Measuring the two-point correlation function of a turbulent fluid with coda wave interferometry

Roel Snieder

Center for Wave Phenomena and Dept. of Geophysics, Colorado School of Mines, Golden CO 80401

ABSTRACT

When a medium changes over time, the multiply scattered waves that propagate through that medium change with a great sensitivity. Coda wave interferometry is a technique that uses this sensitivity to extract information about the change in a medium from the change in the multiply scattered waves. Seeding a turbulent fluid with neutrally buoyant particles, causes acoustic waves to be scattered within the fluid. Coda wave inteferometry, when applied to these multiply scattered acoustic waves, provides a direct measurement of the two-point correlation function of the turbulent fluid at a length scale given by the mean free path of the multiply scattered waves. By varying the properties of the particles or changing the scatterer density, one can vary the mean free path and, as a result, measure the two-point correlation function for different length scales.

1 INTRODUCTION

The statistical character of turbulence is c1haracterized by the two-point correlation function of the velocity. In order to determine this function one must perform a measurement of the velocity within the fluid. A variety of techniques have been developed for this purpose that include various forms of hot-wire anenometry (Wyngaard, 1968; Comte-Bellot and Corrsin, 1971; Saddoughi, S.G. and Veeravalli, 1994) , Particle Image Velocimetry (Raffel et al., 1998; Most and Leipetz, 2001), , Laser-Doppler Anemometry (Durst, 1981), RELIEF flow tagging (Miles et al., 1991), and ultrasonic Doppler tracking (Mordant et al., 2001).

Most of these techniques measure the velocity directly by using either multiple sensors, tracer particles that follow the flow, or a mean flow to provide information on the spatial scale of the velocity variations. The two-point correlation function of a turbulent fluid depends critically on the relative velocity of the fluid at adjacent points, and it is desirable that the technique used to diagnose this function is sensitive to the relative velocity at nearby points.

In this work I propose coda wave interferometry as a technique to monitor the two-point correlation function in a turbulent fluid. The central idea is to place in the fluid neutrally buoyant particles that scatter acoustic waves isotropically. When the scatterers are sufficiently strong, acoustic waves are multiply scattered as they propagate through the fluid, and, because of the turbulent motion in the fluid, the scatterers move. This leads to a change in the multiply scattered waves as the fluid is irradiated repeatedly with the same source of acoustic waves. Scattered waves can thus be used to monitor the displacement of scatterers in a medium over time.

2 PRINCIPLE OF CODA WAVE INTERFEROMETRY

Light has been used to monitor Brownian motion in colloidal suspensions using speckle pattern interferometry (Heckmeier and Maret, 1997). In this technique, a laser beam irradiates a fluid, and the resulting multiply scattered waves form a speckle pattern. The statistical properties of the Brownian motion of the scatterers lead to a temporal change in this speckle pattern over time, which is used to diagnose the Brownian motion.

This process is illustrated with the numerical example shown in Figure 1. An acoustic source indicated with an asterisk radiates acoustic waves into a 2D medium with isotropic point scatterers randomly placed at locations marked by the filled circles. The wave field recorded at a receiver indicated with a triangle is computed using a numerical implementation (Groe-

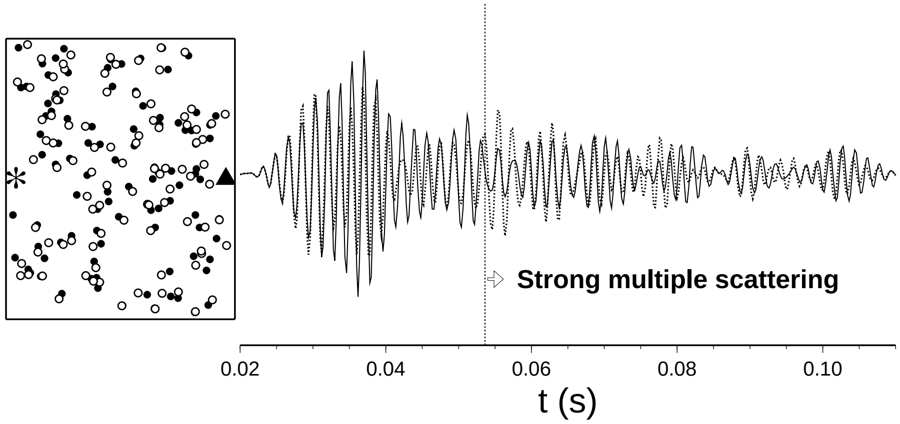

Figure 1. Left: location of 100 scatterers before and after the perturbation (filled dots and open dots respectively) with the source (asterisk) and receiver location (triangle). The scatterers are placed in an area of 40 × 80 m. The waveforms recorded before and after the perturbation at the receiver are shown on the right in solid and dotted lines, respectively.

nenboom and Snieder, 1985; Snieder 1999) of Foldy's method(Foldy, 1945). This wave field is indicated with the solid wiggle trace; it consists of an extended wave-train of multiple scattered waves. The locations of the scatterers are then randomly perturbed to the locations indicated by the open circles. In Figure 1 this displacement relative to the initial position is greatly exaggerated to make it visible. The wave field for this perturbed configuration of scatterers is shown by the dashed wiggle trace in Figure 1. The perturbation of the wave field increases with time because the waves that have bounced back and forth more often are more sensitive to changes in the scatterer locations than are the waves that have been scattered only a few times. As shown by Snieder et al. (2002), the root-mean-square displacement of the scatterer locations can successfully be retrieved from the change in the multiple scattered waves.

Let the unperturbed wave be written as

$$u_{unp} = \sum_P A_P S(t - t_P) \ . \tag{1}$$

In this expression the wave field is written as the Feynman path summation, which expresses the sum over all possible multiple scattering paths P between scatterers (Snieder, 1999). A path is defined as a sequence of scatterers that a wave encounters. The amplitude of the wave that travels along a path with travel time t_P is denoted by A_P, and $S(t)$ is the source time signal. When the scatterer locations are perturbed, the dominant change in expression (1) is that the travel time along each path P is perturbed with an amount τ_P (Snieder et al., 2002). In this approximation the perturbed wave field is given by

$$u_{per} = \sum_P A_P S(t - t_P - \tau_P) \ . \tag{2}$$

The change in the wave field can be measured with the time-shifted correlation function which is defined as

$$R^{(t,T)}(t_s) \equiv \frac{\int_{t-T}^{t+T} u_{unp}(t') u_{per}(t' + t_s) dt'}{\left(\int_{t-T}^{t+T} u_{unp}^2(t') dt' \int_{t-T}^{t+T} u_{per}^2(t') dt' \right)^{1/2}} \ , \tag{3}$$

where the time window has length $2T$ and is centered on time t. The shift time is denoted by t_s.

Snieder et al. (2002) show that the mean travel time perturbation for the paths arriving in the time window is given by the shift time t_s for which the time-shifted correlation function has its maximum:

$$\frac{dR^{(t,T)}(t_s)}{dt_s}(t_s = \langle \tau \rangle^{(t,T)}) = 0 \ . \tag{4}$$

Here the $\langle \cdots \rangle^{(t,T)}$ denotes the average over all paths arriving in the time window $(t - T, t + T)$. The value of the time-shifted correlation function at its maximum is given by

$$R_{\max}^{(t,T)} = 1 - \frac{1}{2} \bar{\omega}^2 \sigma_\tau^2 \ , \tag{5}$$

where σ_τ^2 is the variance of the travel time perturbations of the waves arriving in the time window $(t - T, t + T)$ and the frequency $\bar{\omega}$ is defined by

$$\bar{\omega}^2 \equiv \frac{\int \omega^2 \left| S(\omega) \right|^2 d\omega}{\int \left| S(\omega) \right|^2 d\omega} \ , \tag{6}$$

with $S(\omega)$ the source signal in the frequency domain.

Snieder *et al.* (2002) show that the root-mean-square perturbation of the scatterer locations in Figure 1 can successfully be retrieved from the unperturbed and perturbed wave forms shown in that figure. In that example, the scatterers are perturbed independently. In contrast, when the scatterers are neutrally buoyant particles that are carried around by the flow, the perturbation in the scatterer locations are correlated and depend on the two-point correlation function of the fluid. For this reason we recapitulate the definition of two-point correlation function in section 3. In section 4, I compute the variance of the distance between two points that move apart over a time t_{int} and, in section 5, the variance of the path length when the scatterers move with the flow over a time t_{int}. In section 6, I show that for isotropic scattering the two point correlation function is related to the time-shifted cross correlation function of expression (3), and, in section 7, I show how this can be used in an experiment to measure the two-point correlation function of a turbulent fluid.

3 THE AUTOCORRELATION FOR A RANDOM INCOMPRESSIBLE FLUID

The two-point correlation $\langle v_i(\mathbf{r})v_j(0)\rangle$ describes the instantaneous correlation of the velocity in a turbulent medium. The requirement of incompressibility imposes a constraint on the two-point correlation. As shown by Pope (2000), for three dimensions this requirement leads for isotropic turbulence to the following form of the two-point correlation function

$$\langle v_i(\mathbf{r})v_j(0)\rangle = \left(f(r) + \frac{r}{2}\frac{\partial f}{\partial r}\right)\delta_{ij} - \frac{r}{2}\frac{\partial f}{\partial r}\frac{x_i x_j}{r^2} , \quad (7)$$

where $\langle\cdots\rangle$ denotes an ensemble avarage. So far the function $f(r)$ is arbitrary, but the symmetry of $f(r)$ dictates that (Pope, 2000)

$$\frac{\partial f}{\partial r}(r=0) = 0 . \quad (8)$$

Given that the mean of the velocity is assumed to vanish ($\langle\mathbf{v}\rangle = 0$), the variance of the total displacement is defined by

$$\left\langle |\mathbf{v}|^2 \right\rangle = 3\sigma_v^2 . \quad (9)$$

With (7) this gives the following expression for the variance: $\sigma_v^2 = \left(f(r) + \frac{r}{3}\frac{\partial f}{\partial r}\right)(r=0)$. By virtue of (8) the last term vanishes so that

$$\sigma_v^2 = f(0) . \quad (10)$$

The autocorrelations in the direction of the displacement and the perpendicular direction respectively are given by

$$\langle v_x(x,0,0)v_x(0,0,0)\rangle = f(x) , \quad (11)$$

$$\langle v_x(0,y,0)v_x(0,0,0)\rangle = \left(f(r) + \frac{r}{2}\frac{\partial f}{\partial r}\right)(r=y) . \quad (12)$$

By the assumed isotropy, these expressions hold for the other directions as well. The specific forms of these results for the Gaussian and exponential two point correlation functions are given in appendices A and B, respectively. Because of the assumptions of stationarity and isotropy, these equations can also be written as

$$\hat{\mathbf{n}}\hat{\mathbf{n}} : \langle v(r\hat{\mathbf{n}})v(0)\rangle = f(r) , \quad (13)$$

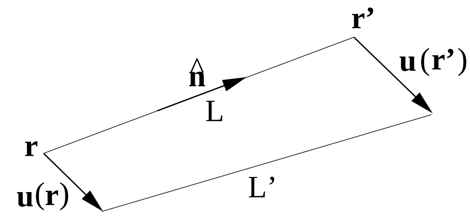

Figure 2. Definition of the geometric variables as the distance between the points $\mathbf{r}$ and $\mathbf{r}'$ changes with the deformation $\mathbf{u}(\mathbf{r})$.

$$\hat{\mathbf{n}}\hat{\mathbf{n}} : \langle v(r\hat{\mathbf{n}}_\perp)v(0)\rangle = \left(f(r) + \frac{r}{2}\frac{\partial f}{\partial r}\right)(r) , \quad (14)$$

where $\hat{\mathbf{n}}_\perp$ is an arbitrary unit vector perpendicular to the line of separation.

4 CHANGE IN THE DISTANCE BETWEEN TWO POINTS

In the application of coda wave interferometry presented here, neutrally buoyant particles scatter acoustic waves, but these scatterers move with the flow. Let the time elapsed between the acoustic measurements described in section 2 be denoted by t_{inc}. We assume here that this time increment is small compared to the decorrelation time of the turbulent motion. (This time is defined as the time over which the velocity of a material point in the flow changes significantly.) Under this assumption, the velocities of the scatterers are constant during the time increment t_{inc} and the associated displacement satisfies

$$\mathbf{u} = \mathbf{v}t_{inc} . \quad (15)$$

With expression (7), this implies that the displacment satisfies the following two-point cross correlation

$$\langle u_i(\mathbf{r})u_j(0)\rangle = t_{inc}^2 \left(\left(f(r) + \frac{r}{2}\frac{\partial f}{\partial r}\right)\delta_{ij} - \frac{r}{2}\frac{\partial f}{\partial r}\frac{x_i x_j}{r^2}\right) . \quad (16)$$

In this section we consider the variance of the distance that separates two points under this deformation.

Consider two points $\mathbf{r}$ and $\mathbf{r}'$ that are separated by a distance L before the deformation, see Figure 2. After the deformation these points are separated by a distance L'. Using that $L'^2 = |\mathbf{r}' - \mathbf{r} + \mathbf{u}(\mathbf{r}) - \mathbf{u}(\mathbf{r}')|^2$ and that $L^2 = |\mathbf{r}' - \mathbf{r}|^2$, we find the first order change $\delta L = L' - L$ in the path length:

$$\delta L = \hat{\mathbf{n}}\cdot\left(\mathbf{u}(\mathbf{r}) - \mathbf{u}(\mathbf{r}')\right) , \quad (17)$$

where $\hat{\mathbf{n}}$ is the unit vector pointing from $\mathbf{r}$ to $\mathbf{r}'$. Since the deformation has zero mean, the change in the path length also has zero mean: $\langle\delta L\rangle = 0$. It follows from expression (17) that the variance of the path length is given by

$$\sigma_L^2 = \hat{\mathbf{n}}\hat{\mathbf{n}} : \langle\mathbf{uu}\rangle - 2\hat{\mathbf{n}}\hat{\mathbf{n}} : \left\langle\mathbf{uu}'\right\rangle + \hat{\mathbf{n}}\hat{\mathbf{n}} : \left\langle\mathbf{u}'\mathbf{u}'\right\rangle , \quad (18)$$

where $\mathbf{u} \equiv \mathbf{u}(\mathbf{r})$ and $\mathbf{u}' \equiv \mathbf{u}(\mathbf{r}')$. By virtue of the assumed stationarity, the last term on the right hand side is equal to the first. Each term on the right hand side gives a two-point correlation as given in equation (13), because the vector $\hat{\mathbf{n}}$ from the points $\mathbf{r}$ to $\mathbf{r}'$ for which the displacement is evaluated. Using this result gives

$$\sigma_L^2 = 2t_{inc}^2 \left(f(0) - f(L)\right) \ . \tag{19}$$

In general the two point correlation function (7) depends on $f(r)$ as well as $\partial f/\partial r$. The change in the distance between two points that move with the flow is to first order caused by the relative displacement of these points along the line of separation, while the relative displacement of these points perpendicular to the line of separation leads to a change in the distance that is of second order in the relative displacment. According to expression (13) the relative displacement along the line of separation depends on $f(r)$ only, so that σ_L^2 does not depend on the derivative $\partial f/\partial r$.

For the special case of the Gaussian autocorrelation (A2), the variance in the distance is given by

$$\sigma_L^2 = 2t_{inc}^2 \sigma_v^2 \left(1 - e^{-L^2/a^2}\right) \ . \tag{20}$$

When the separation of the points is much more than the correlation length ($L \gg a$) the variance is equal to $\sigma_L^2 = 2t_{inc}^2\sigma_v^2$. This is twice the variance in the perturbation in the location of each point, which is the expected value because in this situation the deformations of both are uncorrelated. When ($L < a$) the variance is smaller than this value, and the deformations of the two points are correlated. Consequently, because the points in space are moved in unison, the perturbation in the distance is smaller.

5 VARIANCE OF THE PATH LENGTH

In this section, we consider the variance of the length L of a path that connects N scatterers when scatterer n at location $\mathbf{r}_n$ is displaced over a distance $\mathbf{u}(\mathbf{r}_n)$. The unit vector $\hat{\mathbf{n}}^{(n)}$ points from the $(n-1)$-th scatterer along the path to the n-th; see Figure 5. The path length varies with the location of scatterer n according to

$$\frac{\partial L}{\partial x_i^{(n)}} = -n_i^{(n)} + n_i^{(n-1)} \ , \tag{21}$$

so that the perturbation of the path length due to the scatterer displacement $\mathbf{u}^{(n)}$ becomes

$$\delta L^{(n)} = -\left(\hat{\mathbf{n}}^{(n)} - \hat{\mathbf{n}}^{(n-1)}\right) \cdot \mathbf{u}^{(n)} \ . \tag{22}$$

The variance of the total path length is given by

$$\sigma_L^2 = \sum_{n,m=1}^{N} \sum_{i,j=1}^{3} \frac{\partial L}{\partial x_i^{(n)}} \frac{\partial L}{\partial x_j^{(m)}} \langle u_i(\mathbf{r}_n) u_j(\mathbf{r}_m) \rangle \ , \tag{23}$$

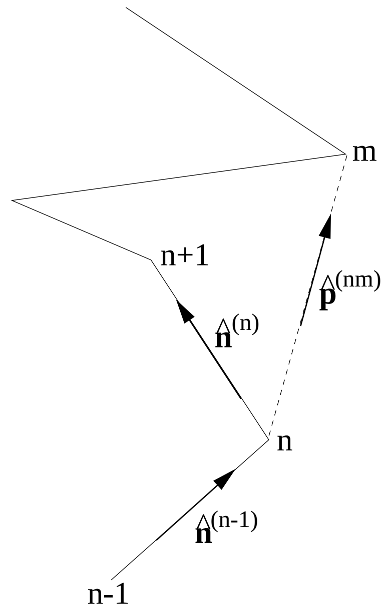

Figure 3. Definition of the geometric variables for the computation of the variance in the length of the path followed by a multiply scattered wave. The path of a multiple scattered wave is shown by a solid line. The unit vector $\hat{\mathbf{n}}^{(n)}$ points from subsequent scatterers n to $n+1$ along the path, while the unit vector $\hat{\mathbf{p}}^{(nm)}$ points from scatterer n to scatterer m, which are not necessarily subsequent scatterers for that path.

which, using (21), is given by

$$\sigma_L^2 = \sum_{n,m=1}^{N} \sum_{i,j=1}^{3} \left(n_i^{(n)} - n_i^{(n-1)}\right) \left(n_j^{(m)} - n_j^{(m-1)}\right) \times \langle u_i(\mathbf{r}_n) u_j(\mathbf{r}_m) \rangle \ . \tag{24}$$

To streamline the notation, let us use the following variables:

$$\hat{\mathbf{p}}^{(nm)} \equiv \frac{\mathbf{r}_n - \mathbf{r}_m}{|\mathbf{r}_n - \mathbf{r}_m|} \ , \tag{25}$$

and

$$r_{nm} \equiv |\mathbf{r}_n - \mathbf{r}_m| \ . \tag{26}$$

With these definitions and expression (7) for the expectation value of the scatterer displacement, the variance of the path length can be written as

$$\begin{aligned} \sigma_L^2 = & -\sum_{n,m=1}^{N} t_{inc}^2 \hat{\mathbf{p}}^{(nm)} \cdot \left(\hat{\mathbf{n}}^{(n)} - \hat{\mathbf{n}}^{(n-1)}\right) \\ & \times \hat{\mathbf{p}}^{(nm)} \cdot \left(\hat{\mathbf{n}}^{(m)} - \hat{\mathbf{n}}^{(m-1)}\right) \frac{r}{2}\frac{\partial f}{\partial r}(r_{nm}) \\ & - \sum_{n,m=1}^{N} t_{inc}^2 \left(\hat{\mathbf{n}}^{(n)} - \hat{\mathbf{n}}^{(n-1)}\right) \cdot \left(\hat{\mathbf{n}}^{(m)} - \hat{\mathbf{n}}^{(m-1)}\right) \\ & \times (f + \frac{r}{2}\frac{\partial f}{\partial r})(r_{nm}) \ . \end{aligned} \tag{27}$$

In general, because there are long-range correlations in the fluid and there is no limitation on the pairs n, m of scatterers that contribute to the sums, the summations here are difficult to evaluate. The summation can be simplified when the scattering is isotropic: this is treated in the next section.

6 VARIANCE OF THE PATH LENGTH FOR ISOTROPIC SCATTERING

Multiplying the various terms in (27), that expression can be written as

$$\begin{aligned}\sigma_L^2 = &-\sum_{n,m=1}^N t_{inc}^2 \Big(\underbrace{\hat{\mathbf{p}}^{(nm)}\cdot\hat{\mathbf{n}}^{(n)}}_{(A)} - \underbrace{\hat{\mathbf{p}}^{(nm)}\cdot\hat{\mathbf{n}}^{(n-1)}}_{(B)}\Big) \\ &\times \Big(\underbrace{\hat{\mathbf{p}}^{(nm)}\cdot\hat{\mathbf{n}}^{(m)}}_{(C)} - \underbrace{\hat{\mathbf{p}}^{(nm)}\cdot\hat{\mathbf{n}}^{(m-1)}}_{(D)}\Big) \frac{r}{2}\frac{\partial f}{\partial r}(r_{nm}) \\ &+\sum_{n,m=1}^N t_{inc}^2 \Big\{ -\underbrace{\hat{\mathbf{n}}^{(n)}\cdot\hat{\mathbf{n}}^{(m)}}_{(E)} + \underbrace{\hat{\mathbf{n}}^{(n)}\cdot\hat{\mathbf{n}}^{(m-1)}}_{(F)} \\ &+\underbrace{\hat{\mathbf{n}}^{(n-1)}\cdot\hat{\mathbf{n}}^{(m)}}_{(G)} - \underbrace{\hat{\mathbf{n}}^{(n-1)}\cdot\hat{\mathbf{n}}^{(m-1)}}_{(H)} \Big\} (f + \frac{r}{2}\frac{\partial f}{\partial r})(r_{nm}) \ .\end{aligned} \tag{28}$$

Let us consider term (E) first. In general this term is nonzero; however, for isotropic scattering the scattering directions are uncorrelated, so that on average

$$\hat{\mathbf{n}}^{(n)}\cdot\hat{\mathbf{n}}^{(m)} = \delta_{nm} \ . \tag{29}$$

Term (E) thus gives a nonzero contribution only when $n = m$. The same reasoning applies to the terms (F) through (H). This has the result that the distance r_{nm} is either equal to zero (when $n = m$) or that it is equal to the mean scatterer distance (when $n = m \pm 1$). Since the mean scatterer distance as seen by the waves defines the mean free path l of the scattered waves, $r_{nn\pm1} = l$.

A similar reasoning can be applied to the terms (A) through (D). In general $\hat{\mathbf{p}}^{(nm)}$ and $\hat{\mathbf{n}}^{(r)}$ are uncorrelated for isotropic scattering and the inner products are on average equal to zero. These vectors are, however, correlated when $m = n \pm 1$ and $r = n$, so that

$$\begin{aligned} m = n+1: &\quad \hat{\mathbf{p}}^{(nm)}\cdot\hat{\mathbf{n}}^{(n)} = -\hat{\mathbf{n}}^{(n)}\cdot\hat{\mathbf{n}}^{(n)} = -1 \\ m = n-1: &\quad \hat{\mathbf{p}}^{(nm)}\cdot\hat{\mathbf{n}}^{(n-1)} = \hat{\mathbf{n}}^{(n-1)}\cdot\hat{\mathbf{n}}^{(n-1)} = 1 \\ n = m+1: &\quad \hat{\mathbf{p}}^{(nm)}\cdot\hat{\mathbf{n}}^{(m)} = \hat{\mathbf{n}}^{(m)}\cdot\hat{\mathbf{n}}^{(m)} = 1 \\ n = m-1: &\quad \hat{\mathbf{p}}^{(nm)}\cdot\hat{\mathbf{n}}^{(m-1)} = -\hat{\mathbf{n}}^{(m-1)}\cdot\hat{\mathbf{n}}^{(m-1)} = -1 \ . \end{aligned} \tag{30}$$

All other contributions to the summation (28) are on average equal to zero. Note that in the four cases in (30) for which $m = n \pm 1$, we evaluate subsequent points along a path. On average these points are separated by a distance given by the mean free path; hence, for the corresponding terms in (28), we should use that $r_{nm} = l$. Using these results gives

$$\begin{aligned}\sigma_L^2 = &\sum_{n=1}^N t_{inc}^2 \Big\{ r\frac{\partial f}{\partial r}(r = l) \\ &+2(f + \frac{r}{2}\frac{\partial f}{\partial r})(r = 0) - 2(f + \frac{r}{2}\frac{\partial f}{\partial r})(r = l) \Big\} \ ,\end{aligned} \tag{31}$$

where the first term comes from the terms (A) through (D), the middle term from the terms (E) and (H), and the last term from the terms (F) and (G). In the summation all the terms are equal, and the variance of the path length given by

$$\sigma_L^2 = 2Nt_{inc}^2\left(f(0) - f(l)\right) \ . \tag{32}$$

Note that the variance in the path length is equal to N times the variance in the distance between two points as given by (19). One might think that this is due to the fact that the perturbations of the path lengths between scatterers are independent random variables whose fluctuations should be added quadratically. This is, however, not the case; when the location of a scatterer is perturbed the length of the incoming and outgoing path from that scatterer are perturbed and these perturbations are correlated. However, when the scattering is isotropic, these correlations cancel when averaged over all scattering angles. Isotropic scattering thus causes the simple relation between the expressions (19) and (32). The requirement of isotropic scattering leads to another simplification as well; for non-isotropic scattering, values other than $m = n \pm 1$ contribute to the double sum (28) and the two-point correlation function needs to be evaluated for distances other than $r = 0$ and $r = l$. I show in the next section how the variance in the path length can be inferred from the change in multiply scattered acoustic waves. This means that, for isotropic scattering, the two point correlation function $f(l)$ evaluated at the mean free path of the scattered waves can be obtained from these waves. For non-isotropic scattering, a contribution from $f(r)$ at other distances than the mean free path would contribute to the variance of the path length as well.

7 HOW TO RETRIEVE THE CORRELATION FUNCTION

The variance σ_L^2 in the path length as the scatterers move with the flow over a time t_{inc} leads to a variance $\sigma_\tau^2 = \sigma_L^2/c^2$ of the arrival times of the multiple scattered waves that arrive in a time window with center time t. In this expression c denotes the propagation velocity of acoustic waves in the fluid. The number of scatterers encountered along the multiple scattering path is on average given by the total path length ct divided by the mean free path: $N = ct/l$. With these relationships and equation (32), the variance of the travel time perturbation for the multiple scattered waves arriving in a time window with center time t is given by

$$\sigma_\tau^2 = \frac{2tt_{inc}^2}{cl}(f(0) - f(l)) \; . \qquad (33)$$

As stated in section 2 the change in the multiple scattered waves with the change of the medium can be used for diagnosing the change in the medium. Expression (5) relates the maximum value of the time windowed cross correlation of the unperturbed and the perturbed waveforms to the variance in the travel time perturbation. With equation (33) this gives

$$R_{\max}^{(t,T)} = 1 - \frac{\bar{\omega}^2 tt_{inc}^2}{cl}(f(0) - f(l)) \; . \qquad (34)$$

This expression can be used in the following way to measure the two-point correlation function. First the variance of the velocity fluctuations is measured at one point to give σ_v^2. According to equation (10) this gives $f(0)$. Coda wave interferometry can then be used to measure $f(l)$ by inserting neutrally buoyant particles in the flow. An acoustic source generates multiply scattered waves, and the unperturbed wave $u_{unp}(t)$ is recorded at one or more receivers. After a time t_{inc}, that is much smaller than the decorrelation time of the turbulence, the same source emits the same signal, and the perturbed wave $u_{per}(t)$ is recorded at the same receiver(s). With expression (3) the maximum $R_{\max}^{(t,T)}$ of the time windowed cross correlation can be computed for several center times t of the time window. A linear regression of this quantity as a function of time t then gives the quantity $(\bar{\omega}^2 t_{inc}^2/cl)(f(0) - f(l))$. If desired, this experiment can be repeated for several time increments t_{inc} in order to obtain a more robust estimate of $(f(0) - f(l))$.

The frequency $\bar{\omega}$ can be computed from the recorded waveforms using expression (6). The wave velocity c is known for many fluids, and can be measured independently. The mean free path l can be computed in the dilute scattering approximation if the scattering properties of a single scatterer are known (Lax, 1951; Groenenboom and Snieder, 1995; Ishimaru, 1997). Alternatively the mean free path can be measured using the principle that the intensity of a plane wave that propagates along the z-axis decays as $\exp(-z/l)$.

This means that all quantities are known and that equation (34) can be used to infer $f(l)$ from the measured value of the cross-correlation of the unperturbed and perturbed wave forms. The mean free path l serves as the length at which $f(l)$ is being measured. By varying the scattering size or the concentration of scatterers, the mean free path l can be varied so that the two-point correlation function can be determined as a function of l.

As shown in section 6, these results are valid for isotropic scattering only. For this reason the particles must have a size much smaller than the wavelength of the acoustic waves. Small scatterers with a perturbed density scatter acoustic waves anisotropically with a dipole radiation pattern (Morse and Ingard, 1968). Since the particles are neutrally buoyant, the density of the scatterers is by definition equal to the density of the fluid and the scattering is caused by the contrast in bulk modulus only. For such a contrast the particles scatter acoustic waves isotropically provided they are much smaller than the wavelength of the acoustic waves (Morse and Ingard, 1968). The neutral buoyancy of the scatterers therefore is not only crucial for keeping the particles suspended in the fluid; it is also essential for ensuring that the scattering is isotropic.

8 DISCUSSION

The development here shows that coda wave interferometry can provide an estimate of the two-point correlation function of a turbulent fluid at a length scale given by the mean free path of acoustic waves that are scattered by neutrally buoyant particles that move with the flow. The basic principle that underlies this technique is that the path length of multiple scattered waves changes over time as the scatterers move with the flow. This change of the path length is caused by the change in the distance between scatterers by the spatial variations in the velocity field. This means that coda wave interferometry yields a direct estimate of the statistical properties of the spatial variations in the velocity. This is reflected by the fact that expression (34) gives a direct estimate of the two-point correlation function at a length given by the mean free path of the scattered waves. By seeding the fluid with different scatterers or by using different scatterer densities one can vary the mean free path of the scattered waves, so that the two-point correlation function can be measured for different values of distance of separation.

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APPENDIX A: THE GAUSSIAN AUTOCORRELATION

The general form of the Gaussian autocorrelation function is given by

$$f(r) = f(0)e^{-r^2/a^2} . \tag{A1}$$

With (10) this can also be written as

$$f(r) = \sigma_v^2 e^{-r^2/a^2} . \tag{A2}$$

The autocorrelation (7) for this case is given by

$$\langle v_i(\mathbf{r})v_j(0)\rangle = \sigma_v^2 \left\{ \delta_{ij}\left(1-\frac{r^2}{a^2}\right) + \frac{x_i x_j}{a^2} \right\} e^{-r^2/a^2} , \tag{A3}$$

while the autocorrelations parallel and perpendicular to the displacement are

$$\langle v_x(x,0,0)v_x(0,0,0)\rangle = \sigma_v^2 e^{-x^2/a^2} , \tag{A4}$$

and

$$\langle v_x(0,y,0)v_x(0,0,0)\rangle = \sigma_v^2 \left(1-\frac{y^2}{a^2}\right) e^{-y^2/a^2} . \tag{A5}$$

This last expression shows that the relative flow on average reverses direction over a perpendicular distance given by the correlation length a.

APPENDIX B: THE EXPONENTIAL AUTOCORRELATION

The exponential autocorrelation function is given by

$$f(r) = \sigma_v^2 e^{-r/a} . \tag{B1}$$

The autocorrelation of the velocity given by (7) for this case is

$$\langle v_i(\mathbf{r})v_j(0)\rangle = \sigma_v^2 \left\{ \delta_{ij}\left(1-\frac{r}{2a}\right) + \frac{x_i x_j}{2ra} \right\} e^{-r/a} . \tag{B2}$$

Note the differences with the corresponding expression (A3) for the Gaussian autocorrelation. The autocorrelations parallel and perpendicular to the displacement are given by

$$\langle v_x(x,0,0)v_x(0,0,0)\rangle = \sigma_v^2 e^{-x/a} , \tag{B3}$$

and

$$\langle v_x(0,y,0)v_x(0,0,0)\rangle = \sigma_v^2 \left(1-\frac{y}{2a}\right) e^{-y/a} , \tag{B4}$$

respectively. Here the relative flow reverses sign over a distance $y = 2a$ equal to twice the correlation length.