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Closed-form expressions for map time-migration in VTI media and the applicability of map depth-migration in the presence of caustics

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ABSTRACT
Provided the velocity of the medium is known and the the medium does not allow different reflectors to have identical surface seismic measurements that persist under small perturbations of the medium, map migration achieves a one-to-one mapping from surface seismic data to the subsurface seismic image by using the slopes in addition to the location (and time) of the events in the data. In this paper we present 3D pre-stack map time-migration in closed form for P-waves in homogeneous isotropic and qP-waves in VTI media, and discuss the condition for applicability of pre-stack map depth-migration and demigration in the presence of caustics. As far as pre-stack time-demigration is concerned, we present closed-form expressions for the mapping in isotropic homogeneous media, while for homogeneous VTI media we derive a system of four nonlinear equations with four unknowns that needs to be solved numerically. In addition we present closed-form expressions for both pre-stack map time-migration and demigration in the common-offset domain for homogeneous isotropic media that use only the slopes in the common-offset domain as opposed to slopes in both the common-shot and common-receiver (or equivalently the common-offset and common-midpoint) domain. All time-migration and demigration equations presented can be used in media with mild lateral and vertical velocity variations, provided the velocity is replaced with the local RMS velocity. The expressions for pre-stack map time-migration in VTI homogeneous media can be used for anisotropic parameter estimation (i.e., the anellipticity parameter η) in the context of time-migration velocity analysis.

Key words: pre-stack map migration, closed-form, homogeneous, VTI, canonical relation, caustics

1 INTRODUCTION

The kinematics and geometry of seismic migration can be described in terms of surfaces of equal traveltime, i.e., isochrons. Migration encompasses the integration of signal processed data along diffraction surfaces corresponding to these isochrons. In terms of linear filter theory, the image in constant media is a convolution of the impulse response of the migration operator shaped in accordance with the isochrons. This approach uses the positions, traveltimes, and amplitudes of the events in the data, and thus uses the information given by the reflection slopes in common-shot or common-receiver gathers only implicitly.

In the high-frequency approximation, seismic waves (or singularities) propagate along rays through the subsurface. Provided the velocity in the earth is known, reflection slopes in the data determine the directions of rays at the recording surface (or the singular direction of the wavefront set of the recorded wavefield). Therefore, once the traveltime and the slopes at the source and receiver are known along with the velocity, the location and local dip of a reflector in the subsurface (i.e., singularity) can in principle be determined with the aid of ray-tracing (Cerveny, 2000). The determination of the reflector position and orientation from the times and slopes of a reflection in the data at the source and receiver locations is generally referred to as *map migration*. In a mathematical context, provided that the velocity is known and the medium does not allow different reflectors to have identical surface seismic measurements that persist under small perturbations of the medium, the use of the slope information in map migration results in a one-to-one mapping from the unmigrated quantities associated with a reflection in the data to the migrated quantities associated with a reflector. Collecting the migrated and unmigrated quantities in a 'table', leads to the notion of *canonical relation*.

Here we aim to elucidate that pre-stack map depth-migration is closely related to the canonical relation of the single scattering modeling or imaging operators in complex media. Guillemin (1985), ten Kroode *et al.* (1998), de Hoop & Brandsberg-Dahl (2000), and Stolk & de Hoop (2002b) have shown that that imaging artefacts (or *imaging phantoms*) are avoided if the projection of this canonical relation on the unmigrated quantities is one-to-one. We explain this, and make clear that this condition provides the applicability condition that allows map depth-migration in the presence of caustics.

The concept of map migration is certainly not new. Weber (1955) gives an early account of map migration, wherein the zero-offset 3D map migration equations are derived for a constant-velocity medium and arbitrary recording surface. Independently Graeser *et al.* (1957) and Haas & Viallix (1976) derive the position of a reflector in 3D from zero-offset data using straight rays by using the slope information. In an early attempt at the use of numerical ray-tracing, Musgrave (1961) uses the slope information to calculate wavefront charts and migration lists, and Sattlegger (1964) derives a series expansion for the coordinates of the raypath. Both methods assume vertically-varying velocity media and can be used for 3D migration of zero-offset data. Reilly (1991) and Whitcombe & Carroll (1994) present successful applications of map migration using post-stack field data.

Map migration has been used for velocity estimation in several different approaches (e.g., Gjoystdal & Ursin (1981), Gray & Golden (1983), and Maher *et al.* (1987)). To improve horizon-based velocity model building, map migration has been used in seismic event-picking schemes consisting of map migration, followed by picking, map demigration, and remigration in the updated velocity model. The initial map-migration step in such a scheme attempts to reduce mispositioning of the velocity picks. The idea to use map migration for velocity analysis stems from the sensitivity of pre-stack map migration to errors in the migration velocity, as pointed out by Sattlegger *et al.*(1980). Sword (1987p.22) develops a controlled directional reception (CDR) tomographic inversion technique, first suggested by Harlan & Burridge (1983), to find interval velocities from pre-stack seismic data. In that method the slopes (or horizontal slownesses) are picked automatically using the CDR picking technique — slant-stack over a short range of offsets with subsequent picking — developed in the former Soviet Union [e.g., Zavalishin (1981) and Riabinkin (1991)] but first introduced by (Rieber, 1936) and later reintroduced by Hermont (1979). Subsequently the estimates of the ray parameters are used to trace rays through the initial estimate of the velocity model, and a depth is found wherein the sum of the traveltime along the downgoing (source) and upgoing (receiver) rays equals the observed traveltime. Then, at this depth, the horizontal distance between the endpoints of two rays in the subsurface is minimized using a modified Gauss-Newton method to yield the velocity model.

Recently Iversen & Gjoystdal (1996) performed 2D map migration in arbitrarily complex media using a layer-stripping approach similar to that of Gray & Golden (1983) to achieve simultaneous inversion of velocity and reflector structure; they later extended this method for 2D anisotropic media (Iversen *et al.*, 2000). Their linearized inversion scheme, which minimizes the projected difference along the reflector normal between events from different offsets, uses derivatives of reflection-point coordinates with respect to model parameters as introduced by van Trier (1990) rather than derivatives of traveltimes with respect to model parameters as used in classical tomographic inversion [e.g., Bishop *et al.* (1985)]. Such an approach allows for more consistent event picking because reflectors can be identified in a geological structure. In addition, an initial imaging step generally improves the signal-to-noise ratio allowing for more accurate event picking. Finally, Billette & Lambare (1998) most recently reiterated the importance of slope information in macro-velocity model estimation while validating that precision in measured slopes, traveltimes, and positions in seismic reflection data is sufficient to recover velocity fields using stereotomography.

The outline of this paper is as follows. First, we develop closed-form expressions for the geometry of pre-stack map time-migration and demigration in 3D for a homogeneous isotropic medium. These equations assume, in addition to the velocity (as is common in seismic imaging), that only the location and the slopes in unmigrated common-offset gathers are known. This is in contrast to the migration equations in 2D presented by Sword (1987p.22), which also require the slope within the common-midpoint gathers (or, laternatively, the slopes in the common-source and common-receiver gathers). Time migration, which uses the assumption of a homogeneous model and the root-mean-square (RMS) velocity remains in large use in practice. In this context

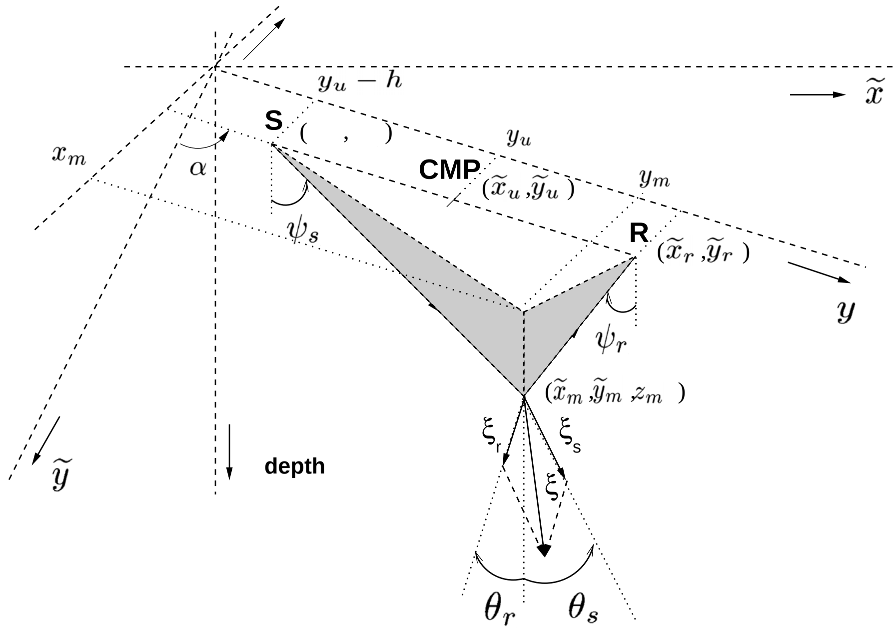

Figure 1. Geometry defining the DSR equation.

our expressions have current applicability, provided the constant velocity in them is replaced by the local RMS velocity. To complement the work of Alkhalifah & Tsvankin (1995) and Alkhalifah (1996) on velocity analysis in transversely isotropic media, we derive closed-form expressions for 3D pre-stack map time-migration for qP waves in VTI media and present a system of four nonlinear equations with four unknowns for the demigration problem, which needs to be solved numerically. These map migration expressions can be used for anisotropic parameter estimation (i.e., the anellipticity parameter η) in the context of time-migration velocity analysis. We then proceed to explain the applicability of map depth-migration in the presence of caustics, and revisit pre-stack map time-migration in homogeneous isotropic media to show that the pre-stack map time-migration and demigration equations *define* the canonical relation of the single scattering modeling and imaging operators in such media. For practical issues such as slope estimation, accuracy and stability of the algorithm, and sampling and grid distortion, we refer to existing literature [e.g., Kleyn (1977) and Maher & Hadley (1985)].

2 ISOTROPIC HOMOGENEOUS MEDIA

2.1 The DSR equation for common-offset and common-azimuth

To illustrate the concept of map migration, we derive explicit 3D pre-stack map migration and demigration equations for isotropic homogeneous media. We assume that preprocessing has already compensated data for any topography on the acquisition surface, and deal with only the kinematics of migration. The results, however, could be extended to take geometrical spreading effects into account. For media with mild lateral and vertical velocity variations, these equations can be used provided the velocity is replaced with the local RMS velocity.

For 2D pre-stack map time-migration, Sword (1987p.22) derived closed-form expressions for the migrated location and reflector dip using the horizontal slownesses at both the source and the receiver. Since the velocity for the isotropic case does not depend on the angle, however, the actual angles at the source and the receiver given by the horizontal slownesses need not be known. We derive closed-form expressions for pre-stack map time-migration and demigration in 3D, using only the slopes in common-offset gathers, which prove it unnecessary to use both horizontal slownesses. These expressions provide a practical advantage over existing closed-form solutions that use both slownesses, since only one slope needs to be measured instead of two. Such a reduction, unfortunately, no longer holds in heterogeneous or anisotropic media.

The double square root (DSR) equation, governing traveltime in a homogeneous medium, in 3D is given by

$$\begin{aligned} t_u &= \frac{1}{v} \left(\sqrt{(\widetilde{x}_u - \widetilde{x}_m - h\sin\alpha)^2 + (\widetilde{y}_u - \widetilde{y}_m - h\cos\alpha)^2 + \left(\frac{vt_m}{2}\right)^2} \right. \\ &\quad \left. + \sqrt{(\widetilde{x}_u - \widetilde{x}_m + h\sin\alpha)^2 + (\widetilde{y}_u - \widetilde{y}_m + h\cos\alpha)^2 + \left(\frac{vt_m}{2}\right)^2} \right) , \end{aligned} \tag{1}$$

where $\widetilde{x}_u$ and $\widetilde{y}_u$ are the common-midpoint (CMP) coordinates, $\widetilde{x}_m$ and $\widetilde{y}_m$ are the reflection-point coordinates, t_m is the two-way migrated traveltime, α is the acquisition azimuth measured positive in the direction of the positive $\widetilde{x}$ axis, v is the velocity, and h is the half-offset (see Figure 1). Rotating the positive $\widetilde{y}$ direction to the source-to-receiver direction, the DSR equation becomes

$$t_u = \frac{1}{v} \left(\sqrt{(x_u - x_m)^2 + (y_u - y_m - h)^2 + \left(\frac{vt_m}{2}\right)^2} + \sqrt{(x_u - x_m)^2 + (y_u - y_m + h)^2 + \left(\frac{vt_m}{2}\right)^2} \right) . \tag{2}$$

To find x_u, y_u, t_u, p_u^x and p_u^y from $\widetilde{x}_u, \widetilde{y}_u, t_u, \widetilde{p}_u^x$ and $\widetilde{p}_u^y$, where $p_u^{x,y}$ and $\widetilde{p}_u^{x,y}$ are the horizontal slownesses of the unmigrated reflection in the rotated and unrotated coordinate systems, respectively, we calculate

$$\begin{pmatrix} x_u \\ y_u \\ t_u \\ p_u^x \\ p_u^y \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \cos\alpha & -\sin\alpha \\ 0 & 0 & 0 & \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \widetilde{x}_u \\ \widetilde{y}_u \\ t_u \\ \widetilde{p}_u^x \\ \widetilde{p}_u^y \end{pmatrix} . \tag{3}$$

To be consistent with the general treatment of time-migration using common-midpoint coordinates and offset, we derive our results in this reference frame. Since we align the positive y-axis with the source-receiver direction, we develop our equations in the common-offset, common-azimuth domain. In the remaining text we assume the velocity to be known and equal to the RMS velocity; i.e. we develop the pre-stack map migration and demigration equations and their solutions in the context of time-migration. Table 1 summarizes our notation throughout the remaining text.

2.2 Pre-stack common-offset migration

Equation (2) has three unknowns — x_m, y_m, and t_m. We obtain two additional equations by calculating the partial derivatives of t_u with respect to x_u and y_u while keeping the reflector location and offset constant, i.e., $p_u^x = \frac{1}{2}\frac{\partial t_u}{\partial x_u}$ and $p_u^y = \frac{1}{2}\frac{\partial t_u}{\partial y_u}$:

$$p_u^x = \frac{1}{2v} \left(\frac{x_u - x_m}{\sqrt{(x_u - x_m)^2 + (y_u - y_m - h)^2 + \left(\frac{vt_m}{2}\right)^2}} + \frac{x_u - x_m}{\sqrt{(x_u - x_m)^2 + (y_u - y_m + h)^2 + \left(\frac{vt_m}{2}\right)^2}} \right) , \tag{4}$$

$$p_u^y = \frac{1}{2v} \left(\frac{y_u - y_m - h}{\sqrt{(x_u - x_m)^2 + (y_u - y_m - h)^2 + \left(\frac{vt_m}{2}\right)^2}} + \frac{y_u - y_m + h}{\sqrt{(x_u - x_m)^2 + (y_u - y_m + h)^2 + \left(\frac{vt_m}{2}\right)^2}} \right) , \tag{5}$$

where the horizontal slownesses $p_u^{x,y}$ can be measured. With these additional equations we arrive at a system of three equations with three unknowns. To derive equations (4) and (5) we used the property that on the pre-stack migration isochrone, defined by the DSR equation, $\frac{\partial t_m}{\partial x_u} = \frac{\partial t_m}{\partial y_u} = 0$ for constant h.

Solving equations (2), (4), and (5) for x_m, y_m and t_m results in

$$x_m = x_u - \frac{v^2 p_u^x t_u}{2}\left(1 - \Lambda_u^2\right) , \tag{6}$$

$$y_m = y_u - \left(\frac{vt_u}{2}\right)^2 \frac{\Lambda_u}{h} , \tag{7}$$

$$t_m = 2\left\{ \left(\frac{t_u}{2}\right)^2 \left(1 - (vp_u^x)^2\right) - \left(\frac{h}{v}\right)^2 + \left(\frac{vt_u\Lambda_u}{4h}\right)^2 \left(8\,(p_u^x h)^2 - t_u^2 + \left(\frac{2h}{v}\right)^2 \left[1 - (vp_u^x\Lambda_u)^2\right]\right) \right\}^{\frac{1}{2}} , \tag{8}$$

Table 1. Summary of notation.

parameter	Description
x, y, z	horizontal location x and y, and depth z
t	two-way traveltime
h	half-offset
v	group velocity
V	phase velocity
ψ	group angle
θ	phase angle
$p^{x,y}$	horizontal slowness in x or y direction
α	acquisition azimuth (measured anti-clockwise with positive y-axis
u, m	subscripts denoting migrated (m) and unmigrated (u) variables
s, r	subscripts denoting source (s) and receiver (r)
$\phi_{x,y}$	reflector dip angle with horizontal in x or y direction
$V_{P0,S0}$	vertical phase velocity qP and qSV waves respectively
$V_{NMO}(0)$	zero-dip NMO velocity
$\boldsymbol{\xi}_m$	dip covector
ϵ, δ	Thomson parameters
η	anellipticity parameter
γ	azimuth angle of slowness vector with positive x-axis measured clock-wise
$(s, r)_\gamma$	$(\sin\gamma_s, \sin\gamma_r)$
$(s, r)_\theta$	$(\sin\theta_s, \sin\theta_r)$

in which

$$\Lambda_u = \Lambda_u\left(p_u^y, \Theta_u, h\right) \equiv \frac{1}{2\sqrt{2}p_u^y h}\sqrt{\Theta_u\left(1-\sqrt{1-\frac{64\left(p_u^y h\right)^4}{\Theta_u^2}}\right)}\ , \tag{9}$$

with

$$\Theta_u = \Theta_u\left(t_u, p_u^y, h\right) \equiv t_u^2 + \left(\frac{2h}{v}\right)^4\frac{1}{t_u^2} - 2\left(\frac{2h}{v}\right)^2\left(1-\left(vp_u^y\right)^2\right)\ , \tag{10}$$

where the signs of the roots are chosen such that migration moves energy up-dip. The local dip of the reflector $p_m^{x,y} = \frac{1}{2}\frac{\partial t_m}{\partial(x,y)_m}$ can be found by calculating the partial derivatives $\left(\frac{\partial}{\partial x_m}, \frac{\partial}{\partial y_m}\right)$ of equation (2), using that $\frac{\partial t_u}{\partial x_m} = \frac{\partial t_u}{\partial y_m} = 0$ for constant h, and using equations (6)-(8) for x_m, y_m and t_m. This yields

$$\begin{aligned} p_m^{x,y} \quad &= \quad p_u^{x,y} t_u\left(\frac{|\Lambda_u - 1|\,|\Lambda_u + 1|}{|\Lambda_u - 1| + |\Lambda_u + 1|}\right) \\ &\times\left\{\left(\frac{t_u}{2}\right)^2\left(1-\left(vp_u^x\right)^2\right) - \left(\frac{h}{v}\right)^2 + \left(\frac{vt_u\Lambda_u}{4h}\right)^2\left(8\left(p_u^x h\right)^2 - t_u^2 + \left(\frac{2h}{v}\right)^2\left(1-\left(vp_u^x\Lambda_u\right)^2\right)\right)\right\}^{-\frac{1}{2}}\ . \end{aligned} \tag{11}$$

Equations (6)-(8) and (11) thus are explicit and exact expressions that determine the migrated reflector coordinates $(x_m, y_m, t_m, p_m^x, p_m^y)$ from the specular reflection coordinates $(x_u, y_u, t_u, p_u^x, p_u^y)$, given h. The 2D case is readily obtained by setting $x_m = x_u = 0$ and $p_m^x = p_u^x = 0$, which reduces equations (7), (8) and (11) to the 2D equivalent expressions. Note that equations (6)-(8) and (11) do not use the offset horizontal slowness $p_h = \frac{1}{2}\frac{\partial t_u}{\partial h}$. This means that, in practice, only p_u^x and p_u^y need to be estimated, and the slope in a common-midpoint gather can be ignored. Usually expressions or algorithms for map migration use the slopes in both the common-offset and midpoint gathers, or, alternatively, the slopes in source and receiver gathers.

2.3 Zero-offset migration

Equations (6)-(8) and (11) are strictly valid when $h \neq 0$ and $p_u^y \neq 0$. For small h or p_u^y, i.e.,

$$\frac{64(p_u^y h)^4}{\Theta_u^2} \ll 1 \,, \tag{12}$$

we use a first-order Taylor expansion for Λ_u such that

$$\Lambda_u \simeq \frac{2p_u^y h}{\sqrt{\Theta_u}} \,. \tag{13}$$

Substituting this approximation for Λ_u into equations (6)-(8) gives

$$x_m \simeq x_u - \frac{v^2 p_u^x t_u}{2}\left(1 - \frac{4\,(p_u^y h)^2}{\Theta_u}\right) , \tag{14}$$

$$y_m \simeq y_u - \frac{(vt_u)^2\, p_u^y}{2\sqrt{\Theta_u}} \,, \tag{15}$$

$$t_m \simeq 2\left\{\frac{t_u^2}{4}\left(1-(vp_u^x)^2\right) - \left(\frac{h}{v}\right)^2 + \frac{1}{\Theta_u}\left(\frac{vt_u p_u^y}{2}\right)^2\left(8\,(p_u^x h)^2 - t_u^2 + \left(\frac{2h}{v}\right)^2\left[1 - \frac{1}{\Theta_u}\,(2vp_u^x p_u^y h)^2\right]\right)\right\}^{\frac{1}{2}} , \tag{16}$$

$$\begin{aligned} p_m^{x,y} \simeq\ & p_u^{x,y} t_u \left(\frac{\left|2p_u^y h - \sqrt{\Theta_u}\right|\left|2p_u^y h + \sqrt{\Theta_u}\right|}{\left|2p_u^y h - \sqrt{\Theta_u}\right| + \left|2p_u^y h + \sqrt{\Theta_u}\right|}\right) \\ & \times\left\{\Theta_u\left(\frac{t_u^2}{4}\left(1-(vp_u^x)^2\right) - \left(\frac{h}{v}\right)^2\right) + \left(\frac{vt_u p_u^y}{2}\right)^2\left(8\,(p_u^x h)^2 - t_u^2 + \left(\frac{2h}{v}\right)^2\left[1 - \frac{1}{\Theta_u}\,(2vp_u^x p_u^y h)^2\right]\right)\right\}^{-\frac{1}{2}} . \end{aligned} \tag{17}$$

For the special case when $h = 0$, these equations reduce to their zero-offset (or post-stack) counterparts, i.e.,

$$x_m = x_u - \frac{v^2 p_u^x t_u}{2} \,, \tag{18}$$

$$y_m = y_u - \frac{v^2 p_u^y t_u}{2} \,, \tag{19}$$

$$t_m = t_u\sqrt{1 - v^2 p_u^2} \,, \tag{20}$$

$$p_m^{x,y} = \frac{p_u^{x,y}}{\sqrt{1 - v^2 p_u^2}} \,, \tag{21}$$

where we have defined

$$p_u \equiv \sqrt{(p_u^x)^2 + (p_u^y)^2} \,. \tag{22}$$

Setting $x_m = x_u = 0$ and $p_m^x = p_u^x = 0$ gives the expressions in 2D.

2.4 Pre-stack common-offset demigration

The demigration equations for x_u and y_u can be found by first solving equation (2) for t_m and evaluating the partial derivatives $\frac{\partial t_m}{\partial x_m}$ and $\frac{\partial t_m}{\partial y_m}$. Using the resulting expressions for x_u and y_u in equation (2) then give the explicit expression for t_u. To find the slopes p_u^x and p_u^y, we then simply substitute the expressions for x_u, y_u, and t_u in equations (4) and

(5). The resulting equations are

$$x_u = x_m + \frac{v^2 p_m^x t_m}{2} , \quad (23)$$

$$y_u = y_m + \frac{v^2 p_m^y t_m}{2} + h\Lambda_m , \quad (24)$$

$$t_u = \sqrt{\frac{4h^2}{v^2} + \frac{2p_m^y h t_m}{\Lambda_m}} , \quad (25)$$

$$p_u^{x,y} = \frac{p_m^{x,y}\frac{t_m}{2}\left(\frac{|\Lambda_m - 1| + |\Lambda_m + 1|}{|\Lambda_m - 1|\,|\Lambda_m + 1|}\right)}{\sqrt{\frac{4h^2}{v^2} + \frac{2p_m^y h t_m}{\Lambda_m}}} , \quad (26)$$

in which

$$\Lambda_m = \Lambda_m\left(p_m^y, \Theta_m, h\right) \equiv \frac{4p_m^y h}{\Theta_m\left(1 + \sqrt{1 + \frac{16(p_m^y h)^2}{\Theta_m^2}}\right)} , \quad (27)$$

with

$$\Theta_m = \Theta_m(t_m, p_m^{x,y}) \equiv t_m\left(1 + v^2\left[(p_m^x)^2 + (p_m^y)^2\right]\right) . \quad (28)$$

Equations (23)-(26) determine the specular reflection $(x_u, y_u, t_u, p_u^x, p_u^y)$ from the migrated reflector $(x_m, y_m, t_m, p_m^x, p_m^y)$. Note that $p_m^{x,y}$ can be estimated from the dip of the imaged reflector using that

$$\tan\phi_{x,y} = \frac{v}{2}\frac{\partial t_m}{\partial\left(x, y\right)_m} = v p_m^{x,y} , \quad (29)$$

where $\phi_{x,y}$ is the reflector dip angle with the horizontal in the x- or y-direction (measured positive clockwise). Again, the 2D case follows by setting $x_m = x_u = 0$ and $p_m^x = p_u^x = 0$.

2.5 Zero-offset demigration

The demigration mapping given by equations (23)-(26) indeed reduces to its zero-offset counterpart if $h = 0$. The resulting expressions are

$$x_u = x_m + \frac{v^2 p_m^x t_m}{2} , \quad (30)$$

$$y_u = y_m + \frac{v^2 p_m^y t_m}{2} , \quad (31)$$

$$t_u = t_m\sqrt{1 + v^2 p_m^2} , \quad (32)$$

$$p_u^{x,y} = \frac{p_m^{x,y}}{\sqrt{1 + v^2 p_m^2}} , \quad (33)$$

where

$$p_m \equiv \sqrt{(p_m^x)^2 + (p_m^y)^2} . \quad (34)$$

With $x_m = x_u = 0$ and $p_m^x = p_u^x = 0$ these expressions reduce to the 2D case.

3 TANSVERSELY ISOTROPIC MEDIA WITH A VERTICAL SYMMETRY AXIS

3.1 The DSR equation for common-offset and common-azimuth

Consider the case of homogeneous transversely isotropic media with a vertical symmetry axis (VTI). In general, the group velocity vector is perpendicular to the slowness surface, whereas the slowness vector is perpendicular to the wavefront. In transversely isotropic (TI) media, the group velocity depends only on the phase angle θ with the axis of rotational symmetry, and for qP (and qSV) waves is given by (Tsvankin, 2001p.29)

$$v = V(\theta)\sqrt{1 + \left(\frac{1}{V(\theta)}\frac{dV}{d\theta}\right)^2} , \quad (35)$$

where v is the group velocity, and V is the phase velocity. The group angle ψ in such media is given by

$$\tan\psi = \frac{\tan\theta + \dfrac{1}{V(\theta)}\dfrac{dV}{d\theta}}{1 - \dfrac{\tan\theta}{V(\theta)}\dfrac{dV}{d\theta}} . \tag{36}$$

The group angle ψ is defined as the angle of the ray with the rotational symmetry axis, and the phase angle θ is the angle of the normal to the wavefront with the symmetry axis. Since the group velocity points in the direction of the ray, the DSR equation for a homogeneous VTI medium is given by

$$t_u = \frac{\sqrt{(x_u - x_{s,r})^2 + (y_s - y_m)^2 + z_m^2}}{v_s} + \frac{\sqrt{(x_u - x_{s,r})^2 + (y_r - y_m)^2 + z_m^2}}{v_r} , \tag{37}$$

where $v_{s,r}$ are the group velocities (which depend on the phase angles) along the source and receiver legs, respectively. Note that the positive y-direction here is in the source-receiver direction, and the coordinate system is right-handed as before.

3.2 Parametrization

For general transversely isotropic (TI) media, the phase velocity for qP waves using the parameterization introduced by Thomsen (1986) is given by (Tsvankin, 2001p.22)

$$V(\theta) = V_{P0}\sqrt{1 + \epsilon\sin^2\theta - \frac{f}{2}\left(1 - \sqrt{\left(1 + \frac{2\epsilon\sin^2\theta}{f}\right)^2 - \frac{2(\epsilon-\delta)\sin^2 2\theta}{f}}\right)} , \tag{38}$$

where V_{P0} is the phase velocity for the qP wave at $\theta = 0$, $f \equiv 1 - \frac{V_{S0}^2}{V_{P0}^2}$ with V_{S0} the phase velocity of the qSV wave at $\theta = 0$, and ϵ and δ are the Thomsen anisotropy parameters. Taking the derivative with respect to the phase angle θ gives

$$\frac{dV}{d\theta} = \frac{V_{P0}^2}{V(\theta)}\left(\epsilon s\sqrt{1-s^2} + \frac{\left(1 + \frac{2\epsilon s^2}{f}\right)\epsilon s\sqrt{1-s^2} - 2(\epsilon-\delta)\,s\sqrt{1-5s^2-4s^6}}{\sqrt{\left(1 + \frac{2\epsilon s^2}{f}\right)^2 - \frac{8(\epsilon-\delta)\,s^2\left(1-s^2\right)}{f}}}\right) , \tag{39}$$

where $s \equiv \sin\theta$. Note that we have $0 \le \theta < \pi/2$, since rays cannot turn in homogeneous media.

The P-wave phase velocity, however, depends only weakly on the vertical shear-wave velocity V_{S0} [e.g., Tsvankin (1996) and Alkhalifah (1998); a precise analysis is contained in Schoenberg & de Hoop (2000)], such that influence of V_{S0} in all kinematic problems involving P-waves can be ignored. Because in this paper we are only dealing with the geometry of map migration, we can for most practical purposes set $f = 1$ in equations (38), (39), and (56); for small V_{P0}/V_{S0} ratios a better choice is to use a typical V_{P0}/V_{S0} ratio. If V_{S0} is known, f can be calculated and subsequently used in equations (38), (39) and (56) to find the phase velocity, its derivative, and the phase angle.

Alkhalifah and Tsvankin (1995) showed that the time-signatures (e.g., reflection move-out, DMO, and time-migration operators) of qP-waves in homogeneous VTI media are characterized mainly by the zero-dip normal-moveout (NMO) velocity $V_{NMO}(0) = V_{P0}\sqrt{1+2\delta}$ and the anellipticity parameter $\eta = (\epsilon-\delta)/(1+2\delta)$, with an almost negligible influence of V_{P0}. Using these expressions for η and $V_{NMO}(0)$, equations (38) and (39) can be rewritten in terms of η, $V_{NMO}(0)$, and V_{P0}. Since in practice $V_{NMO}(0)$ is estimated from move-out velocity analysis, the expressions we derive for map migration in VTI media can be used to estimate the anellipticity parameter η, by using the slope information of one event at two (or more) different offsets and calculating the migrated times for these offsets for an assumed value of η (given arbitrary V_{P0}). The correct value of η (and $V_{NMO}(0)$ should yield the same migrated time for all offsets, since the data have one common reflection point.

3.3 Pre-stack or finite-offset migration

For VTI (and isotropic) media, all vertical planes are medium mirror symmetry planes. Both vertical planes, the one defined by the source position and the reflector position, and the one defined by the receiver position and the

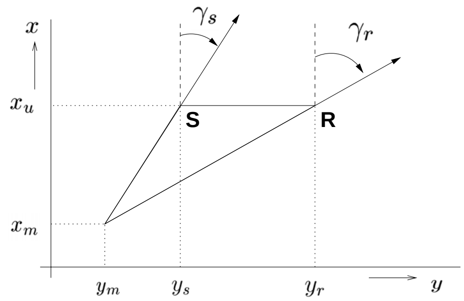

Figure 2. Definition of $\gamma_{s,r}$ as the angles of the horizontal projections of the slowness vectors with the positive x-axis.

reflector position, thus also are symmetry planes. Throughout the remainder, we refer to these planes as the *source* and *receiver* planes (see Figure 1). In the source plane,

$$\tan\psi_s = \frac{\sqrt{(x_s - x_m)^2 + (y_s - y_m)^2}}{z_m} \ , \tag{40}$$

while in the receiver plane, we have

$$\tan\psi_r = \frac{\sqrt{(x_r - x_m)^2 + (y_r - y_m)^2}}{z_m} \ , \tag{41}$$

where $\psi_{s,r}$ are the group angles at the source and receiver, and $z_m = V_{P0}t_m/2$ is the migrated depth. The horizontal slownesses satisfy the relation

$$p_{s,r} = \frac{\sin\theta_{s,r}}{V_{s,r}} \ , \tag{42}$$

where we defined

$$V_{s,r} \equiv V(\theta_{s,r}) \ , \tag{43}$$

$$p_{s,r} \equiv \sqrt{\left(p^x_{s,r}\right)^2 + \left(p^y_{s,r}\right)^2} \ , \tag{44}$$

with $p^{x,y}_{s,r}$ denoting the horizontal slownesses at the source or receiver in the x or y direction, and $\theta_{s,r}$ the phase angle at the source or receiver. Then, using equation (42) in equation (36) and substituting the result in equations (40) and (41), we get

$$\frac{\sqrt{(x_{s,r} - x_m)^2 + (y_{s,r} - y_m)^2}}{z_m} = \frac{\dfrac{V_{s,r}p_{s,r}}{\sqrt{1 - V^2_{s,r}p^2_{s,r}}} + \dfrac{1}{V_{s,r}}\left.\dfrac{dV}{d\theta}\right|_{s,r}}{1 - \dfrac{p_{s,r}}{\sqrt{1 - V^2_{s,r}p^2_{s,r}}}\left.\dfrac{dV}{d\theta}\right|_{s,r}} \ . \tag{45}$$

Here $\left.\frac{dV}{d\theta}\right|_{s,r}$ is the derivative of the phase velocity with respect to the phase angle θ with the vertical symmetry axis, evaluated at the phase angle at the source (θ_s) or receiver (θ_r).

Using equation (45) in (37) then results in an expression for the migrated time

$$t_m = \frac{2t_u}{V_{P0}}\left(\frac{1}{V_s\left(\sqrt{1 - V_s^2p_s^2} - p_s\left.\frac{dV}{d\theta}\right|_s\right)} + \frac{1}{V_r\left(\sqrt{1 - V_r^2p_r^2} - p_r\left.\frac{dV}{d\theta}\right|_r\right)}\right)^{-1} \ . \tag{46}$$

Then, defining $\gamma_{s,r}$ as the angles of the horizontal projection of the slowness vector at the source and receiver with the positive x-axis (see Figure 2), we find

$$\sin\gamma_{s,r} = \frac{y_{s,r} - y_m}{\sqrt{(x_u - x_m)^2 + (y_{s,r} - y_m)^2}} = \frac{p^y_{s,r}}{p_{s,r}} \ . \tag{47}$$

Then, using equation (45) and (46) in equation (47) gives

$$y_m = y_{s,r} - \frac{t_u p^y_{s,r} \left(V_{s,r} + \sqrt{\frac{1}{V^2_{s,r} p^2_{s,r}} - 1} \left. \frac{dV}{d\theta} \right|_{s,r} \right) \left(\sqrt{1 - V^2_{r,s} p^2_{r,s}} - p_{r,s} \left. \frac{dV}{d\theta} \right|_{r,s} \right)}{\frac{1}{V_r} \left(\sqrt{1 - V_s^2 p_s^2} - p_s \left. \frac{dV}{d\theta} \right|_s \right) + \frac{1}{V_s} \left(\sqrt{1 - V_r^2 p_r^2} - p_r \left. \frac{dV}{d\theta} \right|_r \right)} \; . \tag{48}$$

Note the order of the subscripts s, r and r, s.

To find x_m, we first calculate (see Figure 2)

$$\cos \gamma_{s,r} = \frac{x_{s,r} - x_m}{\sqrt{(x_{s,r} - x_m)^2 + (y_{s,r} - y_m)^2}} = \frac{p^x_{s,r}}{p_{s,r}} \; . \tag{49}$$

Then, using equations (45) and (46) in this expression, gives

$$x_m = x_{s,r} - \frac{t_u p^x_{s,r} \left(V_{s,r} + \sqrt{\frac{1}{V^2_{s,r} p^2_{s,r}} - 1} \left. \frac{dV}{d\theta} \right|_{s,r} \right) \left(\sqrt{1 - V^2_{r,s} p^2_{r,s}} - p_{r,s} \left. \frac{dV}{d\theta} \right|_{r,s} \right)}{\frac{1}{V_r} \left(\sqrt{1 - V_s^2 p_s^2} - p_s \left. \frac{dV}{d\theta} \right|_s \right) + \frac{1}{V_s} \left(\sqrt{1 - V_r^2 p_r^2} - p_r \left. \frac{dV}{d\theta} \right|_r \right)} \; . \tag{50}$$

Again note the order of the subscripts s, r and r, s. Thus, equations (46), (48), and (50) are closed-form expressions for the migrated location. Setting $x_m = x_u = 0$ and $p^x_s = p^x_r = 0$ gives the expressions in 2D.

To find the reflector dip covector ξ_m (i.e., the wave-vector associated with the reflector in the image), we use that the slowness vectors $\mathbf{p}_{s,r}$ at the source and receiver locations obey Snell's law upon reflection at the reflector. Since we define the slowness vectors to point in the negative z-direction (i.e., upwards), we have

$$\xi_m = \xi_s + \xi_r = \omega \mathbf{p}_s + \omega \mathbf{p}_r \; , \tag{51}$$

where ω is the angular frequency, and $\xi_{s,r}$ are the wave-vectors associated with the source and receiver rays (see Figure 1). The slowness vectors at the source and receiver are given by

$$\mathbf{p}_{s,r} = \begin{pmatrix} -p^x_{s,r} \\ -p^y_{s,r} \\ \sqrt{\frac{1}{V^2_{s,r}} - p^2_{s,r}} \end{pmatrix} \; . \tag{52}$$

Therefore, the dip covector ξ_m is given by

$$\xi_m = \omega \begin{pmatrix} -\left(p^x_s + p^x_r\right) \\ -\left(p^y_s + p^y_r\right) \\ \sqrt{\frac{1}{V_s^2} - p_s^2} + \sqrt{\frac{1}{V_r^2} - p_r^2} \end{pmatrix} \; . \tag{53}$$

To translate ξ_m to the migrated horizontal slowness components $p^{x,y}_m$, we use that (defining $\nu_{x,y}$)

$$-\nu_{x,y} \equiv \frac{-\xi^{x,y}_m}{\xi^z_m} = \tan \phi_{x,y} = \frac{V_{P0}}{2} \frac{\partial t_m}{\partial \left(x, y\right)_m} = V_{P0} p^{x,y}_m \; , \tag{54}$$

where $\xi^{x,y,z}_m$ are the components of the dip covector, and $\phi_{x,y}$ is again the reflector dip with the horizontal in the x- or y-direction (measured positive clockwise). By using equation (53) in (54) it follows that

$$p^{x,y}_m = \frac{p^{x,y}_s + p^{x,y}_r}{V_{P0} \left(\sqrt{\frac{1}{V_s^2} - p_s^2} + \sqrt{\frac{1}{V_r^2} - p_r^2} \right)} \; . \tag{55}$$

Again, setting $x_m = x_u = 0$ and $p^x_s = p^x_r = 0$ gives the 2D expressions.

3.4 Finding the phase angles

Equations (38) and (39) can be used to calculate the phase velocity and its derivative if the angle θ, and thus $s = \sin\theta$, is known. To find s, we need to solve equation (42) for s, using equation (38) for the phase velocity. This gives

$$\sin\theta_{s,r} = \left\{ P_{s,r}\left[(2-f) - 2P_{s,r}\left(\epsilon - \delta f\right)\right] \frac{\left[1 + \sqrt{1 - \dfrac{4\left(1-f\right)\left(1 - 2\epsilon P_{s,r} - 2P_{s,r}^2 f\left(\epsilon-\delta\right)\right)}{\left(f - 2 + 2P_{s,r}\left(\epsilon - \delta f\right)\right)^2}}\right]}{2\left(1 - 2\epsilon P_{s,r} - 2P_{s,r}^2 f\left(\epsilon - \delta\right)\right)} \right\}^{\frac{1}{2}} , \tag{56}$$

where

$$P_{s,r} \equiv p_{s,r}^2 V_{P0}^2 , \tag{57}$$

with $p_{s,r}$ defined in equation (44).

Therefore, given V_{P0} and the anisotropic parameters ϵ and δ (or η and $V_{NMO}(0)$), we can use equation (56) with the measured $p_{s,r}$ to calculate the phase angles $\theta_{s,r}$. The resulting values can then be used in equations (38) and (39) to find the phase velocity and its derivative at both the source and receiver location.

Note that if the horizontal slowness is used to parametrize for the phase velocity and its derivative, we need not solve for the phase angles. To find the phase velocity as a function of the horizontal slowness, we simply replace $\sin\theta_{s,r}$ with $V_{s,r}p_{s,r}$ in equation (38) and solve for $V_{s,r}$. In Appendix A, we give the resulting expressions for the phase velocity and its derivative as functions of the horizontal slowness are given, using that $f = 1$ for most practical purposes.

3.5 Zero-offset migration

By setting $p_s^{x,y} = p_r^{x,y} = p_u^{x,y}$ and $\theta_s = \theta_r = \theta$, the pre-stack map migration equations reduce to their zero-offset counterparts. Doing this for equations (46), (48), (50), (53), and (55) gives

$$t_m = \frac{V(\theta)t_u}{V_{P0}}\left(\sqrt{1 - V^2(\theta)p_u^2} - p_u \left.\frac{dV}{d\theta}\right|_\theta\right) , \tag{58}$$

$$(x,y)_m = (x,y)_u - \frac{V^2(\theta)p_u^{x,y}t_u}{2} - \frac{V(\theta)p_u^{x,y}t_u}{2}\sqrt{\frac{1}{V^2(\theta)p_u^2} - 1}\left.\frac{dV}{d\theta}\right|_\theta , \tag{59}$$

$$\xi_m = 2\omega\begin{pmatrix} -p_u^x \\ -p_u^y \\ \sqrt{\dfrac{1}{V^2(\theta)} - p_u^2} \end{pmatrix} , \tag{60}$$

$$p_m^{x,y} = \frac{V(\theta)p_u^{x,y}}{V_{P0}\sqrt{1 - V^2(\theta)p_u^2}} , \tag{61}$$

where p_u is defined in equation (22). Again, when we set $x_m = x_u = 0$ and $p_m^x = p_u^x = 0$, the 2D expressions follow from their 3D counterparts. Note that by setting $\dfrac{dV}{d\theta} = 0$ and replacing $V(\theta)$ and V_{P0} with v, these expressions for VTI media reduce to their counterparts for isotropic media.

3.6 Pre-stack or finite-offset demigration

For the demigration problem, we assume the migrated location (x_m, y_m, z_m) and migrated dips $\phi_{x,y}$ are given. To find the unmigrated midpoint location, we need to find the phase angles with the (vertical) symmetry axis and the azimuths of the rays from both the source and the receiver to the reflection point. If we know the angles with the symmetry axis, we can use equations (38), (39), and (42) to find the phase velocity, its derivative and $p_{s,r}$. The projections $p_{s,r}^{x,y}$ are then calculated using

$$p_{s,r}^x = -\mathrm{sgn}(\nu_x)p_{s,r}\sqrt{1 - (s,r)_\gamma^2} , \tag{62}$$

$$p_{s,r}^y = p_{s,r}\left(s,r\right)_\gamma , \tag{63}$$

with $(s,r)_\gamma = \sin\gamma_{s,r}$ the azimuth angles $\gamma_{s,r}$ defined in Figure 2, and ν_x defined in equation (54). The unmigrated location then follows from solving equations (46), (48), and (50) for t_u, y_u, and x_u, using the values for the phase velocity, its derivative, and $p^{x,y}_{s,r}$.

To find the azimuth angles, $\gamma_{s,r}$, and the angles with the vertical symmetry axis, $\theta_{s,r}$, we use the offset and azimuth information, and the dips $\phi_{x,y}$ or $\nu_{x,y} = -\tan\phi_{x,y}$; see equation (54). Since in the rotated coordinate system (the positive y-axis in the source-receiver direction) the projection onto the x-axis of the rays connecting the source and the reflector equals that connecting the receiver and the reflector, we must have

$$\sqrt{1-s_\gamma^2}\,\frac{\left(s_\theta + \dfrac{\sqrt{1-s_\theta^2}}{V_s}\left.\dfrac{dV}{d\theta}\right|_s\right)}{\sqrt{1-s_\theta^2} - \dfrac{s_\theta}{V_s}\left.\dfrac{dV}{d\theta}\right|_s} = \sqrt{1-r_\gamma^2}\,\frac{\left(r_\theta + \dfrac{\sqrt{1-r_\theta^2}}{V_r}\left.\dfrac{dV}{d\theta}\right|_r\right)}{\sqrt{1-r_\theta^2} - \dfrac{r_\theta}{V_r}\left.\dfrac{dV}{d\theta}\right|_r}\ , \tag{64}$$

where $(s,r)_\theta \equiv \sin\theta_{s,r}$ with $\theta_{s,r}$ the phase angles.

Note that since $0 \le \theta_{s,r} < \pi/2$ and $0 \le \gamma_{s,r} < 2\pi$, we have $0 \le (s,r)_\theta < 1$ and $-1 \le (s,r)_\gamma \le 1$. Because the azimuth angles vary between 0 and 2π, we need to keep track of the sign of $\cos\gamma_{s,r}$. In our rotated coordinate system, the sign of ν_x determines the sign of $\cos\gamma_{s,r}$. Therefore, for demigration,

$$\cos\gamma_{s,r} = -\mathrm{sgn}(\nu_x)\sqrt{1-(s,r)_\gamma^2}\ . \tag{65}$$

This explains the form of equation (62).

Furthermore, the difference of the projections onto the y-axis should equal the offset $2h$, i.e.,

$$\left(r_\gamma \frac{\left(r_\theta + \dfrac{\sqrt{1-r_\theta^2}}{V_r}\left.\dfrac{dV}{d\theta}\right|_r\right)}{\sqrt{1-r_\theta^2} - \dfrac{r_\theta}{V_r}\left.\dfrac{dV}{d\theta}\right|_r} - s_\gamma \frac{\left(s_\theta + \dfrac{\sqrt{1-s_\theta^2}}{V_s}\left.\dfrac{dV}{d\theta}\right|_s\right)}{\sqrt{1-s_\theta^2} - \dfrac{s_\theta}{V_s}\left.\dfrac{dV}{d\theta}\right|_s} \right) = \frac{4h}{V_{P0}t_m}\ . \tag{66}$$

From equation (53) for ξ_m and the definitions of $p^{x,y}_{s,r}$ and $\nu_{x,y}$, it follows that

$$\nu_x = \mathrm{sgn}(\nu_x)\left(\frac{V_r s_\theta\sqrt{1-s_\gamma^2} + V_s r_\theta\sqrt{1-r_\gamma^2}}{V_r\sqrt{1-s_\theta^2} + V_s\sqrt{1-r_\theta^2}}\right)\ , \tag{67}$$

$$\nu_y = -\left(\frac{V_r s_\theta s_\gamma + V_s r_\theta r_\gamma}{V_r\sqrt{1-s_\theta^2} + V_s\sqrt{1-r_\theta^2}}\right)\ , \tag{68}$$

with $\nu_{x,y}$ defined in equation (54).

Equation (64) and equations (66)-(68) are four nonlinear equations with four unknowns: $(s,r)_\theta$ and $(s,r)_\gamma$. Attempts to eliminate, for example, r_θ, s_γ, and r_γ to get one equation in s_θ lead to a high order polynomial equation in s_θ. Therefore, to find the unknown angles, a numerical scheme such as Gauss-Newton (Dennis, 1977) or Levenberg-Marquardt (Moré, 1977) can be used, with the isotropic solution as initial value. It goes beyond the purpose of this paper, however, to treat the details of the proper choice of numerical scheme. In Appendix A, we show that the system of equation ((64) and equations (66)-(68) can be rewritten into a somewhat simpler system using the horizontal slowness instead of the phase angles, under the assumption that for most practical purposes we can set $f = 1$. Note that for the 2D problem, $(s,r)_\gamma = 0$ and $-1 < (s,r)_\theta < 1$, so equations (66) and (68) form a set of two nonlinear equations with two unknowns, $(s,r)_\theta$.

3.7 Zero-offset demigration

For zero-offset demigration the source and receiver coincide, which means that the phase angle with the vertical symmetry axis equals the dip of the reflector. Since in the demigration problem this dip is known, we know the phase angle and thus the group angle. Therefore, once the position and orientation of the reflector are known, we can calculate the unmigrated location. From equation (61) we directly find

$$p^{x,y}_u = \frac{V_{P0}p^{x,y}_m}{V(\theta)\sqrt{1+V_{P0}^2 p_m^2}}\ , \tag{69}$$

with p_m defined in equation (34). From this expression we find

$$p_u^2 = \frac{V_{P0}^2 p_m^2}{V^2(\theta)\,(1 + V_{P0}^2 p_m^2)}\ , \tag{70}$$

which, when used in equation (58), gives

$$t_u = \frac{t_m \sqrt{1 + V_{P0}^2 p_m^2}}{\left(\dfrac{V(\theta)}{V_{P0}} - p_m \left.\dfrac{dV}{d\theta}\right|_\theta\right)}\ . \tag{71}$$

Then, using equations (58), (69), and (70) in equation (59), gives

$$(x,y)_u = (x,y)_m + \frac{V_{P0}^2 p_m^{x,y} t_m \left(V(\theta) + \dfrac{1}{V_{P0} p_m} \left.\dfrac{dV}{d\theta}\right|_\theta\right)}{2\left(V(\theta) - V_{P0} p_m \left.\dfrac{dV}{d\theta}\right|_\theta\right)}\ . \tag{72}$$

Note that by setting $\dfrac{dV}{d\theta} = 0$ and replacing $V(\theta)$ and V_{P0} with the constant velocity v, the equations for zero-offset demigration in VTI media reduce to their isotropic equivalents. Also, setting $x_m = x_u = 0$ and $p_m^x = p_u^x = 0$ gives the expressions in 2D.

To find the angle θ, we use that the reflector dip equals the phase angle for zero-offset:

$$\theta = \arctan \sqrt{\nu_x^2 + \nu_y^2}\ . \tag{73}$$

The calculated value of θ can subsequently be used in equation (38) and (39) (with $f = 1$ or some realistic estimate of f) to find the phase velocity and its derivative.

4 THE APPLICABILITY OF MAP DEPTH-MIGRATION IN THE PRESENCE OF CAUSTICS

The closed-form expressions derived for map time-migration in homogeneous isotropic and VTI media thus far explicitly show that the mapping from the surface measurements (i.e., source and receiver positions, traveltimes *and* slopes) to the subsurface (i.e., reflector position *and* orientation), or vice-versa, is *one-to-one* for such media. In this section we explain that this mapping remains one-to-one for arbitrary complex media in which caustics can develop, provided that the medium does not allow different reflectors to have identical surface seismic measurements that persist under small perturbations of the medium [this is the Bolker condition (Guillemin, 1985)]. This section thus treats map depth-migration as opposed to map time-migration which has been treated up to this point. If we assume large frequency, in arbitrary complex media the relation between the reflections measured at the surface and the reflectors in the subsurface is governed by ray-tracing. We capture this relation schematically with the symbol Λ. Since also we assume single-scattering only, this relation is by definition the *canonical relation* of the single scattering modeling or imaging operators that relates the surface seismic measurements to the subsurface reflectors. Throughout the remainder we will refer to this relation as *the* canonical relation.

4.1 Canonical relation

The canonical relation is formed by collecting the unmigrated and migrated quantities in a table, viz.

$$\Lambda = \{(\underbrace{\boldsymbol{x}_s^h, \boldsymbol{x}_r^h, t_u, \omega \boldsymbol{p}_s^h, \omega \boldsymbol{p}_r^h, \omega}_{\text{reflection}}; \underbrace{\boldsymbol{x}_m, \boldsymbol{\xi}_m}_{\text{reflector}})\}, \tag{74}$$

where $\boldsymbol{x}_{s,r}^h \equiv (x_{s,r}, y_{s,r})$ are the source and receiver locations, $t_u = t_s + t_r$ is the two-way traveltime, $\boldsymbol{p}_{s,r}^h \equiv (p_{s,r}^x, p_{s,r}^y)$ are the horizontal slownesses at the sources and receivers, $\boldsymbol{x}_m = (x_m, y_m, z_m)$ is the reflector subsurface position, $\boldsymbol{\xi}_m$ is the dip covector (i.e. the wave-vector associated with the reflector), and ω is the angular frequency; here h denotes horizontal components only. In general media, this table is evaluated with the aid of ray-tracing equations. Writing the solution to the ray-tracing equations, subject to initial conditions $(\boldsymbol{x}_0, \boldsymbol{\xi}_0)$ at time 0, in the general form $(\boldsymbol{x}(\boldsymbol{x}_0, \boldsymbol{\xi}_0, t), \boldsymbol{\xi}(\boldsymbol{x}_0, \boldsymbol{\xi}_0, t))$, the canonical relation becomes, for a horizontal surface,

$$\Lambda = \left\{ \left(\boldsymbol{x}^h(\boldsymbol{s}), \boldsymbol{x}^h(\boldsymbol{r}), t_u, \boldsymbol{\xi}^h(\boldsymbol{s}), \boldsymbol{\xi}^h(\boldsymbol{r}), \omega; \boldsymbol{x}_m, \boldsymbol{\xi}_m\right) \text{ such that } [z(\boldsymbol{s}) = 0, z(\boldsymbol{r}) = 0] \right\}. \tag{75}$$

where we defined $\boldsymbol{s} \equiv (\boldsymbol{x}_m, \boldsymbol{\xi}_s, t_s)$, $\boldsymbol{r} \equiv (\boldsymbol{x}_m, \boldsymbol{\xi}_r, t_r)$, and $\boldsymbol{\xi}^h \equiv (\xi_x, \xi_y)$. This symbolically represents a table evaluated through upward ray-tracing from the reflector at location $\boldsymbol{x}_m$ with normal direction along $\boldsymbol{\xi}_m$, towards the source and receiver in the directions $\boldsymbol{\xi}_s$ and $\boldsymbol{\xi}_r$ respectively, such that $\boldsymbol{\xi}_s + \boldsymbol{\xi}_r = \boldsymbol{\xi}_m$.

Defining $\boldsymbol{u} \equiv (x_s, y_s, x_r, y_r, t_u)$ and $\boldsymbol{v} \equiv (\omega p_s^x, \omega p_s^y, \omega p_r^x, \omega p_r^y, \omega)$, $(\boldsymbol{u}, \boldsymbol{v})$ is a surface seismic measurement characterized by the source and receiver locations, two-way traveltime, *and* slopes in common source and receiver gathers; this vector is an element of phase space U — a mathematical precise notation would be that $(\boldsymbol{u}, \boldsymbol{v}) \in T^*Y\backslash 0$ when $\boldsymbol{u}$ is contained in a set Y; with our notation we sacrifice mathematical precision to gain physical clarity. Similarly, defining $\boldsymbol{m} \equiv \boldsymbol{x}_m$ and $\boldsymbol{\mu} \equiv \boldsymbol{\xi}_m$, we see that the vector $(\boldsymbol{m}, \boldsymbol{\mu})$ is a subsurface reflector defined by its subsurface location *and* orientation; it is an element of phase space M. Let π_U now denote the *projection* of Λ on U and π_M denote the projection of Λ on M so that

$$\begin{array}{ccc} & \Lambda & \\ \pi_M \swarrow & & \searrow \pi_U \\ M & & U \end{array} .$$

These projections extract, respectively, the surface seismic measurements $(\boldsymbol{u}, \boldsymbol{v})$ and the subsurface reflectors $(\boldsymbol{m}, \boldsymbol{\mu})$ from the canonical relation, i.e., the table evaluated using ray-tracing. Guillemin (1985), in his paper on the generalized Radon transform, introduced the *Bolker condition* on the canonical relation, which in simple terms states that the medium does not allow different reflectors to have identical surface seismic measurements that persist in being identical under small perturbations of the medium. This means that, under these conditions, only one subsurface reflector in the table can relate to a particular surface seismic measurement. In our application this reduces to the condition that $\pi_U : \Lambda \to \pi_U(\Lambda)$ is *one-to-one*. But then we can introduce the mapping

$$\pi_M \circ \pi_U^{-1} \; : (\boldsymbol{u}, \boldsymbol{v}) \mapsto (\boldsymbol{m}, \boldsymbol{\mu}) \; ,$$

which is precisely map depth-migration: the traveltimes *and* slopes at the source and receiver determine the reflector location *and* orientation. The Bolker condition is thus the condition of applicability of map depth migration in complex media (i.e., in the presence of caustics).

We can always use $(\boldsymbol{m}, \boldsymbol{\mu})$ as the first four (2D) or six (3D) local coordinates on Λ (Stolk & De Hoop, 2002a). To find the surface seismic measurements from a subsurface reflector through map depth-demigration, however, we need to specifiy the scattering angle and azimuth at the reflector. By parameterizing the subsets $\{(\boldsymbol{m}, \boldsymbol{\mu})\} = \text{const}$ on Λ we can introduce these coordinates and denote them as $\boldsymbol{e}$. In the absence of caustics, $\boldsymbol{e}$ can be chosen to be acquisition offset and azimuth (as was done above in the section on pre-stack map time-demigration in homogeneous VTI media). Map depth-demigration then follows from the mapping

$$(\boldsymbol{m}, \boldsymbol{\mu}, \boldsymbol{e}) \mapsto (\boldsymbol{u}, \boldsymbol{v}) \quad \text{for given } \boldsymbol{e}.$$

4.2 The isotropic problem revisited

To make explicit the connection between the closed-form expressions for pre-stack map time-migration and demigration derived in the previous sections and the canonical relation, we revisit the isotropic case. We next show that for this case, the map time-migration and demigration equations define the canonical relation.

Since we derived all expressions for map time-migration in the rotated coordinate system with the y-axis positive in the source-receiver direction, these expressions implicitly assume common azimuth. Subjecting the canonical relation in equation (74) to the restriction $x_r = x_s = x_u$, it attains the form

$$\Lambda' = \{(x_u, y_s, y_r, t_u, \omega(p_s^x + p_r^x), \omega p_s^y, \omega p_r^y, \omega; \boldsymbol{x}_m, \boldsymbol{\xi}_m)\}. \tag{76}$$

For isotropic homogeneous media, the equations for the migrated location in the canonical relation follow directly from the equations for homogeneous VTI media by setting $\dfrac{dV}{d\theta} = 0$ and $V_s = V_r = V_{P0} = v$. Under this restriction, equations (46), (48) and (50) thus determine $\boldsymbol{x}_m = (x_m, y_m, vt_m/2)$ for isotropic homogeneous media. Because the expressions for the reflector dip in homogeneous VTI media do not contain the derivative of the phase velocity, the equations for the dip covector $\boldsymbol{\xi}_m$ in the canonical relation (76) are given by equation (53), with $V_s = V_r = v$.

To find the remaining parameters of the canonical relation, we need to find x_u, $y_{s,r}$, t_u, and $p_{s,r}^{x,y}$ for homogeneous isotropic media. By setting $\dfrac{dV}{d\theta} = 0$ and $V_s = V_r = v$ in equations (64), (66)-(68), we get the system of equations that need to be solved to find the scattering angles $\theta_{s,r}$ and azimuths $\gamma_{s,r}$ at the reflector. The resulting system of equations and its solution are given in appendix B. Once we find $(s,r)_\theta = (\sin\theta_s, \sin\theta_r)$ and $(s,r)_\gamma = (\sin\gamma_s, \sin\gamma_r)$, we calculate the source and receiver locations (x_u, $y_{s,r}$), two-way traveltime t_u, and the horizontal slownesses $p_{s,r}^{x,y}$

from simple geometry considerations. This gives

$$x_u = x_{s,r} = x_m + \mathrm{sgn}(\nu_x)\frac{vt_m}{2}\frac{(s,r)_\theta\sqrt{1-(s,r)_\gamma^2}}{\sqrt{1-(s,r)_\theta^2}}\ , \tag{77}$$

$$y_{s,r} = y_m + \frac{vt_m}{2}\frac{(s,r)_\gamma\,(s,r)_\theta}{\sqrt{1-(s,r)_\theta^2}}\ , \tag{78}$$

$$t_u = t_m\sqrt{1-s_\theta^2} + t_m\sqrt{1-r_\theta^2}\ , \tag{79}$$

$$p_{s,r}^x = (s,r)_\gamma\,\frac{(s,r)_\theta}{v}\ , \tag{80}$$

$$p_{s,r}^y = \mathrm{sgn}(\nu_x)\sqrt{1-(s,r)_\gamma^2}\,\frac{(s,r)_\theta}{v}\ . \tag{81}$$

For homogeneous isotropic media, therefore, our map time-migration and demigration equations define the canonical relation of the single scattering modeling or imaging operators in such media.

5 DISCUSSION

We have presented closed-form 3D pre-stack map time-migration expressions for qP-waves in homogeneous isotropic and VTI media. Our equations for the isotropic case do not need the slope in the common-midpoint domain (i.e., p_h); only the slope in the common-offset domain needs to be determined. This provides an additional advantage over methods where both p_u and p_h (or equivalently both p_s and p_r, the slopes at the source and receiver position) are required, especially since estimating slopes can be cumbersome in the presence of noise. All the derived pre-stack expressions reduce to their zero-offset equivalents. Since map migration is currently used primarily for velocity estimation, and since such methods are typically iterative, our closed-form expressions for pre-stack map time-migration in homogeneous isotropic and VTI media allow for a significant speed-up of existing velocity-inversion algorithms that use map migration in such media. In addition, our expressions for pre-stack map time-migration in homogeneous VTI media can be used to determine the anellipticity parameter η for such media in a time-migration velocity analysis context. Note that for media with mild lateral and vertical velocity variations, our equations can be used provided the velocity is replaced by the local RMS velocity.

Not surprising, our closed-form expressions for pre-stack map time-migration and demigration in homogeneous isotropic and VTI media, exemplify that, for such media, the mapping from the surface seismic measurements (i.e., the source and receiver locations, two-way traveltime, *and* slopes in common source and receiver gathers) to the subsurface reflectors (i.e., location *and* orientation), is one-to-one. We have explained that this mapping remains one-to-one for pre-stack map depth-migration in arbitrary complex media in which caustics can develop, provided the medium does not allow different reflectors to have identical surface seismic measurements that persist under small perturbations of the medium [this is the Bolker condition (Guillemin, 1985)]. This condition is thus the condition of applicability of pre-stack map depth-migration in the presence of caustics. In addition, we have shown that for homogeneous isotropic media, our pre-stack map time-migration and demigration expressions define the canonical relation of the single scattering modeling or imaging operators.

Map migration and demigration provide a mapping between the wavefront sets of seismic data and the medium perturbations. The tangential directions to the wavefronts recorded at the acquisition surface directly give us the directions locally in which the data are smooth; imaging these wavefronts gives the directions in which the medium perturbations are smooth. For the purpose of data and image compression, the highest compression rate will be accomplished in the smooth directions, i.e., along the wavefronts. Since map migration provides a one-to-one mapping from the singular directions in the data (i.e., the directions normal to the wavefronts) to the singularites in the image (i.e., the normals to the reflectors), one can thus think of map migration and demigration as mappings between optimal compression directions.

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APPENDIX A: FROM PHASE ANGLE TO HORIZONTAL SLOWNESS WHEN $V_{S0} = 0$

The slowness surface for qP-waves is convex, which in combination with a VTI medium, assures that the only branch points occur at $\theta = \pm\pi/2$. In homogeneous media, these branch-points are never reached, since turning waves do not occur in such media. Thus, for qP-waves in such media we can parametrize the phase velocity and its derivative uniquely in terms of the horizontal slowness p.

Substituting $\sin\theta = pV$ in equation (38) and solving for V, leads to

$$V(p) = V_{P0}\sqrt{\frac{1+2p^2V_{P0}^2\,(\delta-\epsilon)}{1-2p^2V_{P0}^2\epsilon+2p^4V_{P0}^4\,(\delta-\epsilon)}}\ , \tag{A1}$$

where we used $V_{S0} = 0$ (i.e., $f = 1$) and that the kinematics of qP-waves in anisotropic media are independent of V_{S0} within the limits of seismic accuracy (Alkhalifah, 1998). The derivative $\frac{dV}{dp}$ is then

$$\frac{dV}{dp} = \frac{2pV_{P0}^3\delta - 4p^3V_{P0}^5\,(\delta-\epsilon) - 4p^5V_{P0}^7\,(\delta-\epsilon)^2}{\left(1-2p^2V_{P0}^2\epsilon+2p^4V_{P0}^4\,(\delta-\epsilon)\right)^{3/2}\sqrt{1+2p^2V_{P0}^2\,(\delta-\epsilon)}}\ . \tag{A2}$$

Note that both expressions can be readily rewritten in terms of η, $V_{NMO}(0)$, and V_{P0}.

Using these expressions, we can write the vertical slowness $q = \sqrt{1/V^2 - p^2}$ as

$$q = \frac{1}{V_{P0}}\sqrt{\frac{1-p^2V_{P0}^2\,(1+2\epsilon)}{1+2p^2V_{P0}^2\,(\delta-\epsilon)}}\ , \tag{A3}$$

or in terms of η, $V_{NMO}(0)$, and V_{P0} [see also Alkhalifah (1998equation A-10)],

$$q = \frac{1}{V_{P0}}\sqrt{1-\frac{p^2V_{NMO}^2(0)}{1-2p^2V_{NMO}^2(0)\eta}}\ , \tag{A4}$$

In order to calculate the group angle, equation (36), we use the chain rule,

$$\frac{dV}{d\theta} = \frac{dV}{dp}\frac{dp}{d\theta}\ . \tag{A5}$$

Using $p = \sin\theta/V$ this becomes

$$\frac{dV}{d\theta} = \left(\frac{dV}{dp}\sqrt{1-p^2V^2}\right) / \left(V + p\frac{dV}{dp}\right)\ . \tag{A6}$$

Then, using this expression in equation (36) we find

$$\tan\psi = \frac{pV_{P0}\,(1+2\delta)}{\left(1+2p^2V_{P0}^2\,(\delta-\epsilon)\right)^{3/2}\sqrt{1-p^2V_{P0}^2\,(1+2\epsilon)}}\ , \tag{A7}$$

or in terms of η, $V_{NMO}(0)$ and V_{P0},

$$\tan\psi = \frac{pV_{NMO}^2(0)}{V_{P0}\left(1-2p^2V_{NMO}^2(0)\eta\right)^{3/2}\sqrt{1-p^2V_{NMO}^2(0)\,(1+2\eta)}}\ . \tag{A8}$$

Using the simplified expressions (A4) and (A8) for the group angle and vertical slowness in equations (64) and

(66)-(68) then leads to the following system of equations for pre-stack map demigration in homogeneous VTI media,

$$\frac{p_r\sqrt{1-r_\gamma^2}}{\left(1-2\eta p_r^2V_{NMO}^2(0)\right)^{3/2}\sqrt{1-p_r^2V_{NMO}^2(0)\left(1+2\eta\right)}} = \frac{p_s\sqrt{1-s_\gamma^2}}{\left(1-2\eta p_s^2V_{NMO}^2(0)\right)^{3/2}\sqrt{1-p_s^2V_{NMO}^2(0)\left(1+2\eta\right)}}, \quad \text{(A9)}$$

$$\frac{V_{NMO}^2(0)t_m r_\gamma p_r}{\left(1-2\eta p_r^2V_{NMO}^2(0)\right)^{3/2}\sqrt{1-p_r^2V_{NMO}^2(0)\left(1+2\eta\right)}} = 4h + \frac{V_{NMO}^2(0)t_m s_\gamma p_s}{\left(1-2\eta p_s^2V_{NMO}^2(0)\right)^{3/2}\sqrt{1-p_s^2V_{NMO}^2(0)\left(1+2\eta\right)}}, \quad \text{(A10)}$$

$$\frac{\mathrm{sgn}(\nu_x)V_{P0}\left(p_s\sqrt{1-s_\gamma^2}+p_r\sqrt{1-r_\gamma^2}\right)}{\sqrt{1-\frac{p_s^2V_{NMO}^2(0)}{1-2\eta p_s^2V_{NMO}^2(0)}}+\sqrt{1-\frac{p_r^2V_{NMO}^2(0)}{1-2\eta p_r^2V_{NMO}^2(0)}}} = \nu_x\ , \quad \text{(A11)}$$

$$\frac{V_{P0}\left(s_\gamma p_s+r_\gamma p_r\right)}{\sqrt{1-\frac{p_s^2V_{NMO}^2(0)}{1-2\eta p_s^2V_{NMO}^2(0)}}+\sqrt{1-\frac{p_r^2V_{NMO}^2(0)}{1-2\eta p_r^2V_{NMO}^2(0)}}} = -\nu_y\ . \quad \text{(A12)}$$

Note, that for the 2D problem we have $(s,r)_\gamma = 0$ and $-1 < V_{s,r}p_{s,r} < 1$, and that equations (A10) and (A12) then form a nonlinear system of two equations with two unknowns, viz. $p_{s,r}$.

APPENDIX B: SOLVING FOR SCATTERING ANGLES AND AZIMUTHS FOR PRE-STACK DEMIGRATION IN HOMOGENEOUS ISOTROPIC MEDIA

To find the scattering angles $\theta_{s,r}$ (see Figure 1) and azimuths $\gamma_{s,r}$ (see Figure 2) for pre-stack demigration in homogeneous isotropic media, we set $\frac{dV}{d\theta} = 0$ and $V_s = V_r = v$ in equations (64), (66)-(68). The resulting system of equations is

$$0 = \sqrt{1-r_\gamma^2}\frac{r_\theta}{\sqrt{1-r_\theta^2}} - \sqrt{1-s_\gamma^2}\frac{s_\theta}{\sqrt{1-s_\theta^2}}\ , \quad \text{(B1)}$$

$$\frac{4h}{vt_m} = \left(r_\gamma\frac{r_\theta}{\sqrt{1-r_\theta^2}} - s_\gamma\frac{s_\theta}{\sqrt{1-s_\theta^2}}\right)\ , \quad \text{(B2)}$$

$$\nu_x = \mathrm{sgn}(\nu_x)\left(\frac{s_\theta\sqrt{1-s_\gamma^2}+r_\theta\sqrt{1-r_\gamma^2}}{\sqrt{1-s_\theta^2}+\sqrt{1-r_\theta^2}}\right)\ , \quad \text{(B3)}$$

$$\nu_y = -\left(\frac{s_\theta s_\gamma + r_\theta r_\gamma}{\sqrt{1-s_\theta^2}+\sqrt{1-r_\theta^2}}\right)\ , \quad \text{(B4)}$$

where $(s,r)_\theta = (\sin\theta_s, \sin\theta_r)$ and $(s,r)_\gamma = (\sin\gamma_s, \sin\gamma_r)$ are unknown. (From these equations it is clear that the pathological cases $(s,r)_\theta = 1$, i.e. 90 degree dipping reflectors, are not included in this system of equations, and that by choosing $0 \le \theta < \pi/2$ we exclude these impractical cases.) In appendix C it is shown that this system of equations reduces to a quadratic equation in s_θ. The 2D problem is in this appendix treated as a special case of the 3D problem.

Once s_θ is found, we need to solve for the remaining parameters r_θ and $(s,r)_\gamma$. To find r_θ we first use equation (B4) in (B2) to eliminate $r_\theta r_\gamma$ to give

$$-\frac{4h}{vt_m} = \nu_y + \frac{\nu_y\sqrt{1-s_\theta^2}}{\sqrt{1-r_\theta^2}} + s_\theta s_\gamma\left(\frac{1}{\sqrt{1-s_\theta^2}} + \frac{1}{\sqrt{1-r_\theta^2}}\right)\ . \quad \text{(B5)}$$

Then, to eliminate s_γ we use equation (C3) and subsequently solve for r_θ to give

$$r_\theta = \sqrt{1-\left(\frac{hvt_m\sqrt{s_\theta^2-\nu_x^2(1-s_\theta^2)} - \sqrt{1-s_\theta^2}\left[\nu_y hvt_m + \Delta\right]}{4h^2+2\nu_y hvt_m+\Delta}\right)^2}\ . \quad \text{(B6)}$$

where we defined

$$\Delta \equiv \frac{v^2t_m^2}{4}\left(\nu_x^2+\nu_y^2-\frac{s_\theta^2}{1-s_\theta^2}\right)\ . \quad \text{(B7)}$$

Once $(s,r)_\theta$ are found, we solve equation (B5) for s_γ to get

$$s_\gamma = -\frac{\sqrt{1-s_\theta^2}}{s_\theta}\,\frac{\left(\nu_y\sqrt{1-s_\theta^2}+\sqrt{1-r_\theta^2}\left(\frac{4h}{vt_m}+\nu_y\right)\right)}{\left(\sqrt{1-s_\theta^2}+\sqrt{1-r_\theta^2}\right)}\;, \tag{B8}$$

provided $s_\theta \neq 0$. Finally, using equation (B4) in (B2) to eliminate $s_\gamma s_\theta$ and solving for r_γ, we find

$$r_\gamma = \frac{\sqrt{1-r_\theta^2}}{r_\theta}\,\frac{\left(\sqrt{1-s_\theta^2}\left(\frac{4h}{vt_m}-\nu_y\right)-\nu_y\sqrt{1-r_\theta^2}\right)}{\left(\sqrt{1-s_\theta^2}+\sqrt{1-r_\theta^2}\right)}\;, \tag{B9}$$

provided $r_\theta \neq 0$.

APPENDIX C: THE QUADRATIC EQUATION IN S_θ^2 FOR PRE-STACK DEMIGRATION IN HOMOGENEOUS ISOTROPIC MEDIA

To find s_θ, we first use equations (B2) and (B4) to eliminate $r_\theta r_\gamma$. Then we eliminate $s_\theta\sqrt{1-s_\gamma^2}$ from equations (B1) and (B3), and combine the results to give an equation with only s_θ, s_γ and r_γ:

$$\left(\frac{\nu_x vt_m}{2}\right)^2\left\{\nu_y\sqrt{1-s_\theta^2}+s_\theta s_\gamma\right\}^2 = \left(1-r_\gamma^2\right)\left\{4h^2+2hvt_m\left(\nu_y+\frac{s_\theta s_\gamma}{\sqrt{1-s_\theta^2}}\right)+\left(\frac{s_\theta vt_m}{2}\right)^2\left(\nu_y+\frac{s_\theta s_\gamma}{\sqrt{1-s_\theta^2}}\right)^2\right\}. \tag{C1}$$

In order to get r_γ^2 as a function of s_θ and s_γ, we use equations (B1) and (B2) to eliminate the term $r_\theta/\sqrt{1-r_\theta^2}$, and subsequently solve for r_γ^2. This gives

$$r_\gamma^2 = \frac{\left(2h\sqrt{1-s_\theta^2}+\frac{s_\theta s_\gamma vt_m}{2}\right)^2}{4h^2\left(1-s_\theta^2\right)+2hs_\theta s_\gamma vt_m\sqrt{1-s_\theta^2}+\left(\frac{s_\theta vt_m}{2}\right)^2}\;. \tag{C2}$$

Before we substitute this expression for r_γ^2 into equation (C1), we first eliminate $r_\theta\sqrt{1-r_\gamma^2}$ from equations (B1) and (B3), and solve for s_γ^2, which gives

$$s_\gamma^2 = 1+\nu_x^2-\frac{\nu_x^2}{s_\theta^2}\;, \tag{C3}$$

where $s_\theta \neq 0$. Then, using equations (C2) and (C3) in (C1), gives

$$\left(1-s_\theta^2\right)\left\{h^2\left[\beta\left(1-s_\theta^2\right)-4\nu_y^2\right]+2h\nu_y\alpha\right\} = \frac{v^2t_m^2}{4}\left\{\beta\left[\left(1+2\nu_x^2\right)s_\theta^2-\left(1+\nu_x^2\right)s_\theta^4-\ \nu_x^2\right]-\nu_y^2\right\}\;, \tag{C4}$$

where we defined

$$\alpha \equiv \frac{vt_m}{2}\left(1+\nu_x^2-\nu_y^2\right)\;, \tag{C5}$$

$$\beta \equiv \left(1+\nu_x^2+\nu_y^2\right)^2\;. \tag{C6}$$

Equation (C4) is a quadratic equation in s_θ^2 that can be solved for s_θ to give

$$s_\theta = \sqrt{\frac{2h^2\left(\beta-2\nu_y^2\right)+2h\nu_y\left(\alpha\pm\gamma\right)+\left[\beta\left(\frac{vt_m}{2}\right)^2\left(1+2\nu_x^2\right)\mp\alpha\gamma\right]}{2\beta\left(h^2+\left(1+\nu_x^2\right)\left(\frac{vt_m}{2}\right)^2\right)}}\;, \tag{C7}$$

with

$$\gamma \equiv \sqrt{4h^2\nu_y^2+\beta^2\left(\frac{vt_m}{2}\right)^2}\;. \tag{C8}$$

The proper root in equation (C7) can be found through substitution in the original system of equations (B1)-(B4).

SPECIAL CASES

In 2D, the system of equations (B1)-(B4) reduces to two equations with two unknowns:

$$\left(\frac{r_\theta}{\sqrt{1-r_\theta^2}} - \frac{s_\theta}{\sqrt{1-s_\theta^2}}\right) = \frac{4h}{vt_m} \ , \tag{C9}$$

$$-\left(\frac{s_\theta + r_\theta}{\sqrt{1-s_\theta^2} + \sqrt{1-r_\theta^2}}\right) = \nu_y \ , \tag{C10}$$

with $-1 < (s, r)_\theta < 1$. In order to solve this system for the unknowns $(s, r)_\theta$, we first rewrite equation (C9) to get

$$\sqrt{1-r_\theta^2} = \frac{r_\theta vt_m \sqrt{1-s_\theta^2}}{4h\sqrt{1-s_\theta^2} + s_\theta vt_m} \ . \tag{C11}$$

Using this expression in equation (C10) to eliminate $\sqrt{1-r_\theta^2}$ then gives

$$r_\theta = -\left(s_\theta + \frac{4h\nu_y\left(1-s_\theta^2\right)}{s_\theta vt_m + \sqrt{1-s_\theta^2}\left(4h + \nu_y vt_m\right)}\right) \ . \tag{C12}$$

Then, squaring both sides of this expression, and using the result in equation (C11) to eliminate r_θ^2, gives a quadratic equation in $\tan\theta_s = s_\theta/\sqrt{1-s_\theta^2} \equiv \tau_{\theta s}$, viz.

$$\nu_y vt_m \tau_{\theta s}^2 + \left(4h\nu_y + vt_m\left(\nu_y^2 - 1\right)\right)\tau_{\theta s} + 2h\left(\nu_y^2 - 1\right) - \nu_y vt_m = 0 \ , \tag{C13}$$

with roots

$$\tau_{\theta s} = \frac{1}{2\nu_y}\left(1-\nu_y^2\right) - \frac{2h}{vt_m} \pm \sqrt{\frac{4h^2}{v^2t_m^2} + \frac{\left(1+\nu_y^2\right)^2}{4\nu_y^2}} \ . \tag{C14}$$

Then, by using this expression in equation (C9) we find

$$\tau_{\theta r} = \frac{1}{2\nu_y}\left(1-\nu_y^2\right) + \frac{2h}{vt_m} \pm \sqrt{\frac{4h^2}{v^2t_m^2} + \frac{\left(1+\nu_y^2\right)^2}{4\nu_y^2}} \ . \tag{C15}$$

Therefore, $(s, r)_\theta$ are then given by

$$(s, r)_\theta = \sin\left(\arctan\left(\tau_{\theta s}, \tau_{\theta r}\right)\right) \ . \tag{C16}$$

The proper roots in equation (C14) and (C15) can be chosen through substitution in the original system of equations (C9) and (C10). Note that the pathological cases $(s, r)_\theta = 0$ mentioned in the previous subsection, are included in this solution.

For the special case $\nu_y = 0$, i.e., the zero dip case, the solution for $(s, r)_\theta$ is given simply by

$$s_\theta = \sin\left(\arctan\left(\frac{-2h}{vt_m}\right)\right) = -r_\theta \ . \tag{C17}$$