

On the relationship between the 2D Beylkin determinant and 2D ray Jacobian: Consequences for inversion formulas

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ABSTRACT

I derive the relationship between the 2D common-shot Beylkin determinant and the 2D ray Jacobian for the rays from the image point to the receiver array. This relationship allows us to simplify the formulas for 2D and 2.5D inversion for the common-shot, common-receiver and common-offset cases, including zero-offset. In particular, the simplification dispenses with the need to calculate the derivative of the traveltime gradient with respect to the receiver position.

Introduction

In Section 5.3 of Bleistein, et al., [2001], referred to below as MMSIM1, the relationship between the 3D Beylkin determinant for common-shot or common-receiver inversion and the corresponding ray Jacobian of ray theory was derived. This section was largely based on an excellent paper by Najmi [1996], with an extension of his results to a curved acquisition geometry. We have since found that the corresponding results for the 2D case can be useful. Here, I derive those and relate them to some more recent results about the ray theoretic Green's function amplitudes [Bleistein, 2001]. Thereafter, I show how these results simplify the inversion formulas for 2.5D common-shot, common-offset and zero-offset inversion. The last of these results is particularly simple. In all cases, this approach dispenses with the need to calculate the derivative of the traveltime gradient with respect to the upper surface parameter—a quantity denoted by $\partial \mathbf{p} / \partial \xi$ in MMSIM1. This last simplification of the inversion formulas was not presented in the 3D discussion in the text and will be the subject of another short note. My intent is that this be a self-contained derivation, perhaps somewhat easier to follow than the corresponding 3D case in the text.

Variables and Terminology.

Suppose that we have an acquisition geometry with coordinates \mathbf{x}_0 that are functions of a parameter ξ ; that is, $\mathbf{x}_0 = \mathbf{x}_0(\xi)$. Ray trajectories are governed by the system of equations [MMSIM1],

$$\frac{d\mathbf{x}}{d\gamma_2} = 2\lambda(\mathbf{x})\mathbf{p} = 2\lambda(\mathbf{x})\nabla\tau, \quad \mathbf{x} = \mathbf{x}_0(\xi) \quad \text{for } \gamma_2 = 0, \quad (1)$$

$$\frac{d\mathbf{p}}{d\gamma_2} = -2\lambda(\mathbf{x})\frac{\nabla c(\mathbf{x})}{c^3(\mathbf{x})}, \quad \mathbf{p} = \mathbf{q}(\gamma_1) \quad \text{for } \gamma_2 = 0.$$

In these equations, we use $\mathbf{x} = (x_1, x_2)$, with x_2 being vertical and downward. We depart from the usual geophysical convention of using x_3 or z for the vertical variable in order to be able to use summation convention more easily below. The scale factor $2\lambda(\mathbf{x})$ defines the particular choice of γ_2 , the running parameter along the ray; for example, for $2\lambda(\mathbf{x}) = c(\mathbf{x})$, γ_2 is arc length, while for $2\lambda(\mathbf{x}) = c^2(\mathbf{x})$, γ_2 becomes traveltime. Dimensional analysis of the product, $\gamma_2\lambda$ is easily done by using the differential equation for \mathbf{x} . We know that the dimensions of $\nabla\tau$ are time/length, while the dimension of \mathbf{x} is length. That leads us to conclude that the dimensions of $\gamma_2\lambda$ are length²/time. It is easy then to check the consistency of dimensions of γ_2 and λ separately for the examples cited. The function, $\mathbf{q}(\gamma_1)$ in (1) defines the initial direction along the ray. Finally, $\tau(\mathbf{x}, \mathbf{x}_0(\xi))$ is the traveltime along the ray, for which the governing problem is

$$\frac{d\tau}{d\gamma_2} = \frac{2\lambda}{c^2(\mathbf{x})}, \quad \tau(\mathbf{x}_0(\xi), \mathbf{x}_0(\xi)) = 0. \quad (2)$$

For either of the choices of common-shot or common-receiver the Beylkin determinant is defined by

$$H(\mathbf{x}, \xi) = \det \begin{bmatrix} \mathbf{p} \\ \frac{\partial \mathbf{p}}{\partial \xi} \end{bmatrix}, \quad (3)$$

and the ray Jacobian is defined by

$$J(\gamma_1, \gamma_2) = \det \begin{bmatrix} \frac{d\mathbf{x}}{d\gamma_1} \\ \frac{d\mathbf{x}}{d\gamma_2} \end{bmatrix}. \quad (4)$$

For the common-shot case, the rays of interest are those from the set of receivers at $\mathbf{x}_0(\xi)$ to \mathbf{x} , while for the common-receiver case, the rays of interest propagate from the set of sources at $\mathbf{x}_0(\xi)$ to \mathbf{x} . For the zero-offset case, the total traveltime is twice τ , as it is defined here. Consequently, the right side of (3) must be multiplied by 4. That is,

$$H_{ZO} = 4H. \quad (5)$$

We will also have occasion to differentiate τ with respect to \mathbf{x}_0 and therefore we define

$$\nabla_{\mathbf{x}_0} \tau(\mathbf{x}, \mathbf{x}_0(\xi)) = \mathbf{p}_0(\mathbf{x}, \mathbf{x}_0(\xi)) \quad (6)$$

and observe that

$$\mathbf{p}_0(\mathbf{x}, \mathbf{x}_0(\xi)) = -\mathbf{q}(\gamma_1). \quad (7)$$

Relating H to J .

We are now prepared to present the fundamental derivation of the relationship between the Beylkin determinant and the ray Jacobian. Let us begin by examining the ξ -derivative in (3). We set

$$\frac{\partial p_i}{\partial \xi} = \frac{\partial}{\partial \xi} \frac{\partial \tau}{\partial x_i} = \frac{\partial x_{0j}}{\partial \xi} \frac{\partial^2 \tau}{\partial x_i \partial x_{0j}} \quad (8)$$

$$= \frac{dx_{0j}}{d\xi} \frac{\partial p_{0j}}{\partial x_i}, \quad i = 1, 2.$$

In this equation and below, we sum over repeated indices from 1 to 2. The key to relating H to a ray Jacobian is to eliminate one of the components of \mathbf{p}_0 by using the eikonal equation:

$$p_{02} = \sqrt{1/c^2(\mathbf{x}_0) - p_{01}^2}, \quad (9)$$

and then writing

$$\frac{\partial p_{02}}{\partial x_i} = -\frac{p_{01}}{p_{02}} \frac{\partial p_{01}}{\partial x_i}. \quad (10)$$

By using this result and the last expression in (8), we find that

$$\begin{aligned} \frac{\partial \mathbf{p}}{\partial \xi} &= \nabla p_{01} \left[\frac{dx_{01}}{d\xi} - \frac{p_{01}}{p_{02}} \frac{dx_{02}}{d\xi} \right] \\ &= \frac{\nabla p_{01}}{p_{02}} \left[p_{02} \frac{dx_{01}}{d\xi} - p_{01} \frac{dx_{02}}{d\xi} \right] = \frac{\nabla p_{01}}{p_{02}} \mathbf{p}_0 \cdot \mathbf{n} \\ &= \frac{\nabla p_{01}}{p_{02}} |\mathbf{p}_0| \left| \frac{d\mathbf{x}_0(\xi)}{d\xi} \right| \cos \psi \end{aligned}$$

$$= \frac{\nabla p_{01}}{p_{02}} \left| \frac{d\mathbf{x}_0(\xi)}{d\xi} \right| \frac{\cos \psi}{c(\mathbf{x}_0(\xi))}. \quad (11)$$

Here, in the first equality in the second line, we have recognized the upward normal to the acquisition surface as

$$\mathbf{n} = \left(-\frac{dx_{02}}{d\xi}, \frac{dx_{01}}{d\xi} \right). \quad (12)$$

In the next expression, we have introduced the acute angle ψ between this vector and \mathbf{p}_0 . In the last expression, we have identified the magnitude of \mathbf{p}_0 as $1/c(\mathbf{x}_0(\xi))$. Returning to (3), we see that the second and third factors in the rightmost equality in (11) simply become multipliers of the second row and we can write

$$\begin{aligned} H &= \frac{1}{p_{02}} \left| \frac{d\mathbf{x}_0(\xi)}{d\xi} \right| \frac{\cos \psi}{c(\mathbf{x}_0(\xi))} \det \begin{bmatrix} \nabla \tau \\ \nabla p_{01} \end{bmatrix} \\ &= \frac{1}{p_{02}} \left| \frac{d\mathbf{x}_0(\xi)}{d\xi} \right| \frac{\cos \psi}{c(\mathbf{x}_0(\xi))} \frac{\partial(\tau, p_{01})}{\partial(x_1, x_2)}. \end{aligned} \quad (13)$$

We have taken care about the signs in H . We need to know these signs for the 2.5D common-offset inversion formula, where we use the absolute value of the sum of the common-shot and common-receiver Beylkin determinants. See equation (33), below. That result depends on the relative signs of the two associated Beylkin determinants. Recall that we will ultimately use H in an integration with respect to ξ to obtain a reflectivity function. We remark that

$$\left| \frac{d\mathbf{x}_0}{d\xi} \right| d\xi = ds, \quad (14)$$

with s being arclength along the acquisition surface. Thus, the integration is really independent of the parameterization of the acquisition curve whether for zero-offset, common-shot or common-receiver, where the Beylkin determinant appearing as a factor in the inversion formula is of the type we are discussing here. The result is somewhat more complicated for the common-offset case, where, as noted, the formula involves a sum of two determinants and, hence two derivatives of the form (14), namely, $d\mathbf{x}_s/d\xi$ in one term and $d\mathbf{x}_g/d\xi$ in the other. Let us now turn to the second factor in (13) and observe that

$$\left[\frac{\partial(\tau, p_{01})}{\partial(x_1, x_2)} \right]^{-1} = \frac{\partial(x_1, x_2)}{\partial(\tau, p_{01})} = -\frac{\partial(x_1, x_2)}{\partial(\tau, q_1)}. \quad (15)$$

The right side here is a ray Jacobian with respect to a particular parameterization, namely τ and q_1 . Now, we need only relate this result to the general parameterization in terms of γ_1 and γ_2 . To do this, we repeatedly use the chain rule for Jacobians as well as the ray equations, (1) and (2). Therefore, we write

$$\begin{aligned}
 \frac{\partial(x_1, x_2)}{\partial(\tau, q_1)} &= \frac{\partial(x_1, x_2)}{\partial(\gamma_2, q_1)} \frac{\partial(\gamma_2, q_1)}{\partial(\tau, q_1)} = \frac{\partial(x_1, x_2)}{\partial(\gamma_2, q_1)} \left[\frac{d\tau}{dq_1} \right]^{-1} \\
 &= \frac{\partial(x_1, x_2)}{\partial(\gamma_2, q_1)} \frac{c^2(\mathbf{x})}{2\lambda} \\
 &= \frac{\partial(x_1, x_2)}{\partial(\gamma_2, \gamma_1)} \frac{\partial(\gamma_2, \gamma_1)}{\partial(\gamma_2, q_1)} \frac{c^2(\mathbf{x})}{2\lambda} \quad (16) \\
 &= \frac{\partial(x_1, x_2)}{\partial(\gamma_2, \gamma_1)} \left[\frac{dq_1}{d\gamma_1} \right]^{-1} \frac{c^2(\mathbf{x})}{2\lambda} \\
 &= -J(\gamma_1, \gamma_2) \left[\frac{dq_1}{d\gamma_1} \right]^{-1} \frac{c^2(\mathbf{x})}{2\lambda}.
 \end{aligned}$$

In the first equation we have replaced the variable τ by the variable γ_2 , using the chain rule for determinants. The second factor here is an ordinary derivative which leads to the third equality. Now the ray equation (2) for τ leads to the fourth equality. In the first expression in the second line, we have replaced the variable q_1 in the Jacobian of the last expression by the variable γ_1 . The next equality again recognizes the second Jacobian of the previous expression as an ordinary derivative. Finally, the first factor of the this last expression is seen to be the negative of a ray Jacobian of \mathbf{x} with respect to the general ray parameters, γ . Now, combining (15) and (16), we can write

$$\frac{\partial(\tau, p_{01})}{\partial(x_1, x_2)} = \frac{2\lambda}{c^2(\mathbf{x})} \frac{dq_1}{d\gamma_1} \frac{1}{J}. \quad (17)$$

This result can now be used in (13) to obtain the relationship between H and J :

$$H = \frac{1}{q_{02}} \left| \frac{d\mathbf{x}_0}{d\xi} \right| \frac{\cos \psi}{c(\mathbf{x}_0)} \frac{2\lambda}{c^2(\mathbf{x})} \frac{dq_1}{d\gamma_1} \frac{1}{J}. \quad (18)$$

All of the factors on the right depend on the geometry of the acquisition surface and the variables of the ray equations. Thus, H can be computed without determining the derivative of \mathbf{p} with respect to ξ . This is the advantage of Najmi's [1996] result. Further, by relating H to J , we can express H in terms of the amplitude along the rays. Since factors of amplitude already appear in the inversion formulas, this will allow us to carry out various quotients that will significantly simplify those inversion formula.

Relationship between H and Green's Function Amplitudes.

Next we set down the relationships between H and the 2D and 2.5D Green's function amplitudes. Here, we rely on results of Bleistein [2001].

H and 2D Green's function amplitude.

Equation (38) of Bleistein [2001] can be rewritten in the current notation as

$$A_{2D}^2 = \frac{2\lambda}{|J|} \left| \frac{\partial \mathbf{q}}{\partial \gamma_1} \right| \frac{c(\mathbf{x}_0)}{8\pi} = \frac{2\lambda}{J} \left| \frac{\partial \mathbf{q}}{\partial \gamma_1} \right| \frac{c(\mathbf{x}_0)}{8\pi} \text{sgn}(J), \quad (19)$$

or, equivalently,

$$\frac{2\lambda}{J} = \frac{8\pi A_{2D}^2}{\left| \frac{\partial \mathbf{q}}{\partial \gamma_1} \right| c(\mathbf{x}_0)} \text{sgn}(J). \quad (20)$$

Now, by using this result in (18), we find that

$$H = \frac{8\pi A_{2D}^2 \cos \psi \text{sgn}(J)}{q_2 c^2(\mathbf{x}) c^2(\mathbf{x}_0)} \frac{dq_1}{d\gamma_1} \left| \frac{d\mathbf{x}_0}{d\xi} \right|. \quad (21)$$

We see here that the final result does not depend on the choice of 2λ . That is to be expected: this result is a relationship among elements of the acquisition geometry and the rays, and should not depend on the parameters that are chosen to describe them. The only remnant of the parameterization is the derivative with respect to ξ and that has been explained in (14).

H and 2.5D Green's function amplitude.

In 2.5D, the most sensible choice of parameter, γ_1 is the dip angle of the ray. Thus, let us specialize to that case. That is,

$$\mathbf{q} = \frac{1}{c(\mathbf{x}_0)} (\sin \gamma_1, \cos \gamma_1), \quad (22)$$

$$\frac{d\mathbf{q}}{d\gamma_1} = \frac{1}{c(\mathbf{x}_0)} (\cos \gamma_1, -\sin \gamma_1).$$

For this choice, note that

$$\frac{1}{q_2} \frac{dq_1}{d\gamma_1} = 1,$$

which simplifies (18). Also, since we have also seen that the choice of 2λ does not enter into the final result, we make the convenient choice, $2\lambda = 1$. Then, γ_2 has been traditionally denoted by σ , with

$$\frac{d\mathbf{x}}{d\sigma} = \mathbf{p},$$

in the ray equations (1). This parameter has the dimensions of length²/time and is also the factor that is traditionally used to describe the out-of-plane geometrical spreading of the 2.5D amplitude. Consequently, (18) now becomes

$$H = \left| \frac{d\mathbf{x}_0}{d\xi} \right| \frac{\cos \psi}{c(\mathbf{x}_0)} \frac{1}{c^2(\mathbf{x})} \frac{1}{J}. \quad (23)$$

The relationship between J and the 2.5D amplitude is given in equation (27) of Bleistein, [2001]. In the current notation that result leads to

$$\frac{1}{|J|} = \frac{\text{sgn}(J)}{J} = 16\pi^2 \sigma A_{2.5D}^2, \quad (24)$$

so that the previous equation becomes

$$H = \frac{16\pi^2 \sigma A_{2.5D}^2 \cos \psi}{c(\mathbf{x}_0)c^2(\mathbf{x})} \left| \frac{\partial \mathbf{x}_0}{\partial \xi} \right| \text{sgn}(J). \quad (25)$$

Result (23) might actually be more useful for computation. The reason is the appearance of the extra factor of σ in the last result. Its only role is to compensate for a division by σ in the representation of $A_{2.5D}^2$. On the other hand, (23) depended on a particular choice of γ_2 because we set $2\lambda = 1$. To restore its generality, we need only re-introduce the factor of 2λ in the numerator of the right side.

2.5D Inversion Formulas.

We now exploit the results derived here to write down formulas for the reflectivity β totally in terms of ray theoretic Green's function amplitudes or Green's function Jacobians, rather than in terms of the Beylkin determinant and amplitudes. We start from the general 2.5D inversion formula in MMSIM1, equation (6.2.8):

$$\beta(\mathbf{y}) = \frac{1}{[2\pi]^{3/2}} \int d\xi \frac{|H(\mathbf{y}, \xi)|}{A(\mathbf{y}, \mathbf{x}_s)A(\mathbf{y}, \mathbf{x}_g(\xi))} \frac{\sqrt{\sigma_s + \sigma_g}}{\sqrt{\sigma_s \sigma_g}} \frac{D(\mathbf{y}, \xi)}{|\nabla_{\mathbf{y}}[\tau(\mathbf{y}, \mathbf{x}_s) + \tau(\mathbf{y}, \mathbf{x}_g(\xi))]|}. \quad (26)$$

In this equation

$$D(\mathbf{y}, \xi) = \frac{1}{2\pi} \int \sqrt{|\omega|} d\omega e^{i\text{sgn}(\omega)[\pi/4 + \pi/2(K_s + K_g)]} \cdot u_S(\mathbf{x}_g(\xi), \mathbf{x}_s, \omega) e^{-i\omega[\tau(\mathbf{y}, \mathbf{x}_s) + \tau(\mathbf{y}, \mathbf{x}_g(\xi))]}. \quad (27)$$

Here, K_s and K_g are the KMAH indices associated with the rays from \mathbf{y} to \mathbf{x}_s and $\mathbf{x}_g(\xi)$, respectively. Each counts the number of caustics on the trajectory between the initial point and the final point. Thus, the distinct choices of phase shift arise from the choices, 0, 1, 2, 3, with the latter two merely producing the negatives of the former two. So, where, in the absence of caustics we might have to compute only the transform corresponding to the choice, 0, it is now necessary to compute the Fourier transform associated with the choice, 1, as well. The result for $K = 1$ is just the Hilbert transform of the result for $K = 0$. The addition of the KMAH indices is a feature not discussed in MMSIM1.

Common-shot inversion.

We now specialize the above formula to common-shot inversion. The total Beylkin determinant for this case can be computed from MMSIM1, equation (6.3.7), with

$H_s = 0$ because the shotpoint is fixed and not a function of ξ . The result is

$$|H(\mathbf{y}, \xi)| = 2 \cos^2 \theta |H_g(\mathbf{y}, \xi)|. \quad (28)$$

Here, θ is the half angle between the traveltimes gradients, $\tau(\mathbf{y}, \mathbf{x}_s)$ and $\tau(\mathbf{y}, \mathbf{x}_g(\xi))$; H_g is one of the choices of the Beylkin determinant discussed in the previous section, when we make the identification, $\mathbf{x}_0(\xi) = \mathbf{x}_g(\xi)$. Further, it was shown in MMSIM1 that

$$|\nabla_{\mathbf{y}}[\tau(\mathbf{y}, \mathbf{x}_s) + \tau(\mathbf{y}, \mathbf{x}_g(\xi))]| = \frac{2 \cos \theta}{c(\mathbf{y})}. \quad (29)$$

Now, (26) can be rewritten as

$$\beta(\mathbf{y}) = \frac{c(\mathbf{y})}{[2\pi]^{3/2}} \int d\xi \frac{\cos \theta |H_g(\mathbf{y}, \xi)|}{A(\mathbf{y}, \mathbf{x}_s)A(\mathbf{y}, \mathbf{x}_g(\xi))} \frac{\sqrt{\sigma_s + \sigma_g}}{\sqrt{\sigma_s \sigma_g}} \cdot D(\tau(\mathbf{y}, \xi), \xi). \quad (30)$$

We can use (25) in this last result to eliminate H_g . Then, we find that

$$\beta(\mathbf{y}) = \frac{8}{c(\mathbf{y})} \sqrt{\frac{\pi}{2}} \int \left| \frac{d\mathbf{x}_g}{d\xi} \right| \frac{\cos \theta d\xi}{c(\mathbf{x}_g(\xi))} \frac{A(\mathbf{y}, \mathbf{x}_g(\xi))}{A(\mathbf{y}, \mathbf{x}_s)} \cdot \cos \psi_g \sqrt{\frac{\sigma_g}{\sigma_s}} \sqrt{\sigma_s + \sigma_g} D(\tau(\mathbf{y}, \xi), \xi). \quad (31)$$

Alternatively, we can use (24), or (23) on the previous equation, to write down an inversion in terms of the $2D$ ray Jacobians of those earlier equations, as follows.

$$\beta(\mathbf{y}) = \frac{8}{c(\mathbf{y})} \sqrt{\frac{\pi}{2}} \int \left| \frac{d\mathbf{x}_g}{d\xi} \right| \frac{\cos \theta d\xi}{c(\mathbf{x}_g(\xi))} \sqrt{\frac{|J_s|}{|J_g|}} \cdot \cos \psi_g \sqrt{\sigma_s + \sigma_g} D(\tau(\mathbf{y}, \xi), \xi). \quad (32)$$

Common-offset inversion.

We use (26) again to develop the formulas for common-offset inversion. As above, the Beylkin determinant for this case is also given by MMSIM1 equation (6.3.7). Now, however, both H_s and H_g are nonzero. Thus,

$$|H(\mathbf{y}, \xi)| = 2 \cos^2 \theta |H_s(\mathbf{y}, \xi) + H_g(\mathbf{y}, \xi)|. \quad (33)$$

It is in this application that we have to pay attention to the relative signs of the two separate Beylkin determinants. Thus, we use (23) for each of H_s and H_g , but use the middle expression in (24) for $1/J$. The result is

$$\begin{aligned}
 \beta(\mathbf{y}) &= \frac{8}{c(\mathbf{y})} \sqrt{\frac{\pi}{2}} \int \frac{\cos \theta d\xi}{A(\mathbf{y}, \mathbf{x}_s)A(\mathbf{y}, \mathbf{x}_g)} \frac{\sqrt{\sigma_s + \sigma_g}}{\sqrt{\sigma_s \sigma_g}} \\
 &\quad \cdot D(\tau(\mathbf{y}, \xi), \xi) \\
 &\quad \cdot \left| \frac{\sigma_s A^2(\mathbf{y}, \mathbf{x}_s) \cos \psi_s}{c(\mathbf{x}_s)} \left| \frac{d\mathbf{x}_s}{d\xi} \right| \operatorname{sgn}(J_s) \right. \\
 &\quad \left. + \frac{\sigma_g A^2(\mathbf{y}, \mathbf{x}_g) \cos \psi_g}{c(\mathbf{x}_g)} \left| \frac{d\mathbf{x}_g}{d\xi} \right| \operatorname{sgn}(J_g) \right| \\
 &= \frac{8}{c(\mathbf{y})} \sqrt{\frac{\pi}{2}} \int \cos \theta d\xi \frac{\sqrt{\sigma_s + \sigma_g}}{\sqrt{\sigma_s \sigma_g}} \\
 &\quad \cdot D(\tau(\mathbf{y}, \xi), \xi) \\
 &\quad \cdot \left| \frac{\sigma_s A(\mathbf{y}, \mathbf{x}_s) \cos \psi_s}{A(\mathbf{y}, \mathbf{x}_g) c(\mathbf{x}_s)} \left| \frac{d\mathbf{x}_s}{d\xi} \right| \operatorname{sgn}(J_s) \right. \\
 &\quad \left. + \frac{\sigma_g A(\mathbf{y}, \mathbf{x}_g) \cos \psi_g}{A(\mathbf{y}, \mathbf{x}_s) c(\mathbf{x}_g)} \left| \frac{d\mathbf{x}_g}{d\xi} \right| \operatorname{sgn}(J_g) \right|.
 \end{aligned} \tag{34}$$

This last result can also be written totally in terms of the Jacobians, themselves. That is,

$$\begin{aligned}
 \beta(\mathbf{y}) &= \frac{8}{c(\mathbf{y})} \sqrt{\frac{\pi}{2}} \int \cos \theta d\xi \sqrt{\sigma_s + \sigma_g} \\
 &\quad \cdot D(\tau(\mathbf{y}, \xi), \xi) \\
 &\quad \cdot \left| \sqrt{\frac{|J_g|}{|J_s|}} \frac{\cos \psi_s}{c(\mathbf{x}_s)} \left| \frac{d\mathbf{x}_s}{d\xi} \right| \operatorname{sgn}(J_s) \right. \\
 &\quad \left. + \sqrt{\frac{|J_s|}{|J_g|}} \frac{\cos \psi_g}{c(\mathbf{x}_g)} \left| \frac{d\mathbf{x}_g}{d\xi} \right| \operatorname{sgn}(J_g) \right|.
 \end{aligned} \tag{35}$$

Finally, if this result is specialized to a horizontal acquisition line and a constant wave speed c_0 on that line, this result becomes

$$\begin{aligned}
 \beta(\mathbf{y}) &= \frac{8}{c(\mathbf{y})c_0} \sqrt{\frac{\pi}{2}} \int \cos \theta d\xi \sqrt{\sigma_s + \sigma_g} D(\tau(\mathbf{y}, \xi), \xi) \\
 &\quad \cdot \left| \sqrt{\frac{|J_g|}{|J_s|}} \cos \psi_s \operatorname{sgn}(J_s) \right. \\
 &\quad \left. + \sqrt{\frac{|J_s|}{|J_g|}} \cos \psi_g \operatorname{sgn}(J_g) \right|.
 \end{aligned} \tag{36}$$

Zero-offset.

The previous results is readily specialized to zero-offset 2.5D inversion. We need only set all subscript g quantities equal to their corresponding subscript s counterparts and set $\cos \theta = 1$. Thus, (35) is replaced by

$$\beta(\mathbf{y}) = \frac{16\sqrt{\pi}}{c(\mathbf{y})} \int \left| \frac{d\mathbf{x}_0}{d\xi} \right| d\xi \frac{\sqrt{\sigma} \cos \psi}{c(\mathbf{x}_0(\xi))} D(\tau(\mathbf{y}, \xi), \xi). \tag{37}$$

In this case, the formula for $D(\tau(\mathbf{y}, \xi), \xi)$ in (27) also simplifies, becoming

$$\begin{aligned}
 D(\tau(\mathbf{y}, \xi), \xi) &= \frac{1}{2\pi} \int \sqrt{|\omega|} d\omega e^{i \operatorname{sgn}(\omega) [\pi/4 + \pi K]} \\
 &\quad \cdot u_S(\mathbf{x}_0(\xi), \omega) e^{-2i\omega\tau(\mathbf{y}, \mathbf{x}_0)}.
 \end{aligned} \tag{38}$$

That is, the sum of the traveltimes and the sum of KMAH indices just double in this case. Consequently the only change as a result of phase shifts through caustics is a possible minus sign when K is odd. Finally, the specialization of (37) to a horizontal acquisition surface with a constant wave speed c_0 , reduces to

$$\beta(\mathbf{y}) = \frac{16\sqrt{\pi}}{c(\mathbf{y})c_0} \int d\mathbf{x}_0 \sqrt{\sigma} \cos \psi D(\tau(\mathbf{y}, \mathbf{x}_0), \mathbf{x}_0), \tag{39}$$

with D again given by (38).

Summary and Conclusions.

We have derived the basic relationship between the 2D Beylkin determinant and the 2D ray Jacobian. Then, we rewrote those results in terms of the 2D and 2.5D ray theoretic amplitudes. Following that, 2.5D inversion formulas were recast in terms of the 2.5D amplitude and also in terms of the 2D Jacobian. This was done for the cases of common-shot, common-offset and zero-offset acquisition geometries. The zero-offset formulas turn out to be particularly simple, not involving and ray Jacobian or amplitude calculation at all. That result requires only the traveltime, geometrical spreading factor σ , and the angle between the gradient and the normal to the acquisition surface. These results all dispense with the need to calculate the derivative of the gradient of the traveltime with respect to the parameter ξ that defines the acquisition geometry. In that regard, these are simpler formulas than were originally stated in MMSIM1 and should prove easier to code, even for heterogeneous media.

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