Radiative transfer in 1D and connection to the O'Doherty-Anstey formula

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ABSTRACT
There is a growing interest in incorporating multiply scattered waves into modeling the Earth's interior using radiative transfer. We examine radiative transfer in a layered medium with general scattering and directional properties of the source. This allows us to demonstrate in detail the nature of energy propagation in the presence of strong scattering. At its most basic level, radiative transfer predicts that, after a distance known as the mean free path, the wavefield breaks into a coherent, or wave-like part and an incoherent, or diffusive flow. The dynamic properties of both aspects are linked.
For 1D point scatterers, or thin beds, we derive the equivalence of the exponential decay of the transmitted wave predicted by the O'Doherty-Anstey formula with the coherent, or direct, wave obtained from the radiative transfer equation. The equivalence shows an underlying relationship between mean field theory and radiative transfer.
Turning to the incoherent wave intensity, we make the well-known diffusion approximation to the late-time radiative transfer behavior. A finite-difference simulation of the wave equation with random scatterers corroborates the theoretical results for the incoherent energy.

Key words: multiple scattering, attenuation, diffusion

1 INTRODUCTION

Radiative transfer has its origins in the kinetic theory of gases and is sometimes referred to as the Boltzmann transport equation in honor of its earliest proponent. In the earth sciences, it first appeared within the context of light propagation through the atmosphere (Schuster, 1905). Recently, geophysicists have begun to address the applicability of radiative transfer to multiply-scattered seismic waves (Hennino *et al.*, 2001; Campillo & Paul, 2003; van Wijk *et al.*, 2003).

By squaring a wavefield and averaging over many realizations of random disorder, the phase information of the underlying wavefield is, for the most part, lost. What remains is the average intensity, or squared amplitude. Radiative transfer is a phenomenological theory for the spatial and temporal evolution of a wavefield's average intensity. The theory's strengths lie in the ability to provide statistical information about the structure of a medium at scales less than a wavelength and the description of the decoupling of scattering and absorption effects for incoherent wave energy.

Here, we give the complete solution of the radiative transfer equation in one dimension (1D) for general directional sources and general scattering. Such generality is relevant for plane wave propagation in layered media, and recently became important for describing physical experiments of surface wave propagation through 1D disordered grooves with a directional source (van Wijk *et al.*, 2003). We derive results from radiative transfer that agree with results from mean field theory, namely the O'Doherty-Anstey formula. Such an equivalence suggests that radiative transfer is a proper extension of mean field theory (a "variance field" theory) for the fluctuating, multiply-scattered waves.

At late times, we demonstrate that radiative transfer can be simplified even further by approximating its behavior as the solution to a diffusion equation. Results of finite-difference simulations of the 1D wave equation with random scatterers are presented to support the accuracy of this approximation. Using correct values for the parameters needed to describe the scattering, the average intensity of the numerical simulations is seen to approach the diffusive limit with time.

2 THE RADIATIVE TRANSFER EQUATION

The radiative transfer equation can be derived from energy balance considerations (Morse & Feshbach, 1953; Ishimaru, 1978; Turner, 1994). Heuristically, the equation takes the form:

$$[\partial_t + v \cdot \nabla]\, Intensity = source - loss + gain. \quad (1)$$

The left-hand side of equation (1) is the total time derivative of the intensity. On the right-hand side, loss and gain mechanisms in addition to sources determine the dynamic behavior. In the absence of loss or gain, this equation becomes the advection, or one-way wave, equation. Scattering and absorption show up as loss mechanisms since both remove energy from the forward direction. Only scattering can put energy back into the original direction of propagation. Hence, scattering and absorption enter equation (1) in fundamentally different ways. This fact leads to the ability to separate their effects within radiative transfer theory.

Using the same form as equation (1), here is a general radiative transfer equation valid for any dimension:

$$\frac{\partial I(\vec{r},\Omega,t)}{\partial t} + v\hat{n}(\Omega)\cdot\nabla I(\vec{r},\Omega,t) = S(\vec{r},\Omega,t) - \frac{1}{\tau_s}I(\vec{r},\Omega,t) - \frac{1}{\tau_a}I(\vec{r},\Omega,t) + \frac{1}{\tau_s}\int \frac{1}{\sigma_s}\frac{\partial\sigma_s}{\partial\Omega'}I(\vec{r},\Omega',t)\,d\Omega', \quad (2)$$

where $I(\vec{r},\Omega,t)$ is the intensity, or average squared wavefield, at position $\vec{r}$ propagating in direction Ω, v is the group velocity of the average (coherent) wavefield, $\hat{n}$ is the unit vector in the direction of propagation, and $S(\vec{r},\Omega,t)$ is the angle-resolved source function. The differential scattering cross section, $\partial\sigma_s/\partial\Omega'$, describes the exchange of energy traveling from direction Ω into direction Ω'. The characteristic time between these exchanges is τ_s, the scattering mean free time. The total scattering cross section, σ_s, is the energy exchanged into all directions:

$$\sigma_s = \int \frac{\partial\sigma_s}{\partial\Omega'}d\Omega'. \quad (3)$$

We have allowed for attenuation by including the characteristic absorption time τ_a.

Using terminology originally coined by Clausius in 1858, it is common to define mean free paths for scattering and absorption, ℓ_s and ℓ_a, according to the relations $\ell_s = v\tau_s$ and $\ell_a = v\tau_a$. The scattering mean free path, ℓ_s, can be thought of as the typical distance a wave travels between scatterings. Under most circumstances, ℓ_s is inversely proportional to the number density of scatterers, ρ, and their scattering cross section:

$$\ell_s = \frac{1}{\rho\sigma_s}. \quad (4)$$

This equation is called the independent scattering approximation (ISA) and it holds when the scatterers are separated by more than a wavelength. It can be obtained from a stationary phase argument applied to the average wavefield in random media (Ishimaru, 1978). From equation (4), ℓ_s contains information about the product of ρ and σ_s in a way analogous to a wave reflected from an interface containing information about the acoustic impedance.

3 RADIATIVE TRANSFER IN 1D

Since in 1D only two directions of propagation exist, a general expression for the differential scattering cross section, appearing under the integral in equation (2), is:

$$\frac{\partial\sigma_s(\Omega,\Omega')}{\partial\Omega'} = E_f\delta(\Omega'-\Omega) + E_b\delta(\Omega'-\Omega-180^\circ), \quad (5)$$

where E_b and E_f represent amounts of energy backscattered and forward-scattered divided by the energy of the incident wave. Their sum is equal to the total scattering cross section:

$$\sigma_s = E_b + E_f. \quad (6)$$

Hence, in equation (2), the differential scattering cross section divided by the total scattering cross section becomes:

$$\frac{1}{\sigma_s}\frac{\partial\sigma_s(\Omega,\Omega')}{\partial\Omega'} = \frac{E_f}{E_b+E_f}\delta(\Omega'-\Omega) + \frac{E_b}{E_b+E_f}\delta(\Omega'-\Omega-180^\circ). \quad (7)$$

For the rest of this paper, we denote the ratios $E_f/(E_b+E_f)$ and $E_b/(E_b+E_f)$ by F and B respectively. These ratios satisfy $B+F=1$. In the case of isotropic scattering, $B=F=\frac{1}{2}$ (Paaschens, 1997).

For a general 1D scatterer, B and F can be related to the total transmission and reflection coefficients of a thin bed, T_t and R_t (Sheng, 1995):

$$B = \frac{|R_t|^2}{|R_t|^2+|T_t-1|^2}$$
$$F = \frac{|T_t-1|^2}{|R_t|^2+|T_t-1|^2}. \quad (8)$$

Note that a thin bed consists of two interfaces, and hence R_t and T_t are not simple reflection and transmission coefficients. The quantities R_t and T_t can be related to a geometric summation of the interface reflection and transmission coefficients via generalized rays(Aki & Richards, 1980).

Inserting equation (7) into equation (2), we obtain:

$$\frac{\partial I(x,\Omega,t)}{\partial t} + v\hat{n}(\Omega)\frac{\partial I(x,\Omega,t)}{\partial x} =$$
$$S(x,\Omega,t) - \frac{1}{\tau_s}I(x,\Omega,t) - \frac{1}{\tau_a}I(x,\Omega,t) +$$
$$\frac{1}{\tau_s}\int \left[F\delta(\Omega'-\Omega) + B\delta(\Omega'-\Omega-180^\circ)\right] I(\vec{r},\Omega',t)\, d\Omega'$$
$$= \frac{B}{\tau_s}I(x,\Omega+180^\circ,t) - \frac{B}{\tau_s}I(x,\Omega,t) -$$
$$\frac{1}{\tau_a}I(x,\Omega,t) + S(x,\Omega,t), \qquad (9)$$

where we have used the fact that $B + F = 1$. Equation (9) can be evaluated for the two possible directions in 1D, $\Omega = 0^\circ$ or 180°. In this paper, we will refer to these directions as right and left, respectively. For simplicity, the total intensity propagating in direction $\Omega = 0^\circ$, $I(\vec{r},0^\circ,t)$, will be represented by I_r, $I(\vec{r},180^\circ,t)$ will be I_l, and the source function will be split into S_r and S_l. The coordinate system is defined such that $\hat{n}(0^\circ) = 1$ and $\hat{n}(180^\circ) = -1$. The two equations that describe the propagation of right-going and left-going intensities are:

$$\frac{\partial I_r}{\partial t} + v\frac{\partial I_r}{\partial x} = \frac{B}{\tau_s}(I_l - I_r) - \frac{I_r}{\tau_a} + S_r, \qquad (10)$$

$$\frac{\partial I_l}{\partial t} - v\frac{\partial I_l}{\partial x} = \frac{B}{\tau_s}(I_r - I_l) - \frac{I_l}{\tau_a} + S_l. \qquad (11)$$

This system of partial differential equations comprises radiative transfer in 1D and has been derived by other methods (Goedecke, 1977). In Appendix A, the system of partial differential equations is solved for both I_r and I_l. For now, we solve for the total intensity, $I_t = I_r + I_l$, since this is commonly measured in practice.

Two new quantities emerge from adding and subtracting equations (10) and (11). In addition to the total intensity, I_t, the net right-going intensity, $I_n = I_r - I_l$, appears. Similarly, the source function can be expressed as its total and net right-going components: $S_t = S_r + S_l$ and $S_n = S_r - S_l$. The result of adding equations (10) and (11) is:

$$\frac{\partial I_t}{\partial t} + v\frac{\partial I_n}{\partial x} = -\frac{I_t}{\tau_a} + S_t, \qquad (12)$$

Subtracting equations (10) and (11) yields:

$$\frac{\partial I_n}{\partial t} + v\frac{\partial I_t}{\partial x} = -\frac{2B}{\tau_s}I_n - \frac{I_n}{\tau_a} + S_n. \qquad (13)$$

From these two equations, we derive a single partial differential equation in terms of what we measure, I_t, by taking the spatial derivative of equation (13):

$$\frac{\partial}{\partial t}\frac{\partial I_n}{\partial x} + v\frac{\partial^2 I_t}{\partial x^2} = \left[-\frac{2B}{\tau_s} - \frac{1}{\tau_a}\right]\frac{\partial I_n}{\partial x} + \frac{\partial S_n}{\partial x}. \qquad (14)$$

But we know from equation (12) that:

$$\frac{\partial I_n}{\partial x} = \frac{1}{v}\left[-\frac{I_t}{\tau_a} - \frac{\partial I_t}{\partial t} + S_t\right]. \qquad (15)$$

Substituting equation (15) into equation (14) yields a single partial differential equation in I_t. Omitting some algebraic manipulation, we obtain:

$$\frac{\partial^2 I_t}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 I_t}{\partial t^2} + \left[\frac{2B}{v\ell_s} + \frac{2}{v\ell_a}\right]\frac{\partial I_t}{\partial t} + \frac{1}{\ell_a}\left[\frac{2B}{\ell_s} + \frac{1}{\ell_a}\right] I_t - \left[\frac{2B}{v\ell_s} + \frac{1}{v\ell_a}\right] S_t - \frac{1}{v^2}\frac{\partial S_t}{\partial t} + \frac{1}{v}\frac{\partial S_n}{\partial x}, \qquad (16)$$

Equation (16) encapsulates a wealth of information. First of all, in the absence of a source, the scattering and attenuation show up in both the first and zeroth order time derivatives of the total intensity. For a medium with no scattering or attenuation, $\ell_s^{-1} = \ell_a^{-1} = 0$, we are left with the 1D wave equation. Also, in order to solve for the Green's function of the total intensity, we cannot simply insert a δ-source into the homogeneous form of equation (16). Instead, a more complicated combination of the source and its time and spatial derivatives must be inserted.

4 THE GREEN'S FUNCTION OF THE TOTAL INTENSITY

To solve for the Green's function of the total intensity, we find the Green's function of the homogeneous form of equation (16) and construct the total intensity Green's function from it. First, take an impulsive total source function:

$$S_t = \delta(x)\delta(t), \qquad (17)$$

and a general form for its net right-going component:

$$S_n = cS_t, \qquad (18)$$

where $c \in [-1, 1]$. The parameter c allows the radiation pattern of the impulsive source function to directionally vary from left-going ($c = -1$), to isotropic ($c = 0$), to right-going ($c = 1$), and to any combination in between. After inserting this source into equation (16), we find that the "effective source", denoted S_e, is a combination of a δ-function in time, its time derivative, and its x-derivative:

$$S_e = \left[\frac{2B}{v\ell_s} + \frac{1}{v\ell_a}\right]\delta(x)\delta(t) + \frac{1}{v^2}\delta(x)\delta'(t) - \frac{c}{v}\delta'(x)\delta(t). \qquad (19)$$

This effective source can be constructed from the knowledge of the Green's function, P of the homogeneous form of equation (16):

$$\frac{\partial^2 P}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 P}{\partial t^2} + \left[\frac{2B}{v\ell_s} + \frac{2}{v\ell_a}\right]\frac{\partial P}{\partial t} + \frac{1}{\ell_a}\left[\frac{2B}{\ell_s} + \frac{1}{\ell_a}\right] P - \delta(x)\delta(t). \qquad (20)$$

Note that P is not the Green's function for the total intensity. This equation is a variation of the telegraph equation, there being a zeroth order derivative appearing due to the presence of attenuation. In Morse and Feshbach (1953), the Green's function of the telegraph equation is solved via a spatial Fourier transform and a Laplace transform over time. Applying the same technique, the Green's function of equation (20) can be readily obtained by generalizing the solution stated in Morse and Feshbach (1953):

$$P(x,t) = \frac{v}{2}\exp(-Bvt/\ell_s - vt/\ell_a) \times J_0\left(\frac{B}{\ell_s}\sqrt{x^2 - v^2t^2}\right) u(vt - |x|), \quad (21)$$

where $u(vt-|x|)$ is the unit step-function, guaranteeing causality. This Green's function only differs from the one for the telegraph equation by the exponential damping factor due to attenuation. The Green's function for the total intensity, denoted I_t, can be expressed in terms of the above Green's function through equation (19):

$$I_t = \left[\frac{2B}{v\ell_s} + \frac{1}{v\ell_a}\right] P + \frac{1}{v^2}\frac{\partial P}{\partial t} - \frac{c}{v}\frac{\partial P}{\partial x} \quad (22)$$

Taking the necessary derivatives of P, we obtain for $B \in [0,1]$ and $c \in [-1,1]$:

$$I_t(x,t) = \frac{1}{2}\exp(-Bvt/\ell_s - vt/\ell_a) \times [(1-c)\delta(vt+x) + (1+c)\delta(vt-x) + \frac{B}{\ell_s}u(vt-|x|)\left[I_0\left(\frac{B}{\ell_s}\sqrt{v^2t^2 - x^2}\right) + \frac{vt + cx}{\sqrt{v^2t^2 - x^2}} I_1\left(\frac{B}{\ell_s}\sqrt{v^2t^2 - x^2}\right)\right]], \quad (23)$$

where I_0 and I_1 are the modified Bessel functions of the zeroth and first orders. These should not be confused with the symbols used for the various intensities (I_t, I_r, I_l, and I_n). A previous result by Hemmer (1961) is obtained from equation (23) for the case of an isotropic source ($c = 0$) and isotropic scattering, $B = \frac{1}{2}$.

The Green's function for the total intensity can be broken up into two parts. The term with the δ-function propagates like a wave and is called the *coherent* intensity. The rest of the total intensity is referred to as the *incoherent* intensity. It does not propagate ballistically, and, in later sections of this paper, we show that at late times it propagates diffusively. Also, in Appendix A we show that each Bessel function represents a different direction of propagation for the incoherent energy.

An interesting result in equation (23) is that the decay of coherent intensity due to scattering, described by the first exponential term, goes with distance by the factor ℓ_s/B and not ℓ_s. This new length scale, determining the decay of the coherent energy, is called the *extinction* mean free path, ℓ_{ext}. The fact that $\ell_{ext} \neq \ell_s$ is unique to 1D (Paaschens, 1997).

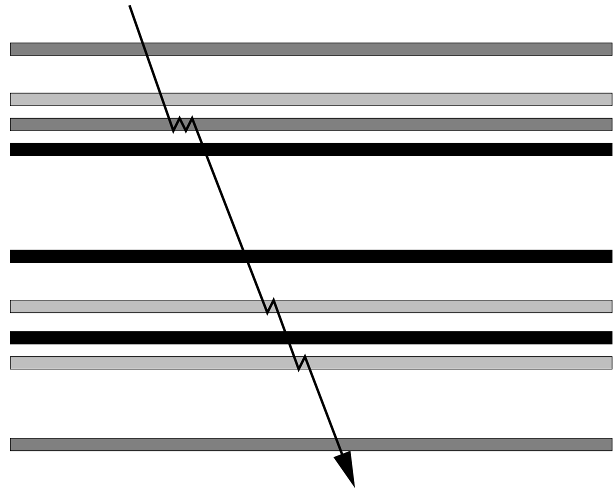

Figure 1. A wave transmitted through a random sequence of thin beds of varying strength. The thin beds are embedded in a constant background medium.

5 THE COHERENT INTENSITY AND THE O'DOHERTY-ANSTEY FORMULA

In the field of exploration geophysics, a well known result for waves multiply scattered by a 1D layering is that obtained by O'Doherty and Anstey (O'Doherty & Anstey, 1970). The O'Doherty-Anstey formula has subsequently been derived from mean field theory (Banik *et al.*, 1985). One outcome of O'Doherty-Anstey is that the amplitude of a wave transmitted through a stack of layers decays exponentially with distance as (Shapiro & Zien, 1993):

$$|T| \sim \exp(-\tilde{R}(k)x), \quad (24)$$

where $\tilde{R}(k)$ represents the power spectrum of the average reflection coefficient series normalized by two-way travel distance (Banik *et al.*, 1985). From the solution for the total intensity obtained in the last section, equation (23), radiative transfer also predicts an exponential decay for the transmitted, or coherent, wave with distance:

$$|T| \sim \exp(-Bx/2\ell_s), \quad (25)$$

where the distance x has replaced vt in equation (23) since the δ-function is only non-zero at $x = vt$. The factor of 1/2 in the exponent of this equation shows up since radiative transfer predicts decay of the transmitted intensity - the square of the true transmission coefficient. We investigate the equivalence of these two theories for the transmission of normally incident waves through assemblages of weak 1D point scatterers (thin beds). The two theories are equivalent if:

$$\tilde{R}(k) = B/2\ell_s. \quad (26)$$

Depicted in Fig. 1 is the random medium we will consider: a series of thin layers of varying strength are embedded in a constant velocity background medium. In the parlance of O'Doherty-Anstey, this would be called a "cyclic" sequence. It happens to be the type of medium that radiative transfer, and scattering theory, are geared for. The reflection coefficient series, $RC(x)$, for such a medium would be a series of delta functions of oscillating plus and minus sign:

$$RC(x) = \sum_{j=1}^{N} R_j \left[\delta(x - d_j) - \delta(x - h - d_j)\right], \tag{27}$$

where h is the thickness of the beds, R_j and d_j represent the reflection coefficient and location of the j-th bed, respectively, and N is the number of beds.

To calculate $\tilde{R}(k)$, we take the Fourier transform of equation (26), square its magnitude to get the power spectrum, and divide by the two-way travel distance:

$$\tilde{R}(k) = \frac{|\int_{-\infty}^{\infty} RC(x) e^{-i2kx} dx|^2}{2L}. \tag{28}$$

Note that the Fourier transform is with respect to $2k$ and not k, similar to a Born inversion formula in 1D (Bleistein *et al.*, 2001). This is evident from standard references in the literature (Banik *et al.*, 1985; Shapiro & Zien, 1993).

Inserting equation (27) into equation (28) results in:

$$\tilde{R}(k) = \frac{1}{2L} |\sum_{j=1}^{N} R_j e^{2ikd_j} (1 - e^{2ikh})|^2. \tag{29}$$

For thin layers, $kh << 1$ and a first order Taylor series expansion in h leads to $1 - e^{2ikh} \approx -2ikh$. Pulling it out of the summation yields:

$$\tilde{R}(k) = \frac{1}{2L} 4k^2h^2 |\sum_{j=1}^{N} R_j e^{2ikd_j}|^2. \tag{30}$$

We now use a standard argument from the theory of multiple scattering: if d_j, the spacing of the thin beds, is a random variable, the cross terms in the square of the summation in equation (30) cancel in the *average* and the squaring can be brought inside the summation:

$$\tilde{R}(k) = \frac{1}{2L} 4k^2h^2 \sum_{j=1}^{N} |R_j e^{2ikd_j}|^2. \tag{31}$$

Now, inside the summation, the exponential does not contribute to the magnitude and we are left with:

$$\tilde{R}(k) = \frac{1}{2L} 4k^2h^2 \sum_{j=1}^{N} |R_j|^2 = \frac{1}{L} 2k^2h^2 N \langle |R_j|^2 \rangle, \tag{32}$$

where $\langle |R_j|^2 \rangle$ is the mean-square of the interface reflection coefficients.

Returning to equation (26), to prove that radiative transfer and the O'Doherty-Anstey formula predict the same exponential decay for the transmitted wave, we set equation (32) to:

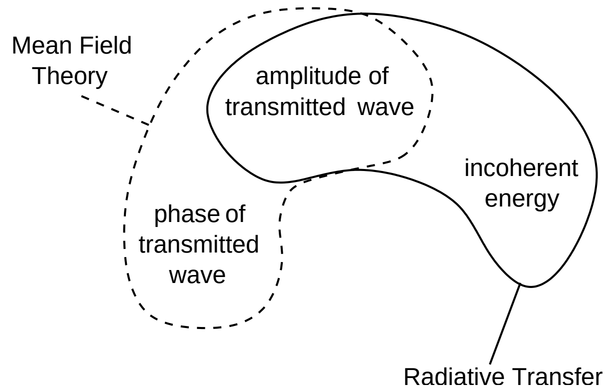

Figure 2. A diagram representing the overlap of mean field theory and radiative transfer for the amplitude of the transmitted wave through a medium like that depicted in Fig. 1

$$\frac{B}{2\ell_s} = \frac{1}{L} 2k^2h^2 N \langle |R_j|^2 \rangle. \tag{33}$$

For (Rayleigh) point scatterers in 1D, the radiation is isotropic. Hence, $B = 1/2$. Rearranging equation (33):

$$\ell_s = \frac{1}{8k^2h^2 \langle |R_j|^2 \rangle \frac{N}{L}}. \tag{34}$$

The quantity N/L is simply the number density of the thin beds, ρ. In the limit of weak scatterers (such that $R_j << 1$) $8k^2h^2 \langle |R_j|^2 \rangle = \sigma_s$, the scattering cross section (Sheng, 1995). The presence of weak reflection coefficients is an underlying assumption in the O'Doherty-Anstey result (Banik *et al.*, 1985), so that equation (34) can now be rewritten in a familiar form:

$$\ell_s = \frac{1}{\rho \sigma_s}. \tag{35}$$

This is recognized as equation (4), the independent scattering approximation. Previously, we stated that for this relation to hold, the scatterers (thin beds) had to be separated by at least a wavelength. Hence, in this model, no reflections from below the recording depth interfere with the transmitted wave. All the interference resulting in the exponential decay of the direct wave originates from peg-leg multiples within the thin beds, not between them (Fig. 1). Equation (35) demonstrates that, for this model, the exponential decay of the transmitted wave from O'Doherty-Anstey, or mean-field theory, is equivalent to that predicted by radiative transfer.

A conceptual diagram of this equivalence is shown in Fig. 2. From mean field theory, both the phase and the amplitude of the transmitted wave can be obtained; however, the incoherent energy, for which the mean is zero, falls out. Similarly, 1D radiative transfer can address the amplitude of the transmitted wave and the behavior of the incoherent intensity, but phase information is lost. Both theories agree in their region of overlap, as demonstrated by the case of random layering we considered here.

6 THE DIFFUSION APPROXIMATION IN INFINITE 1D MEDIA

In addition to the coherent intensity, physical insight can be gained on the incoherent part of the total intensity. The general expression for the Green's function for radiative transfer in 1D, equation (23), shows that for late times the coherent term is zero and the incoherent field, defined by a combination of Bessel functions, approximates the solution to the diffusion equation (Ishimaru, 1978). Especially in optics, where it is hard to obtain phase information, inferences on the statistical properties of the medium are often based on this late-time diffusive behavior (Boas *et al.*, 1995). In elastic wave-scattering, the incoherent field is used to decipher the different mechanisms of attenuation (Margerin *et al.*, 1999).

To derive the diffusion approximation from equation (23), all we need is that $vt >> x$. Noting that the zeroth and first modified Bessel functions have the asymptotic forms:

$$I_0(z) \approx I_1(z) \approx (2\pi z)^{-\frac{1}{2}} \exp(z) \qquad \text{for } z >> 1, \tag{36}$$

we can write equation (23) in the late-time limit as:

$$I_t(x,t) = \exp(-Bvt/\ell_s - vt/\ell_a) \times \frac{B}{\ell_s}\left[\frac{\exp(\frac{B}{\ell_s}\sqrt{v^2t^2-x^2})}{\sqrt{2\pi\frac{B}{\ell_s}\sqrt{v^2t^2-x^2}}}\right]. \tag{37}$$

In this expression, the delta functions from equation (23) have fallen out.

Organizing terms in equation (37), expanding the square root in the exponential as a Taylor series in the small parameter x/vt, and keeping the lowest order in x/vt, we get:

$$I_t(x,t) = \exp(-Bvt/\ell_s - vt/\ell_a)\frac{\exp(\frac{Bvt}{\ell_s}(1-\frac{1}{2}(x/vt)^2))}{\sqrt{2\pi\frac{\ell_s}{B}vt}} \tag{38}$$

Two of the exponentials cancel in equation (38) and, after isolating the term $\ell_s/2B$, the late-time limit of the radiative transfer equation can finally be written:

$$I_t(x,t) = \frac{\exp\left(-\frac{x^2}{4(\frac{\ell_s}{2B})vt} - \frac{vt}{\ell_a}\right)}{\sqrt{4\pi\left(\frac{\ell_s}{2B}\right)vt}}. \tag{39}$$

In the case of no attenuation ($\ell_a \to \infty$), equation (39) can be identified as the Green's function for the 1D diffusion equation with the diffusion constant $D = (\ell_s/2B)v$. This implies that the movement of energy at late times has an effective mean free path different from ℓ_s or ℓ_{ext}. This effective mean free path is called the *transport* mean free path, $\ell_{tr} = \ell_s/2B$. In 1D, $\ell_{tr} = \frac{1}{2}\ell_{ext}$, since $\ell_{ext} = \ell_s/B$. Note that the transport mean free path can be determined from the extinction mean free path without knowledge of the underlying details of the scattering.

It is common to relate ℓ_{tr} to ℓ_s via:

$$\ell_{tr} = \frac{\ell_s}{1-\langle cos\theta\rangle}, \tag{40}$$

where $\langle cos\theta\rangle$ represents the average scattered energy in all directions weighted by the cosine of that direction. For isotropic scattering, $\langle cos\theta\rangle = 0$ and the two mean free paths are identical. However, using the general relation $\langle cos\theta\rangle = F - B$ (Hendrich *et al.*, 1994) and the fact that $F + B = 1$, equation (40) can be rewritten:

$$\ell_{tr} = \frac{\ell_s}{1-F+B} = \frac{\ell_s}{2B}, \tag{41}$$

which is exactly the relationship we have derived from the diffusion approximation.

7 THE DIFFUSION APPROXIMATION IN FINITE 1D MEDIA

The above derivation of the diffusion approximation showed how the solution of the radiative transfer equation approaches that of the diffusion equation at late times. In this section, we prove that the underlying governing equation for the total intensity at late times also becomes the diffusion equation. While the radiative transfer equation cannot be analytically solved for in a finite geometry, its late time equivalent - the diffusion equation - can be solved with boundary conditions.

Neglecting absorption ($\ell_a \to \infty$), we can rearrange equation (13) as:

$$\frac{\partial I_n}{\partial t} + \frac{2B}{\tau_s}I_n = -v\frac{\partial I_t}{\partial x}. \tag{42}$$

In the diffusive regime, we assume that (Morse & Feshbach, 1953):

$$\frac{2B}{\tau_s}I_n >> \frac{\partial I_n}{\partial t}, \tag{43}$$

meaning that the time rate of change of the right and left-going intensities is relatively small. Under this condition, equation (42) becomes:

$$\frac{2B}{\tau_s}I_n = -v\frac{\partial I_t}{\partial x}. \tag{44}$$

Substituting equation (44) into equation (12) for I_n yields:

$$\frac{\partial I_t}{\partial t} + v\frac{\partial}{\partial x}\left[-\frac{\tau_s v}{2B}\frac{\partial I_t}{\partial x}\right] = 0. \tag{45}$$

Under the assumption that v and τ_s do not depend on position, equation (45) takes the form:

$$\frac{\partial I_t}{\partial t} = v\left(\frac{\ell_s}{2B}\right)\frac{\partial^2 I_t}{\partial x^2}, \tag{46}$$

which we recognize as the 1D diffusion equation with the same diffusion constant $D = v(\ell_s/2B) = v\ell_{tr}$ we obtained in the previous section.

Now assume there is a boundary at $x = 0$ where scattering occurs to the right (positive values of x), but not to the left (negative values of x). Then, at $x = 0$, there is no intensity coming *into* the scattering region, i.e. the right-going intensity is zero. We can express the right-going intensity as the sum of the total intensity and the net right-going intensity (flux) and set it to zero at $x = 0$:

$$I_r = \frac{1}{2}I_t + \frac{1}{2}I_n = 0 \qquad \text{at } x = 0. \tag{47}$$

Using the approximation we derived in equation (44), the I_n-term can be replaced by a spatial derivative of I_t:

$$\frac{1}{2}I_t + \frac{1}{2}\left(-\frac{\ell_s}{2B}\frac{\partial I_t}{\partial x}\right) = 0. \tag{48}$$

From this equation, we learn that:

$$I_t = \frac{\ell_s}{2B}\frac{\partial I_t}{\partial x} = \ell_{tr}\frac{\partial I_t}{\partial x}. \tag{49}$$

The solution to equation (49) states that, near $x = 0$, I_t has the form:

$$I_t \sim x + \ell_{tr}. \tag{50}$$

Extrapolating away from the boundary according to equation (50), $I_t = 0$ at $x = -\ell_{tr}$. Hence, the presence of a boundary that radiates energy out of a finite scattering region can be approximated by a Dirichlet boundary condition a distance ℓ_{tr} *outside* the scattering region.

Suppose there is a region of length L extending from $x = 0$ to $x = L$. Then, at late times, the Green's function for the total intensity should obey the boundary value problem:

$$\begin{aligned} &\frac{\partial I_t}{\partial t} = D\frac{\partial^2 I_t}{\partial x^2} + \delta(x - x')\delta(t) \\ &I_t = 0 \text{ at } x = -\ell_{tr} \text{ and } x = \ell_{tr} + L. \end{aligned} \tag{51}$$

where $D = v\ell_{tr}$. In 1D, this PDE can be solved by expanding over the modes of the Laplacian:

$$\begin{aligned} I_t(x, x', t) = &\sum_{m=1}^{\infty} \exp\left(-\frac{m^2\pi^2 Dt}{(L + 2\ell_{tr})^2}\right) \times \\ &\sin\left(\frac{m\pi(x + \ell_{tr})}{L + 2\ell_{tr}}\right)\sin\left(\frac{m\pi(x' + \ell_{tr})}{L + 2\ell_{tr}}\right). \end{aligned} \tag{52}$$

The PDE could have equivalently been solved by the method of images.

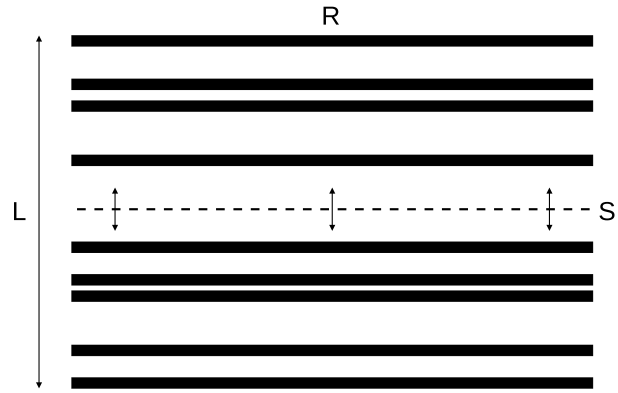

Figure 3. The geometry of the 1D numerical scattering experiments. The source was at the center of a region with thin random layers and a receiver was positioned above the layers for each experiment. The size L of the scattering region varied between experiments with the values 80m, 120m, 160m, 200m, and 240m.

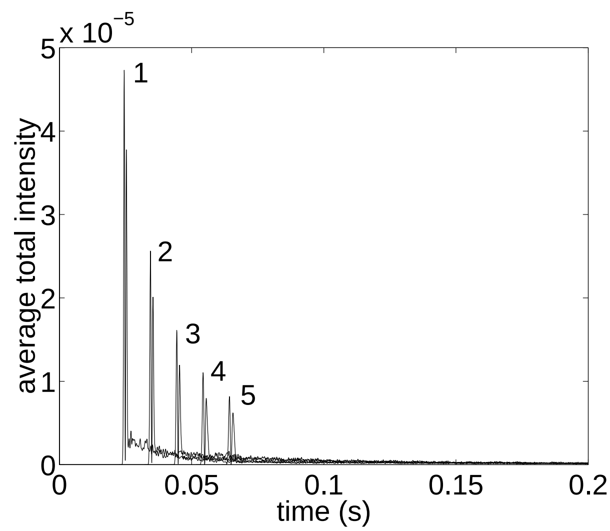

Figure 4. The average intensity measured as a function of time for the 5 experiments. Note how the direct wave decays due to the scattering with offset.

8 NUMERICAL SIMULATIONS

These late-time solutions of the total intensity have been tested with finite-difference simulations of the wave equation in the presence of random discrete scatterers. The setup of the numerical experiment is shown in Fig. 3. A source S is excited in the center of a finite 1D random medium, of size L, containing identical low velocity (1 km/s) thin beds. By "thin" in this experiment, we mean that their thickness is approximately one-tenth of the dominant wavelength. The background medium in which they are embedded has a velocity of 2 km/s. A receiver R is placed just outside the scattering region. The experiment is repeated for $L = 80$m,

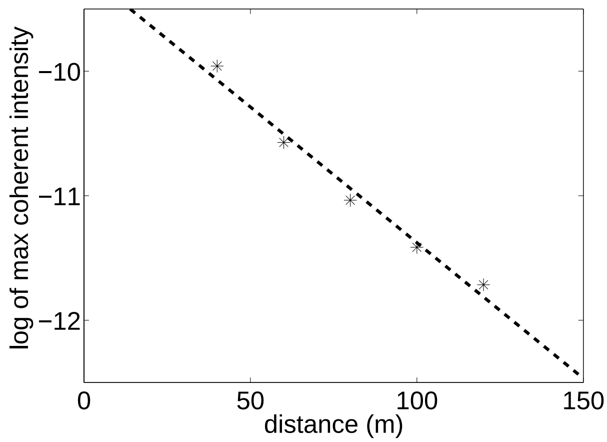

Figure 5. The maximum of the direct wave (coherent intensity) as a function of offset for the 5 experiments. Note that this is a log-linear plot. A linear fit to the data gives the characteristic exponential decay due to scattering - the extinction mean free path, ℓ_{ext}. We estimate $\ell_{ext} = 45.9$m.

120m, 160m, 200m, and 240m. The number of scatterers per unit length is constant for each size of the scattering region and the number of scatterers per dominant wavelength is 2. All interfaces are numerically put into welded contact (Boore, 1970) and there is no intrinsic absorption ($\ell_a \to \infty$). For each of the five sizes of the scattering region, the average intensity was obtained by performing the experiment for 20 realizations of the randomness, squaring each of the 20 wavefields, and adding them. Of course, in practice we only have one realization of the model, the Earth. By assuming the Earth to be ergodic at some scale, we can obtain ensemble averages from earthquake data, seismic exploration, or rock physics (Scales & Malcolm, 2003).

The results are plotted in Fig. 4. The average intensities contain a large direct wave traveling at nearly the background velocity (2 km/s). This is the coherent intensity. Following the coherent intensity is the incoherent multiply scattered energy. If, in the averaging process, the wavefields were added (stacked) before they were squared, the incoherent energy would cancel out and leave only the coherent intensity. From the moveout of the direct wave, we know that the group velocity entering the radiative transfer equation, v, is the background velocity (2 km/s).

To fully characterize the scattering in the radiative transfer model, the extinction mean free path, ℓ_{ext}, must be measured. This parameter describes the exponential decay of the maximum of the coherent intensity with distance. The decay is depicted in Fig. 5. By doing a linear regression on a logarithmic plot, the characteristic distance over which the direct wave decays exponentially can be estimated. For this model, $\ell_{ext} = 45.9 \pm 2.1$m. In the previous sections, we showed that $\frac{1}{2}\ell_{ext} = \ell_{tr}$. Therefore $\ell_{tr} = 22.9 \pm 1.0$m. Getting an estimate of ℓ_s would require knowledge of the degree of back-scattering the individual scatterers radiated relative to their forward-scattering (van Wijk *et al.*, 2003).

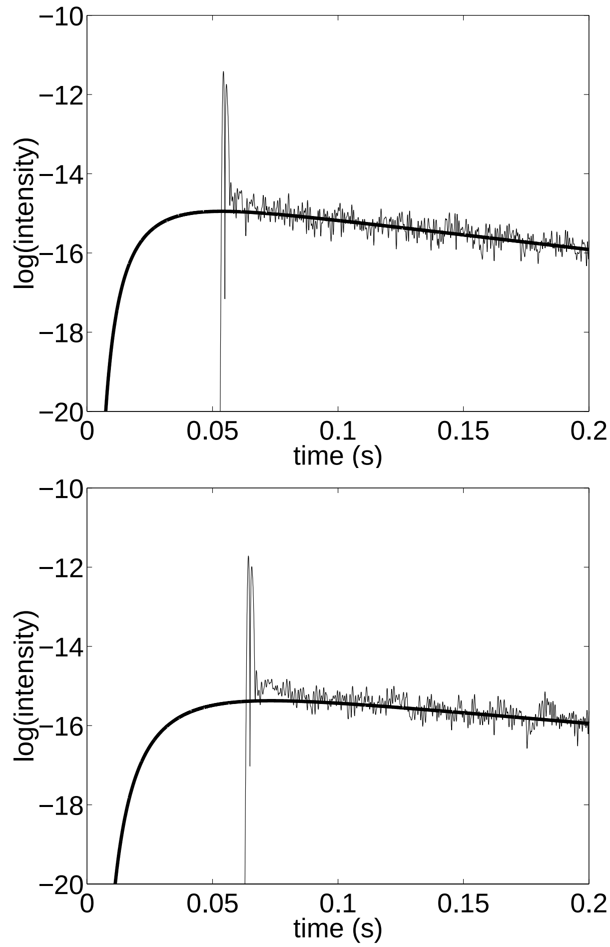

Figure 6. A logarithmic plot of the total intensity of experiment 4, at an offset of 100m (top) and 120m (bottom) compared with the diffusion approximation, equation (52). At early times, the analytic solution to the diffusion equation (thick line) differs from the numerical observations, because the diffusion solution is acausal, and does not account for the coherent field.

With v and l_{tr} estimated from the numerical results, the theoretical prediction of equation (52) can be compared to the simulated total intensities. In Fig. 6, the solution of the diffusion equation asymptotically approaches the numerical intensities with time, as it should. Note that the approximation fails severely for early times since it is acausal. The late time exponential decay is correctly predicted by the diffusion approximation. Such behavior verifies our radiative transfer model for late times and hints at the fact that, in 1D systems, there is an intermediate range of distances where the incoherent intensity can be described by diffusion instead of localization (Sheng, 1995).

9 DISCUSSION

In higher dimensions, the radiative transfer equation becomes considerably more difficult since there are an infinite number of directions to scatter into, as compared to 2 in 1D (Paaschens, 1997). However, even in 1D, the rich character of radiative transfer is evident. Exponential decay is experienced by the direct wave due to scattering and absorption. Aspects of both wave and diffusive behavior emerge in the average total intensity, and, in the presence of both, a "mesoscopic" picture of the scattering medium can be formed.

The theory of radiative transfer has its limitations. The most severe is that it does not include wave interference. As a result of this, there exists a distance between source and receiver, known as the localization length, past which radiative transfer is incorrect. Sheng (1995) estimates that in 1D the localization length is approximately 4 mean free paths. This offers the possibility of an intermediate range (1 to 4 mean free paths) when radiative transfer holds. Future work should attempt to find good bounds on this range in practice.

10 CONCLUSIONS

Radiative transfer is a relative newcomer to the field of exploration seismology. By formulating the theory in 1D, we have attempted to make the connection with familiar concepts such as reflection/transmission coefficients, thin beds, and the O'Doherty-Anstey formula. In the process, new features have emerged, such as the diffusion approximation and incoherent intensity.

The link between radiative transfer and the O'Doherty-Anstey formula can be extended beyond the 1D point scatterer approximation we made in this paper. To do so implies moving into the more complicated Mie scattering regime, where the wavelength is on the order of the size of the scatterer. Additionally, we considered a "cyclic" sequence, which radiative transfer and scattering theory are designed for. It remains to be seen what radiative transfer can do for "transitional" sequences when the interfaces cannot be grouped into pairs that define scatterers.

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APPENDIX A: THE GREEN'S FUNCTION FOR THE DIRECTIONAL INTENSITY

Expressions (10) and (11) show that the 1D radiative transfer equation can be split into a system of PDEs in terms of the left and right-going intensities. So far, only the reduced PDE governing the total intensity has been studied. This is due to the fact that measuring either the left or right-going intensity entails splitting the wavefield into left and right-going waves. Such a decomposition requires dense spatial sampling to perform the type of filtering routinely done in Vertical Seismic Profiling: separating up from down-going waves. Here, we show that knowledge of the individual left and right-going energies can give us more detailed insight into the incoherent energy.

Assuming that the wavefield has been decomposed into left and right-going waves, we now solve the system of 2 partial differential equations that comprise the full radiative transfer equation. To begin, we write equations (10) and (11) in matrix form:

$$\frac{\partial \vec{I}}{\partial t} + M\frac{\partial \vec{I}}{\partial x} = N\vec{I} + \vec{S}, \tag{A1}$$

where $\vec{I}$, M, N, and $\vec{S}$ are:

$$\vec{I} = \begin{bmatrix} I_r \\ I_l \end{bmatrix}, M = \begin{bmatrix} v & 0 \\ 0 & -v \end{bmatrix},$$
$$N = \begin{bmatrix} -\frac{B}{\tau_s} - \frac{1}{\tau_a} & \frac{B}{\tau_s} \\ \frac{B}{\tau_s} & -\frac{B}{\tau_s} - \frac{1}{\tau_a} \end{bmatrix},$$
$$\vec{S} = \begin{bmatrix} S_r \\ S_l \end{bmatrix}. \tag{A2}$$

There exists no general theory for solving systems of PDEs as there is for systems of ODEs. Hence, we proceed by Fourier transforming equation (A1) over space, solving the system of ODEs, and inverse Fourier transforming back to spatial coordinates. With the Fourier conventions:

$$\vec{I}(x) = \int_{-\infty}^{\infty} \tilde{I}(k)e^{-ikx}dk \;\; \tilde{I}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \vec{I}(x)e^{ikx}dx, \tag{A3}$$

equation (A1) becomes a system of 2 ODEs:

$$\frac{\partial \tilde{I}}{\partial t} = (N + ikM)\tilde{I} + \tilde{S}. \tag{A4}$$

For the source function, we again take a general directional point source with right and left-going components S_r and S_l. Allowing the parameter c to govern the directivity of the source as we did previously, the source vector is:

$$\vec{S} = \begin{bmatrix} 1+c \\ 1-c \end{bmatrix} \frac{\delta(x)\delta(t)}{2}. \tag{A5}$$

The solution of the system of ODEs follows that given in standard texts on differential equations (Boyce & DiPrima, 1997). Here we give the solution in the k-domain:

$$I_r(k,t) = \frac{1}{4\pi}e^{-Bvt/\ell_s}e^{-vt/\ell_a}$$
$$\left((1-c)\frac{B}{\tau_s} + i(1+c)kv\right)\frac{\sinh\left(t\sqrt{\frac{B^2}{\tau_s^2} - k^2v^2}\right)}{\sqrt{\frac{B^2}{\tau_s^2} - k^2v^2}} +$$
$$(1+c)\cosh\left(t\sqrt{\frac{B^2}{\tau_s^2} - k^2v^2}\right) \tag{A6}$$

$$I_l(k,t) = \frac{1}{4\pi}e^{-Bvt/\ell_s}e^{-vt/\ell_a}$$
$$\left((1+c)\frac{B}{\tau_s} - i(1-c)kv\right)\frac{\sinh\left(t\sqrt{\frac{B^2}{\tau_s^2} - k^2v^2}\right)}{\sqrt{\frac{B^2}{\tau_s^2} - k^2v^2}} +$$
$$(1-c)\cosh\left(t\sqrt{\frac{B^2}{\tau_s^2} - k^2v^2}\right) \tag{A7}$$

To get the directional intensities in the spatial domain, we must inverse Fourier transform equations (A6) and (A7). Two identities are needed for this inversion:

$$ix\int_{-\infty}^{\infty} \tilde{I}(k)e^{-ikx}dk = \int_{-\infty}^{\infty} \frac{\partial \tilde{I}(k)}{\partial k}e^{-ikx}dk, \tag{A8}$$

and from the theory of Bessel functions (Hemmer, 1961):

$$\int_{-\infty}^{\infty} \cos(kx)\frac{\sin\left(t\sqrt{k^2v^2 - \frac{B^2}{\tau_s^2}}\right)}{\sqrt{k^2v^2 - \frac{B^2}{\tau_s^2}}}dk =$$
$$\frac{\pi}{v}I_0\left[\frac{B}{\ell_s}\sqrt{v^2t^2 - x^2}\right]u(vt - |x|). \tag{A9}$$

After inverting the Fourier transform, we obtain for the right-going intensity:

$$I_r(x,t) = \frac{1}{4}e^{-Bvt/\ell_s}e^{-vt/\ell_a}[2(1+c)\delta(vt - x) +$$
$$\frac{B}{\ell_s}u(vt - |x|)\left[(1-c)I_0\left(\frac{B}{\ell_s}\sqrt{v^2t^2 - x^2}\right) +\right.$$
$$\left.(1+c)\sqrt{\frac{vt + x}{vt - x}}I_1\left(\frac{B}{\ell_s}\sqrt{v^2t^2 - x^2}\right)\right]], \tag{A10}$$

and for the left-going intensity:

$$I_l(x,t) = \frac{1}{4}e^{-Bvt/\ell_s}e^{-vt/\ell_a}[2(1-c)\delta(vt + x) +$$
$$\frac{B}{\ell_s}u(vt - |x|)\left[(1+c)I_0\left(\frac{B}{\ell_s}\sqrt{v^2t^2 - x^2}\right) +\right.$$
$$\left.(1-c)\sqrt{\frac{vt - x}{vt + x}}I_1\left(\frac{B}{\ell_s}\sqrt{v^2t^2 - x^2}\right)\right]]. \tag{A11}$$

These equations for the two intensities show that the two Bessel functions that make up the incoherent intensity are sensitive to different aspects of the source radiation pattern. For instance, if the source were unidirectional, $c = -1$ or $c = 1$ and the zero order Bessel function would come from one direction and the first order Bessel from the other. It can also be verified that adding equations (A10) and (A11) gives the total intensity, equation (23). In the absence of phase information, perhaps the directional intensities can yield important information about spatial variations in the material properties.