Nonhyperbolic moveout analysis in VTI media using rational interpolation

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ABSTRACT
We present a rational interpolation approach to nonhyperbolic moveout correction in the $t-x$ domain, for qP-waves in homogeneous transversely isotropic media with a vertical symmetry axis. This method has no additional computational overhead compared to using expressions explicit in the relevant parameters, i.e., the anellipticity parameter η and the (zero-dip) normal moveout velocity V_{NMO}. The lack of such additional overhead can be attributed to the observation that, for a fixed value of η and a fixed zero-offset two-way traveltime t_0, the moveout curve for different values of V_{NMO} can be calculated by simple stretching and squeezing of the offset axis, where the amount of stretch or squeeze depends on the change in V_{NMO}. This observation is based on the generally accepted assumptions that the traveltimes of qP-waves in transversely isotropic media, depend mainly on η and V_{NMO}, and that the shear-wave velocity along the symmetry axis has a negligible influence on these traveltimes. The accuracy obtained with this method is as good as that of these approximations. The method can be tuned to be accurate to any offset range of interest, by increasing the order of the interpolation, making it accurate for arbitrary offsets.
We test the method using both synthetic and field data, and compare it with the nonhyperbolic moveout equation of Alkhalifah and Tsvankin (1995). Both data types confirm that for $\eta \gtrsim 0.1$ our method significantly outperforms the nonhyperbolic moveout equation in terms of combined unbiased parameter estimation with accurate moveout correction. A comparison with the shifted hyperbola equation of Fomel (2004) establishes almost identical accuracy of the rational interpolation method and his equation. Under the above-mentioned approximations, we show that the correction factor for the Alkhalifah-Tsvankin equation introduced by Grechka and Tsvankin (1998), is independent of V_{NMO}, and we present a method to estimate its optimal value in practice. This factor can be used to maximize the performance of the Alkhalifah-Tsvankin approximation.

Key words: Moveout analysis, nonhyperbolic, anisotropy, anellipticity, VTI, rational interpolation

Introduction

Over the past two decades, the importance of anisotropy and its influence on seismic data processing have become increasingly appreciated. Since 75% of the clastic infill of sedimentary basins consists of shale formations (Tsvankin, 2001, p.11), and since the transversely isotropic (TI) model adequately describes the intrinsic anisotropy of shales (Sayers, 1994), wave propagation in TI media has attracted much attention. Because the dispersion relations govern the propagation velocities of the different wave modes (and hence the traveltimes used in seismic data processing), and because these relations are nonlinear in the elastic coefficients, many authors have worked on approximations of the dispersion relations in TI media (Dellinger *et al.*, 1993; Tsvankin & Thomsen, 1994; Alkhalifah, 1998; Schoenberg & de Hoop, 2000; Zhang & Uren, 2001; Stovas & Ursin, 2004; Fomel, 2004), with varying levels of accuracy; Fomel's (2004) shifted hyperbola approach seemingly the most accurate of all. Fowler (2003) gives a comparative review of some of these approximations with an emphasis on TI media with a vertical symmetry axis (VTI media).

Here, we propose a rational interpolation scheme for traveltimes of qP-waves in homogeneous VTI media that requires no additional computational overhead compared to methods using approximations explicit in the physical relevant parameters, such as those just mentioned. The choice to use a rational interpolation was motivated by the observation that several of the aforementioned approximations achieve high accuracy through the use of rational approximations [e.g., Schoenberg and de Hoop (2000) and Stovas and Ursin (2004)], and by the form of the nonhyperbolic moveout equation of Tsvankin and Thomsen (1994), which resembles a rational approximation. We refrain from an attempt to derive yet another approximation to traveltimes of qP-waves in such media that is explicit in the relevant parameters. Instead, we simply make use of the facts that the shear-wave velocity along the (vertical) symmetry axis, V_{S0} has negligible influence on the traveltimes of qP-waves in TI media (Tsvankin & Thomsen, 1994; Tsvankin, 1996; Alkhalifah, 1998; Alkhalifah, 2000), and that these traveltimes depend mainly on the anellipticity parameter η and the (zero-dip) normal-moveout velocity V_{NMO} (Alkhalifah & Tsvankin, 1995). That is, the influence of Thomsen parameter δ is only small. We show that these two assumptions cause the influence of V_{NMO} on the nonhyperbolic moveout of qP-waves in a horizontal homogeneous VTI layer, for fixed anellipticity parameter η and two-way zero-offset traveltime t_0, to be a simple horizontal stretch (or squeeze), i.e., along the offset axis, of the moveout curve for some reference value of V_{NMO}, where the amount of stretch (or squeeze) depends on the change in V_{NMO}. This simple observation allows the traveltimes needed for the rational interpolation to be calculated from a small number of precomputed traveltimes stored in a table. Within the limits of the accuracy of the above two approximations, our rational interpolation approach can be tuned to almost arbitrary accuracy at any offset-to-depth ratio (ODR) (or group angle) of interest. A comparison of our approach with the shifted hyperbola approach of Fomel (2004) shows that our method has accuracy almost identical to that of Fomel.

For velocity analysis in VTI media using qP-waves, the nonhyperbolic moveout equation for a single horizontal VTI layer, derived by Tsvankin and Thomsen (1994), is the current standard in seismic data processing. Alkhalifah and Tsvankin (1995) have rewritten this equation in terms of V_{NMO} and η. Even though this approximation is exact at zero offset and infinite offset, Grechka and Tsvankin (1998) mention that at intermediate offsets "this approximation can be somewhat improved by empirically changing the denominator of the nonhyperbolic term." This limited accuracy at intermediate offsets was also noted by other authors [e.g., Zhang and Uren (2001), van der Baan and Kendall (2002), and Stovas and Ursin (2004)]. In an attempt to overcome this limitation in accuracy, Grechka and Tsvankin (1998) introduce a correction factor C in the denominator of the nonhyperbolic term. We show that within the two approximations mentioned above, this correction factor depends only on the ODR and η, and we present a figure that shows C as a function of both ODR and η, for ODR up to four and $0 \leq \eta \leq 1$. We explain how this figure can be used in practice to determine the value of C, that minimizes the bias in the estimated value of η when the nonhyperbolic moveout equation of Alkhalifah and Tsvankin (1995) is used. Through a study of the accuracy of this equation, we establish its limits of applicability.

The motivation for higher accuracy at larger ODR stems from the observation that a larger ODR provides better resolution for the anellipticity parameter η (Alkhalifah, 1997; Grechka & Tsvankin, 1998; Wookey *et al.*, 2002). Hence, improved accuracy at larger ODR can help reduce the uncertainty in inversion for η [within the limits of the trade-off between η and V_{NMO}, as observed by Grechka and Tsvankin (1998)] when large ODR is available in the data. We show that our rational interpolation approach provides significantly more accurate traveltimes than does the currently standard nonhyperbolic moveout equation, especially for larger ODR (larger than two), and for arbitrary levels of anellipticity, with no additional computational overhead. The lack of such overhead stems from the influence of V_{NMO} being limited to a stretch (or squeeze) along the offset axis of the moveout curve for some reference value of V_{NMO}, and for fixed η and t_0.

The outline of this paper is as follows. First, we study the accuracy of the nonhyperbolic moveout equation from Tsvankin and Thomsen (1994) and analyze the influence of the introduction of the correction factor of Grechka and Tsvankin (1998) on this accuracy. Subsequently, we introduce the rational interpolation approach to nonhyperbolic moveout for qP waves in VTI media, and explain the stretch-squeeze influence of V_{NMO} on the moveout curve of qP reflections in a single horizontal homogeneous VTI layer. Subsequently we explain the influence of this observation on the correction factor C in the nonhyperbolic moveout equation. Synthetic tests, for both a single homogeneous horizontal VTI layer and a horizontally layered VTI medium, verify the improved accuracy of the method when compared to the nonhyperbolic moveout equation. Finally these findings are confirmed by an application of the method to field data. Also, a comparison between our

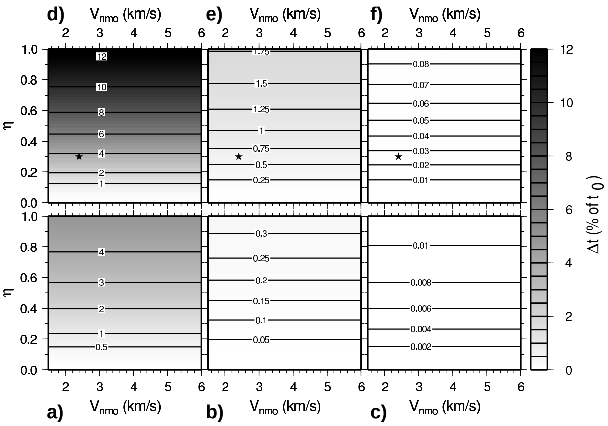

Figure 1. Comparison of the accuracy of the nonhyperbolic moveout equation without the correction factor, i.e., $C = 1$ (a), with the optimal correction factor C (b), and of the [2/2] rational interpolation (c), for a range of models (combinations of η and V_{NMO}) that spans most models of practical interest. Contours are drawn for the maximum absolute percentage error in traveltime (compared to ray-traced traveltimes where ray tracing was done with $V_{S0} = 0$ km/s and $\delta = 0$) for a single horizontal VTI layer, over a range of ODR values up to two. The contoured values are shown in percentage of t_0. Subfigures (d) through (f) show the same contours as in subfigures (a) through (c), except for a range of ODR values up to four. The stars in subfigures (d) through (f) indicate the model parameters used for the residual moveout plots in Figure 4.

method and the shifted hyperbola method of Fomel (2004) establishes the almost identical accuracy of the two methods.

Accuracy of the nonhyperbolic moveout equation

The nonhyperbolic moveout equation of Tsvankin and Thomsen (1994), rewritten in terms of the anellipticity parameter η by Alkhalifah and Tsvankin (1995), is given by

$$t^2(x) = t_0^2 + \frac{x^2}{V_{NMO}^2} - \frac{2\eta x^4}{V_{NMO}^2\left[t_0^2 V_{NMO}^2 + (1+2\eta)\,x^2\right]}, \quad (1)$$

where t_0 is the two-way traveltime at zero offset, x is offset, and V_{NMO} is the (zero-dip) NMO velocity. Note how the moveout reduces to hyperbolic moveout for elliptical anisotropy ($\eta = 0$), as pointed out by Helbig (1983). This equation is exact at both zero offset and infinite offset. In an attempt to increase the accuracy at intermediate offsets, Grechka and Tsvankin (1998) introduce a correction factor. Noting that for a single horizontal VTI layer the absolute error in η is at least twice as large as the relative error in the horizontal velocity $V_{hor} = V_{NMO}\sqrt{1+2\eta}$, they rewrite equation (1) in terms of V_{NMO} and V_{hor},

$$t^2(x) = t_0^2 + \frac{x^2}{V_{NMO}^2} - \frac{\left(V_{hor}^2 - V_{NMO}^2\right)x^4}{V_{NMO}^2\left(t_0^2 V_{NMO}^4 + C V_{hor}^2 x^2\right)}, \quad (2)$$

where C is the above-mentioned correction factor.

Figure 1a show contours of the maximum absolute traveltime difference (in percentage of t_0) between ray-traced traveltimes for a single horizontal VTI layer and the nonhyperbolic moveout equation without the correction factor, for virtually all combinations of η and V_{NMO} of practical interest. The maximum ODR is two, and $t_0 = 1$ s for all models considered. The ray-traced traveltimes were determined with $V_{S0} = 0$ km/s and $\delta = 0$. The differences in the traveltimes are of the order of a percent, which in this case amounts to 10 ms. For a dominant frequency of 50 Hz in surface seismic data, this amounts to a traveltime error of about half a dominant period. Such errors are not negligible, can lead to substantial reduction in stacking power, and thus cause errors in the estimation of η and V_{NMO} in velocity analysis. Note that here we use the true values of η and V_{NMO} for the moveout analysis rather than the best-fit values, since we want to analyze the accuracy of the nonhyperbolic moveout equation for a known model. It is known that the traveltime differences shown in Figure 1a can be reduced by using the best-fit values of η and V_{NMO}, rather than the true ones.

Grechka and Tsvankin (1998) state that, for a single horizontal VTI layer, introducing the coefficient $C = 1.2$ minimizes deviations from the exact traveltimes for the range of offset-to-depth ratio (ODR) $1.5 - 2.5$. They also mention that the correction factor can be used as an optimization parameter by comparison of traveltimes obtained using equation (2) and ray-tracing. Figure 1b is as Figure 1a, except now the optimized correction factor was determined for each combination of η and V_{NMO} shown and then used in equation (2). Note that indeed the introduction of the correction factor reduced the maximum errors in the traveltimes substantially, resulting in errors of the order of a tenth of a percent. A straightforward application of the $C = 1.2$ for all models (not shown here), did reduce the errors somewhat, but the errors were still of the order of a percent. This indicates that a straightforward application of $C = 1.2$ is in general not recommended, except for models where $C = 1.2$ may happen to be the optimal correction factor. It is worth mentioning that in practice the optimal value of C can be determined only approximately because the true values of η and V_{NMO} are unknown. Therefore, Figure 1b represents the best accuracy that can be obtained using the nonhyperbolic moveout equation with the correction factor.

As pointed out by Alkhalifah (1997), a larger maximum ODR (say larger than 1.5) provides increased stability and resolution for the inversion for η using nonhyperbolic moveout. With current acquisition systems, such values of ODR are feasible, especially for shallow targets. Moreover, the need for cost-effective acquisition systems could, with time, increase the maximum ODR in acquisition systems. Also, in near-surface (or shallow) geophysical application, large ODR is common. Figures 1d and 1e are as Figures 1a and 1b, respectively, except that here the maximum ODR is four. The introduction of the optimal correction factor again reduces the errors in traveltime significantly, but the errors are of the order of a percent; that is, they are sizable.

The dependence of nonhyperbolic moveout for a single VTI layer on η and V_{NMO}

From simple geometric considerations, it follows that for a homogeneous horizontal VTI layer, the traveltime is given by

$$t = \frac{V_{P0}\ t_0}{v \cos\psi}\ , \tag{3}$$

where V_{P0} is the P-wave traveltime along the vertical symmetry axis, ψ is the group angle, and v is the group velocity for propagation in direction ψ. The offset x associated with this traveltime is given by

$$x = V_{P0}\ t_0 \tan\psi\ . \tag{4}$$

In a TI medium, the phase velocity is given by (Tsvankin, 2001, p.22)

$$V(\theta) = V_{P0}\left\{1 + \epsilon\sin^2\theta - \frac{f}{2}\left(1 - \ldots \sqrt{\left(1 + \frac{2\epsilon\sin^2\theta}{f}\right)^2 - \frac{2\,(\epsilon-\delta)\sin^2 2\theta}{f}}\right)\right\}^{1/2} \tag{5}$$

where θ is the phase angle, ϵ and δ are the Thomsen parameters, and $f \equiv 1 - V_{S0}^2/V_{P0}^2$, with V_{S0} the S-wave phase velocity along the symmetry axis. For such a medium, the group angle is related to the phase angle through (Tsvankin, 2001, p.29)

$$\tan\psi = \frac{\tan\theta + \dfrac{1}{V(\theta)}\dfrac{dV}{d\theta}}{1 - \dfrac{\tan\theta}{V(\theta)}\dfrac{dV}{d\theta}}\ , \tag{6}$$

while the group velocity v is given by

$$v = V(\theta)\sqrt{1 + \left(\frac{1}{V(\theta)}\frac{dV}{d\theta}\right)^2}\ . \tag{7}$$

It is known that the qP-wave phase-velocity in TI media depends only weakly on V_{S0} (Tsvankin & Thomsen, 1994; Tsvankin, 1996; Alkhalifah, 1998). For all kinematic problems regarding qP waves, V_{S0} is therefore usually ignored. We likewise set $V_{S0} = 0$ (or $f = 1$) in equation (5). This is the *acoustic approximation* from Alkhalifah (1998; 2000). Note that Alkhalifah and Tsvankin (1995) obtained equation (1) from the nonhyperbolic moveout equation of Tsvankin and Thomsen (1994), by also setting $V_{S0} = 0$. In addition, Alkhalifah and Tsvankin (1995) showed that the time signatures of qP-waves in homogeneous VTI media depend mainly on the (zero-dip) normal-moveout velocity V_{NMO} and the anellipticity parameter η, with an almost negligible influence of V_{P0}. Since we are interested only in traveltime calculations, we can choose $\delta = 0$, and thus $V_{P0} = V_{NMO}$ and $\epsilon = \eta$, in equation (5). That equation then becomes

$$V(\theta) = V_{NMO}\left\{\eta\sin^2\theta + \frac{1}{2}\left(1 + \ldots \sqrt{\left(1 + 2\eta\sin^2\theta\right)^2 - 2\eta\sin^2 2\theta}\right)\right\}^{1/2}\ , \tag{8}$$

while equations (3) and (4) for the traveltime t and the associated offset x, become

$$t = \frac{V_{NMO}\ t_0}{v\cos\psi}\ , \tag{9}$$

and

$$x = V_{NMO}\ t_0 \tan\psi\ , \tag{10}$$

respectively. Note that the phase velocity $V(\theta)$ now depends linearly on V_{NMO}. This linearity causes the term $\dfrac{1}{V(\theta)}\dfrac{dV}{d\theta}$ in equations (6) and (7) to be independent of V_{NMO}. Since the dependence of the group angle on the anisotropic parameters is governed by the term $\dfrac{1}{V(\theta)}\dfrac{dV}{d\theta}$ [cf. equation (6)], the group angle ψ is independent of V_{NMO} and depends only on η. In addition, it follows from equation (7) that the group velocity v depends linearly on V_{NMO} since the phase velocity depends linearly on V_{NMO}. From equations (9) and (10) it then follows that *the traveltime t becomes independent of V_{NMO} and that the associated offset x depends linearly on V_{NMO}*. Also, t_0 is a simple scaling factor for both the traveltime t and the associated offset x. In a single horizontal VTI layer, this means that *for fixed η and t_0, the moveout curve for different values of V_{NMO} can be calculated by simple horizontal stretching and squeezing along the offset axis* (see Figure 2). This important observation is a straightforward consequence of the negligible influence of V_{S0} on qP-wave traveltimes in TI media and the fact that the kinematics of qP-waves in homogeneous VTI media depend mainly on V_{NMO} and η. To make explicit the independence of the traveltimes of V_{NMO}, we rewrite equation (9) as

$$t = \frac{t_0}{v|_{V_{NMO}=1}\cos\psi}\ , \tag{11}$$

where $v|_{V_{NMO}=1}$ denotes the group velocity calculated for $V_{NMO} = 1$ km/s.

Figure 1a (as well as b-e) shows that the contoured maximum traveltime differences between ray-traced traveltimes and traveltimes calculated using the

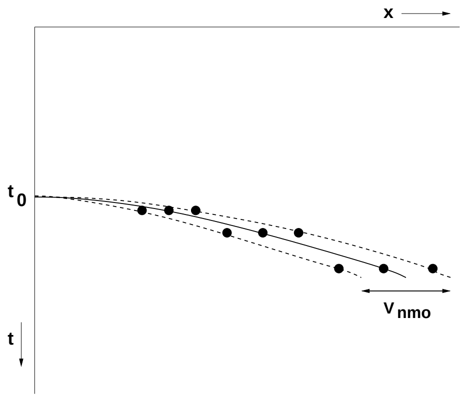

Figure 2. Under the customary assumptions that traveltimes of qP-waves in VTI media depend mainly on η and V_{NMO}, and that the shear-wave velocity along the symmetry axis (V_{S0}) has negligible influence on the traveltimes of qP-waves in such media, the nonhyperbolic moveout curve for fixed η and t_0, but varying V_{NMO}, can be calculated by simple horizontal stretching (or squeezing) along the offset axis. The amount of stretch is determined by the change in V_{NMO}.

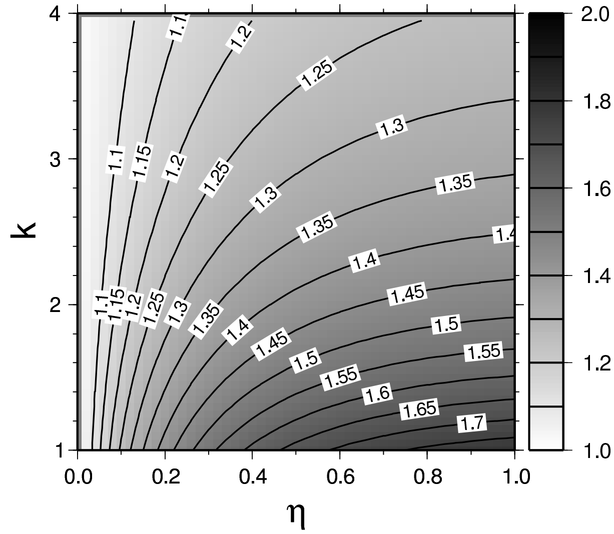

Figure 3. Optimal correction factor C as a function of ODR (k) and η.

nonhyperbolic moveout equation without the correction factor are independent of V_{NMO}. This is now easily understood in light of the previous observation that the nonhyperbolic moveout for fixed η (and t_0) but different values of V_{NMO}, are simply horizontally stretched or squeezed versions of each other. Note that we used $V_{S0} = 0$ and $\delta = 0$ for the ray tracing to determine the traveltime differences shown in Figure 1.

For hard rocks, setting $V_{S0} = 0$ is not as good an approximation as for softer rocks (Tsvankin, personal communication). For such geology (say carbonate reservoirs) it is better to use an 'intelligent estimate' of the V_p/V_s ratio. Note that in this case the traveltimes t are still independent of V_{NMO} because f is simply equal to some appropriately chosen constant value. Therefore, the moveout curves for fixed values of η and t_0 but different values of V_{NMO} are again horizontally stretched or squeezed versions of each other.

Implications for the correction factor C

Rewriting equation (2) in terms of η and V_{NMO} gives

$$t^2(x) = t_0^2 + (x/V_{NMO})^2 - \ldots \frac{2\eta\,(x/V_{NMO})^4}{\left[t_0^2 + C\,(1+2\eta)\,(x/V_{NMO})^2\right]} \,. \qquad (12)$$

Using the observation that the offsets x are linear in V_{NMO}, i.e., using equation (10), it follows that

$$t^2(k) = t_0^2 \left\{ 1 + \frac{k^2}{4} - \frac{\eta k^4}{2\,(4 + C\,[1+2\eta]\,k^2)} \right\} \,, \qquad (13)$$

where we used $k/2 = \tan\psi$, with k the ODR. This equation shows explicitly that, under the above-mentioned approximations, the traveltimes for a given ODR k (or group angle ψ) are independent of V_{NMO}. Note that using $k/2 = \tan\psi$ in equation (10) gives

$$k = \frac{2x}{t_0 V_{NMO}} \,. \qquad (14)$$

Strictly, k is a true ODR only if $\delta = 0$. For convenience, we refer to k as the ODR throughout the remainder of this paper.

Under the above-mentioned two approximations, the traveltimes $t(k)$ on the left-hand side of equation (13) are independent of V_{NMO}. It follows that under these approximations the correction factor C is independent of V_{NMO}. Therefore we need only study the dependence of C on η and k. We determine the *optimal* C by minimizing the maximum traveltime difference between ray-traced traveltimes (with $V_{S0} = 0$ km/s and $\delta = 0$) and the traveltimes calculated using equation (13) for different values of η and k. Figure 3 shows a contour plot of C as a function of η and k. Clearly, the optimal C value is not a constant, but rather varies with η and k. Grechka and Tsvankin (1998, p.959) state that, for a single horizontal VTI layer, "introducing the coefficient $C = 1.2$ ... minimizes deviations from the exact traveltimes for the most practical range of offsets $1.5h < x < 2.5h$," where h is the depth of the reflector. Figure 3 shows that for this range of ODR, $C = 1.2$ is optimal only for values of η ranging from 0.08 to 0.18. Therefore, their statement should be understood with the added qualifier that the optimal value of C is close to 1.2 for these particular values of ODR range *in combination* with these particular values of η only.

In practice, neither η or k are known. However, an estimate of V_{NMO} and η can be obtained using equation (12) [or (2)] with $C = 1$. Subsequently, an estimate of

k can be obtained using the estimated value of V_{NMO} in equation (14). Then, the optimal C value can be obtained from Figure 3 using the estimated k and η.

Nonhyperbolic moveout using rational interpolation

Fitting function values at various points (that are not necessarily distinct) using a rational function is usually referred to as *multipoint Padé approximation* (Baker & Graves-Morris, 1981a). Such approximation is also referred to as *N-point Padé approximation*, *Newton Padé approximation*, or *rational interpolation*, depending on the context. Since in this paper, we do not use coincident interpolation points (often referred to as *confluent interpolation points*), we prefer to use the term *rational interpolation*.

A rational approximation to a function $T(x)$ is written as

$$T(x) \approx \frac{N_L(x)}{D_M(x)} , \qquad (15)$$

where $N_L(x)$ is a polynomial of maximum order L, and $D_M(x)$ a polynomial of maximum order M. We denote such an approximation as $[L/M]$ and use the normalization $N_L(0) = T(0)$ and $D_M(0) = 1.0$, after Baker (1975, pp. 5-6). Given $n = L + M$ function values T_i at points x_i, with $i = 1, \dots, n$, we arrive at a linear system of n equations with n unknowns, the coefficients of the polynomials. Once the coefficients are found, the resulting $[L/M]$ approximant can be used to find the function values $T(x)$ at values of x different from the interpolation points x_i. The solution to this system for the [2/2] rational approximation is given in appendix A.

As with any linear system of equations, the system may be singular. This is a known hazard of rational interpolation. Hence, blind use of rational interpolation can be problematic. There exist reliable algorithms, in the sense that if an interpolant to the function values t_i exists, they find it, whereas if no interpolant exists because the linear system is degenerate, the algorithm exits with an error. An example of such a reliable algorithm is the modified Thacher-Tukey algorithm, e.g., Graves-Morris and Hopkins (1981). Baker and Graves-Morris (1981b, pp. 7-17) give an overview of different algorithms for rational approximations.

Here we use rational interpolation to approximate nonhyperbolic moveout in a horizontal transversely isotropic homogeneous layer with a vertical symmetry axis. Rewriting equation (2) in the form of a rational approximation using the definition of Baker (1975, pp. 5-6), gives

$$t^2(x) = \left\{ t_0^2 + \left(A + V_{NMO}^{-2}\right) x^2 + V_{NMO}^{-2} t_0^{-2} \left(A + \dots \right.\right. \\ \left.\left. \left[V_{NMO}^2 - V_{hor}^2\right] / V_{NMO}^4\right) x^4 \right\} / \left(1 + A t_0^{-2} x^2\right) , \qquad (16)$$

where $A \equiv C V_{hor}^2 V_{NMO}^{-4}$. This expression reveals that the nonhyperbolic moveout equation can be viewed as a [2/1] rational approximation for t^2 as a function of x^2. Therefore, we could try a [2/1] rational interpolation to approximate nonhyperbolic moveout. In this paper, we choose a [2/2] rational interpolation in an attempt to gain extra accuracy.

To calculate the four unknown coefficients for the [2/2] rational interpolation using equations (A2)-(A5), we need four traveltimes and four associated offsets. Let t_i $(i = 1, \dots, 4)$ be the traveltime for a fixed ODR k_i (or group angle ψ_i), and let the associated offset be x_i. From the previous section, we know that the traveltimes t_i are independent of V_{NMO} and depend on t_0 through a simple scaling only. This means that the traveltimes t_i needed for the rational interpolation can be calculated from a small subset of traveltimes calculated for a fixed reference value of t_0, denoted as t_0^{ref}, and a range of η values, say from -0.2 to 1.0 in steps of 0.01. This subset can be precomputed and stored in a table. Hence, when the traveltimes t_i for a particular combination of t_0, V_{NMO}, and η are desired, a simple lookup in this table for the particular η combined with scaling with t_0/t_0^{ref} [cf. equation (11)], gives the traveltimes t_i needed for the rational interpolation. That is, the desired traveltime t_i is obtained from

$$t_i = t_i^{table} \left(\frac{t_0}{t_0^{ref}} \right) , \qquad (17)$$

where t_i^{table} is the value of t_i obtained from the table (evaluated for $t_0 = t_0^{ref}$), for ODR k_i and the desired value of η. Evaluating the table for $t_0^{ref} = 1$ s allows the calculation of t_i to be done by scaling of t_i^{table} with t_0 only. Using the given values of V_{NMO} and t_0, the corresponding offsets x_i are then found simply from

$$x_i = \frac{V_{NMO}\ t_0\ k_i}{2} . \qquad (18)$$

The traveltimes and offsets obtained using the method outlined above, allow us to perform velocity-analysis in VTI media using our [2/2] rational interpolation. The efficiency of this approach is comparable to that of current velocity analysis using nonhyperbolic moveout equations (1) or (2), since the small subset of traveltimes for $t_0 = 1$ s and a range of η values, is precomputed and stored in a table. Hence, no computational overhead is required compared to that of current methods. In other words, the nonhyperbolic moveout equation is replaced simply with the rational interpolation formula, and the needed traveltimes are read from the precomputed table.

To precompute the table of traveltimes, a standard anisotropic ray-tracing algorithm can be used. Here, to calculate the traveltimes, we first solve

$$\frac{k_i}{2} = \frac{\tan\theta_i + \frac{1}{V_i} \left.\frac{dV}{d\theta}\right|_{\theta_i}}{1 - \frac{\tan\theta_i}{V_i} \left.\frac{dV}{d\theta}\right|_{\theta_i}} , \qquad (19)$$

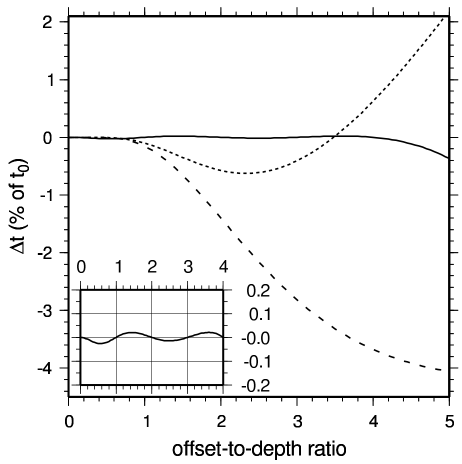

Figure 4. Residual moveout as a function of ODR using the nonhyperbolic moveout equation without the correction factor, i.e., $C = 1$ (dashed), with the optimal correction factor (dotted), and [2/2] rational interpolation (solid). For all three cases the model parameters used are $\eta = 0.3$ and $V_{NMO} = 2.4$ km/s (indicated by the star in Figures 1a-c). The inset shows the residual moveout for the [2/2] rational interpolation on a larger scale. At the interpolated ODR values 1, 2, 3, and 4, the residual moveout is identical zero.

for θ_i numerically, using the Matlab function 'fsolve' which uses an interior-reflective Newton method to solve nonlinear equations (Coleman & Li, 1994; Coleman & Li, 1996). To obtain equation (19) we used $\tan\psi_i = k_i/2$. In equation (19), V_i is the phase velocity associated with ODR k_i. The traveltime is then found through calculation of the group velocity $v_i|_{V_{NMO}=1}$ using equations (8) and (7) with $V_{NMO} = 1$ km/s, and subsequent use of this velocity in

$$t_i^{table} = \frac{t_0^{ref}}{v_i|_{V_{NMO}=1} \cos\left(\tan^{-1}\dfrac{k_i}{2}\right)}\,, \tag{20}$$

[cf. equation (11)]. We found that using the group angle as an initial guess for the phase angle generally worked well. We did not investigate different methods to solve for the phase angles θ_i. Using the 'fsolve' function in Matlab, the calculation of a table with four traveltimes for about 100 values of η takes on the order of one minute on a modern PC.

Accuracy comparison between rational interpolation and the nonhyperbolic moveout equation

Figure 1c shows contours of the maximum absolute percentage traveltime difference between ray-traced traveltimes for a single horizontal VTI layer and traveltimes resulting from the [2/2] rational interpolation. Here $t_0 = 1$ s, and the maximum ODR is two. The ODR values used for the rational interpolation are $k_1 = \frac{1}{2}$, $k_2 = 1$, $k_3 = \frac{3}{2}$, and $k_4 = 2$. Here the maximum traveltime errors are of the order 10^{-3}% of t_0 (or 0.01 ms in this case), which is one-to-two orders of magnitude more accurate than that using the nonhyperbolic moveout equation with the optimal correction factor, and two-to-three orders of magnitude more accurate than the accuracy of this equation without the correction factor. With a dominant frequency of 50 Hz, the traveltime errors are about 0.05% of the dominant period, and are hence negligible. Figure 1f shows the same contours, but now for a maximum ODR of four (the ODR values used for the rational interpolation are $k_1 = 1$, $k_2 = 2$, $k_3 = 3$, and $k_4 = 4$). The traveltime errors are now somewhat larger and of the order of 10^{-2}% of t_0 (or 0.1 ms in this case), but still one-to-two orders of magnitude smaller than those for the nonhyperbolic moveout with or without the correction factor. Again, compared to a dominant period of 20 ms, these errors are negligible (0.5% of the dominant period). Hence, rational interpolation achieves a significant improvement in accuracy up to large ODR, and is highly accurate for all models of practical interest, without the use of an optimization parameter such as the correction factor C in the nonhyperbolic moveout equation (2).

We explained earlier that rational interpolation can lead to a degenerate linear system. For our [2/2] rational interpolation, this happens if the moveout is purely hyperbolic. Figure 1, however, shows that for virtually all anisotropic models of practical interest such degeneracy does not occur. Figure 1 was calculated using discrete offsets and discrete values of V_{NMO} and η. If degeneracy would somehow occur for values in the continuous range between the discrete values we used for η and V_{NMO}, adding a small amount of numerical noise (say on the order of 10^{-2} ms) should overcome. Numerical tests showed that adding such a small amount of numerical noise, indeed removed the degeneracy for purely hyperbolic moveout.

Figure 4 shows the residual moveout as a function of ODR for the nonhyperbolic moveout equation without (dashed) and with (dotted) the optimized correction factor, respectively, and for the [2/2] rational interpolation (solid). Here we used $V_{NMO} = 2.4$ km/s*, $\eta = 0.3$ (indicated by the star in Figures 1d-f), and $t_0 = 1$s. Notice that the correction factor introduces a bias in the residual moveout; the undercorrection at intermediate offsets is compensated by an overcorrection at large offsets. The inset shows the residual moveout for the [2/2] rational interpolation at a larger scale. The residual moveout is strictly zero at the specified ODR values of $k_1 = 1$, $k_2 = 2$, $k_3 = 3$, and $k_4 = 4$. We note that

*Because of our observation that the traveltime errors do not depend on V_{NMO}, the actual value of V_{NMO} is irrelevant.

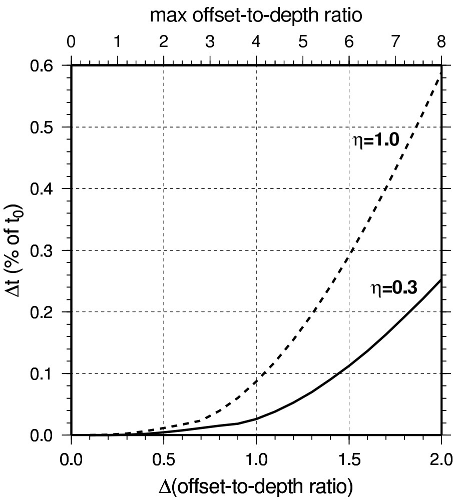

Figure 5. Accuracy of the [2/2] rational interpolation as a function of the separation in ODR of the interpolation traveltimes for $\eta = 0.3$ and $\eta = 1.0$. In both cases $V_{NMO} = 2.4$ km/s. Even if the separation in ODR is two, i.e., the maximum ODR is 8, the percentage errors in traveltime are of the order 10^{-1}.

extrapolation beyond k_4 is dangerous and can result in large errors in traveltimes because the rational approximation starts oscillating with typically increasing amplitudes. The onset of such oscillation can be seen in Figure 4 for ODR values beyond $k_4 = 4$. Rational interpolation should never be used for extrapolation. That is, k_4 should be chosen such that it exceeds the largest offset in the data.

Comparing Figures 1c and 1f, it seems that use of larger intervals of Δk between the k_i causes the maximum traveltime errors to increase. To analyze the influence of Δk, and to see what maximum ODR can be reached with [2/2] rational interpolation with considerable accuracy, Figure 5 shows the maximum absolute percentage traveltime error as a function of Δk. The top horizontal scale of the figure shows the maximum ODR. Here we used $V_{NMO} = 2.4$ km/s and $t_0 = 1$ s, and plot the traveltime errors for two different values of η, i.e., $\eta = 0.3$ (solid) and $\eta = 1.0$ (dashed). Note again that the value of V_{NMO} is irrelevant; errors do not depend on V_{NMO}, as explained previously. The high η value for the dashed curve can be taken as a worst-case scenario with respect to accuracy. For practical values of η, the maximum error in traveltime for ODR values up to 8 (achieved with $k_1 = 2$, $k_2 = 4$, $k_3 = 6$, and $k_4 = 4$), is of the order of a tenth of a percent of t_0. Therefore, for almost all practical cases, our [2/2] rational interpolation provides an accurate nonhyperbolic moveout approximation. If needed, higher-order rational interpolation, combined with $\Delta k = 1$, would obtain extra accuracy.

Figure 6 shows semblance scans (at fixed t_0) as a function of V_{NMO} and V_{hor} calculated using the nonhyperbolic moveout equation without the correction factor (a, d, and g), with the optimal correction factor (b, e, and h), and the [2/2] rational interpolation (c, f, and i). In each row of figures the true model parameters (indicated by the star) vary, and are, respectively, $V_{NMO} = 2000$ m/s and $V_{hor} = 2300$ m/s (or $\eta = 0.16$) for a-c, $V_{NMO} = 2892$ m/s and $V_{hor} = 3745$ m/s (or $\eta = 0.34$) for d-f, and $V_{NMO} = 2464$ m/s and $V_{hor} = 3880$ m/s (or $\eta = 0.74$) for g-i; the semblance maxima are indicated by the squares. Note that the synthetic gathers were generated using ray-traced travel-

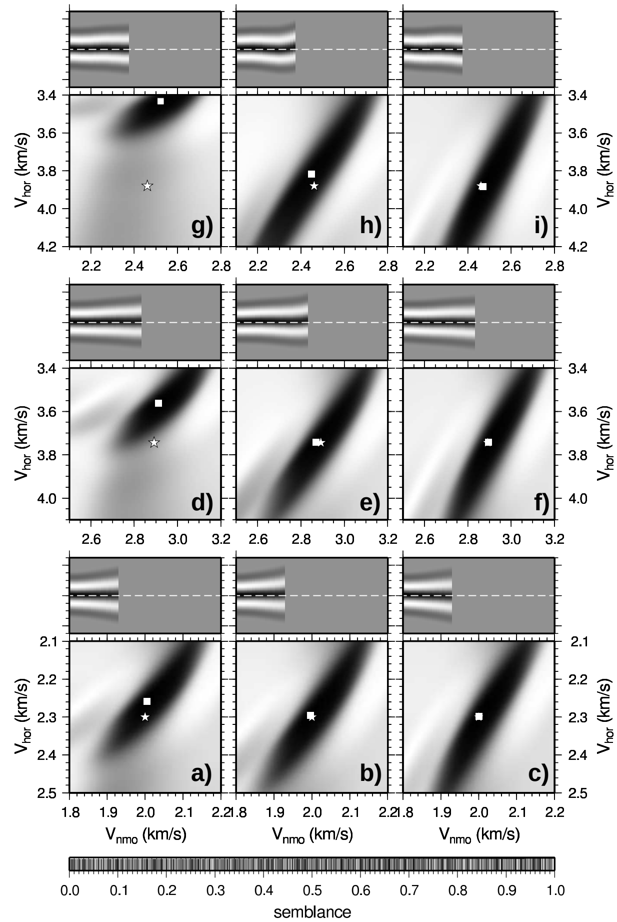

Figure 6. Semblance scans and moveout-corrected gathers, where the moveout correction was done with the parameters related to the maximum semblance values. The model for each subfigure consists of a single horizontal VTI layer, and the maximum ODR is two. For all models, $t_0 = 1$ s. The true model parameters are indicated by the star, and are: $V_{NMO} = 2.0$ km/s and $V_{hor} = 2.3$ km/s (i.e., $\eta = 0.16$) in subfigures a - c; $V_{NMO} = 2.892$ km/s and $V_{hor} = 3.745$ km/s (i.e., $\eta = 0.34$) in d - f (shale under zero confining pressure); and $V_{NMO} = 2.46$ km/s and $V_{hor} = 3.88$ km/s (i.e., $\eta = 0.74$), in g - i (Green River Shale). In the first column, the differences are calculated using equation (2) with $C = 1$, while in the second column the optimal correction factor was used. In the third column the [2/2] rational interpolation was used.

times, where the ray tracing was done using the true values of δ, i.e., $\delta = 0$ for a-c, $\delta = -0.05$ for d-f, and $\delta = -0.22$ for g-i, while we used $V_{S0} = 0$ km/s for all models.

The model parameters for the lower row of figures correspond to the model parameters used in Figures 1 and 2 of Grechka and Tsvankin (1998). [These parameters were originally chosen because such values of η were observed on field data (Alkhalifah *et al.*, 1996)]. For the middle row, the model parameters correspond to a shale under zero confining pressure, and for the top row, the parameters correspond to Green River shale [see Table 1 in Thomsen (1986) for these two cases]. For all subfigures (i.e., a - i) we used $t_0 = 1$ s and a maximum ODR of two. For the rational interpolation we used $k_1 = \frac{1}{2}$, $k_2 = 1$, $k_3 = \frac{3}{2}$, and $k_4 = 2$. Since for the lower row in Figure 6 the model parameters are identical to those in Figures 1 (and 2) of Grechka and Tsvankin (1998), our Figure 6a is the semblance scan equivalent of Figure 1 in their paper. The C values used in Figures 6b, e, and h were determined in the following way: the estimates of V_{NMO} and V_{hor} determined without the correction factor were used to determine the optimal C-value for this model from Figure 3. This method mimics the way C would be determined in practice.

Figures 6a, d, and g show that the the nonhyperbolic moveout equation (2) without the correction factor obtains high semblance values, but for the wrong values of V_{hor}; V_{NMO} seems largely unbiased. This means that the associated common midpoint (CMP) gathers are well flattened using the wrong value of V_{hor}, hence introducing a bias in the estimated value of V_{hor}, and thus η. The associated moveout corrected gathers for all three models are shown above the semblance scans; the gathers are well flattened with the biased estimate of V_{hor}. Note that for all three models, the semblance maximum (indicated by a square) for the nonhyperbolic moveout equation without the correction factor (Figures 6a, d, and g) indicates a value of η smaller than the true value (a smaller difference between V_{hor} and V_{NMO} than the difference between their true respective values). Hence, this method seems to underestimate η. Without the correction factor, accurate η values are obtained only when the true η values are small (≤ 0.1).

Figures 6b, e, and h indicate that using the correction factor in equation (2) gives maximum semblance values for values of V_{NMO} and V_{hor} that almost coincide with the true values, except for the model with extreme anisotropy (Figure 6h); for this model the method slightly underestimates V_{hor} (and thus η). The associated moveout-corrected CMP gathers are well flattened although careful analysis of the moveout corrected gather for the model with extreme anisotropy indicates a slight residual moveout. Overall, we conclude that even for large levels of anellipticity and ODR up to two, the correction factor in the nonhyperbolic moveout equation (2) allows for accurate estimation of the model parameters V_{NMO} and V_{hor}, even though use of the correction factor leaves some distortion in the moveout correction that becomes more noticeable with increasing anisotropy. Straightforward application of a correction factor $C = 1.2$ (not shown here) provided accurate estimates for the model shown in the bottom row only, for which the optimum C value was close to 1.2. The optimum C values for Figures 6b, e, and h, found from Figure 3, were $C = 1.23$, $C = 1.36$, and $C = 1.46$, respectively. Straightforward application of the [2/2] rational interpolation method provides, for all models, maximum semblances that coincide with the true model parameters, and accurately flattens gathers without any residual moveout (see Figures 6c, f, and i). Of course no optimization is required.

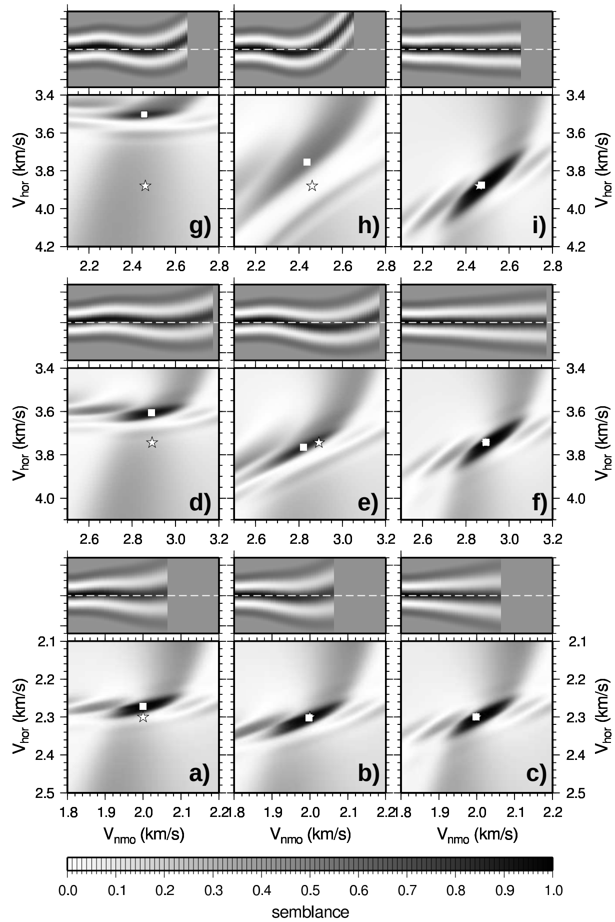

Figure 7. Same as Figure 6, except that the maximum ODR is now four.

Figure 7 shows the same semblance scans and moveout corrected gathers, for the same models and methods as in Figure 6, except that here the maximum ODR is four. For the rational interpolation we used $k_1 = 1$, $k_2 = 2$, $k_3 = 3$, and $k_4 = 4$. Notice that for all models and methods, the peak of the semblance scans is much better defined than in Figure 6; that is, the large semblance values span a much smaller range of V_{hor} (and thus η) values, and a somewhat smaller range of V_{NMO} values. Indeed the higher resolution for η for larger offset ranges was mentioned by Alkhalifah (1997), Grechka and Tsvankin (1998), and Wookey *et al.* (2002). Aside from the improved resolution in η, Figures 7a, d, and g, show again the underestimation of η when the nonhyperbolic moveout equation is used without the correction factor, just as in Figures 6a, d, and g. The maximum semblance, however, now deteriorates with increasing levels of anellipticity, and the associated moveout-corrected CMP gathers clearly indicate a distortion in the moveout, even for the model with $\eta = 0.16$ (see Figure 7a). For the optimum correction factor C (Figures 7b, e, and h), the bias in the estimated values of η is much reduced, but the associated residual moveout is more pronounced than without the use of the correction factor. Again, this is clearly observed even on the model with $\eta = 0.16$ (Figures 7a and b). The rational interpolation method (Figures 7c, f, and i) gives unbiased estimates of η and V_{NMO} *combined with no residual moveout.* This is a direct consequence of the high accuracy of the rational interpolation method shown in Figure 1f for this ODR.

From Figures 6 and 7, for ODR up to two, the nonhyperbolic moveout equation without the correction factor allows for accurate moveout correction for arbitrary levels of anellipticity, but does so with a biased estimate of V_{hor} (and thus η). The bias in η can subsequently be corrected for by determining the optimal value of the correction factor C using Figure 3, and redoing the semblance scans. The value of η determined in this way, is largely unbiased, and results in accurate moveout correction except for the model with extreme anisotropy (see Figures 7g and h), where a slight bias in η and residual moveout correction remains. With increase of ODR up to four, the nonhyperbolic moveout equation no longer gives accurate moveout correction, not even for the model with $\eta = 0.16$. Use of the optimal correction factor C reduces the bias in the estimated value of η, but increases the residual moveout. The rational interpolation method we propose, combines accurate moveout correction with unbiased parameter estimation, for arbitrary levels of anellipticity and ODR up to four. If accuracy is desired for larger offsets, higher-order rational interpolation can be used, or [2/2] rational interpolation can be used with an increased interval between the ODR k_i (say $k_1 = 2$, $k_2 = 4$, $k_3 = 6$, $k_4 = 8$). Figure 5 shows that with the [2/2] rational interpolation we can achieve reasonable accuracy up to ODR of 8.

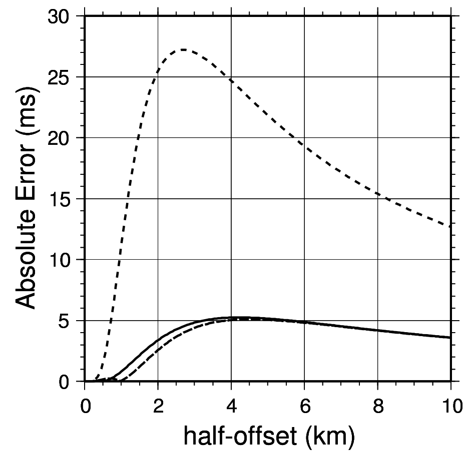

Figure 8. Comparison of accuracy between the nonhyperbolic moveout equation without the correction factor (dotted), the shifted hyperbola method of Fomel (2004) (dashed), and our squeeze-stretch approach (solid), for Greenhorn-shale anisotropy ($\eta = 0.34$ and $V_{NMO} = 2935$ m/s). The absolute traveltime errors are calculated with respect to ray-traced traveltimes. The model parameters are: $V_{P0} = 3094$ m/s, $V_{S0} = 1510$ m/s, $\delta = -0.05$, and $\epsilon = 0.256$ (Greenhorn shale).

Accuracy comparison with the nonhyperbolic moveout approximation of Fomel

Recently, Fomel (2004) proposed a shifted-hyperbola approximation for the group velocity and converted this into the following moveout equation for a single homogeneous horizontal VTI layer,

$$t^2(x) = \frac{3+4\eta}{4(1+\eta)} H(x) + \frac{1}{4(1+\eta)} \times \dots \sqrt{H^2(x) + 16\eta(1+\eta)\frac{t_0^2 x^2}{(1+2\eta)V_{NMO}^2}}\,, \quad (21)$$

where $H(x)$ represents the hyperbolic part,

$$H(x) = t_0^2 + \frac{x^2}{(1+2\eta)V_{NMO}^2}\,. \quad (22)$$

Fomel (2004) showed that this approximation is significantly more accurate than the nonhyperbolic moveout equation (1) of Alkhalifah and Tsvankin (1995).

Our rational interpolation approach is based on the approximations that (1) the influence of V_{S0} on qP-wave traveltimes in TI media is negligible, and (2) the kinematics of qP-waves in homogeneous VTI media depend mainly on V_{NMO} and η; i.e., we can set $\delta = 0$. Figure 8 shows the traveltime difference between ray-traced traveltimes for a single horizontal VTI layer with Greenhorn-shale anisotropy (Jones & Wang, 1981) and the nonhyperbolic moveout equation (1) form Alkhalifah and Tsvankin (dotted), the shifted hyperbola equation (21) from Fomel (dashed), and traveltimes resulting from ray tracing with $V_{S0} = 0$ km/s and $\delta = 0$

ODR = 2

	C=1.0				**optimal C**				**RI**			
layer	s	V_{NMO}	V_{hor}	η	s	V_{NMO}	V_{hor}	η	s	V_{NMO}	V_{hor}	η
1	1.00	2097	2101	0.00	1.00	2097	2101	0.00	1.00	2099	2101	0.00
2	0.98	2058	2179	0.06	0.98	2056	2188	0.07	0.98	2058	2188	0.07
3	1.00	2271	2839	0.28	0.98	2253	2996	0.38	1.00	2252	3022	0.40
4	0.99	2313	3012	0.35	0.93	2290	3255	0.51	1.00	2282	3285	0.54

ODR = 4

	C=1.0				**optimal C**				**RI**			
layer	s	V_{NMO}	V_{hor}	η	s	V_{NMO}	V_{hor}	η	s	V_{NMO}	V_{hor}	η
1	1.00	2094	2099	0.00	1.00	2094	2099	0.00	1.00	2096	2099	0.00
2	0.95	2029	2207	0.09	0.95	2025	2218	0.10	0.96	2035	2218	0.09
3	0.57	2238	2894	0.34	0.54	2299	2892	0.29	0.84	2189	3158	0.54
4	0.57	2320	3038	0.36	0.57	2300	3152	0.44	0.87	2241	3391	0.64

Table 1. Comparison of obtained semblance values (s), V_{NMO}, V_{hor}, and η,for the three methods, for maximum ODR of two (top) and four (bottom), for a layered medium. The model consists of four horizontal homogeneous VTI layers with the following values of V_{NMO} and η. The parameters for each layer are given in the main text.

but the same values of V_{NMO} and η (solid). The anisitropy parameters are $V_{P0} = 3094$ m/s, $V_{S0} = 1510$ m/s, $\delta = -0.05$, and $\epsilon = 0.256$ (i.e., $V_{NMO} = 2935$ m/s and $\eta = 0.34$), and $t_0 = 646.5$ ms. Except for the solid line, Figure 8 reproduces Figure 7 of Fomel's (2004) paper. Note that the accuracy of the shifted hyperbola approximation is marginally better than the ray-traced traveltimes with $V_{S0} = 0$ km/s and $\delta = 0$ but the same values of V_{NMO} and η. The maximum difference between these two traveltime approximations is about 1 ms (i.e., 0.15% of t_0) for this particular model. From a practical point of view, the two approximations are therefore identical. Even though we have not calculated the solid line with a rational interpolation, we can approximate the solid line to almost arbitrary precision with rational interpolation (i.e., with $[M/N]$ rational interpolation where $M, N > 2$). For ODR up to four, we showed this in Figure 1f using a [2/2] rational interpolation. Since the rational interpolation method that we propose uses traveltimes calculated with $V_{S0} = 0$ km/s and $\delta = 0$, we conclude that the accuracy of our method is basically identical to that of the shifted hyperbola approximation of Fomel (2004). Comparison of semblance scans and moveout corrections calculated using equation (21) and the rational interpolation method (not shown here), for the models and offsets studied in Figures 6 and 7, showed basically no difference between both methods.

Application to horizontally layered VTI media

Up to this point, we have treated only a single horizontal VTI layer. In this section we test the applicability of the rational interpolation method to a horizontally layered VTI medium, and compare its accuracy to that of the nonhyperbolic moveout equation with and without the correction factor. Based on the work of Tsvankin and Thomsen (1994), Grechka and Tsvankin (1998) showed that the nonhyperbolic moveout equation (1) remains valid in horizontally layered media provided the parameters η and V_{NMO} are replaced by 'effective' values that are some average over the vertically heterogeneous overburden. Also, they rewrite the Dix-type inversion procedure, originally introduced by Tsvankin and Thomsen (1994), in terms of η and V_{NMO}. In that method, the effective values of η and V_{NMO}, obtained from applying the single-layer equation (1) to data from a layered medium, are used to estimate the interval values of η and V_{NMO}.

We compare the effective values of η and V_{NMO} obtained using the nonhyperbolic moveout equation, with

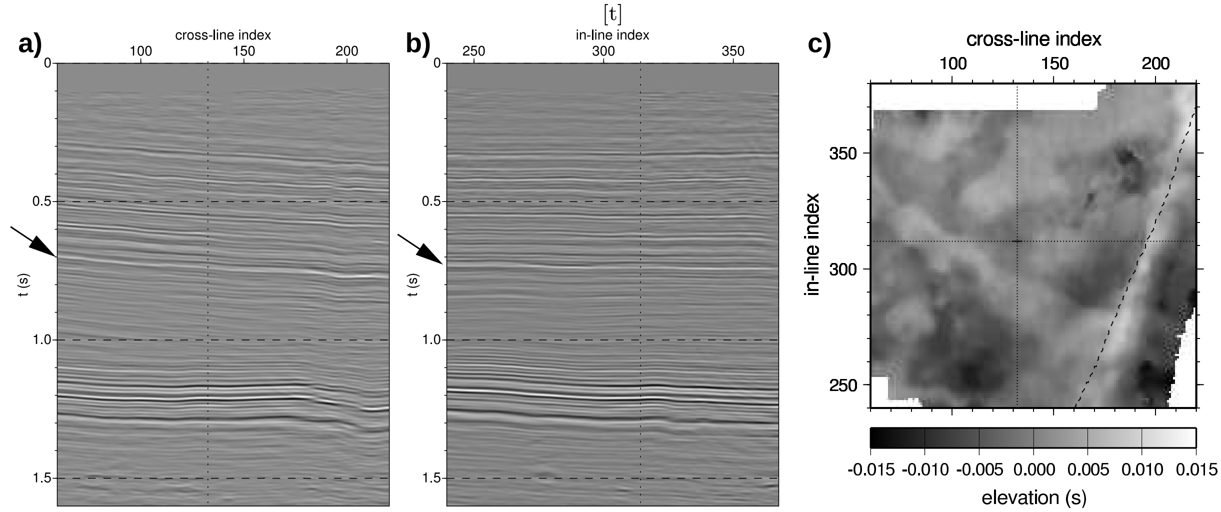

Figure 9. Field data used to test the rational interpolation method. Inline (a), crossline (b), and plan view of the residual topography of the event used for testing (indicated by the arrows in a and b) after removal of the regional dip (c). The dotted lines in c denote the locations of the inline and crossline sections shown in a and b. The dashed line in c denotes the location of a regional fault.

and without the correction factor, and the rational interpolation method, through application of all methods to a horizontally layered medium with the following parameters:

- layer 1 : $h = 1$ km (h is the depth of the bottom of the layer), $V_{NMO} = 2098$ m/s, $V_{hor} = 2098$ m/s, $\eta = 0$, ($V_{P0} = 2000$ m/s, $\epsilon = 0.05$, $\delta = 0.05$),
- layer 2 : $h = 2$ km, $V_{NMO} = 2000$ m/s, $V_{hor} = 2300$ m/s, $\eta = 0.16$, ($V_{P0} = 2000$ m/s, $\epsilon = 0.16$, $\delta = 0$),
- layer 3 : $h = 3$ km, $V_{NMO} = 2892$ m/s, $V_{hor} = 3745$ m/s, $\eta = 0.34$, ($V_{P0} = 3048$ m/s, $\epsilon = 0.255$, $\delta = -0.05$),
- layer 4 : $h = 4$ km, $V_{NMO} = 2460$ m/s, $V_{hor} = 3880$ m/s, $\eta = 0.74$, ($V_{P0} = 3292$ m/s, $\epsilon = 0.195$, $\delta = -0.22$).

For all models we used $V_{S0} = 0$ km/s. Note that here the first layer is elliptically anisotropic, and the second through fourth layers have the same model parameters as those in the models studied in the first through third row of Figures 6 and 7, respectively.

Table 1 shows the results from all three methods for a maximum ODR of two (top) and four (bottom). These results closely resemble the results obtained from the single layer numerical tests shown in Figures 6 and 7. Since all methods are for a single-layer VTI medium only, and because we apply these methods to layered VTI media, the η, V_{NMO}, and V_{hor} values in Table 1 are all 'effective' values. For maximum ODR=2, the nonhyperbolic moveout equation without the correction factor gives consistently lower estimates of η than do the other methods, and the differences increase with depth, i.e., as we increase the level of anellipticity. For that method, the semblance values (indicated with s in Table 1) are high for all layers, indicating high quality moveout correction. Use of the optimal correction factor gives comparable values of η and V_{NMO} to those obtained with the rational interpolation method, for all layers, just as for the single-layer case (see Figure 6). Also, the semblance values from the nonhyperbolic moveout equation with the correction factor slightly decrease with increasing depth (or level of anellipticity), just as for the single-layer case. For maximum ODR=4, the nonhyperbolic moveout equation with and without the correction factor result in decreasing semblance with depth (i.e., increasing level of anellipticity), indicating a lack of ability to accurately moveout-correct the data. Again this is analogous to the results from the single-layer case (Figure 7). That the rational interpolation method has substantially larger semblance values than do the other two methods, for all layers, indicates that this method is able to flatten the gathers best, with the least residual moveout. From a practical point of view, the method that flattens the gathers best, is expected to give the most confidence in the estimated values of η and V_{NMO}. From this point of view, the rational interpolation therefore provides more robust estimates of η and V_{NMO} than do the other two methods, for maximum ODR=4.

It thus seems that the rational interpolation method, which is based on a single horizontal VTI layer, is suitable for application to a horizontally layered medium, at least up to maximum ODR=4 and arbitrary levels of anellipticity. For maximum ODR=2, the nonhyperbolic moveout equation with the correction factor gives comparable estimates of the effective η and V_{NMO} and comparable semblance values to those of the rational interpolation approach. This indicates that for maximum ODR=2, even for extreme levels of anellipticity the effective values of η and V_{NMO} estimated

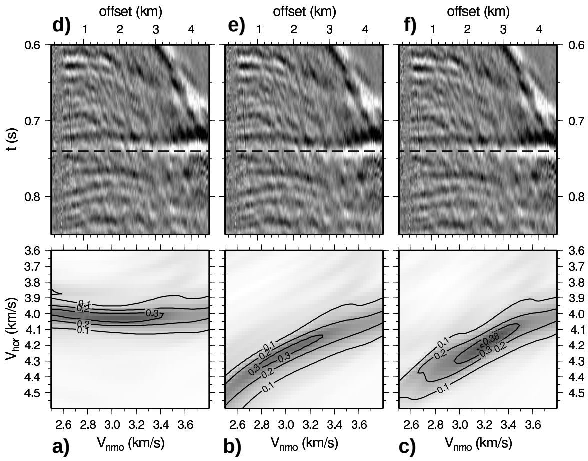

Figure 10. Semblance scans (a-c) for one CMP gather, located at the crossing of the horizontal lines in Figure 9c; its location is also indicated by the dotted lines in Figures 9a and b. The associated moveout corrected gathers are shown in subfigures d-e. The methods used for parameter estimation (i.e., to calculate the semblance scans) and moveout correction are, respectively, nonhyperbolic moveout without (a,d) and with (b,e) the optimal correction factor, and the [2/2] rational interpolation (c,f). A 100-ms window centered around $t_0 = 750$ ms was used in the computation. The contours in a-c indicate the semblance values. Maximum semblance values for the different methods are, respectively, 0.37 (a), 0.32 (b), and 0.40 (c), and the resulting estimates for η and V_{NMO} are, respectively, $\eta = 0.34$ and $V_{NMO} = 3100$ m/s (a), $\eta = 0.45$ and $V_{NMO} = 3100$ m/s (b), and $\eta = 0.33$ and $V_{NMO} = 3250$ m/s (c).

with the rational interpolation method can be used in the Dix-type inversion procedure given by Grechka and Tsvankin (1998). The role of the correction factor in this procedure, however, is currently unclear to us. For maximum ODR=4, the effective values of η and V_{NMO} for substantial anellipticity (say effective η values larger than 0.2) differ substantially between all methods, with the rational interpolation method uniformly giving the best moveout correction (i.e., highest semblance values). This indicates room for extending the rational interpolation method to the vertically heterogeneous case, and thus for developing an inversion procedure based on rational interpolation. It remains to be seen if such an extension is possible.

Field data example

Figures 9a and b show inline and crossline stacked time sections from a land dataset that contains a reflector at about 750-ms two-way traveltime (indicated by the arrows), illuminated with ODR ranging to larger than four. We focus attention on this event throughout the remainder of this section. The geology consists of relatively flat (dip less than two degrees), predominantly shale layers, such that a layered VTI model seems, at first sight, appropriate for these data. Although the structure on the horizon of interest is limited to within ±20 ms of a best-fit planar dip, a subtle NNE-SSW structural trend, associated with deeper faulting (Figure 9c), exists. The dashed line indicates the location of a regional fault. The inline and crossline spacings are both 110 ft.

To test our method, we calculated semblance as a function of V_{NMO} and V_{hor} for the whole dataset over a 100-ms window centered on the event of interest. The offsets used in the analysis were limited to offsets with a maximum ODR of approximately four. Figure 10a-c show the semblance scans for all three methods: (a) the nonhyperbolic equation without and (b) with the optimal correction factor, and (c) the [2/2] rational interpolation, for a randomly selected CMP gather. The location of this gather is indicated by the intersection of the dotted lines in Figure 9c; the location is also indicated by the vertical dotted lines in Figures 9a and b. This gather was generated by collecting traces from a 3-by-3 super bin of adjacent CMPs, and subsequent offset-binning. The change of shape of the semblance contours observed on Figures 10a-c, resembles the change observed from the synthetic data tests; the nonhyperbolic moveout equation without the correction factor exhibits no clear evidence of the inherent trade-off relation between η and V_{NMO}, whereas the other two methods do display this known trade-off. The semblance peak is most clearly defined for our [2/2] rational interpolation method because of its higher accuracy for such a large ODR range. The derived η, V_{NMO}, and maximum semblance values for the three methods are respectively: 0.34, 3100 m/s, 0.38 (no correction factor); 0.45, 3050 m/s, 0.33 (correction factor); 0.33, 3250 m/s, 0.40 (rational interpolation).

Figure 10d-f shows the moveout-corrected gather for the three methods, whit the semblance-derived values of η and V_{NMO} used for the moveout correction. Note that the residual moveout for the nonhyperbolic moveout equation with and without the correction factor is substantial, whereas the rational interpolation method gives well corrected moveout. The estimated values of η are close to the η value for the modeled results shown in Figure 7d-f. Notice the striking resemblance between the semblance scans and the residual moveouts obtained from both the synthetic and field data (cf. Figure 7d-f). We found that straightforward application of the correction factor $C = 1.2$ resulted in even larger residual moveout than that shown in Figure 10e.

Figure 11a, b, and c show mapviews of the values of η obtained for the event of interest over the entire dataset, using, respectively, the (a) nonhyperbolic moveout equation without and (b) with the optimal correction factor, and (c) the [2/2] rational interpolation method, for maximum ODR of two. The η values ob-

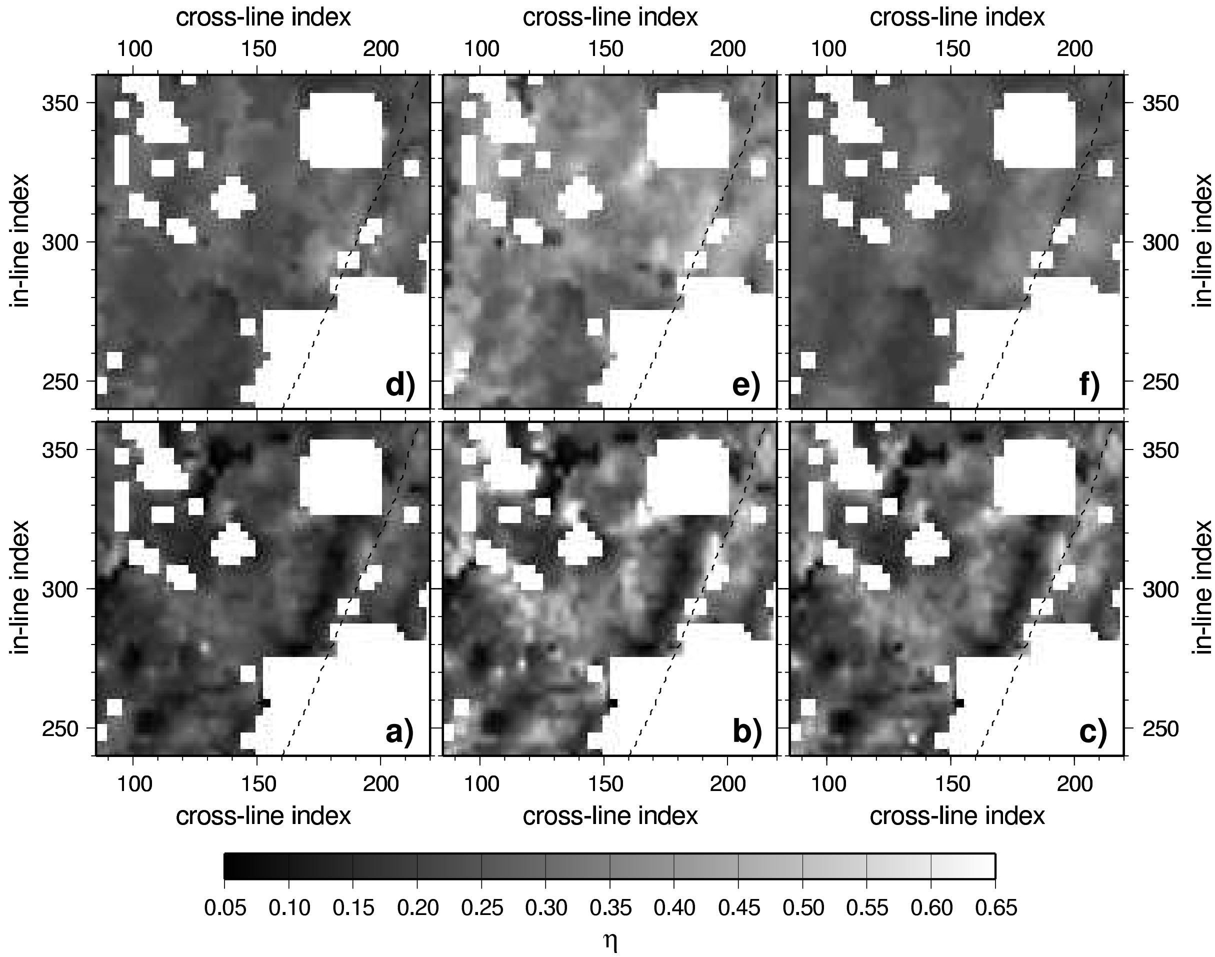

Figure 11. Map view of η derived from the event of interest in the field data with maximum offset of 2250 m, i.e., ODR$\approx$ 2 (a-c), using semblance scans with the nonhyperbolic moveout equation without (a) and with the optimal correction factor (b), and the [2/2] rational interpolation method (c). Subfigures d-f are as a-c, except that the maximum offset is 4500 m, i.e., ODR$\approx$ 4. Estimates of η at locations with poor offset distributions were set to white.

tained using nonhyperbolic moveout equation without the correction factor (a), are generally smaller than the estimated η values from both other methods, just as in our synthetic data examples (cf. Figure 6). The estimates of η obtained using the rational interpolation method (c) and the optimal correction factor approach (b) are quite comparable. This striking similarity is consistent with the results from the synthetic data also. The same comparison for V_{NMO} (not shown) showed that all three methods gave similar estimates of V_{NMO}, just as in the synthetic examples.

Figure 11d, e, and f, are as a, b and c, but here the maximum ODR is four. Notice, that the estimated η values are spatially less variable, for all methods. This can be understood in light of the improved resolution in η for larger ODR, together with the lack of evidence of substantial lateral heterogeneity from the seismic data (cf. Figure 9). Here, the correction-factor approach results in large estimates of η, compared to those of the other methods. From the synthetic tests, we would expect this method to give η values comparable to the estimated η values obtained using rational interpolation. None of our synthetic tests, however, included amplitude-versus-offset (AVO) variations, whereas the field data example clearly does (cf. Figure 10). Some additional synthetic tests (not shown), indicate that the rational interpolation method is less sensitive to such AVO variations than is the nonhyperbolic moveout equation method, with or without the correction factor; this can be explained by the high accuracy of the rational interpolation method†. Therefore, the lack of resemblance between the estimated η values for the correction factor approach and the rational interpolation may be due to

†We did not include phase changes with offset in the tests concerning sensitivity to AVO variation.

the AVO variations in the field data. Note that the presence of AVO variations causes semblance based moveout correction to be biased towards the offsets with higher amplitudes. Such bias is especially noticeable when an inaccurate moveout approximation is used to moveout correct the data.

The η values resulting from the rational interpolation method are on average slightly higher than those from the nonhyperbolic moveout equation method without the correction factor; $\eta_{av} = 0.28$ for the rational interpolation method, and $\eta_{av} = 0.25$ for the nonhyperbolic moveout equation method. The lower values of η for the nonhyperbolic moveout equation method without the correction factor are consistent with the results from the synthetic-data tests (cf. Figure 7). For all methods, the η values are high compared to values typically reported from nonhyperbolic moveout analysis [e.g. Toldi *et al.* (1999)]. Of course the relatively high η values are likely a result of the particular lithology of this area. However, most studies reporting estimates of η have been done on marine data, where a substantial waterlayer reduces the effective values of η, hence introducing a bias in what are traditionally considered acceptable values of η.

The [2/2] rational interpolation resulted in more spatially smooth and continuous values of both η (and V_{NMO}) that in some parts correlate somewhat with the geologic trend (cf. Figure 9c). It is important to note that this spatial continuity was not imposed, but followed from a straightforward application of the rational interpolation method presented here. Since the seismic data show no indication of substantial lateral heterogeneity (cf. Figure 9), the relatively smooth spatial variation of η is consistent with the seismic data. This increases our confidence in the η values obtained using the rational interpolation method.

Figure 12a shows the normalized semblance difference between nonhyperbolic moveout equation method with and without the optimal correction factor, for maximum ODR=2. The normalized semblance difference is the difference between the semblances from both methods divided by the semblance for the nonhyperbolic moveout equation method without the correction factor. Figure 12b is as Figure 12a, but it shows the normalized semblance difference between the [2/2] rational interpolation method and the nonhyperbolic moveout equation without the correction factor. For these offsets, all methods result in similar semblances, indicating similar ability to flatten the gathers. This supports our findings from the numerical tests that all methods obtain comparable results in terms of moveout correction, for an ODR range up to two (cf. Figure 6). Figures 12c and d are as 12a and b, except that here the maximum ODR is four. For this range of ODR, the correction-factor approach has on average 7% lower semblance values than does the nonhyperbolic moveout equation without the correction factor, whereas the [2/2] rational interpolation has on average 10% higher semblance values. This supports (for the whole dataset) our findings from the synthetic results that the correction factor approach increases distortion in the moveout correction and that the rational interpolation method results in more accurate moveout correction.

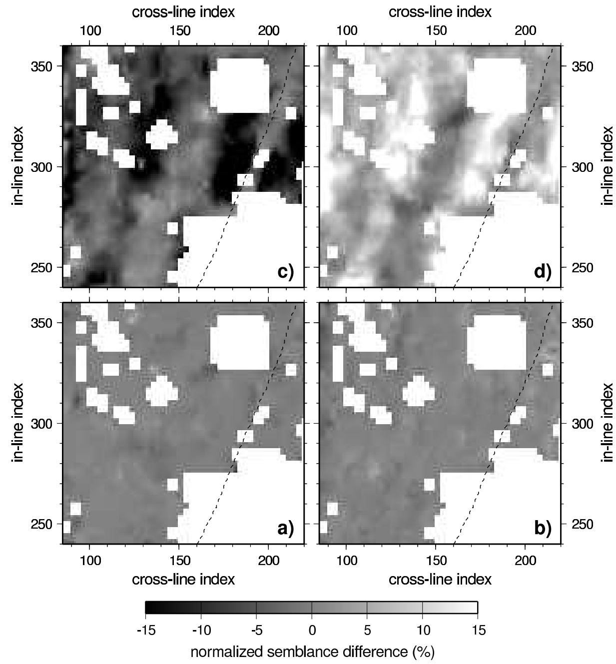

Figure 12. Normalized semblance difference between the nonhyperbolic moveout equation with, and without the correction factor (a), and between the [2/2] rational interpolation method and the nonhyperbolic moveout equation method without the correction factor (b), for a maximum offset of 2250 m, i.e., ODR≈ 2. Subfigures c and d are as a and b, respectively, except for a maximum offset of 4500 m, i.e., ODR≈ 4.

In conclusion, for a maximum ODR larger than two, the [2/2] rational interpolation method provided generally improved moveout correction, improved semblance values, and more spatially continuous estimates of η for this field data example. This supports the applicability of the rational interpolation method to a horizontally layered VTI medium, observed on synthetic data in the previous section. As the numerical tests showed similar results, we expect these findings to generalize to other datasets. For maximum ODR up to two, all methods had similar ability to flatten the gathers, but the nonhyperbolic moveout equation approach gave comparable estimates of η to those of the the rational interpolation method only when the optimal correction factor was used.

Discussion

We have shown that for ODR values up to 8, the [2/2] rational interpolation results in an accuracy of $O(10^{-1})$% of t_0 for most models of practical interest. If higher accuracy is desired, or accuracy up to larger ODR is needed (e.g., in near-surface seismic experiments), higher-order polynomials can be used in the rational approximation. The added computation time for inclusion of several extra terms is negligible; hence the efficiency of the proposed method remains essentially the same. We have not done any numerical testing for polynomials of order higher than two.

In the field data example, we treat the overburden of the reflection event of interest as a single horizontal homogeneous VTI layer. As a result, the estimated values of η and V_{NMO} are effective parameters. The geological significance of such effective quantities is difficult to establish, and an approach assuming a layered overburden and resulting interval estimates of η and V_{NMO} would remove this difficulty. Although we do not demonstrate such an approach in this paper, we believe that such an approach is feasible by applying rational interpolation in a layer-stripping fashion. We leave the verification of this idea to a future study. Meanwhile, the single-layer approach outlined here can be used to obtain more accurate estimates of the average (or effective) values of η and V_{NMO} in layered media. Using these values in current Dix-type averaging procedures could lead to better interval estimates of these parameters.

We showed that our rational interpolation approach achieves accuracy almost identical to that of the shifted hyperbola approximation of Fomel (2004). Since we show the applicability of the rational interpolation method to horizontally layered VTI media, the almost identical accuracy of both methods implies the applicability of the shifted hyperbola approximation of Fomel to such media also. For horizontally layered media, a plane-wave decomposition, obtained through a $\tau - p$ transform, is the natural decomposition of the data. This fact was successfully used by Van der Baan and Kendall (2002) and Van der Baan (2004), to obtain interval estimates of η and V_{NMO} from moveout in the $\tau - p$ domain. Even though they successfully estimate interval values of η and V_{NMO}, the acquisition geometry does not always allow a straightforward $\tau - p$ transform. Therefore, there is room to try to extend our $t - x$ based rational interpolation method to render interval estimates of η and V_{NMO}.

The geometry of straight rays involved in velocity analysis for a horizontal homogeneous layer and the geometry of the rays associated with point scattering in a medium with constant velocity are identical. Therefore, the rational interpolation method proposed here is immediately applicable to the problem of post-stack and pre-stack time-migration in VTI media. Larger offsets in the context of moveout velocity analysis are the equivalents of steeper dips in time-migration. Therefore, the increased accuracy for large maximum ODR provided by the rational interpolation for traveltimes of qP-waves in such media suggests improved accuracy when the rational interpolation scheme is used in imaging of steep reflectors in the context of pre and post-stack time-migration in VTI media.

Conclusions

We have presented a rational interpolation approach to nonhyperbolic moveout correction of qP-waves in VTI media. The accuracy of the method was tested using both synthetic and field data and compared with that of the nonhyperbolic moveout equation, which is the current standard in seismic data processing. Both synthetic and field data results confirm that our method significantly outperforms the nonhyperbolic moveout equation in both unbiased parameter estimation and the quality of moveout correction when a maximum ODR larger than two is used. For a single horizontal VTI layer, and for a maximum ODR up to four, the errors from the [2/2] rational interpolation are $O(10^{-2})$% of t_0 or less; this is one to two orders of magnitude more accurate than the nonhyperbolic moveout equation of Tsvankin and Thomsen (1994). Even for a maximum ODR up to 8, the traveltime errors resulting from the [2/2] rational interpolation are $O(10^{-1})$% of t_0, for virtually all models of practical interest.

Under the customary assumptions that traveltimes of qP-waves in VTI media depend mainly on η and V_{NMO}, and that the influence of V_{S0} on traveltimes of qP-waves in TI media is negligible, we found that the traveltimes in a single horizontal VTI layer, for fixed group-angle, η, and t_0, are independent of V_{NMO}, while the associated offsets are linear in V_{NMO}. As a consequence, therefore, the nonhyperbolic moveout curve for different values of V_{NMO}, but fixed η and t_0, can be calculated by simple horizontal stretching or squeezing, i.e., along the offset axis, where the amount of stretch or squeeze is determined by the change in V_{NMO}. This observation allows us to calculate the traveltimes needed for the interpolation from a small number of traveltimes for a reference value of t_0 (conveniently $t_0^{ref} = 1$ s) and η values ranging from -0.2 to 1.0 in steps of, say, 0.01. This range of η covers most models of practical interest. The few hundred traveltimes for a reference value of t_0 and η values ranging from -0.2 to 1.0, can be precomputed in about a minute on a modern PC and stored in a table. Therefore, the rational interpolation method has no additional computational overhead compared to that of the nonhyperbolic moveout equation method.

We show that the observation of the stretch-squeeze influence of V_{NMO} on the nonhyperbolic moveout causes the correction factor C in the Alkhalifah-Tsvankin nonhyperbolic moveout equation (2) to be independent of V_{NMO}, under the above-mentioned approximations.

This correction factor therefore depends on η and maximum ODR only. We calculated this correction factor for $0 \leq \eta \leq 1$ and maximum ODR between one and four, and presented a contour plot of $C(\eta, ODR)$. If one wants to get the best use out of the Alkhalifah-Tsvankin nonhyperbolic moveout equation (2), this plot can be used in practice to determine the optimal correction factor in the following way; first estimate η and V_{NMO} with the nonhyperbolic moveout equation without the correction factor, then use the estimated V_{NMO} to estimate the maximum ODR using equation (14), and subsequently determine the optimal value of C from Figure 3 using the estimated k and η. For maximum ODR up to four and large levels of anellipticity, tests with synthetic data showed that this correction factor works well to reduce the bias in η that would be obtained if the nonhyperbolic moveout equation without this correction factor was used. The field-data example confirmed this for maximum ODR up to two, but the presence of AVO variation for offsets with maximum ODR between two and four caused this method to fail for ODR up to four. For maximum ODR less than two, this reduction in bias goes together with largely accurate moveout correction, even for strong anellipticity. For ODR up to four, however, the reduction in bias of η goes together with significant distortion in the moveout correction caused by the inaccuracy of the nonhyperbolic moveout equation. In this case the Alkhalifah-Tsvankin nonhyperbolic moveout equation fails to accurately flatten the CMP gathers. The rational interpolation method combines unbiased estimates of η and V_{NMO} with accurate moveout correction in all cases.

From synthetic tests and field data, we found that straightforward application of the correction factor $C = 1.2$, recommended by Grechka and Tsvankin (1998), typically leads to less than satisfactory results. For maximum ODR=4 and $\eta \lesssim 0.1$, the shortcomings in moveout correction and estimated values of η obtained using the nonhyperbolic moveout equation approach without the correction factor are small and probably negligible from a practical point of view. Even though in this particular case satisfactory results can be obtained with the Alkhalifah-Tsvankin nonhyperbolic moveout equation, straightforward application of the rational interpolation method combines unbiased parameter estimation with accurate moveout correction, for arbitrary ODR ranges and arbitrary levels of anellipticity.

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APPENDIX A: [2/2] RATIONAL INTERPOLATION FOR NONHYPERBOLIC MOVEOUT IN A SINGLE HORIZONTAL VTI LAYER

Using the definition and normalization of the rational approximation outlined in the main text, we can write the [2/2] rational approximation for squared traveltimes T as a function of squared offset X as

$$T(X) \approx \frac{T_0 + n_1 X + n_2 X^2}{1 + d_1 X + d_2 X^2} \quad , \tag{A1}$$

where $T_0 = t_0^2$ is the squared zero-offset two-way traveltime, and $n_{1,2}$ and $d_{1,2}$ are the coefficients of the numerator and denominator of the rational approximant, respectively. Using four squared traveltimes $T_i = t_i^2$, with $i = 1, \ldots, 4$, and four accompanying squared offsets $X_i = x_i^2$ as interpolation points, we arrive at a linear system of four equations with four unknowns, the coefficients $n_{1,2}$ and $d_{1,2}$. Since there are only four coefficients, we simply solve (using Mathematica) for the coefficients in terms of T_i, X_i, and T_0. The resulting expressions for the coefficients are given by

$$\begin{aligned}
d_1 \;=\; & ((T_0 - T_4)X_1X_2X_3(T_1X_1(X_2 - X_3) + \ldots \\
& T_3X_3(X_1 - X_2) + T_2X_2(X_3 - X_1)) - \ldots \\
& ((T_1 - T_2)(T_0 - T_3)X_1^2X_2^2 + \ldots \\
& ((T_2 - T_0)(T_1 - T_3)X_1^2 + \ldots \\
& (T_0 - T_1)(T_2 - T_3)X_2^2)X_3^2)X_4 + \ldots \\
& ((T_0 - T_3)X_1X_2(T_1X_1 - T_2X_2 + \ldots \\
& T_4(X_2 - X_1)) - ((T_0 - T_2)(T_1 - T_4)X_1^2 - \ldots \\
& (T_0 - T_1)(T_2 - T_4)X_2^2)X_3 + \ldots \\
& (T_3 - T_4)(T_1X_2 - T_2X_1 + \ldots \\
& T_0(X_1 - X_2))X_3^2)X_4^2)/\ldots \\
& (X_1X_2X_3(T_2(T_3(X_2 - X_3)(X_1 - X_4) - \ldots \\
& T_4(X_1 - X_3)(X2 - X4)) + \ldots \\
& T_1(T_4(X_2 - X_3)(X_1 - X_4) - \ldots \\
& T_3(X_1 - X_3)(X_2 - X_4) + \ldots \\
& T_2(X_1 - X_2)(X_3 - X_4)) + \ldots \\
& T_3T_4(X_1 - X_2)(X_3 - X_4))X_4) \;,
\end{aligned} \tag{A2}$$

$$\begin{aligned}
d_2 \;=\; & (-T_0(X_1 - X_2)(X_1 - X_3)(X_2 - X_3) + \ldots \\
& T_3X_1X_2(X_1 - X_2)(1 + d_1X_3) + \ldots \\
& X_3(T_1X_2(1 + d_1X_1)(X_2 - X_3) + \ldots \\
& T_2X_1(1 + d_1X_2)(X_3 - X_1)))/\ldots \\
& (X_1X_2X_3(T_2X_2(X_1 - X_3) + \ldots \\
& T_3X_3(X_2 - X_1) + T_1X_1(X_3 - X_2))) \;,
\end{aligned} \tag{A3}$$

$$\begin{aligned}
n_2 \;=\; & T_0(X_1 - X_2) + \ldots \\
& T_1(1 + X_1(d_1 + d_2X_1))X_2 - \ldots \\
& T_2X_1(1 + X_2(d_1 + d_2X_2))/\ldots \\
& X_1X_2(X_1 - X_2) \;,
\end{aligned} \tag{A4}$$

$$n_1 = T_1 d_1 - \frac{T_0 - T_1}{X_1} - (n_2 - d_2 T_1) X_1 \;. \tag{A5}$$

Note that only the expression for d_1 is explicit in just T_i, X_i, and T_0, whereas d_2 also depends on d_1, n_2 on $d_{1,2}$, and n_1 on n_2 and $d_{1,2}$. Calculating the coefficients $n_{1,2}$ and $d_{1,2}$ using the above expressions and the interpolation traveltimes t_i and offsets x_i, we can use the resulting values of $n_{1,2}$ and $d_{1,2}$ in equation (A1) to evaluate interpolated traveltimes t for offsets x between offsets x_i.