end{aligned}$$

In this equation, $\hat{\boldsymbol{t}}$ is the unit tangent in the direction of $\partial\boldsymbol{x}/\partial\gamma_1$. Consequently, in place of (9), we now have the equation

$$\begin{aligned} A_{2D}(\boldsymbol{x}(\boldsymbol{\gamma})) &= \frac{K_{2D}(\gamma_1)\sqrt{2\lambda(\boldsymbol{x}(\boldsymbol{\gamma}))}}{\sqrt{\left|\dfrac{\partial(\boldsymbol{x})}{\partial(\boldsymbol{\gamma})}\right|}} \\ &= \frac{K_{2D}(\gamma_1)\sqrt{2\lambda(\boldsymbol{x}(\boldsymbol{\gamma}))}}{\sqrt{J_{2D}(\boldsymbol{\gamma})}}. \end{aligned} \tag{33}$$

Determining $K_{2D}(\gamma_1)$ for the 2D Green's Function.

This discussion follows along the lines of Section . That is, we examine the solution of the ray equations (30) for $\boldsymbol{x}$ near $\boldsymbol{x}_0$. Thus, the analog of equations (10) and (11) are

$$\begin{aligned} \boldsymbol{x} &\approx \boldsymbol{x}_0 + 2\lambda(\boldsymbol{x}_0)\boldsymbol{p}_0\gamma_3, \\ \frac{\partial\boldsymbol{x}}{\partial\gamma_1} &\approx 2\lambda(\boldsymbol{x}_0)\frac{\partial\boldsymbol{p}_0}{\partial\gamma_1}\gamma_3. \end{aligned} \tag{34}$$

Note that from the first equation here

$$|\boldsymbol{x}-\boldsymbol{x}_0| \approx 2\lambda(\boldsymbol{x}_0)|\boldsymbol{p}_0|\gamma_3. \tag{35}$$

Now, following the same procedure as in going from (10) to (11), one can verify that

$$\begin{aligned} \frac{\partial(\boldsymbol{x})}{\partial(\boldsymbol{\gamma})} &\approx 4\lambda^2(\boldsymbol{x}_0)\gamma_3 \det\begin{bmatrix} \boldsymbol{p}_0 \\ \dfrac{\partial\boldsymbol{p}_0}{\partial\gamma_1}\end{bmatrix} \\ &= 4\lambda^2(\boldsymbol{x}_0)\gamma_3 p_0\left|\frac{\partial\boldsymbol{p}_0}{\partial\gamma_1}\right|. \end{aligned} \tag{36}$$

We now proceed as in (12):

$$\begin{aligned} A_{2D} &\approx \frac{1}{2\sqrt{2\pi|\boldsymbol{x}-\boldsymbol{x}_0|/c(\boldsymbol{x}_0)}} \approx \frac{c(\boldsymbol{x}_0)}{4\sqrt{\pi\lambda(\boldsymbol{x}_0)\gamma_3}} \\ &\approx \frac{K_{2D}(\gamma_1)\sqrt{2\lambda(\boldsymbol{x}_0)}}{2\lambda(\boldsymbol{x}_0)\sqrt{\gamma_3 p_0\left|\dfrac{\partial\boldsymbol{p}_0}{\partial\gamma_1}\right|}} \approx \frac{K_{2D}(\gamma_1)}{\sqrt{2\lambda(\boldsymbol{x}_0)\gamma_3 p_0\left|\dfrac{\partial\boldsymbol{p}_0}{\partial\gamma_1}\right|}}. \end{aligned} \tag{37}$$

Here, the last expression in the first line follows from (34). The next line arises from using (36) in (33) and the last line is just a simplification of the previous one. By comparing the second and fourth expressions on the right side here, we conclude that

$$K_{2D}(\gamma_1) = \frac{\sqrt{c(\boldsymbol{x}_0)\left|\dfrac{\partial\boldsymbol{p}_0}{\partial\gamma_1}\right|}}{2\sqrt{2\pi}} \tag{38}$$

and

$$A_{2D}(\boldsymbol{x}) = \frac{\sqrt{2\lambda(\boldsymbol{x}(\boldsymbol{\gamma}))c(\boldsymbol{x}_0)\left|\dfrac{\partial\boldsymbol{p}_0}{\partial\gamma_1}\right|}}{2\sqrt{2\pi J_{2D}(\boldsymbol{\gamma})}} \tag{39}$$

As in the previous cases, we see here that $K_{2D}(\gamma_1)$ is independent of λ.

Summary and Conclusions.

We have derived the constants of acoustic 3D, 2.5D and 2D Green's functions for arbitrary choice of the running parameter along the ray. Surprisingly, we found that in all cases, the constant of the amplitude was independent of the scaling—λ, in this discussion. Since there are good reasons to use τ or σ or arc length in different applications, this is a useful result. Also, there are times when we prefer angles to characterize different ray directions or initial values of the components of $\boldsymbol{p}$ itself. The K's here are given as derivatives with respect any choice of ray parameter, also a useful result to have available. Further, I believe that corresponding derivations for isotropic and anisotropic elastic Green's function amplitudes, as well as for electromagnetic Green's function amplitudes are also worth carrying out, with these derivations as a useful guide. However, MMSIMI address only acoustic waves. Thus, this acoustic case is an appropriate addendum to that text.

References

Bleistein, N., J. K. Cohen and J. W. Stockwell, Jr., 2001, *Mathematics of Multidimensional Seismic Imaging, Migration and Inversion*: Springer-Verlag, New York.