The downward continuation approach to modeling and inverse scattering of seismic data in the Kirchhoff approximation

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ABSTRACT
In this paper we use methods from microlocal analysis and the theory of Fourier integral operators, to study the downward continuation approach to seismic inverse scattering in the Kirchhoff approximation. Furthermore, we explain, analyze and connect different notions and processing procedures that appear in seismic imaging-inversion. These are 'downward continuation' with the 'double-square-root equation', 'controlled illumination', 'common-focus-point technology', (wave-equation) 'angle transform', and the 'Bremmer coupling series'.

Key words: seismic inversion, Kirchhoff approximation, downward continuation, microlocal analysis

1 INTRODUCTION

In reflection seismology one places point sources and point receivers on the earth's surface. The source generates acoustic waves in the subsurface, that are reflected where the medium properties vary discontinuously. The recorded reflections that can be observed in the data are used to reconstruct these discontinuities. This reconstruction is called seismic imaging-inversion.

The *first* key objective of this paper is to explain, analyze and connect different notions and processing procedures that occur in seismic imaging-inversion, in the framework of microlocal analysis. These are 'downward continuation' (Clay, 1978; Claerbout, 1985; Popvici, 1996), 'controlled illumination' (Rietveld, Berkhout and Wapenaar, 1992), 'common-focus-point technology' (Thorbecke, 1997), '(wave-equation) angle transform' (De Bruin et al. 1990; De Haas, 1992; Prucha, Biondi and Symes, 1999; Fomel and Prucha, 1999), 'Bremmer coupling series' (Mendel et al. 1981; De Hoop, 1996), and 'Kirchhoff approximation' (Bleistein et al. 2001). From a mathematical perspective this paper is a follow-up and an application of the work by Stolk and De Hoop (2001) on the downward continuation approach to seismic inverse scattering in the Born approximation.

Seismic data are commonly modeled by a high-frequency single scattering approximation. In what follows, we distinguish the vertical coordinate $z \in \mathbb{R}$ from the horizontal coordinates $x \in \mathbb{R}^{n-1}$ and write $(x,z) \in \mathbb{R}^n$. In these coordinates the scalar acoustic wave equation is given by

$$Pu = f\ , \qquad P = c(x,z)^{-2}\frac{\partial}{\partial t}^{2} + \sum_{j=1}^{n-1} D_{x_j}^2 + D_z^2\ , \quad (1)$$

where $D_x = -\mathrm{i}\frac{\partial}{\partial x}, D_z = -\mathrm{i}\frac{\partial}{\partial z}$. The equation is considered for $(x,z) \in \mathbb{R}^n$, and t in an open time interval $(0,T)$.

By Duhamel's principle, a causal solution operator for the inhomogeneous equation (1) is given by

$$u(x,z,t) = \int_0^t \int G(x,z,t-t_0,x_0,z_0) f(x_0,z_0,t_0)\,\mathrm{d}x_0\mathrm{d}z_0\mathrm{d}t_0, \quad (2)$$

where (when the coefficient c is in C^∞) G defines a Fourier integral operator with canonical relation that is essentially a union of bicharacteristics. The source f is a distribution in $\mathcal{E}'(X_0 \times (0,T))$ where X_0 is a bounded open subset of $\mathbb{R}^n$. The kernel of the Fourier integral operator can be written as a sum of contributions

$$G(x,z,t,x_0,z_0) = \sum_{j\in J} \int_{\mathbb{R}^{N(i)}} a^{(j)}(x,z,t,x_0,z_0,\theta) \exp[\mathrm{i}\phi^{(j)}(x,z,x_0,z_0,t,\theta)]\,\mathrm{d}\theta, \quad (3)$$

where the $\phi^{(j)}$ are non-degenerate phase functions and the $a^{(j)}$ are suitable symbols, see section 4.2.1 and chapter 5 in Duistermaat (1996); J is a finite set. Away from endpoint caustics, the only phase variable is frequency, $\theta = \tau$, and the phase function takes the form $\phi^{(m)} = \tau(T^{(m)}(x,z,x_0,z_0) - t)$, where $T^{(m)}$ denotes traveltime along a ray connecting (x, z) with (x_0, z_0) on branch m.

Scattering from singularities in the subsurface is modeled in the high-frequency, single-scattering approximation: The coefficient c is written as the superposition of a perturbation δc and a background c_0, the latter being smooth. The Born approximation to the scattered waves defines a linear forward scattering map that models (part of) the data from δc.

The reconstruction of the perturbation δc given the background c_0 is essentially done by applying the adjoint of the above mentioned forward scattering map. Applying the adjoint coincides with the process of seismic imaging. The modeling operator is a Fourier integral operator (Rakesh, 1988) (for general references of Fourier integral operators (Treves, 1980; Hörmander, 1983a; Hörmander, 1985a; Hörmander, 1985b; Duistermaat, 1996)). If the composition of adjoint and modeling operators, the normal operator, is pseudodifferential, then the positions of the singularities of the perturbation are recovered by applying the adjoint to the data. By computing explicitly the symbol of the normal operator, and applying its inverse (or by adding suitable factors to the expression for the adjoint operator) a microlocal reconstruction can be carried out. Under various assumptions on the background, concerning the presence of caustics and the geometry of the rays, and various acquisition geometries, results concerning the normal operator have been obtained (Beylkin, 1985; Hansen, 1991; Nolan and Symes, 1997; ten Kroode, 1998; Stolk, 2000).

An alternative approach to the Born approximation is the Kirchhoff approximation. This approximation applies to configurations containing smooth interfaces and describes the single scattered wave field. The Kirchhoff approximation ensures, asymptotically, that the wavefield boundary conditions at the interfaces are satisfied. The development of inverse scattering in the Kirchhoff approximation follows the one of the Born approximation closely (Hansen, 1991; Bleistein et al. 2001). In the Kirchhoff approximation the reflection coefficient associated with an interface is mapped to the data rather than the perturbation δc. Basically, in the case of reflecting interfaces, the Born approximation approaches the Kirchhoff approximation for narrow scattering angles and 'small' perturbations.

The *second* key objective of this paper is to develop the downward continuation approach (Clay, 1978; Claerbout, 1985; Popvici, 1996) to inverse scattering in the Kirchhoff approximation. This approach has been developed in the Born approximation by Stolk and De Hoop (2001) and analyzed in the context of semi-group theory and applied to seismic data by De Hoop *et al.* (2003) and Le Rousseau *et al.* (2003). The results of Stolk and De Hoop are applied here. In the downward continuation approach, the data are downward continued, leading to data from fictitious experiments below the surface at varying depths.

The outline of this paper is as follows. We first introduce the pseudodifferential symbol representation of the reflection coefficient in the Kirchhoff approximation for single scattered waves (Section 2). Using a one-way wave equation (Section 3) for the propagation of waves in the background, the downward continuation is described by the so-called double-square-root (DSR) equation. In Section 4 the solution operator (which is closely connected to the 'double focusing procedure') of the DSR equation is exploited to develop the modeling with the Kirchhoff approximation in the downward continuation approach. The results hold, essentially, in the case that the rays in the background that are associated with the reflections are nowhere horizontal. In Section 5 we derive a microlocal reconstruction equation for the reflection coefficient modeled before as the principal symbol of a pseudodifferential operator. The reconstruction is based on a combination of downward continuation with what seismologists call beamforming; this combination defines the wave-equation angle transform. The kernel of the reflection operator is precisely the distribution that appears in the generalized Bremmer series expansion for multiple scattered waves coupling up to downgoing wave constituents. It is only in the framework of the Bremmer series that the Kirchhoff approximation admits a downward continuation approach to inverse scattering.

The main results are representations (16) and (66)-(67) for the modeling, and Theorem 5.5 for the reconstruction.

2 HIGH-FREQUENCY MODELING IN THE SINGLE SCATTERING APPROXIMATION

In this section, we summarize some results in the literature about the seismic modeling map in the single scattering approximation given a background velocity model in C^∞. We denote this map by F.

The solution operator (2) propagates singularities along bicharacteristics. Denote by $p(z, x, \zeta, \xi, \tau) = -c(z, x)^{-2}\tau^2 + \zeta^2 + \|\xi\|^2$ the principal symbol of P. Propagating singularities are in the characteristic set, given by the points $(z, x, t, \zeta, \xi, \tau) \in T^*\mathbb{R}^{n+1}$ satisfying

$$p(z, x, \zeta, \xi, \tau) = -c(z, x)^{-2}\tau^2 + \zeta^2 + \|\xi\|^2 = 0. \quad (4)$$

The bicharacteristics are the solution curves of a Hamilton system with Hamiltonian given by p,

$$\frac{\mathrm{d}(z, x, t)}{\mathrm{d}\lambda} = \frac{\partial p}{\partial(\zeta, \xi, \tau)}, \quad \frac{\mathrm{d}(\zeta, \xi, \tau)}{\mathrm{d}\lambda} = -\frac{\partial p}{\partial(z, x, t)}. \quad (5)$$

They will be parameterized by initial position (z_0, x_0), take-off direction $\alpha \in S^{n-1}$, frequency τ, which together define the initial cotangent vector $(\zeta_0, \xi_0) = -\tau c(z_0, x_0)^{-1}\alpha$, and time t, and are denoted as

$$\eta(t, z_0, x_0, \alpha, \tau) = (\eta_z(t, z_0, x_0, \alpha, \tau), \eta_x(t, z_0, x_0, \alpha, \tau), t, \eta_\zeta(t, z_0, x_0, \alpha, \tau), \eta_\xi(t, z_0, x_0, \alpha, \tau), \tau); \quad (6)$$

τ is invariant along the Hamilton flow. The evolution parameter is the time t.

BORN APPROXIMATION

First, to develop the appropriate mathematical framework, we adopt the linearized scattering or Born approximation. The linearization is in the wavespeed c around a smooth (C^∞) background c_0, $c = c_0 + \delta c$. The perturbation δc may contain singularities. The perturbation in G is given by (see e.g. Beylkin (1985))

$$\delta G(z_r,r,t,z_s,s) = \int_{\mathbb{R}_+\times\mathbb{R}^{n-1}} \int_0^t G(z_r,r,t-t_0,z_0,x_0)\, 2c_0^{-3}(z_0,x_0)\delta c(z_0,x_0) \times \partial_{t_0}^2 G(z_0,x_0,t_0,z_s,s)\, \mathrm{d}t_0\mathrm{d}z_0\mathrm{d}x_0, \quad (7)$$

where both $s, r \in \mathbb{R}^{n-1}$. The Born modeling map F is then defined through (7) as the map from δc to δG evaluated at the *acquisition surface*, here $z_r = z_s = z = 0$. We assume that the acquisition manifold Y, which contains the set of points (s,r,t) used in the acquisition, is a bounded open subset of $\mathbb{R}^{2n-2} \times \mathbb{R}_+$. Since Y is bounded and the waves propagate with finite speed we may assume that δc is supported in a bounded open subset X of $\mathbb{R}_+ \times \mathbb{R}^{n-1}$. Furthermore, we assume that $\overline{X} \cap \{z=0\} = \emptyset$.

To ensure that δG defines a continuous map from $\mathcal{E}'(X)$ to $\mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n \times (0,T))$, and that the restriction of δG to Y is a Fourier integral, operator we make the following assumption on c_0

Assumption 1. There are no rays from $(0,s)$ to $(0,r)$ with travel time t such that $(s,r,t) \in Y$. For all ray pairs connecting $(0,r)$ via some $(z,x) \in X$ to $(0,s)$ with total time t such that $(s,r,t) \in Y$, the rays intersect the plane $z=0$ transversally at r and s.

We also assume that rays from such a point $(z,x) \in X$ intersect the surface $z=0$ only once, because all reflections must come from the subsurface $z>0$. The first part of the assumption excludes rays that scatter over π; the second part of the assumption excludes rays grazing the plane $z=0$. Concerning the second part, strictly only caustics grazing the plane $z=0$ have to be excluded. We have

Theorem 2.1. (Rakesh, 1988; ten Kroode, 1998) With Assumption 1 the map F is a Fourier integral operator $\mathcal{D}'(X) \to \mathcal{D}'(Y)$ with canonical relation

$$\begin{aligned}\big\{(\eta_x(t_\mathrm{s},z,x,\beta,\tau), \eta_x(t_\mathrm{r},z,x,\alpha,\tau), t_\mathrm{s}+t_\mathrm{r}, \eta_\xi(t_\mathrm{s},z,x,\beta,\tau), \eta_\xi(t_\mathrm{r},z,x,\alpha,\tau), \tau; z,x,\zeta,\xi) \mid \\ t_\mathrm{s}, t_\mathrm{r} > 0,\ \eta_z(t_\mathrm{s},z,x,\beta,\tau) = \eta_z(t_\mathrm{r},z,x,\alpha,\tau) = 0,\ (\zeta,\xi) = -\tau c_0(z,x)^{-1}(\alpha+\beta), \\ (z,x,\alpha,\beta,\tau) \in \text{ subset of } X \times (S^{n-1})^2 \times \mathbb{R}\backslash 0\big\} \subset T^*\mathbb{R}^{2n-1}_{(s,r,t)} \times T^*\mathbb{R}^n_{(z,x)}. \quad (8)\end{aligned}$$

Assumption 1 is microlocal. One can identify the set of points $(s,r,t,\sigma,\rho,\tau) \in T^*Y\backslash 0$ where this assumption is violated. If the symbol $\psi = \psi(s,r,t,\sigma,\rho,\tau)$ vanishes on a neighborhood of this set, then the composition ψF of the pseudodifferential cutoff $\psi = \psi(s,r,t,D_s,D_r,D_t)$ with F is a Fourier integral operator as in the theorem.

Kirchhoff approximation

We make use of the above insights in the development, here, of the theory for seismic modeling and inverse scattering in the downward continuation approach (Claerbout, 1985) rather than the reverse time approach, and in the Kirchhoff approximation (Bleistein et al. 2001) rather than in the Born approximation. We discuss the Kirchhoff approximation in this section and integrate it in the downward continuation approach to modeling seismic data in the next two sections.

Typically, in the Kirchhoff approximation we assume δc to be a conormal distribution representative of interfaces reflecting waves off sedimentary layers, faults and so on. Let

$$\kappa:\ (z,x) \mapsto (\underbrace{z'(z,x)}_{\kappa_1(z,x)}, \underbrace{x'(z,x)}_{\kappa_2(z,x)}), \quad (9)$$

be a coordinate transformation such that a reflecting interface, Σ say, is given by the zero level set, $z'(z,x)=0$; thus (z',x') are 'interface normal' coordinates. The acoustic Kirchhoff approximation (Bleistein et al. 2001, E.6.13, E.8.17, 5.1.45-5.1.46), (Stolk and De Hoop, 2002, Thm. 3.6.1) can be written in the form

$$\begin{aligned}\delta G(z_r,r,t,z_s,s) = \int_0^t \int G(z_r,r,t-t_0,z(z',x'),x(z',x'))\, 2\partial_{t_0} R(z',x',D_{x'},D_{t_0}) \\ G(z(z',x'),x(z',x'),t_0,z_s,s) \underbrace{\left|\det \frac{\partial(z,x)}{\partial(z',x')}\right| \left\|\frac{\partial z'}{\partial(z,x)}\right\| \delta(z')\mathrm{d}z'\mathrm{d}x'}_{\left\|\frac{\partial z'}{\partial(z,x)}\right\| \delta(z'(z,x))\,\mathrm{d}z\mathrm{d}x} \mathrm{d}t_0. \quad (10)\end{aligned}$$

Here, we assume the presence of a single interface but the extension to multiple interfaces is straightforward. In expression (10), R is a pseudodifferential operator. Its principal symbol, $r = r(z',x',\xi',\tau)$ say, is given by the product of the reflection coefficient

$$r_0(0,x',\xi',\tau) = \frac{\sqrt{\frac{1}{c_1^2} - \tau^{-2}\|\xi'\|^2} - \sqrt{\frac{1}{c_2^2} - \tau^{-2}\|\xi'\|^2}}{\sqrt{\frac{1}{c_1^2} - \tau^{-2}\|\xi'\|^2} + \sqrt{\frac{1}{c_2^2} - \tau^{-2}\|\xi'\|^2}}, \quad (11)$$

where $c_1 = c_1(z(-0,x'),x(-0,x'))$ coincides with c_0 above the interface while $c_2 = c_2(z(+0,x'),x(+0,x'))$ represents the velocity below the interface, and a normalizing factor

$$\left\|\frac{\partial z'}{\partial(z,x)}\right\|^{-1}\left\langle\frac{\partial z'}{\partial(z,x)},\frac{\partial}{\partial(z,x)}t_{\rm s}\right\rangle = \sqrt{\frac{1}{c_1(z(z',x'),x(z',x'))^2}-\tau^{-2}\|\xi'\|^2}.$$

The Kirchhoff modeling map F is now defined through (10) as the map

$$F\,:\,\left\|\frac{\partial z'}{\partial(z,x)}\right\|\delta(z'(z,x))\mapsto\delta G(0,r,t,0,s).$$

Assumption 2. There are no rays tangent to the interface at $z'(z,x)=0$ microlocally at (x',t,ξ',τ) in the canonical relation of F.

This assumption basically requires precritical angle reflection at the interface and the absence (or removal) of head waves. Under Assumption 2 the map F defines a Fourier integral operator as in Theorem 2.1 (but of different order).

Reflection operator kernel in Cartesian coordinates: Extension and structure

The integrand of expression (10) can be written entirely in the original coordinates (z,x) on $\mathbb{R}^n$. This is accomplished by subjecting the kernel of reflection operator $R = R(z',x',D_{x'},D_{t_0})$ to a change of coordinates according to mapping κ (cf. (9)), which is a diffeomorphism. This, however, requires an extension of the operator, and its symbol $r = r(z',x',\xi',\tau)$ to full phase space $T^*\mathbb{R}^n_{(z',x')}\backslash 0$.

Let $\varphi = \varphi(\zeta',\xi')$ be a cutoff in $S^0(\{0\}\times(\mathbb{R}\times\mathbb{R}^{n-1}))$ that is 0 for $|\zeta'|\geq\frac{1}{\epsilon}\|\xi'\|$ and $|\zeta'|>\frac{1}{\epsilon}$, and that is 1 for $|\zeta'|<\frac{1}{\epsilon}\|\xi'\|$ for some small $\epsilon>0$. Applying Thm. 18.1.35 in Hörmander(1985a) it then follows that

$$\widetilde{r} := \varphi\, r$$

is a symbol in all variables of $T^*\mathbb{R}^n_{(z',x')}\backslash 0$ and τ. (We observe that operator R is convolutional in time.)

Applying the coordinate transformation to pseudodifferential operator $\widetilde{R}$ with principal symbol $\widetilde{r}$ yields the operator $R_1 = R_1(z,x,D_z,D_x,D_{t_0})$ with kernel,

$$R_1(z,x,t_0-\bar{t}_0,\bar{z},\bar{x}) = (2\pi)^{-(n+1)}\int r_1(z,x,\zeta,\xi,\tau)$$
$$\times\exp[\mathrm{i}((z-\bar{z})\zeta+\langle(x-\bar{x}),\xi\rangle+(t_0-\bar{t}_0)\tau)]\,|\det\left(\frac{\partial(z',x')}{\partial(z,x)}\right)^{-1}|\,\mathrm{d}(\zeta,\xi,\tau),$$

and with symbol

$$r_1(z,x,\zeta,\xi,\tau) = \widetilde{r}(z'(z,x),x'(z,x),\left(\left(\frac{\partial(z',x')}{\partial(z,x)}\right)^{-1}\right)^t(\zeta,\xi),\tau)+\text{l.o.t.} \tag{12}$$

The general expression, up to any order, for this transformation can be found, for example, in Shubin (1987(4.24)).

Equation (10) attains the form

$$\delta G(0,r,t,0,s) = \int_{\mathbb{R}^{n-1}\times\mathbb{R}_+}\int_{-\infty}^{t}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} G(0,r,t-t_0,z,x)\times 2\partial_{t_0}R_1(z,x,t_0-\bar{t}_0,\bar{z},\bar{x})$$
$$\times\, G(\bar{z},\bar{x},\bar{t}_0,0,s)\left\|\frac{\partial z'}{\partial(z,x)}\right\|\delta(z'(z,x))\,\mathrm{d}\bar{t}_0\mathrm{d}\bar{z}\,\mathrm{d}\bar{x}\,\mathrm{d}t_0\,\mathrm{d}x\mathrm{d}z. \tag{13}$$

Remark 2.2. Away from the diagonal in $\mathbb{R}^n_{(z,x)}\times\mathbb{R}^n_{(z,x)}$ the kernel $R_1 = R_1(z,x,t_0-\bar{t}_0,\bar{z},\bar{x})$ induces a regularizing contribution. Hence the integrations over (z,x) and $(\bar{z},\bar{x})$ can be restricted to a bounded open subset X of $\mathbb{R}_+\times\mathbb{R}^{n-1}$ as in the Born approximation.

In preparation of the downward/upward continuation approach to seismic data modeling, we rewrite this equation in the form

$$\delta G(0,r,t,0,s) = \int_{\mathbb{R}^{n-1}\times\mathbb{R}_+}\int_{-\infty}^{t}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} G(0,r,t-t_0,z,x)\times 2\partial_{t_0}R_1(z,x,t_0-\bar{t}_0,\bar{z},\bar{x})\left\|\frac{\partial z'}{\partial(z,x)}\right\|\delta(z'(z,x))$$
$$\times\, G(\bar{z},\bar{x},\bar{t}_0,0,s)\,\mathrm{d}\bar{t}_0\mathrm{d}\bar{z}\,\mathrm{d}\bar{x}\,\mathrm{d}t_0\,\mathrm{d}x\mathrm{d}z. \tag{14}$$

Changing variables of integration, i.e. $t_0 \mapsto t_0' = t_0 - \bar{t}_0$, (14) can be written in the form of an integral operator acting on a distribution,

$$\delta G(0,r,t,0,s) = \int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\left\{\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}}\left(\int_{\mathbb{R}_+} G(0,r,t-t_0'-\bar{t}_0,z,x)\times G(\bar{z},\bar{x},\bar{t}_0,0,s)\,\mathrm{d}\bar{t}_0\right)\right.$$
$$\left.\times\, 2\partial_{t_0} R_1(z,x,t_0',\bar{z},\bar{x})\left\|\frac{\partial z'}{\partial(z,x)}\right\|\delta(z'(z,x))\,\mathrm{d}\bar{x}\,\mathrm{d}x\,\mathrm{d}t_0'\right\}\mathrm{d}\bar{z}\mathrm{d}z. \quad (15)$$

Using the reciprocity relation of the time-convolution type for the Green's function, we arrive at the integral representation

$$\delta G(0,r,t,0,s) = 2\partial_t \int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\left\{\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}}\left(\int_0^{t-t_0} G(0,r,t-t_0-\bar{t}_0,z,x)\times G(0,s,\bar{t}_0,\bar{z},\bar{x})\,\mathrm{d}\bar{t}_0\right)\right.$$
$$\left.\times\, R_1(z,x,t_0,\bar{z},\bar{x})\left\|\frac{\partial z'}{\partial(z,x)}\right\|\delta(z'(z,x))\,\mathrm{d}\bar{x}\,\mathrm{d}x\,\mathrm{d}t_0\right\}\mathrm{d}\bar{z}\mathrm{d}z. \quad (16)$$

The associated operator kernel appears to propagate singularities from two different scattering points, $(\bar{z},\bar{x})$ and (z,x) to the surface at $z=0$.

Microlocally, the extension $\widetilde{R}$ with kernel $\widetilde{R}(z',x',t_0-\bar{t}_0,\bar{z}',\bar{x}')$ of the reflection operator R with kernel $R = R(z',x',t_0-\bar{t}_0,\bar{x}')$ can be thought of as

$$\widetilde{R}(z',x',t_0-\bar{t}_0,\bar{z}',\bar{x}') \sim (2\pi)^{-(n+1)}\int r(z',x',\xi',\tau)\times\exp[\mathrm{i}((z'-\bar{z}')\zeta' + \langle(x'-\bar{x}'),\xi'\rangle + (t_0-\bar{t}_0)\tau)]\,\mathrm{d}(\zeta',\xi',\tau)$$
$$= \delta(z'-\bar{z}')R(z',x',t_0-\bar{t}_0,\bar{x}'). \quad (17)$$

This implies that

$$\widetilde{R}(z',x',t_0-\bar{t}_0,\bar{z}',\bar{x}')\,\delta(z') \sim R(0,x',t_0-\bar{t}_0,\bar{x}')\,\delta(z')\delta(\bar{z}'),$$

where the product of δ's is to be interpreted as a tensor product. Subject to the assumption that $\frac{\partial\kappa_1}{\partial z} \neq 0$ (cf. (9)), we now write the solution to $z'(z,x)=0$ as $z = \boldsymbol{z}(x)$. Then

$$\delta(\kappa_1(z,x)) = \left|\frac{\partial\kappa_1}{\partial z}\right|^{-1}_{z=\boldsymbol{z}(x)}\delta(z-\boldsymbol{z}(x)),$$

and similarly for $\delta(\kappa_1(\bar{z},\bar{x}))$.

Substituting the change of coordinates (9), yields

$$R_1(z,x,t_0-\bar{t}_0,\bar{z},\bar{x})\delta(z'(z,x))$$
$$\sim (2\pi)^{-n}\int r(0,\kappa_2(z,x),\xi',\tau)\exp[\mathrm{i}(\langle\kappa_2(z,x)-\kappa_2(\bar{z},\bar{x}),\xi'\rangle + (t_0-\bar{t}_0)\tau)]\,\mathrm{d}(\xi',\tau)$$
$$\times\,(2\pi)^{-1}\int\exp[\mathrm{i}(-\kappa_1(\bar{z},\bar{x}))\zeta']\,\mathrm{d}\zeta'\;\delta(\kappa_1(z,x))$$
$$= (2\pi)^{-n}\int r(0,\kappa_2(\boldsymbol{z}(x),x),\xi',\tau)\left|\frac{\partial\kappa_1}{\partial z}\right|^{-1}_{z=\boldsymbol{z}(x)}\left|\frac{\partial\kappa_1}{\partial z}\right|^{-1}_{z=\boldsymbol{z}(\bar{x})}$$
$$\times\exp[\mathrm{i}(\langle\kappa_2(\boldsymbol{z}(x),x)-\kappa_2(\boldsymbol{z}(\bar{x}),\bar{x}),\xi'\rangle + (t_0-\bar{t}_0)\tau)]\,\mathrm{d}(\xi',\tau)\;\delta(z-\boldsymbol{z}(x))\delta(\bar{z}-\boldsymbol{z}(\bar{x})). \quad (18)$$

Let us analyze the contribution to the phase function,

$$\langle\underbrace{\kappa_2(\boldsymbol{z}(x),x)-\kappa_2(\boldsymbol{z}(\bar{x}),\bar{x})}_{\Phi(x,\bar{x})},\xi'\rangle,$$

that has the property that

$$\forall_{\xi'\neq 0}\,\exists_{k\in\{1,\ldots,n-1\}}\;\left\langle\left.\frac{\partial\Phi}{\partial x_k}\right|_{x=\bar{x}},\xi'\right\rangle \neq 0,$$

where

$$\frac{\partial\Phi}{\partial x_k} = \frac{\partial\kappa_2}{\partial x_k} + \frac{\partial\kappa_2}{\partial z}\frac{\partial\boldsymbol{z}}{\partial x_k}, \quad \frac{\partial\boldsymbol{z}}{\partial x_k} = -\left(\frac{\partial\kappa_1}{\partial z}\right)^{-1}\frac{\partial\kappa_1}{\partial x_k}.$$

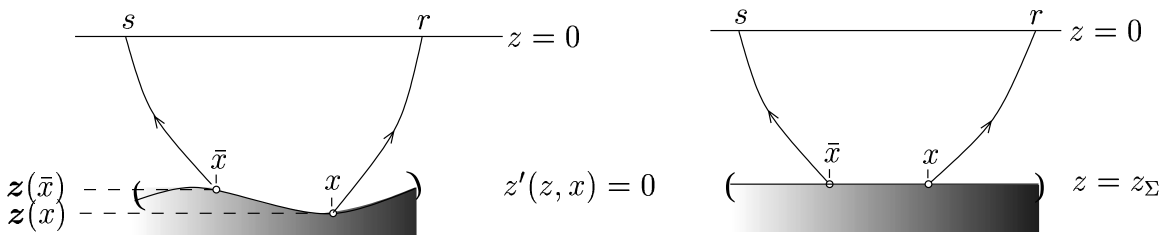

Figure 1. Action of the reflection operator.

We, now, introduce the non-singular $(n-1)\times(n-1)$ matrix ϕ,

$$\phi_{kj}(x) = \left.\frac{\partial \Phi_j}{\partial x_k}\right|_{x=\bar{x}}. \tag{19}$$

Then

$$\langle \kappa_2(\boldsymbol{z}(x), x) - \kappa_2(\boldsymbol{z}(\bar{x}), \bar{x}), \xi'\rangle = \langle x - \bar{x}, \xi\rangle + \cdots, \quad \xi = \phi(x)\xi'.$$

Applying Theorem 4.1 in Shubin (1987), it follows that

$$R_1(z, x, t_0 - \bar{t}_0, \bar{z}, \bar{x})\delta(z'(z,x)) \sim \widehat{R}_1(z, x, t_0 - \bar{t}_0, \bar{z}, \bar{x})\delta(z - \boldsymbol{z}(x))\delta(\bar{z} - \boldsymbol{z}(\bar{x})), \tag{20}$$

with

$$\widehat{R}_1(\boldsymbol{z}(x), x, t_0 - \bar{t}_0, \boldsymbol{z}(\bar{x}), \bar{x}) = (2\pi)^{-n}\int r(0, \kappa_2(\boldsymbol{z}(x), x), \phi(x)^{-1}\xi, \tau)\left|\frac{\partial \kappa_1}{\partial z}\right|^{-1}_{z=\boldsymbol{z}(x)}\left|\frac{\partial \kappa_1}{\partial z}\right|^{-1}_{z=\boldsymbol{z}(\bar{x})} \times \exp[\mathrm{i}(\langle x - \bar{x}, \xi\rangle + (t_0 - \bar{t}_0)\tau)]\,|\det \phi(x)^{-1}|\,\mathrm{d}(\xi, \tau); \tag{21}$$

see Figure 1 for an illustration of the kernel action.

Remark 2.3. *Schwartz reflection kernel* for the Born approximation. We start from (7), and introduce the distribution $\widetilde{\delta c}$ derived from δc via pull back with κ,
$\delta c(z, x) = \widetilde{\delta c}(\kappa(z, x)) = \widetilde{\delta c}(z'(z, x), x'(z, x))$. Then

$$\frac{\partial}{\partial(z, x)}\,\widetilde{\delta c} = c'\,\frac{\partial z'}{\partial(z, x)} + \text{l.o.t.}, \quad c' = \frac{\partial}{\partial z'}\,\widetilde{\delta c}.$$

Assuming a jump discontinuity across the interface, c' contains a factor $\delta(z'(z, x))$. Up to leading order, in the Born approximation, $\widetilde{\delta c}\;\partial^2_{t_0}$ in (7) gets replaced by

$$c'\left\|\frac{\partial z'}{\partial(z, x)}\right\|\,\partial_{t_0}$$

upon carrying out integration by parts. Hence, we identify

$$R_1(z, x, t_0 - \bar{t}_0, \bar{z}, \bar{x})\delta(z'(z, x)) = \delta(z - \bar{z})\;\delta(t_0 - \bar{t}_0)\delta(x - \bar{x})\left(\frac{c'}{c_0^3}\right)\left(\frac{\bar{z} + z}{2}, \frac{\bar{x} + x}{2}\right). \tag{22}$$

This kernel is diagonal.

Flat, horizontal interface

For a horizontal interface, $x' = x$, $z' = z'(z) = z - z_\Sigma$ and the *Schwartz reflection kernel* reduces to (cf. (17))

$$R_1(z, x, t_0 - \bar{t}_0, \bar{z}, \bar{x}) = \delta(z - \bar{z})R(z, x, t_0 - \bar{t}_0, \bar{x}).$$

Thus, (16) reduces to

$$\delta G(0,r,t,0,s) = 2\partial_t \int_{\mathbb{R}_+} \left\{ \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \left(\int_0^{t-t_0} G(0,r,t-t_0-\bar{t}_0,z,x) \times G(0,s,\bar{t}_0,z,\bar{x})\,\mathrm{d}\bar{t}_0 \right) \times R(z,x,t_0-\bar{t}_0,\bar{x})\delta(z-z_\Sigma)\,\mathrm{d}\bar{x}\,\mathrm{d}x\,\mathrm{d}t_0 \right\} \mathrm{d}z \quad (23)$$

if the interface is at depth $z_\Sigma > 0$; see Figure 1 for an illustration of the kernel action.

Remark 2.4. In the flat interface case, we can invoke the Weyl calculus for symbols defined on $T^*\mathbb{R}^n_{(z,x)}$. Let r denote the principal part of the Weyl symbol associated with reflection operator R in (23); r is homogeneous of degree zero in (ξ,τ). Upon substituting $\xi = \tau p$ in $r = r(z,x,\xi,\tau)$, it follows that the dependence on τ drops out: $r = r(z,x,p,1)$. The kernel representation attains the form

$$R(z,x,t_0-\bar{t}_0,\bar{x}) = (2\pi)^{-n} \int \left[r\left(z, \frac{x+\bar{x}}{2}, p, 1\right) + \text{l.o.t.} \right] \times \exp[\mathrm{i}\langle(x-\bar{x}),p\rangle\,\tau] \exp[\mathrm{i}(t_0-\bar{t}_0)\tau]\,|\tau|^{n-1}\mathrm{d}(p,\tau), \quad (24)$$

which can be written as

$$R = \Lambda \mathrm{E} r + \text{l.o.t.}, \quad (25)$$

where Λ is a convolutional operator with symbol $|\tau|^{n-1}$, and E is the transform defined by

$$(\mathrm{E}r)(z,x,t,\bar{x}) = (2\pi)^{-(n-1)} \int \delta(t - \langle(x-\bar{x}),p\rangle)\, r\left(z, \frac{x+\bar{x}}{2}, p, 1\right)\,\mathrm{d}p. \quad (26)$$

In (24) or (26), inside the p-integral, we observe a separation of *midpoint*, $\frac{x+\bar{x}}{2}$, and *offset*, $x-\bar{x}$, variables. This representation is closely related to Gel'fand's plane-wave expansion. In fact, ΛE maps the Weyl reflection symbol to fictituous reflection data.

3 THE ONE-WAY WAVE EQUATIONS

In this section, we discuss the solution of the wave equation (1) in the background model ($c(z,x) = c_0(z,x)$) by evolution in one of the space variables (wave field extrapolation). This evolution problem is in general not well posed, but the propagation of the singularities of the solution can be obtained microlocally, when the propagation direction of the corresponding rays stays somewhat away from horizontal.

Singularities of solutions to the wave equation, that propagate with non-zero vertical velocity are described by a first order evolution equation in z. This follows from a well known factorization argument, see e.g. Taylor (1975) and is at the basis of the generalized Bremmer coupling series. In Stolk (in press) the approximation of solutions to the wave equation, by solutions to an evolution equation in z is discussed, for the acoustic case. Such an equation is called a one-way wave equation. We summarize the results we need for the upward/downward continuation approach to modeling seismic data.

To determine whether the velocity vector at some point of the ray is close to horizontal we use the angle with the vertical, defined to be in $[0,\pi/2]$ and given by $\tan(\theta) = \frac{\|\xi\|}{|\zeta|}$. Recall that the propagating singularities are microlocally in the characteristic set given by (4). Given a point (z,x,ξ,τ) with $\|\xi\| < c(z,x)^{-1}|\tau|$, there are two solutions ζ to (4), given by $\zeta = \pm b$, where $b = b(z,x,\xi,\tau)$ is defined by

$$b(z,x,\xi,\tau) = -\tau\sqrt{c(z,x)^{-2} - \tau^{-2}\xi^2}; \quad (27)$$

in seismology, b/τ is known as the *vertical slowness*. The sign is chosen such that $\zeta = \pm b$ corresponds to propagation with $\pm\frac{\mathrm{d}\eta_z}{\mathrm{d}t} > 0$. There is an angle (phase angle) associated with (z,x,ξ,τ), given by the solution $\theta \in [0,\pi/2]$ of the equation

$$\sin(\theta) = c(z,x)\|\tau^{-1}\xi\|. \quad (28)$$

When this angle is strictly smaller than $\pi/2$ along a ray segment, then the vertical velocity $\frac{\mathrm{d}\eta_z}{\mathrm{d}t}$ does not change sign, and the ray segment can be parameterized by z rather than time. The maximal z-interval such that $\arcsin(c(z,x)\|\tau^{-1}\xi\|) < \theta$ for given θ along the bicharacteristic (cf. (6)) determined by the initial values $(z,x,\pm b,\xi,\tau)$, with $(\pm b,\xi) = -\tau c(z,x)^{-1}\alpha$, will be denoted by

$$(z_{\min,\pm}(z,x,\xi,\tau,\theta), z_{\max,\pm}(z,x,\xi,\tau,\theta)). \quad (29)$$

We also define a subset

$$I_\theta = \{(z,x,t,\zeta,\xi,\tau) \mid \arcsin(c(z,x)\|\tau^{-1}\xi\|) < \theta, |\tau^{-1}\zeta| < C\} \subset T^*(\mathbb{R}^n \times \mathbb{R}_+), \quad (30)$$

where $c(z,x)^{-1} < C$.

To obtain a system of one-way wave equations, the wave equation is written as a first order system in z

$$\frac{\partial}{\partial z}\begin{pmatrix} u \\ \frac{\partial u}{\partial z}\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -A(z,x,D_x,D_t) & 0\end{pmatrix}\begin{pmatrix} u \\ \frac{\partial u}{\partial z}\end{pmatrix} + \begin{pmatrix} 0 \\ -f\end{pmatrix}, \tag{31}$$

where $D_x = -\mathrm{i}\frac{\partial}{\partial x}$, $D_z = -\mathrm{i}\frac{\partial}{\partial z}$ and $A(z,x,D_x,D_t) = c(z,x)^{-2}D_t^2 - D_x^2$. This system is a direct representation of the conservation of momentum and constitutive equations, retaining the pressure and vertical particle velocity. This system is transformed, by using a family of matrix pseudodifferential operators $\mathsf{Q}(z) = \mathsf{Q}(z,x,D_x,D_t)$ according to

$$\begin{pmatrix} u_+ \\ u_-\end{pmatrix} = \mathsf{Q}(z)\begin{pmatrix} u \\ \frac{\partial u}{\partial z}\end{pmatrix}, \qquad \begin{pmatrix} f_+ \\ f_-\end{pmatrix} = \mathsf{Q}(z)\begin{pmatrix} 0 \\ -f\end{pmatrix}. \tag{32}$$

Let $\theta_2 < \pi/2$ be some given angle. With suitable chosen Q, system (31) transforms to a diagonal form on I_{θ_2}, which is equivalent to two equations of the form

$$\left(\tfrac{\partial}{\partial z} - \mathrm{i}B_\pm(z,x,D_x,D_t)\right)u_\pm = f_\pm, \tag{33}$$

microlocally on I_{θ_2}, the one-way wave equations.

The principal part of the symbol of $B_\pm$ is equal to $\pm b$, while its subprincipal part depends on the normalization of $\mathsf{Q}(z)$. We choose the vertical acoustic power flux normalization, when $B_\pm$ are selfadjoint. Then

$$\mathsf{Q}(z,x,\xi,\tau)^{-1} = \begin{pmatrix} a^{-1/4} & a^{-1/4} \\ \mathrm{i}\,\mathrm{sgn}(\tau)a^{1/4} & -\mathrm{i}\,\mathrm{sgn}(\tau)a^{1/4}\end{pmatrix} + \text{l.o.t.}, \tag{34}$$

with $a = a(z,x,\xi,\tau) = c(z,x)^{-2}\tau^2 - \xi^2$ denoting the symbol of A. It follows that

$$u = Q_+^* u_+ + Q_-^* u_-, \quad f_\pm = \pm\tfrac{1}{2}\mathrm{i}Q_\pm f, \tag{35}$$

where $Q_\pm = Q_\pm(z) = Q_\pm(z,x,D_x,D_t)$ have principal symbol $a^{-1/4}$ and are the entries of the second column of Q.

In the further analysis, we will restrict to the $-$ sign. The operator B_- and its symbol are not yet prescribed for $\arcsin(c(z,x)\|\tau^{-1}\xi\|) > \theta_2$ on $T^*(\mathbb{R}^n_{(z,x)} \times \mathbb{R}_+)$. Assume first that B_- is a first order family of pseudodifferential operators with real homogeneous principal symbol. Let $G_{0,-}$ be the solution operator to the initial value problem for $P_{0,-} = \frac{\partial}{\partial z} - \mathrm{i}B_-$,

$$P_{0,-}u_- = 0, \quad z < z_0, \quad Q_-^* u_-(z_0,\cdot) = u(z_0,\cdot). \tag{36}$$

This equation admits propagation of singularities also for $\arcsin(c(z,x)\|\tau^{-1}\xi\|) \geq \theta_2$. Let $J_-(z_0,\theta)$ be defined by

$$J_-(z_0,\theta) = \{(z,x,t,\zeta,\xi,\tau) \in I_\theta \mid \tau^{-1}\zeta > 0 \text{ and } z_{\max,-}(z,x,\xi,\tau,\theta) \geq z_0\}. \tag{37}$$

The solutions to (36) are microlocally correct on the set $J_-(z_0,\theta_2)$ in the following way. Suppose that $\mathrm{WF}(u) \cap \{z = z_0, \tau^{-1}\zeta < 0\} = \emptyset$ (at depth z_0 all singularities are propagating in the $-$ direction), and let u_- be a solution to (36), then it follows from the propagation of regularity/propagation of singularities that

$$u = Q_-^* u_-, \tag{38}$$

microlocally on the set $J_-(z_0,\theta_2)$. This set and the angles θ_1, θ_2 in the context of propagation of singularities are illustrated in Figure 2.

Let θ_1 be given with $0 < \theta_1 < \theta_2$. Suppose we have a pseudodifferential cutoff $\psi_1 = \psi_1(z,z_0,x,D_x,D_t)$ with symbol satisfying

$$\psi_1(z,x,\xi,\tau) = 1 \text{ on } J_-(z_0,\theta_1), \tag{39}$$

$$\psi_1(z,x,\xi,\tau) \in S^\infty \text{ outside } J_-(z_0,\theta_2), \text{ if } z - z_0 > \delta > 0. \tag{40}$$

Then we have

$$\psi_1 u = \psi_1 Q_- u_-. \tag{41}$$

For the suppression of singularities outside J_-, seismologists have added a dissipative term (local 'dip filter'), C say, to $P_{0,-}$,

$$P_- = \frac{\partial}{\partial z} - \mathrm{i}B_\pm(z,x,D_x,D_t) - C(z,x,D_x,D_t), \tag{42}$$

with C of first order with homogeneous, non-negative real principal symbol. It was shown in Stolk (in press) that with C suitably chosen the solution operator to the initial value problem

$$P_- u_- = 0, \quad z < z_0, \quad u_-(z_0,\cdot) = v. \tag{43}$$

is of the form

$$G_-(z,z_0) = \psi_1(z,z_0)G_{0,-}(z,z_0). \tag{44}$$

[h]

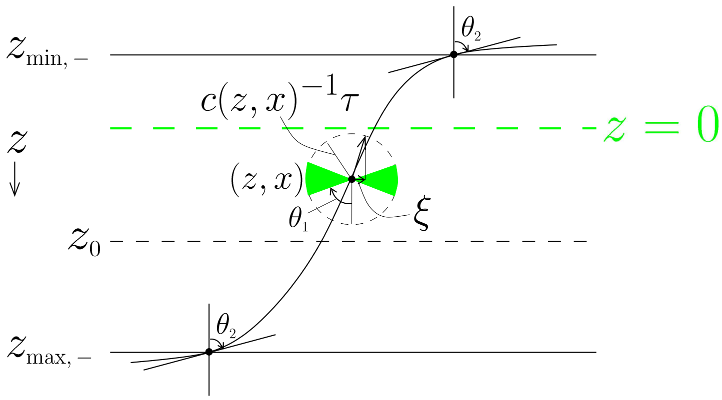

Figure 2. Illustration of θ_1, θ_2, the ψ_1 operator, and propagation of singularities.

The solution operator can also be written as a pseudodifferential cutoff, $\psi_1(z_0, z)$, applied prior to $G_{0,-}$ ⋆,

$$G_-(z, z_0) = G_{0,-}(z, z_0)\psi_1(z_0, z). \tag{45}$$

We use the notation $\gamma(z, z_0, x_0, t_0, \xi_0, \tau)$ for the bicharacteristics of $P_{0,-}$ parameterized by z. In components, we write

$$\gamma(z, z_0, x_0, t_0, \xi_0, \tau) \quad = \quad (z, \gamma_x(z, z_0, x_0, \xi_0, \tau), \gamma_t(z, z_0, x_0, \xi_0, \tau) \;+\; t_0, -b(z, \gamma_x, \gamma_\xi, \tau), \gamma_\xi(z, z_0, x_0, \xi_0, \tau), \tau). \tag{46}$$

Remark 3.1. The operators $B_\pm$ are selfadjoint. It follows that $G_{0,-}(z, z_0)$ is unitary. We have that

$$G_-(z, z_0)^* G_-(z, z_0) = \psi_1(z_0, z)^* \psi_1(z_0, z), \tag{47}$$

and $G_-(z, z_0)^* G_-(z, z_0)$ is one microlocally where $\psi_1(z_0, z)$ is one.

Let the singularities in f be such that $\tau^{-1}\zeta > 0$ (corresponding to propagation direction $\frac{\partial \eta_z}{\partial t} < 0$). Consider u_- defined by (Duhamel's principle)

$$u_-(z, \cdot) = \int_z^\infty G_-(z, z_0)(-\tfrac{1}{2}\mathrm{i}Q_-(z_0))f(z_0, \cdot)\,\mathrm{d}z_0, \tag{48}$$

assuming also that $f = 0$ on a neighborhood of the plane given by z. We have that $Q_-^\star u_-(z, \cdot) \equiv u(z, \cdot)$, where u is the solution to (1) with f replaced by $Q_-^{-1}\psi_1(z_0, z)Q_- f$. The contribution to the original Green's function $G(z, x, t - t_0, z_0, x_0)$ from upgoing propagating singularities thus follows to be (compare (2))

$$-\frac{1}{2}\mathrm{i}Q_-^\star(z, x, D_x, D_t)G_-Q_-(z_0, x_0, D_{x_0}, D_{t_0}). \tag{49}$$

This is precisely the substitution to be made in (16) and (23).

4 MODELING IN THE KIRCHHOFF APPROXIMATION WITH THE DOWNWARD CONTINUATION APPROACH

We show that the Kirchhoff modeling operator can be written, modulo smoothing terms, in terms of solution operators to a one-way wave equation; in case of *horizontal* reflectors, this modeling operator can be written, modulo smoothing terms, in terms of downward/upward data continuation.

⋆The operator $\psi_1(z_0, z)$ is a (z_0, z)-family of pseudodifferential operators with symbol in $S^0_{\rho,1-\rho}(\mathbb{R}^n \times \mathbb{R}^n)$, such that the derivatives $\frac{\partial^{j+k}\psi_1}{\partial z_0^j \partial_z^k}$ are in $S^{(j+k)(1-\rho)}_{\rho,1-\rho}(\mathbb{R}^n \times \mathbb{R}^n)$ for $z \neq z_0$, where ρ can be any number satisfying $\frac{1}{2} < \rho < 1$ ((Stolk, in press; Stolk and De Hoop, 2003a)). For the theory of such operators, see Taylor (1981) and H¨ormander (1985a).

UPWARD/DOWNWARD CONTINUATION

Motivated by (16) and (23) (the convolution in between parentheses), we define an operator $H(z, z_0)$, $z < z_0$ on functions of (s, r, t), by its kernel

$$(H(z, z_0))(s, r, t, s_0, r_0, t_0) = \int_{\mathbb{R}} (G_-(z, z_0))(s, t - t_0 - \bar{t}_0, s_0)(G_-(z, z_0))(r, \bar{t}_0, r_0) \, \mathrm{d}\bar{t}_0. \tag{50}$$

Here $(G_-(z, z_0))(r, \bar{t}_0, r_0, 0)$ denotes the distribution kernel of $G_{-,r}(z, z_0)$, the operator $G_-(z, z_0)$ in the receiver coordinates, while $(G_-(z, z_0))(s, t, s_0, 0)$ denotes the distribution kernel of $G_{-,s}(z, z_0)$, the operator $G_-(z, z_0)$ in the source coordinates; $(H(z, z_0))(s, r, t, s_0, r_0, t_0)$ denotes the distribution kernel of $H(z, z_0)$. The map $H(z, z_0)$, $z < z_0$ is called the upward continuation operator.

If ψ_1 is an operator on functions of (x, t) and is time translation invariant, then $\psi_{1,s}$ and $\psi_{1,r}$ commute. The factors $G_{-,s}$ and $G_{-,r}$ can be written as compositions $\psi_{1,s} G_{0,-,s}$ or $G_{0,-,s} \psi_{1,s}$, and similarly for r, using (44), (45). It follows that the operator H can be written as a composition $\psi_2(z, z_0) H_0(z, z_0)$, where H_0 is given by (50) with G_- replaced by $G_{-,0}$, and $\psi_2(z, z_0) = \psi_{1,s} \psi_{1,r}$. The operator $\psi_2(z, z_0)$ is pseudodifferential with symbol

$$\psi_2(z, z_0, s, r, \sigma, \rho, \tau) = \psi_1(z, z_0, s, \sigma, \tau) \psi_1(z, z_0, r, \rho, \tau). \tag{51}$$

As with G_-, we can also write $H(z, z_0) = H_0(z, z_0) \psi_2(z_0, z)$, with ψ_2 defined by (51) as well.

Let $g(z, s, r, t)$ be supported in the set $0 < \delta < z < Z$. Motivated by (16) and (23) and substitution (49) (g plays the role of $R \, \delta(\cdot - z_\Sigma)$), we define an operator L as follows,

$$Lg = Q^*_{-,s}(0) Q^*_{-,r}(0) \int_0^Z (H(0, z) Q_{-,s}(z) Q_{-,r}(z) g(z, \cdot))(s, r, t) \, \mathrm{d}z. \tag{52}$$

In seismic applications, one encounters also the operator $\bar{L}$, that follows from L upon omitting the (de)composition operators $Q^*_{-,s}$, $Q^*_{-,r}$ and $Q_{-,s}$, $Q_{-,r}$,

$$\bar{L} g = \int_0^Z (H(0, z) g(z, \cdot))(s, r, t) \, \mathrm{d}z.$$

In Stolk and de Hoop (2003a) it is shown that $H(z, z_0)$ and L are Fourier integral operators. They also give a representation of the kernel of $H(z, z_0)$ as an oscillatory integral that we will use below. The operator $H(z, z_0)$ propagates singularities along bicharacteristics, in the notation of (46), given by

$$\begin{aligned} \Gamma(z, z_0; s_0, r_0, t_0, \sigma_0, \rho_0, \tau) = (&\gamma_x(z, z_0, s_0, \sigma_0, \tau), \gamma_x(z, z_0, r_0, \rho_0, \tau), t_0 \\ &+ \gamma_t(z, z_0, s_0, \sigma_0, \tau) + \gamma_t(z, z_0, r_0, \rho_0, \tau), \gamma_\xi(z, z_0, s_0, \sigma_0, \tau), \gamma_\xi(z, z_0, r_0, \rho_0, \tau), \tau). \end{aligned} \tag{53}$$

These are defined on the intersection of the maximal intervals associated with source ray coordinates (z, s, σ, τ) and receiver ray coordinates (z, r, ρ, τ): Let θ be given as in the previous section. The intersection will be denoted by $(Z_{\min}, Z_{\max})$, where

$$Z_{\min} = Z_{\min}(z, s, r, \sigma, \rho, \tau, \theta) = \max\{z_{\min,-}(z, s, \sigma, \tau, \theta), z_{\min,-}(z, r, \rho, \tau, \theta)\} \tag{54}$$

$$Z_{\max} = Z_{\max}(z, s, r, \sigma, \rho, \tau, \theta) = \min\{z_{\max,-}(z, s, \sigma, \tau, \theta), z_{\max,-}(z, r, \rho, \tau, \theta)\}. \tag{55}$$

Now, consider the set,

$$\begin{aligned} \{(\Gamma(0, z, s, r, t, \sigma, \rho, \tau); z, s, r, t, -b(z, s, \sigma, \tau) - b(z, r, \rho, \tau), \sigma, \rho, \tau) \mid \\ (s, r, t, \sigma, \rho, \tau) \in T^* \mathbb{R}^{2n-1}_{(s,r,t)}, Z_{\min}(z, s, r, t, \sigma, \rho, \tau, \theta_2) < 0\}. \end{aligned} \tag{56}$$

This set is a canonical relation in $T^* \mathbb{R}^{2n-1}_{(s,r,t)} \times T^* \mathbb{R}^{2n}_{(z,s,r,t)}$. Let $y_0 = (s_0, r_0, t_0)$, $\eta_0 = (\sigma_0, \rho_0, \tau)$. A convenient choice of phase function for the canonical relation is described in Maslov and Fedoriuk (1981). They state that one can always use a subset of the cotangent vector components as phase variables. There is always a set of local coordinates for the canonical relation of the form,

$$(z, y_{0I}, \eta_{0J}, s, r, t), \tag{57}$$

where $I \cup J$ is a partition of $\{1, \ldots, 2n - 1\}$. It follows from Theorem 4.21 in Maslov and Fedoriuk (1981) that there is a function $S = S(z, y_{0I}, \eta_{0J}, s, r, t)$, such that locally the canonical relation (56) is given by

$$y_{0J} = -\frac{\partial S}{\partial \eta_{0J}}, \qquad \zeta = \frac{\partial S}{\partial z}, \tag{58}$$

$$\eta_{0I} = \frac{\partial S}{\partial y_{0I}}, \qquad (\sigma, \rho, \tau) = -\frac{\partial S}{\partial (s, r, t)}. \tag{59}$$

Lemma 4.1. (Stolk and De Hoop, 2003a) $H(z, z_0)$ is a Fourier integral operator with canonical relation

$$\{(\Gamma(z, z_0, s, r, t, \sigma, \rho, \tau); s, r, t, \sigma, \rho, \tau)\,|(s, r, t, \sigma, \rho, \tau) \in T^*\mathbb{R}^{2n-1}_{(s,r,t)}\backslash 0, Z_{\min}(z, s, r, t, \sigma, \rho, \tau, \theta_2) < 0\}. \tag{60}$$

The operator L is a Fourier integral operator with canonical relation (56). The kernel of $H(0, z)$ admits microlocally an oscillatory integral representation with phase variables η_{0J} given by

$$(H(0, z))(s_0, r_0, t_0, s, r, t) = \\ (2\pi)^{-(2n-1+|I|)/2} \int A(z, y_0, \eta_{0J}, s, r, t) \exp[\mathrm{i}(S(z, y_{0I}, \eta_{0J}, s, r, t) + \langle \eta_{0J}, y_{0J} \rangle)]\, \mathrm{d}\eta_{0J}, \tag{61}$$

such that the principal part a of the amplitude A satisfies

$$|a(z, y_0, \eta_{0J}, s, r, t)| = \left| \frac{\partial(\sigma, \rho, \tau)}{\partial(y_{0I}, \eta_{0J})} \right|^{1/2} \tag{62}$$

with

$$(\sigma(z, y_{0I}, \eta_{0J}, s, r, t), \rho(z, y_{0I}, \eta_{0J}, s, r, t), \tau(z, y_{0I}, \eta_{0J}, s, r, t)) = -\frac{\partial S}{\partial(s, r, t)}(z, y_{0I}, \eta_{0J}, s, r, t) \tag{63}$$

in accordance with (59).

Scattering

We return to scattering problem described by (16) and (66) and Theorem 2.1. We will assume that the tangent vectors to the rays that connect source and receiver to a scattering point in X stay away some finite distance from horizontal. We make this precise by using some angle θ, $0 < \theta < \pi/2$, an angle with the vertical, in

Assumption 3. (DSR assumption (Stolk and De Hoop, 2001; Stolk and De Hoop, 2003a)) If $(z, x) \in X$ and $\alpha, \beta \in S^{n-1}$, $t_{\mathrm{s}}, t_{\mathrm{r}} > 0$ depending on (z, x, α, β) are such that $\eta_z(t_{\mathrm{s}}, z, x, \beta, \tau) = \eta_z(t_{\mathrm{r}}, z, x, \alpha, \tau) = 0$, then

$$c(z, x)^{-1} \frac{\partial \eta_z}{\partial t}(t, z, x, \beta, \tau) < -\cos(\theta), t \in [0, t_{\mathrm{s}}], \tag{64}$$

$$c(z, x)^{-1} \frac{\partial \eta_z}{\partial t}(t, z, x, \alpha, \tau) < -\cos(\theta), t \in [0, t_{\mathrm{r}}]. \tag{65}$$

The assumption is microlocal (and restricts to a *common* scattering point (z, x), see Figure 3(left)); given the background medium, a pseudodifferential cutoff can be applied to the data to remove microlocally the part of the data where Assumption 3 is violated.

Under Assumption 3 and the assumption that the medium perturbation (a conormal distribution) is supported outside a neighborhood of $z = 0$, the singular part of the modeled data is unchanged when G in (16) or (23) is replaced by (49).

Modeling operator in terms of the reflection operator kernel

Using (50) and (52) together with (49) in (23) yields the operator F_{K} defined by

$$F_{\mathrm{K}} = -\tfrac{1}{2}\partial_t L R \;:\; \delta(\cdot - z_\Sigma) \to -\tfrac{1}{2}\partial_t L(R\delta(\cdot - z_\Sigma)). \tag{66}$$

Modeling operator in terms of the Weyl symbol

Upon substituting (24) or (25) into (66), we obtain the equivalent map

$$F_{\mathrm{K}} = -\tfrac{1}{2}\partial_t \Lambda L \mathrm{E} r \;:\; \delta(\cdot - z_\Sigma) \to -\tfrac{1}{2}\partial_t\, \Lambda \int L\left(\delta(\stackrel{t}{\cdot} - \langle(\stackrel{r}{\cdot} - \stackrel{s}{\cdot}), p\rangle)\, r\left(\stackrel{z}{\cdot}, \frac{\stackrel{s}{\cdot} + \stackrel{r}{\cdot}}{2}, p, 1\right) \delta(\stackrel{z}{\cdot} - z_\Sigma)\right) \mathrm{d}p. \tag{67}$$

In the later analysis we consider the operator $-\frac{1}{2}\partial_t \Lambda L \mathrm{E}$ that maps functions of (z, x, p) to functions of (s, r, t).

Remark 4.2. In the Born approximation, essentially, r is p-independent, viz.,

$$r = \frac{c'}{c_0^3}$$

(cf. (22)). Then

$$\Lambda \mathrm{E} r = E_2 E_1 r,$$

[h]

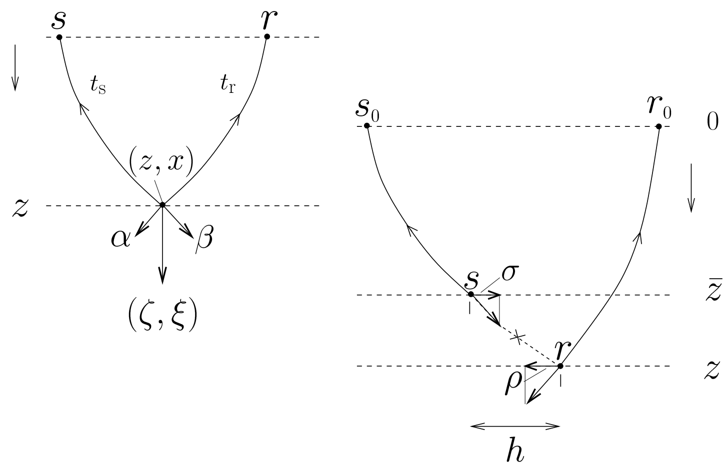

Figure 3. Ray geometry and canonical relation: Left (8), right (70).

where

$$\begin{aligned} E_1 &: \mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^{2n-1}) : (c_0^{-3} c')(z, x) \mapsto h(z, \bar{x}, x) = \delta(x - \bar{x})(c_0^{-3} c')(z, \tfrac{\bar{x}+x}{2}), \\ E_2 &: \mathcal{D}'(\mathbb{R}^{2n-1}) \to \mathcal{D}'(\mathbb{R}^{2n}) : h(z, \bar{x}, x) \mapsto \delta(t) h(z, \bar{x}, x). \end{aligned}$$

The symbol yields the normal incidence, linearized reflection coefficient (cf. (11)).

A theorem can be formulated that essentially coincides with Theorem 5.1 in (2003a) and that describes how F in the Kirchhoff approximation (cf. (23)) formulation of Theorem 2.1 is related to F_{K}: $F_{\mathrm{K}} \delta\genfrac{(}{)}{0pt}{}{z}{\cdot} = \psi_{\mathrm{D}} F \delta c$ in the context of Remarks 2.3 and 4.2.

Remark 4.3. Expression (66) has been postulated in various forms by different authors, for example De Bruin et al. (1990) and Haas (1992). The Kirchhoff modeling formula for the flat, horizontal interface case given by (23), upon substituting (49) leading to (66), matches the first term in the generalized Bremmer coupling series (De Hoop, 1996), hence its importance.

Remark 4.4. For the general, curved interface, we return to modeling formula (16). In analogy with (50) we define the operator kernel

$$(J(z; \bar{z}_0, z_0))(s, r, t, s_0, r_0, t_0) = \int_{\mathbb{R}} (G_-(z, \bar{z}_0))(s, t - t_0 - \bar{t}_0, s_0)(G_-(z, z_0))(r, \bar{t}_0, r_0)\, \mathrm{d}\bar{t}_0 \tag{68}$$

which continues sources and receivers in depth, separately from *different* levels. In general, J does not solve a 'double-square-root' equation, like H does, but, $J(0; z, z) = H(0, z)$.

We define

$$\begin{aligned} \widetilde{L}\left(R_1 \left\| \frac{\partial z'}{\partial(x, z)} \right\|_{\left(\genfrac{}{}{0pt}{}{r}{\cdot}, \genfrac{}{}{0pt}{}{z}{\cdot}\right)} \delta\left(z'\left(\genfrac{}{}{0pt}{}{r}{\cdot}, \genfrac{}{}{0pt}{}{z}{\cdot}\right)\right) \right) &= Q^*_{-,s}(0) Q^*_{-,r}(0) \int_0^Z \int_0^Z J(0; \bar{z}, z) Q_{-,s}(\bar{z}) Q_{-,r}(z) \\ &\quad \times R_1(\bar{z}, \genfrac{}{}{0pt}{}{s}{\cdot}, \genfrac{}{}{0pt}{}{t}{\cdot}, z, \genfrac{}{}{0pt}{}{r}{\cdot}) \left\| \frac{\partial z'}{\partial(x, z)} \right\|_{\left(\genfrac{}{}{0pt}{}{r}{\cdot}, \genfrac{}{}{0pt}{}{z}{\cdot}\right)} \delta\left(z'\left(\genfrac{}{}{0pt}{}{r}{\cdot}, \genfrac{}{}{0pt}{}{z}{\cdot}\right)\right) \Bigg) \mathrm{d}\bar{z}\mathrm{d}z, \end{aligned} \tag{69}$$

which replaces L in the flat interface case (cf. (66)). The propagation of singularities by $\widetilde{L}$ is governed by

$$\begin{aligned} \{ (\gamma_x(0, \bar{z}, s, \sigma, \tau), \gamma_x(0, z, r, \rho, \tau), t_0 \\ + \gamma_t(0, \bar{z}, s, \sigma, \tau) + \gamma_t(0, z, r, \rho, \tau), \gamma_\xi(0, \bar{z}, s, \sigma, \tau), \gamma_\xi(0, z, r, \rho, \tau), \tau; \\ \bar{z}, s, z, r, t, -b(\bar{z}, s, \sigma, \tau), \sigma, -b(z, r, \rho, \tau), \rho, \tau) \,|\, (s, r, t, \sigma, \rho, \tau) \in T^* \mathbb{R}^{2n-1}_{(s,r,t)}, \\ z_{\min,-}(\bar{z}, s, \sigma, \tau, \theta_2) < 0,\ z_{\min,-}(z, r, \rho, \tau, \theta_2) < 0 \} \subset T^* \mathbb{R}^{2n-1}_{(s,r,t)} \times T^* \mathbb{R}^{2n+1}_{(\bar{z},s,z,r,t)}. \end{aligned} \tag{70}$$

Remark 4.5. Since, in the Kirchhoff approximation, L and $\widetilde{L}$ act on delta distributions in z and $\bar{z}, z$, respectively, in the modeling of data the integration over z and $\bar{z}, z$, respectively, disappears.

5 RECONSTRUCTION

The inverse problem can be split into an imaging problem and an inverse scattering problem. For example, the depth z_Σ of an interface could be established by imaging (using, for example,the Born approximation). Once z_Σ is known (and, hence, the modeling no longer contains an integration over z), the operator $H(0, z_\Sigma)^*$ could be applied to the data, and as a consequence of Remark 3.1, the kernel of the reflection operator is obtained.

As mentioned below (67) we consider the operator $-\frac{1}{2}\partial_t \Lambda L E$ as the point of departure for developing an inverse scattering formula. The reconstruction of the symbol r given the background (c_0) is essentially done by applying the adjoint of this operator to the data d. We make use of the results for reconstruction in the Born approximation (Stolk and De Hoop, 2001; Stolk and De Hoop, 2003b).

Definition 5.1. Let L be as defined in (52), and let R denote the adjoint of E, given by

$$\mathrm{R} \,:\, g(z,s,r,t) \mapsto (\mathrm{R}g)(z,x,p) = \int_{\mathbb{R}^{n-1}} g(z, x - \tfrac{h}{2}, x + \tfrac{h}{2}, ph) \chi(z,x,h)\,\mathrm{d}h. \tag{71}$$

Here, $h \mapsto \chi(z,x,h)$ is a compactly supported cutoff function the support of which contains $h = 0$. We define the (wave-equation) angle transform, denoted by A_{WE}, as the composition of adjoints

$$A_{\mathrm{WE}} = \mathrm{R}L^*. \tag{72}$$

In the above, R is closely related to what seismologists call *beamforming*. In (Stolk and De Hoop, 2003b) the properties of A_{WE} (up to a time derivative) were analyzed; they are summarized in the following theorem. A map similar to A_{WE} was introduced in (De Bruin et al. 1990) for the purpose of imaging angle dependent reflection coefficients, see also (Prucha, Biondi and Symes, 1999). For each x, $(A_{\mathrm{WE}}d)(z,x,p)$ is a so-called *common-image-point gather*.

Theorem 5.2. (Stolk and De Hoop, 2003b) Suppose Assumption 3 holds. Let C_0 be an upper bound for c_0. Assume that

$$\|p\| < p_{\max} < \tfrac{1}{2} C_0^{-1}. \tag{73}$$

Then A_{WE} is a Fourier integral operator. Let C_1 be an upper bound for $\frac{\partial c_0^{-2}}{\partial x}$, C_2 an upper bound for c_0^{-1}. If in addition the function $h \mapsto \chi(z,x,h)$, contained in R_3, is supported in $B(0,d)$, where d depends on θ_2, C_0, C_1, C_2, then the canonical relation of A_{WE} corresponds to an invertible map from a subset of $T^*\mathbb{R}^{2n-1}_{(s,r,t)}$ to a subset of $T^*\mathbb{R}^{2n-1}_{(z,x,p)}$ that has nonempty intersection with the set $\vartheta = 0$ (where ϑ denotes the p-covector).

In L, the operator $H(0,z)$ contains an operator $\psi_2(0,z)$. To account for limited acquisition aperture, we introduced a smooth cutoff function $\psi_Y = \psi_Y(s,r,t)$ on Y that is zero near the boundary of Y. The key component operator in L^* is $H(0,z)^*$.

Remark 5.3. Through the convolution in (50), $H(0,z)^*$ represents the so-called ‘double-focusing operator’ (Thorbecke, 1997): It retrofocuses the data in source and receiver arrays. The reference to ‘double’ arises from the following observation: While replacing F in Theorem 2.1 by an operator (66) or (67) containing $H(0,z)$, we have ‘uncoupled’ (except in time) the source and receiver bicharacteristics in the canonical relation of L (cf. (56)) at the scattering point (z,x), see also Figure 3.

Remark 5.4. Data are modeled as $\delta G(0,r,t,0,s)$. Viewing the data as a function of r and t for fixed s yields what seismologists call a *shot record*. Shot records can be ‘synthesized’ to yield what seismologists call an areal shot record: Each shot record is convolved in time with a single time trace (fixed source location) out of the synthesis distribution, and subsequently the shot records are stacked (integrated) per common receiver location. The synthesis can be formulated as an operator acting on the data $\delta G(0,r,t,0,s)$ and is at the basis of *controlled illumination*, see Rietveld *et al.* (1992). (An example of controlled illumination is beamforming.) A particularly interesting choice of synthesis is obtained by requiring focus point (delta) illumination at the reflector depth. For a focus point at $(z,\bar{s})$, say this is achieved when the ‘synthesis operator’ is given by $G_{-,s}(0,z)^* Q^*_{-,s}(0)^{-1}$; indeed, the kernel of the composition $G_{-,s}(0,z)^* H(0,z)$ follows to be

$$\begin{aligned}
&\iint (G_-(0,z))^*(\bar{s}, \bar{t} - t, s)(H(0,z))(s,r,t,s_0,r_0,t_0)\,\mathrm{d}s\mathrm{d}t = \\
&\qquad \iiint (G_-(0,z))^*(\bar{s}, \bar{t} - t, s)(G_-(0,z))(s, t - t_0 - \bar{t}_0, s_0)\,\mathrm{d}s\mathrm{d}t\,(G_-(0,z))(r, \bar{t}_0, r_0)\,\mathrm{d}\bar{t}_0 \\
&\qquad = \int \delta(\bar{s} - s_0)\delta(\bar{t} - t_0 - \bar{t}_0)(G_-(0,z))(r, \bar{t}_0, r_0)\,\mathrm{d}\bar{t}_0 = \delta(\bar{s} - s_0)\,(G_-(0,z))(r, \bar{t} - t_0, r_0),
\end{aligned} \tag{74}$$

which should be modified to include the cutoffs $\psi_1(z,0)^* \psi_1(z,0)$ of Remark 3.1.

INVERSION FORMULA

From the fact that the canonical relation of A_{WE} is invertible between subsets of $T^*\mathbb{R}^{2n-1}$ (Theorem 5.2), it follows that $A_{\mathrm{WE}}(-\frac{1}{2})\partial_t \Lambda L\mathrm{E}$ is pseudodifferential. We calculate its symbol. First we modify the angle transform according to

$$(\widehat{A}_{\mathrm{WE}}\,\psi_Y d)(x,z,p) = \int (Q_{-,s}(z)^{-1} Q_{-,r}(z)^{-1} H(0,z)^* \\ Q^*_{-,s}(0)^{-1} Q^*_{-,r}(0)^{-1} \psi_Y(-2\partial_t^{-1}) d)(x - \tfrac{1}{2}h, x + \tfrac{1}{2}h, \langle p, h\rangle)\,\chi(x,z,h)\,\mathrm{d}h.$$

In the construction below, we omit the cutoff functions that are part of the symbols; the evaluation of $\widehat{A}_{\mathrm{WE}}(-\frac{1}{2})\partial_t \Lambda L\mathrm{E}$ is valid microlocally on the support of the cutoffs.

We consider the map $\mathrm{R}\bar{L}^*\bar{L}\mathrm{E}$ defined by

$$r\delta(\cdot - z_\Sigma) \mapsto (2\pi)^{-(n-1)} \int \Bigg\{ \int (H(0,z)^* \int H(0,\tilde{z})\,\delta(\overset{t}{\cdot} - \langle(\overset{r}{\cdot} - \overset{s}{\cdot}), \tilde{p}\rangle) \\ r\left(\tilde{z}, \frac{\overset{s}{\cdot} + \overset{r}{\cdot}}{2}, \tilde{p}, 1\right) \delta(\tilde{z} - z_\Sigma))(x - \tfrac{1}{2}h, x + \tfrac{1}{2}h, \langle p, h\rangle)\mathrm{d}\tilde{z}\,\mathrm{d}h \Bigg\} \mathrm{d}\tilde{p}. \quad (75)$$

We use the oscillatory integral representation for $H(0,z)$ from Lemma 4.1. Furthermore, we change the variables $(\overset{s}{\cdot}, \overset{r}{\cdot})$ of the kernel input to

$$\tilde{x} = \frac{s+r}{2}, \quad h' = r - s.$$

We find that the principal contribution to the kernel of the map inside the braces of (75), as a function of $(z, x, p, \tilde{z}, \tilde{x}, \tilde{p})$, can be written as

$$(2\pi)^{-(2n-1)} \int \overline{A(z, x - \tfrac{1}{2}h, x + \tfrac{1}{2}h, \langle p, h\rangle, y_0, \eta_{0J})} \times A(\tilde{z}, \tilde{x} - \tfrac{1}{2}h', \tilde{x} + \tfrac{1}{2}h', \langle \tilde{p}, h'\rangle, y_0, \eta_{0J}) \\ \times \exp(\mathrm{i}[\Phi(z, x, h, p, \tilde{z}, \tilde{x}, h', \tilde{p}, y_{0I}, \eta_{0J})])\mathrm{d}y_{0I}\mathrm{d}\eta_{0J}\,\mathrm{d}h'\mathrm{d}h, \quad (76)$$

where

$$\Phi(z, x, h, p, \tilde{z}, \tilde{x}, h', \tilde{p}) \;=\; -S(z, x - \tfrac{1}{2}h, x + \tfrac{1}{2}h, \langle h, p\rangle, y_{0I}, \eta_{0J}) + S(\tilde{z}, \tilde{x} - \tfrac{1}{2}h', \tilde{x} + \tfrac{1}{2}h', \langle h', \tilde{p}\rangle, y_{0I}, \eta_{0J}). \quad (77)$$

We identify the gradient

$$-\frac{\partial S}{\partial(z,s,r,t)}(z,s,r,t,y_{0I},\eta_{0J}) = (\zeta(z,s,r,t,y_{0I},\eta_{0J}), \sigma(z,s,r,t,y_{0I},\eta_{0J}), \\ \rho(z,s,r,t,y_{0I},\eta_{0J}), \tau(z,s,r,t,y_{0I},\eta_{0J}))$$

at

$$s = x - \tfrac{1}{2}h, \;\; r = x + \tfrac{1}{2}h, \;\; t = \langle h, p\rangle \quad (78)$$

and, basically, linearize the phase Φ,

$$\zeta(\tilde{z} - z) + \langle \sigma + \rho, \tilde{x} - x\rangle + \langle \frac{\rho - \sigma}{2}, h' - h\rangle + \tau\,(\langle h', \tilde{p}\rangle - \langle h, p\rangle). \quad (79)$$

We then, for given (z, x, h), apply a change of integration variables, $(y_{0I}, \eta_{0J}) \mapsto (\zeta, \sigma, \rho)$. This encompasses that

$$\tau = \Upsilon(z, x, h, p, \zeta, \sigma, \rho),\; \Upsilon(z, x, h, p, \zeta, \sigma, \rho) = -\frac{\partial S}{\partial t}(z, x - \tfrac{1}{2}h, x + \tfrac{1}{2}h, \langle h, p\rangle, (y_{0I}, \eta_{0J})(\zeta, \sigma, \rho)); \quad (80)$$

we can view Υ as a map from ζ to τ. (At $p = 0$, the map Υ reduces to the map Θ^{-1}: $\Upsilon(z, x, h, 0, \zeta, \sigma, \rho) = \Theta^{-1}(z, x - \frac{1}{2}h, x + \frac{1}{2}h, \zeta, \sigma, \rho)$ (Stolk and De Hoop, 2003aLemma 4.1); also note that Υ becomes p independent if $h = 0$.)

We compute the amplitude. Using Lemma 4.1 we find that

$$|A(z, x - \tfrac{1}{2}h, x + \tfrac{1}{2}h, \langle p, h\rangle, y_0, \eta_{0J})|^2 = \left| \frac{\partial(\sigma, \rho, \tau)}{\partial(y_{0I}, \eta_{0J})} \right|$$

up to leading order, while the change of integration variables induces a factor

$$\left| \frac{\partial(\zeta, \sigma, \rho)}{\partial(y_{0I}, \eta_{0J})} \right|^{-1};$$

we define

$$\Xi := \left|\frac{\partial(\zeta,\sigma,\rho)}{\partial(y_{0I},\eta_{0J})}\right|^{-1} \cdot \left|\frac{\partial(\sigma,\rho,\tau)}{\partial(y_{0I},\eta_{0J})}\right|_{t=\langle h,p\rangle} \quad \text{at } (z,s,r) = (z, x - \tfrac{1}{2}h, x + \tfrac{1}{2}h);$$

note that $\Xi = \left|\frac{\partial \Upsilon}{\partial \zeta}\right|$.

Upon changing integration variables, again, $\sigma = \frac{1}{2}\xi - \theta$, $\rho = \frac{1}{2}\xi + \theta$, the linearized phase attains the form

$$\zeta(\tilde{z}-z) + \langle \xi, \tilde{x}-x\rangle + \langle \theta, h'-h\rangle + \tau\left(\langle h',\tilde{p}\rangle - \langle h,p\rangle\right) = \zeta(\tilde{z}-z) + \langle \xi,\tilde{x}-x\rangle + \langle \theta+\tau p, h'-h\rangle + \tau\,\langle h',\tilde{p}-p\rangle,$$
$$\tau = \Upsilon(z,x,h,p,\zeta,\tfrac{1}{2}\xi-\theta,\tfrac{1}{2}\xi+\theta). \quad (81)$$

The leading order contribution to the oscillatory integral of the kernel associated with the map $\mathrm{R}\bar{L}^*\Lambda\bar{L}\mathrm{E}$ as in (75) then becomes, microlocally (cf. (76))

$$(2\pi)^{-(2n-1)}\int \Xi\,\exp[\mathrm{i}(\zeta(\tilde{z}-z) + \langle\xi,\tilde{x}-x\rangle + \langle\theta,h'-h\rangle + \tau\left(\langle h',\tilde{p}\rangle - \langle h,p\rangle\right))]|\tau|^{n-1}\,\mathrm{d}\zeta\mathrm{d}\xi\,\mathrm{d}\theta\mathrm{d}h\mathrm{d}h'$$
$$= (2\pi)^{-n}\int \exp[\mathrm{i}(\zeta(\tilde{z}-z) + \langle\xi,\tilde{x}-x\rangle)]$$
$$\times\left\{(2\pi)^{-(n-1)}\int \Xi\,\exp[\mathrm{i}(\langle\theta+\tau p,h'-h\rangle + \tau\,\langle h',\tilde{p}-p\rangle)]|\tau|^{n-1}\,\mathrm{d}h'\mathrm{d}\theta\mathrm{d}h\right\}\mathrm{d}\zeta\mathrm{d}\xi,$$
$$\tau = \Upsilon(z,x,h,p,\zeta,\tfrac{1}{2}\xi-\theta,\tfrac{1}{2}\xi+\theta). \quad (82)$$

Upon substituting Υ for τ, omitting the symbol Ξ, and changing the integration variable h' by $\widehat{h}' = \Upsilon(z,x,h,p,\zeta,\frac{1}{2}\xi-\theta,\frac{1}{2}\xi+\theta)\,h'$, the integral in between the braces becomes

$$(2\pi)^{-(n-1)}\int\left\{\int \exp[\mathrm{i}(\langle\,\Upsilon(z,x,h,p,\zeta,\tfrac{1}{2}\xi-\theta,\tfrac{1}{2}\xi+\theta)^{-1}\theta + p, \widehat{h}' - \Upsilon(z,x,h,p,\zeta,\tfrac{1}{2}\xi-\theta,\tfrac{1}{2}\xi+\theta)\,h\rangle)]\,\mathrm{d}\theta\mathrm{d}h\right\}$$
$$\times\exp[\mathrm{i}\langle\widehat{h}',\tilde{p}-p\rangle]\,\mathrm{d}\widehat{h}'. \quad (83)$$

The integral in between braces defines a symbol in $(z,x,p,\zeta,\xi,\widehat{h}')$. The principal part, Π say, of this symbol can be found by changing variables of integration, $(h,\theta)\mapsto(\widehat{h},\widehat{\theta})$ with $\widehat{h} = \Upsilon(z,x,h,p,\zeta,\frac{1}{2}\xi-\theta,\frac{1}{2}\xi+\theta)h$ and $\widehat{\theta} = \Upsilon(z,x,h,p,\zeta,\frac{1}{2}\xi-\theta,\frac{1}{2}\xi+\theta)^{-1}\theta$ for given (z,x,ζ,ξ), and applying the method of stationary phase.

The projection of (56), the canonical relation of L, on the τ variable is non-degenerate; we can always choose τ_0 to be a component of η_{0J}, while $\tau = \tau_0$. But then $\tau = \frac{\partial S}{\partial t}$ is t independent. This implies that Υ becomes p independent.

We summarize these results in the following theorem. The canonical relation of A_{WE} defines a map $(s,r,t,\sigma,\rho,\tau)\mapsto(z,x,p,\zeta,\xi,\vartheta)$ (where ϑ is the p-covector); there is also an associated value of $h = r - s$. By pull back with the inverse of the mentioned map, we map the symbols ψ_Y,ψ_{D} to symbols in the variables $(z,x,p,\zeta,\xi,\vartheta)$. By the evaluation of h one obtains by pull back the cutoff χ in these variables also. We define Ψ as the product of these symbols and cutoff.

Theorem 5.5. Let the modified angle transform be given by

$$(\widehat{A}'_{\mathrm{WE}}\,\psi_Y d)(x,z,p) = \int(\Xi^{-1}Q_{-,s}(z)^{-1}Q_{-,r}(z)^{-1}H(0,z)^*$$
$$Q^*_{-,s}(0)^{-1}Q^*_{-,r}(0)^{-1}\psi_Y(-2\partial_t^{-1})d)(x-\tfrac{1}{2}h,x+\tfrac{1}{2}h,\langle p,h\rangle)\,\chi(x,z,h)\,\mathrm{d}h;$$

then

$$(\Psi(z,x,p,D_z,D_x,D_p) + \text{l.o.t.})\,(r\delta(^{z}_{\cdot})) = (\Pi^{-1}\widehat{A}'_{\mathrm{WE}}d)(z,x,p) \quad (84)$$

if $d = F_{\mathrm{K}}\delta(^{z}_{\cdot})$ is the Kirchhoff modeled data in accordance with (67).

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