Constraining relative source locations with the seismic coda

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ABSTRACT

The relative location of seismic sources is of importance for the location of aftershocks on a fault, for the positioning of sources in repeat seismic surveys, and for monitoring induced seismicity. In this paper I show theoretically how the seismic coda can be used to infer a measure of the relative source location of two identical seismic sources from the correlation of the waveforms recorded at a single receiver. The theory is applicable to an explosive source in an acoustic or elastic medium, and for a point force or double couple in an elastic medium. For an explosive source the relative source location is constrained to be located on a sphere, while for a point force and a double couple the relative source location can be constrained to be located on an ellipsoid whose symmetry axis is determined by the point force or double couple.

1 INTRODUCTION

In a number of applications it is useful to determine the relative location of seismic sources. Aftershocks after a large earthquake help constrain the location and extent of the fault plane (Lay & Wallace, 1995). Earthquake clusters have been used to locate faults planes, both in a tectonic setting (Fuis *et al.*, 2001) and in hydrocarbon reservoirs (Maxwell & Urbancic, 2001). Seismicity has also been used to monitor the fluid transport properties in reservoirs (Shapiro *et al.*, 2002). In repeat seismic surveys with down hole sources, it is essential that the relative source locations in the two surveys are known with great accuracy in order to reduce the imprint of errors in the source location in time lapse measurements.

In principle, the relative position of two source locations can be found by locating each of the sources, and subsequently computing their relative location. The disadvantage of this approach is that errors in the velocity model along the whole path from the sources to the receivers are erroneously mapped into location errors. The relative position computed by comparing the absolute locations may be dominated by the location errors for the individual sources (Pavlis, 1992). For this reason it is advantageous to determine the relative location of the source directly from the recorded waveforms.

Earthquakes that occur within the same cluster of events often generate waveforms that are highly repeatable. Such highly repeatable waveforms have been used to constrain the relative positions of these events (Poupinet *et al.*, 1984; Frémont & Malone, 1987; Got *et al.*, 1994; Nadeau & MvEvilly, 1997; Bokelmann & Harhes, 2000; Waldhauser & Ellsworth, 2000) by measuring the delay times between the arriving P and S waves of the different events. Such a measurement of the delay time of P and S waves has also been used to find the relative location between one master event and a number of smaller events (Ito, 1985; Scherbaum & Wendler, 1986; Frémont & Malone, 1987; VanDecar & Crosson, 1990; Deichmann & Garcia-Fernandez, 1992; Lees, 1998). Usually this delay time is measured by a cross-correlation of the direct P and S arrivals for the different events. A particularly robust variation of this idea is based on a measurement of the cross-correlation of the direct P and S waves that is based on an L_1 norm and a nearest neighbor approach (Shearer, 1997; Astiz & Shearer, 2000).

In this paper I propose a technique to obtain a measure of the relative source location of two events that is based on coda waves. The main idea is that the energy that constitutes the coda waves is radiated in all directions with a radiation pattern that is determined by the source position. When the source position changes, some wave paths will be longer while other wave paths become shorter. The interference pattern of the scattered waves that contribute to the coda thus changes. Here we use the change in the coda waves to constrain the relative locations of the events. This technique is a new application of coda wave interferometry (Snieder *et al.*, 2002; Snieder, 2002).

The principles of coda wave interferometry are introduced in section 2. The displacement of an isotropic source in an acoustic medium is treated in section 3. The generalization to an elastic medium that is excited by a point force and of a double couple is presented in sections 4 and 5.1. The source displacement of an explosive source in an elastic medium is treated in section 5.2.

2 CODA WAVE INTERFEROMETRY AND SOURCE DISPLACEMENT

In this section, I review the elements of coda wave interferometry. A more detailed description is given by Snieder (2002). The idea of coda wave interferometry is based on path summation (Snieder, 1999). In this approach the wavefield is written as a superposition of the waves that follow all the different paths in the medium:

$$\mathbf{u}^{(u)}(t) = \sum_T \mathbf{A}_T(t) \ . \tag{1}$$

The subscript T labels the different trajectories along which the waves have traveled. A trajectory not only specifies the path that a wave has taken through space, it also specifies which of the segments along that path have been traversed as a P-wave or as an S-wave. The summation over trajectories therefore also contains a summation over the different wave modes (P or S) that an elastic wave can take while propagating between the scatterers along each path. The function $\mathbf{A}_T(t)$ denotes the contribution of the trajectory T to the waveform recorded at the station under consideration. The superscript (u) in equation (1) denotes that this is the unperturbed waveform, i.e., that associated with the reference source position.

Now suppose that the source location is perturbed. When the source displacement is small, the dominant change to the wave field comes from the travel time perturbation τ_T for the wave along each trajectory T. (In reality the geometrical spreading and radiation of the waves that propagate from the displaced source to a fix scatterer change as well, but we assume that this contribution is sub-dominant.) Under this assumption the perturbed wave field is given by

$$\mathbf{u}^{(p)}(t) = \sum_T \mathbf{A}_T(t - \tau_T) \ . \tag{2}$$

The change in the waveform can be measured by a time-shifted cross-correlation, with shift time t_s computed over a time window of length $2t_w$ and center time t:

$$R^{(t,t_w)}(t_s) \equiv \frac{\int_{t-t_w}^{t+t_w} u_i^{(u)}(t') u_i^{(p)}(t'+t_s) dt'}{\left(\int_{t-t_w}^{t+t_w} u_i^{(u)2}(t') dt' \int_{t-t_w}^{t+t_w} u_i^{(p)2}(t') dt' \right)^{1/2}} \ . \tag{3}$$

As shown by Snieder (2002), this cross-correlation attains its maximum value for

$$t_s = \langle \tau \rangle_{(t,t_w)} \ , \tag{4}$$

where the average in this expression is given by

$$\langle \tau \rangle = \frac{\sum_T A_T^2 \tau_T}{\sum_T A_T^2} \ . \tag{5}$$

In this expression the summation is over the trajectories with an arrival time within the interval $(t - t_w, t + t_w)$. The maximum value of the cross-correlation is given by

$$R_{max}^{(t,t_w)} = 1 - \frac{1}{2} \overline{\omega^2} \sigma_\tau^2 \ , \tag{6}$$

where σ_τ is the variance of the travel time perturbation defined as

$$\langle \sigma_\tau^2 \rangle = \frac{\sum_T A_T^2 (\tau_T - \langle \tau \rangle)^2}{\sum_T A_T^2} \ . \tag{7}$$

The frequency $\overline{\omega^2}$ is given by

$$\overline{\omega^2} \equiv - \frac{\int_{t-t_w}^{t+t_w} u^{(u)}(t') \ddot{u}^{(u)}(t') dt'}{\int_{t-t_w}^{t+t_w} u^{(u)2}(t') dt'} \ . \tag{8}$$

The cross-correlation (3) can readily be computed given the measured unperturbed and the perturbed waveforms, so both the location of its maximum and the peak value can be measured. With expressions (4) and (6) the mean and variance of the travel time perturbations can therefore be determined from the observations.

When the source location is perturbed, only the length of the wave path to the first scatterer along each path is perturbed, because a perturbation in the source location does not change the relative positions of the scatterers along a path. The travel time change due to a perturbation δ in the source source location leads to a change in the travel time given by

$$\tau_T = -\frac{1}{v} (\hat{\mathbf{r}}_T \cdot \delta) \ . \tag{9}$$

In this expression the unit vector $\hat{\mathbf{r}}_T$ points in the direction in which the trajectory T takes off at the source and the velocity v is the velocity of the trajectory as it leaves the source. This can either be the P-velocity or the S-velocity. I assume throughout this paper that the velocity is constant over the region over which the source is displaced and that the velocity is isotropic.

When the scatterers are distributed homogeneously, the summation over all trajectories that leave the source can be replaced by an angular integration over all directions toward which a wave can leave the source. Since the averages (5) and (7) are taken with a weight given by energy of the wave that travels along each the trajectory, the integration over all take-off directions at the source is to be weighted with the radiated energy in those directions. This principle will be used in the following section to compute the mean and variance of the travel time caused by changes in the source location.

3 AN ISOTROPIC SOURCE IN AN ACOUSTIC MEDIUM

In this section we consider the simplest case of a displacement of an isotropic source in an acoustic medium. I assume that the propagation of the waves from the source to the first scatterer along each path can be described by the Green's function for a homogeneous medium For an isotropic source at the origin with source spectrum $S(\omega)$, the waves that propagate to the first scatterer along each path are thus given by

$$u(r) = -\frac{e^{ikr}}{4\pi r} S(\omega) \; . \tag{10}$$

When the scatterers in the medium are distributed homogeneously, the travel time perturbation (5) due to a source displacement δ for each path is given by (9) and the mean travel time perturbation (5) is given by

$$\langle\tau\rangle = \frac{-\int\int \left| -\frac{e^{ikr}}{4\pi r} \right|^2 \frac{1}{v} \left(\hat{\mathbf{r}}\cdot\delta\right) |S(\omega)|^2 \, d\Omega d\omega}{\int\int \left| -\frac{e^{ikr}}{4\pi r} \right|^2 |S(\omega)|^2 \, d\Omega d\omega} \; , \tag{11}$$

where $\int \cdots d\Omega$ denotes the angular integration over all outgoing directions, $\int \cdots d\omega$ denotes an integration over frequency, and r is the distance to the first scatterer in each direction. When the scatterers are distributed homogeneously, this distance is on average the same for each direction, and expression (11) can be rewritten as

$$\langle\tau\rangle = \frac{-\int \left(\hat{\mathbf{r}}\cdot\delta\right) d\Omega \int |S(\omega)|^2 \, d\omega}{4\pi v \int |S(\omega)|^2 \, d\omega} \; . \tag{12}$$

The frequency integrals cancel so that

$$\langle\tau\rangle = -\frac{1}{4\pi v}\int \left(\hat{\mathbf{r}}\cdot\delta\right) d\Omega \; . \tag{13}$$

The integrand is an odd function of the location $\hat{\mathbf{r}}$ on the unit sphere. Since we integrate over the full unit sphere this integral vanishes:

$$\langle\tau\rangle = 0 \; . \tag{14}$$

Physically this reflects the fact that as the source location is moved, some paths become longer while others become shorter. On average the associated imprint on the travel time is zero.

Since the mean travel time perturbation vanishes, $\sigma_\tau^2 = \langle\tau^2\rangle$. This quantity can be computed using the same reasoning that led to (13), the only difference being that $\tau = -v^{-1}\left(\hat{\mathbf{r}}\cdot\delta\right)$ needs to be replaced by $\tau^2 = v^{-2}\left(\hat{\mathbf{r}}\cdot\delta\right)^2$. This gives

$$\sigma_\tau^2 = \frac{1}{4\pi v^2}\int \left(\hat{\mathbf{r}}\cdot\delta\right)^2 d\Omega \; . \tag{15}$$

This expression is most easily evaluated by using

$$\hat{\mathbf{r}} = \begin{pmatrix} \cos\varphi\sin\theta \\ \sin\varphi\sin\theta \\ \cos\theta \end{pmatrix} \; , \tag{16}$$

with θ and φ the colatitude and longitude used in a system of spherical coordinates. Aligning the z-axis of the integration variable with the source displacement δ reduces the angular integral in (15) to $\delta^2 \int \cos^2\theta d\Omega = 4\pi\delta^2/3$, and hence

$$\sigma_\tau^2 = \frac{1}{3}\frac{\delta^2}{v^2} \; . \tag{17}$$

Using coda wave interferometry, we can infer σ_τ^2 from the changes in the waveform. Then, from (17), we can infer the magnitude δ of the source displacement, but not the direction of the source displacement. The change in the arrival time of the first-arriving wave constrains $(\hat{\mathbf{t}}\cdot\delta)$, with $\hat{\mathbf{t}}$ the unit vector in the take-off direction of a ray at the source that propagates directly to the receiver. The first-arriving waves and the later-arriving waves thus impose complementary information on the perturbation of the source position by constraining $(\hat{\mathbf{t}}\cdot\delta)$ and δ, respectively.

4 A POINT FORCE IN AN ELASTIC MEDIUM

Let us now consider the displacement of a point force in an elastic medium. The far-field displacement due to a point force $\mathbf{F}$ at the origin in a homogeneous medium given by Aki & Richards (1980):

$$\mathbf{u} = \frac{e^{ik_\alpha r}}{4\pi\rho\alpha^2 r}\hat{\mathbf{r}}\left(\hat{\mathbf{r}}\cdot\mathbf{F}\right) + \frac{e^{ik_\beta r}}{4\pi\rho\beta^2 r}\left(\mathbf{F} - \hat{\mathbf{r}}\left(\hat{\mathbf{r}}\cdot\mathbf{F}\right)\right) \; . \tag{18}$$

In this expression α and β are the P and S-velocity respectively, while the wave numbers that appear are given by $k_\alpha = \omega/\alpha$ and $k_\beta = \omega/\beta$. For an arbitrary source spectrum $S(\omega)$, this expression should be multiplied with $S(\omega)$, but using the same reasoning that led to (13) we conclude that the source spectrum cancels out. For this reason we suppress the presence of the source spectrum $S(\omega)$ altogether.

When the source is displaced over a distance δ, the perturbation of the arrival time for the P-waves is given by $-\alpha^{-1}\left(\hat{\mathbf{r}}\cdot\delta\right)$, while the perturbation of the arrival of the S-waves is given by $-\beta^{-1}\left(\hat{\mathbf{r}}\cdot\delta\right)$. In the averages (5) and (7), the averages are taken with the intensities of each path as weight function. Since the P-waves and the S-waves can be considered to be different paths, the mean travel time perturbation is given by expression (T1) of Table 1. This expression can be simplified with the following identities: $\left|\hat{\mathbf{r}}\left(\hat{\mathbf{r}}\cdot\mathbf{F}\right)\right|^2 = \left(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}\right)^2\left(\hat{\mathbf{r}}\cdot\mathbf{F}\right)^2 = \left(\hat{\mathbf{r}}\cdot\mathbf{F}\right)^2$ and $\left|\left(\mathbf{F} - \hat{\mathbf{r}}\left(\hat{\mathbf{r}}\cdot\mathbf{F}\right)\right)\right|^2 = F^2 - 2\left(\hat{\mathbf{r}}\cdot\mathbf{F}\right)\left(\hat{\mathbf{r}}\cdot\mathbf{F}\right) + \left(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}\right)\left(\hat{\mathbf{r}}\cdot\mathbf{F}\right)^2 = F^2 - \left(\hat{\mathbf{r}}\cdot\mathbf{F}\right)^2$. Using these identities gives equation (T2) of Table 1. Both integrals in the numerator of that expression vanish because the integrands are odd functions of $\hat{\mathbf{r}}$ and the integration is carried out over the full unit sphere; therefore $\langle\tau\rangle = 0$.

The variance in the travel time can be computed by replacing $-\alpha^{-1}\left(\hat{\mathbf{r}}\cdot\delta\right)$ in the numerator by $\alpha^{-2}\left(\hat{\mathbf{r}}\cdot\delta\right)^2$ and $-\beta^{-1}\left(\hat{\mathbf{r}}\cdot\delta\right)$ by $\beta^{-2}\left(\hat{\mathbf{r}}\cdot\delta\right)^2$. This gives expression

$$\langle\tau\rangle = \frac{-\int \left|\frac{e^{ik_\alpha r}}{4\pi\rho\alpha^2 r}\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{F})\right|^2 \frac{1}{\alpha}(\hat{\mathbf{r}}\cdot\delta)d\Omega - \int \left|\frac{e^{ik_\beta r}}{4\pi\rho\beta^2 r}(\mathbf{F}-\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{F}))\right|^2 \frac{1}{\beta}(\hat{\mathbf{r}}\cdot\delta)d\Omega}{\int \left|\frac{e^{ik_\alpha r}}{4\pi\rho\alpha^2 r}\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{F})\right|^2 d\Omega + \int \left|\frac{e^{ik_\beta r}}{4\pi\rho\beta^2 r}(\mathbf{F}-\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{F}))\right|^2 d\Omega} \tag{T1}$$

$$\langle\tau\rangle = \frac{-\int \frac{1}{\alpha^4}(\hat{\mathbf{r}}\cdot\mathbf{F})^2 \frac{1}{\alpha}(\hat{\mathbf{r}}\cdot\delta)d\Omega - \int \frac{1}{\beta^4}\left(F^2-(\hat{\mathbf{r}}\cdot\mathbf{F})^2\right)^2 \frac{1}{\beta}(\hat{\mathbf{r}}\cdot\delta)d\Omega}{\int \frac{1}{\alpha^4}(\hat{\mathbf{r}}\cdot\mathbf{F})^2 d\Omega + \int \frac{1}{\beta^4}\left(F^2-(\hat{\mathbf{r}}\cdot\mathbf{F})^2\right)d\Omega} \tag{T2}$$

$$\sigma_\tau^2 = \frac{\int \frac{1}{\alpha^6}(\hat{\mathbf{r}}\cdot\mathbf{F})^2(\hat{\mathbf{r}}\cdot\delta)^2 d\Omega + \int \frac{1}{\beta^6}\left(F^2-(\hat{\mathbf{r}}\cdot\mathbf{F})^2\right)^2(\hat{\mathbf{r}}\cdot\delta)^2 d\Omega}{\int \frac{1}{\alpha^4}(\hat{\mathbf{r}}\cdot\mathbf{F})^2 d\Omega + \int \frac{1}{\beta^4}\left(F^2-(\hat{\mathbf{r}}\cdot\mathbf{F})^2\right)d\Omega} \tag{T3}$$

$$\langle\tau\rangle = \frac{-\int \frac{1}{\alpha^6}^{2}(\hat{\mathbf{r}}\hat{\mathbf{r}}:\mathbf{M})^2 \frac{1}{\alpha}(\hat{\mathbf{r}}\cdot\delta)d\Omega - \int \frac{1}{\beta^6}\left((\hat{\mathbf{r}}\cdot\mathbf{M})^2-(\hat{\mathbf{r}}\hat{\mathbf{r}}:\mathbf{M})^2\right)\frac{1}{\beta}(\hat{\mathbf{r}}\cdot\delta)d\Omega}{\int \frac{1}{\alpha^6}(\hat{\mathbf{r}}\hat{\mathbf{r}}:\mathbf{M})^2 d\Omega + \int \frac{1}{\beta^6}\left((\hat{\mathbf{r}}\cdot\mathbf{M})^2-(\hat{\mathbf{r}}\hat{\mathbf{r}}:\mathbf{M})^2\right)d\Omega} \tag{T4}$$

$$\sigma_\tau^2 = \frac{\int \frac{1}{\alpha^8}(\hat{\mathbf{r}}\hat{\mathbf{r}}:\mathbf{M})^2(\hat{\mathbf{r}}\cdot\delta)^2 d\Omega + \int \frac{1}{\beta^8}\left((\hat{\mathbf{r}}\cdot\mathbf{M})^2-(\hat{\mathbf{r}}\hat{\mathbf{r}}:\mathbf{M})^2\right)(\hat{\mathbf{r}}\cdot\delta)^2 d\Omega}{\int \frac{1}{\alpha^6}(\hat{\mathbf{r}}\hat{\mathbf{r}}:\mathbf{M})^2 d\Omega + \int \frac{1}{\beta^6}\left((\hat{\mathbf{r}}\cdot\mathbf{M})^2-(\hat{\mathbf{r}}\hat{\mathbf{r}}:\mathbf{M})^2\right)d\Omega} \tag{T5}$$

Table 1. Expressions for the mean and the variance of the travel time.

(T3) of Table 1. This expression can also be rewritten as

$$\sigma_\tau^2 = \frac{\left(\frac{1}{\alpha^6}-\frac{1}{\beta^6}\right)\int(\hat{\mathbf{r}}\cdot\mathbf{F})^2(\hat{\mathbf{r}}\cdot\delta)^2 d\Omega + \frac{F^2}{\beta^6}\int(\hat{\mathbf{r}}\cdot\delta)^2 d\Omega}{\left(\frac{1}{\alpha^4}-\frac{1}{\beta^4}\right)\int(\hat{\mathbf{r}}\cdot\mathbf{F})^2 d\Omega + \frac{F^2}{\beta^4}\int d\Omega} \,. \tag{19}$$

The integrations over the unit sphere can be carried out using representation (16) for $\hat{\mathbf{r}}$. It is convenient to align the z-axis with the point force $\mathbf{F}$, so that $(\hat{\mathbf{r}}\cdot\mathbf{F}) = F\cos\theta$. For that coordinate system, the first integral in the numerator is given by

$$\begin{aligned}
&\int(\hat{\mathbf{r}}\cdot\mathbf{F})^2(\hat{\mathbf{r}}\cdot\delta)^2 d\Omega \\
&= F^2\delta_x^2\int_0^{2\pi}\int_0^{\pi}\sin^2\theta\cos^2\varphi\cos^2\theta\sin\theta d\theta d\varphi \\
&+F^2\delta_y^2\int_0^{2\pi}\int_0^{\pi}\sin^2\theta\sin^2\varphi\cos^2\theta\sin\theta d\theta d\varphi \\
&+F^2\delta_z^2\int_0^{2\pi}\int_0^{\pi}\cos^2\theta\cos^2\theta\sin\theta d\theta d\varphi \\
&= \frac{4\pi}{15}F^2\left(\delta_x^2+\delta_y^2\right)+\frac{4\pi}{5}F^2\delta_z^2 \,.
\end{aligned} \tag{20}$$

The last expression holds for the special case of a coordinate system that is aligned with the point force. However, the last line can be rewritten as

$$\begin{aligned}
&\frac{4\pi}{15}F^2\left(\delta_x^2+\delta_y^2\right)+\frac{4\pi}{5}F^2\delta_z^2 \\
&= \frac{4\pi}{15}F^2\left(\delta_x^2+\delta_y^2+\delta_z^2\right)+4\pi\left(\frac{1}{5}-\frac{1}{15}\right)F^2\delta_z^2 \\
&= \frac{4\pi}{15}F^2\delta^2+\frac{8\pi}{15}^2(\mathbf{F}\cdot\delta)^2 \,.
\end{aligned} \tag{21}$$

The last identity is invariant under unitary coordinate transformations such as rotations; therefore this expression holds in any coordinate system.

Applying a similar analysis to all terms in (19) gives, after dividing by F^2 and some rearrangement,

$$\sigma_\tau^2 = \frac{\left(\frac{1}{\alpha^6}+\frac{4}{\beta^6}\right)\delta^2 - 2\left(\frac{1}{\beta^6}-\frac{1}{\alpha^6}\right)\left(\hat{\mathbf{F}}\cdot\delta\right)^2}{5\left(\frac{1}{\alpha^4}+\frac{2}{\beta^4}\right)} \,, \tag{22}$$

where $\hat{\mathbf{F}} \equiv \mathbf{F}/F$ is the unit vector in the direction of the point force.

Note that for the elastic wave generated by a point force, $\langle\sigma_\tau^2\rangle$ depends on not just δ^2 but also on $(\hat{\mathbf{F}}\cdot\delta)$, the projection of the source displacement along the point force. Therefore, the direction of the point force is needed in order to relate the observed value of $\langle\sigma_\tau^2\rangle$ to the source displacement. This contrasts with the acoustic case treated in the previous section where the variance of the travel time was dependent on δ only.

For a Poisson medium ($\alpha = \sqrt{3}\beta$), expression (22) results in

$$\sigma_\tau^2 \approx \frac{\delta^2}{\beta^2}\left(0.382 - 0.182\left(\hat{\mathbf{F}}\cdot\hat{\delta}\right)^2\right) \quad \text{(Poisson medium)}. \tag{23}$$

Note that the last term is rewritten in terms of the unit vector $\hat{\delta}$. The first term gives the contribution that is direction-independent. For the acoustic case the corresponding result (17) is $\langle\sigma_\tau^2\rangle = \delta^2/3v^3 \approx 0.333\delta^2/v^2$, which is close to the coefficient 0.382 in (23) when the shear velocity β is equated to the velocity v in the acoustic medium. There is a simple reason for this. In expression (22) the velocities α and β are raised to a fairly high negative power (-4 and -6 respectively). Since $\beta < \alpha$ this leads to a dominance of the terms that are dependent on β. In fact, when the α-dependent terms in (22) are ignored altogether, expression (22) reduces to

$$\sigma_\tau^2 \approx \frac{\delta^2}{\beta^2}\left(0.4 - 0.2\left(\hat{\mathbf{F}}\cdot\hat{\delta}\right)^2\right) \qquad (\alpha \text{ ignored}). \tag{24}$$

This crude approximation leads to a result that is close to (23) for a Poisson medium. Physically this happens because a point force excites much stronger S-waves than P-waves. Since the travel time averages are weighted with the intensity of both wave types, the contribution of the shear waves dominates. The dominance of the S-wave energy over the P-wave energy has been noted before in different contexts (Aki & Chouet, 1975; Weaver, 1982; Papanicolaou & Ryzhik, 1999; Trégourès & van Tiggelen, 2002; Snieder, 2002).

In some situations one can eliminate the dependence of the variance of the travel time on the direction of the source displacement. As an example, consider a vibrator in a bore hole that vibrates in a direction perpendicular to the bore hole. The relative location of different deployments of the vibrator depends only on the distance along the bore hole. In that case the force $\mathbf{F}$ and the source displacement δ are perpendicular, and equation (22) reduces to:

$$\sigma_\tau^2 = \frac{\left(\dfrac{1}{\alpha^6} + \dfrac{4}{\beta^6}\right)}{5\left(\dfrac{1}{\alpha^4} + \dfrac{2}{\beta^4}\right)}\delta^2 \qquad (\mathbf{F} \perp \delta)\,. \tag{25}$$

Of course one may question the validity of the employed model for a source in a bore hole since the presence of the bore hole and the properties of its walls modify the radiation of elastic waves.

In this section and the following section I assume that the velocity is isotropic. For an anisotropic elastic medium, the velocity depends on the direction of propagation. In that case the velocity cannot be taken outside the angular integrals. However, the angular integrals can in that case be extended to take the anisotropy of the velocity into account.

5 A MOMENT TENSOR SOURCE IN AN ELASTIC MEDIUM

In this section we consider the displacement caused by a moment tensor source in an elastic medium. In section 5.1 we consider the case of a double couple and in section 5.2 that of an explosive source. According to Aki and Richards (1980), the displacement in an elastic medium due to a moment tensor source $\mathbf{M}$ is given by

$$\begin{aligned} u_i(\mathbf{r}) = &-\frac{i\omega e^{ik_\alpha r}}{4\pi\rho\alpha^3 r}\hat{r}_i\hat{r}_j\hat{r}_k M_{jk} \\ &-\frac{i\omega e^{ik_\beta r}}{4\pi\rho\beta^3 r}\left(\hat{r}_i\hat{r}_j\hat{r}_k M_{jk} - \delta_{ij}\hat{r}_k M_{jk}\right)\,. \end{aligned} \tag{26}$$

According to expression (5) the travel time perturbation for each trajectory is weighted by the intensity for that trajectory. The intensity corresponding to the different terms in (26) can be computed using the identities $\left(\hat{r}_i\hat{r}_j\hat{r}_k M_{jk}\right)^2 = (\hat{\mathbf{r}}\cdot\hat{\mathbf{r}})^2\left(\hat{\mathbf{r}}\hat{\mathbf{r}}:\mathbf{M}\right)^2 =^2 \left(\hat{\mathbf{r}}\hat{\mathbf{r}}:\mathbf{M}\right)^2$, and $\left(\hat{r}_i\hat{r}_j\hat{r}_k M_{jk} - \delta_{ij}\hat{r}_k M_{jk}\right)^2 = (\hat{\mathbf{r}}\cdot\mathbf{M})^2 - (\hat{\mathbf{r}}\hat{\mathbf{r}}:\mathbf{M})^2$. By analogy with expression (T2) the mean travel time change due to a perturbation δ in the source location is given by equation (T4) in Table 1. Just as is the integral (T2), the integrand in expression (T4) is an odd function of $\hat{\mathbf{r}}$, and the integral vanishes upon integration over the unit sphere so the mean travel time perturbation vanishes: $\langle\tau\rangle = 0$. Using the same reasoning as that used for (T2), the variance of the travel time is given by equation (T5) in Table 1. The integration over the unit sphere is most easily carried out when a simplified form of the moment tensor is assumed.

5.1 A double couple source

In this section we analyze the variance of the travel time for a double couple source. In the integration we use a coordinate system with the z-axis perpendicular to the double couple. In that coordinate system the moment tensor is given by

$$\mathbf{M} = \begin{pmatrix} 0 & M & 0 \\ M & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,. \tag{27}$$

With representation (16) for the unit vector $\hat{\mathbf{r}}$, the contractions that appear in (T5) are given by

$$(\hat{\mathbf{r}}\cdot\mathbf{M})^2 = M^2\sin^2\theta\,, \tag{28}$$

$$(\hat{\mathbf{r}}\hat{\mathbf{r}}:\mathbf{M})^2 = 4M^2\sin^4\theta\sin^2\varphi\cos^2\varphi\,. \tag{29}$$

With these representations the angular integrals that appear in (T5) can be carried out. The resulting integrals are tedious and are given by

$$\int(\hat{\mathbf{r}}\cdot\mathbf{M})^2\,d\Omega = \frac{8\pi}{3}M^2\,, \tag{30}$$

$$\int(\hat{\mathbf{r}}\hat{\mathbf{r}}:\mathbf{M})^2\,d\Omega = \frac{16\pi}{15}M^2\,, \tag{31}$$

$$\int (\hat{\mathbf{r}} \cdot \mathbf{M})^2 (\hat{\mathbf{r}} \cdot \delta)^2 \, d\Omega = \frac{16\pi}{15} M^2 (\delta_x^2 + \delta_y^2) + \frac{8\pi}{15} M^2 \delta_z^2 \, , \tag{32}$$

$$\int (\hat{\mathbf{r}}\hat{\mathbf{r}} : \mathbf{M})^2 (\hat{\mathbf{r}} \cdot \delta)^2 \, d\Omega = \frac{16\pi}{35} M^2 (\delta_x^2 + \delta_y^2) + \frac{16\pi}{105} M^2 \delta_z^2 \, . \tag{33}$$

Inserting these results in expression (28) and using the identity $\delta_x^2 + \delta_y^2 = \delta^2 - \delta_z^2$ gives after a rearrangement of terms

$$\sigma_\tau^2 = \frac{\left(\frac{6}{\alpha^8} + \frac{7}{\beta^8} \right) \delta^2 - \left(\frac{4}{\alpha^8} + \frac{3}{\beta^8} \right) \delta_z^2}{7 \left(\frac{2}{\alpha^6} + \frac{3}{\beta^6} \right)} \, . \tag{34}$$

This expression is not quite satisfactory yet because it does not contain the moment tensor in a covariant form. (The only information of the orientation of the double couple is captured in the choice of the z-direction, which is orthogonal to the double couple.) A covariant formulation can be obtained by defining the following norm of the moment tensor

$$\|\mathbf{M}\|^2 \equiv (\mathbf{M} : \mathbf{M}) \, . \tag{35}$$

Since this quantity is the double contraction of a tensor of rank two, it is a tensor of rank zero; hence $\|\mathbf{M}\|$ is invariant for unitary coordinate transforms. With this norm we can define a normalized moment tensor

$$\hat{\mathbf{M}} \equiv \frac{\mathbf{M}}{\|\mathbf{M}\|} \, . \tag{36}$$

Since $\mathbf{M}$ is a tensor of rank two and $\|\mathbf{M}\|$ is a scalar, $\hat{\mathbf{M}}$ is a tensor of rank two. For the moment tensor (27)

$$\left(\hat{\mathbf{M}} \cdot \delta \right)^2 = \frac{1}{2} \left(\delta^2 - \delta_z^2 \right) \, . \tag{37}$$

This result can be used to eliminate δ_z from expression (34) so that

$$\sigma_\tau^2 = \frac{\left(\frac{2}{\alpha^8} + \frac{4}{\beta^8} \right) \delta^2 + 2 \left(\frac{4}{\alpha^8} + \frac{3}{\beta^8} \right) \left(\hat{\mathbf{M}} \cdot \delta \right)^2}{7 \left(\frac{2}{\alpha^6} + \frac{3}{\beta^6} \right)} \, . \tag{38}$$

This expression is invariant for rotations of the coordinate system and can therefore be applied to a double couple with an arbitrary orientation.

Just as with expression (22) for a point force, the variance of the travel time depends on the magnitude δ of the source displacement as well as on the direction of the source displacement. Therefore, it is necessary to know the orientation of the double couple in order to relate the variance in the travel time perturbation as inferred from coda wave interferometry to the source displacement.

For a Poisson medium

$$\sigma_\tau^2 \approx \frac{\delta^2}{\beta^2} \left(0.187 + 0.283 \left(\hat{\mathbf{M}} \cdot \hat{\delta} \right)^2 \right) \quad \text{(Poisson medium)}. \tag{39}$$

A comparison with expression (23) for a point force shows that the ratio of the isotropic term to the direction-dependent term is, relatively speaking, smaller for the double couple than for the point force. The reason is that for the double couple the angular variation in the radiation pattern is larger than that for a point force.

Equation (38) contains terms that depend on the P-velocity and those that depend on the S-velocity. When the contributions of the P-waves are ignored altogether, the variance of the travel time change is given by

$$\sigma_\tau^2 \approx \frac{\delta^2}{\beta^2} \left(0.190 + 0.285 \left(\hat{\mathbf{M}} \cdot \hat{\delta} \right)^2 \right) \qquad (\alpha \text{ ignored}). \tag{40}$$

Note that this result is close to the variance of the travel time for a Poisson medium given in (39). For a double couple the radiated energy varies as β^{-6} whereas for a point force it varies as β^{-4} . For this reason the dominance of the S-waves in coda wave interferometry is even more pronounced for a double couple than for a point force.

As a special case let us consider the application of this theory to the relative location of aftershocks on a fault. In that case, the relative source location δ lies in the fault plane. In the coordinate system used in expression (34), the x, y-plane is aligned with the fault plane and the component δ_z perpendicular to the fault plane is equal to zero so that

$$\sigma_\tau^2 = \frac{\left(\frac{6}{\alpha^8} + \frac{7}{\beta^8} \right)}{7 \left(\frac{2}{\alpha^6} + \frac{3}{\beta^6} \right)} \delta^2 \qquad \text{(displacement in fault plane)}. \tag{41}$$

Just as in expression (25) the variance of the travel time perturbation is now related to the absolute value of the source displacement only.

5.2 An explosive source

For an explosive source, the moment tensor is given by

$$\mathbf{M} = \begin{pmatrix} M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & M \end{pmatrix} \, . \tag{42}$$

For such a moment tensor $(\hat{\mathbf{r}}\hat{\mathbf{r}} : \mathbf{M})^2 = M^2 \hat{r}_i \hat{r}_j \delta_{ij} = M^2 (\hat{\mathbf{r}} \cdot \hat{\mathbf{r}})^2 = M^2$, and $(\hat{\mathbf{r}} \cdot \mathbf{M})^2 = (\hat{\mathbf{r}} \cdot \hat{\mathbf{r}})^2 M^2 = M^2$, so that (T5) is given by

$$\sigma_\tau^2 = \frac{\int \frac{1}{\alpha^8} (\hat{\mathbf{r}} \cdot \delta)^2 \, d\Omega}{\int \frac{1}{\alpha^6} d\Omega} \, . \tag{43}$$

Note that the terms that depend on the shear velocity β have disappeared; physically this is because an explosive source does not excite shear waves. The angular integration can be carried out in the same way as in section 3, so that

$$\sigma_\tau^2 = \frac{1}{3}\frac{\delta^2}{\alpha^2} \ . \quad (44)$$

This result is identical to expression (17) for acoustic waves.

6 DISCUSSION, APPLICATIONS TO SOURCE RELOCATION

As shown in the previous sections, the decorrelation of the coda waves carries information of the source displacement. The cross-correlation (3) can be applied to a number of non-overlapping time windows in the seismic coda. This gives a number of independent measurements on the source displacement that allow for a consistency check of the method and that make it possible to derive an error estimate.

Let us first consider the character of the constraints on the source displacement that follow from coda wave interferometry. For an explosive source in either an acoustic or elastic medium, expressions (17) and (44) state that the source displacement is located on a sphere with radius $\delta^2/3v^2$, with v the velocity of acoustic waves for an acoustic medium and the P-wave velocity for an elastic medium respectively.

For a point force in an elastic medium the constraint on the source location is slightly more complicated. It is possible to decompose the source displacement into a component parallel to the point force and a component perpendicular to the point force:

$$\delta = \delta_{\perp F} + \delta_{//F}\hat{\mathbf{F}} \ . \quad (45)$$

With this decomposition, equation (22) can be written as

$$\sigma_\tau^2 = \frac{\left(\frac{1}{\alpha^6}+\frac{4}{\beta^6}\right)\delta_{\perp F}^2 + \left(\frac{3}{\alpha^6}+\frac{2}{\beta^6}\right)\delta_{//F}^2}{5\left(\frac{1}{\alpha^4}+\frac{2}{\beta^4}\right)} \ . \quad (46)$$

This expression states the the source displacement is located on an ellipsoid whose symmetry axis is aligned with the point force.

For a double couple the source displacement can be decomposed in a component $\delta_{//fault}$ parallel to the fault and a component $\delta_{\perp fault}$ perpendicular to the fault. Following equation (34), the corresponding constraint on the source displacement is given by

$$\sigma_\tau^2 = \frac{\left(\frac{6}{\alpha^8}+\frac{7}{\beta^8}\right)\delta_{//fault}^2 + 2\left(\frac{1}{\alpha^8}+\frac{2}{\beta^8}\right)\delta_{\perp fault}^2}{7\left(\frac{2}{\alpha^6}+\frac{3}{\beta^6}\right)} \ . \quad (47)$$

This expression states that the source displacement is located on an ellipsoid with a symmetry axis perpendicular to the fault plane. In the special case of aftershocks that occur on the same fault, the source displacement is in general in the plane of the fault. In that case the source displacement is constrained to be located on a circle in the fault plane:

$$\sigma_\tau^2 = \frac{\left(\frac{6}{\alpha^8}+\frac{7}{\beta^8}\right)}{7\left(\frac{2}{\alpha^6}+\frac{3}{\beta^6}\right)}\delta_{//fault}^2 \ . \quad (48)$$

In all these situations the source displacement is constrained by coda wave interferometry to be located on a sphere, an ellipsoid, or a circle. This constraint can be used in addition to constraints on the relative source location as inferred from the differential arrival times for the P- and S-waves. Coda wave interferometry thus adds an additional geometrical constraint to the relative source locations of multiple events.

Suppose one knows the location of n events and that one seeks the relative location of the next event. The location of that event is described by three position variables. Coda wave interferometry gives n constraints on these variables, which implies that with at least three known events one can locate the subsequent events. In practice one would probably seek an iterative approach that employs the differential arrival times of the P- and S-waves as well. Iterative techniques have been developed to find the relative location of events given constraints on the source displacement between different pairs of events (Menke, 1999; Waldhauser & Ellsworth, 2000). Coda wave interferometry provides additional constraints on the relative location of events.

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