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The theory of coda wave interferometry

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ABSTRACT

Coda waves are sensitive to changes in the subsurface because the strong scattering that generates these waves causes coda waves to repeatedly sample a limited region of space. Coda wave interferometry is a technique that exploits this sensitivity to estimate weak changes in the medium from a comparison of the coda waves before and after the perturbation. Here I present the general theory of coda wave interferometry, and show how the time-shifted correlation coefficient can be used to estimate the mean and variance of the travel time perturbation caused by the perturbation of the medium. This mean and average are defined based on the intensity of the coda waves. I show how this general theory can be used to estimate changes in the wave velocity, in the location of scatterer positions, and in the source location.

Key words: coda wave interferometry

Preface

In December 2004, a symposium was held at the Fall Meeting of the American Geophysical Union in honor of Keiiti Aki. He is known not only for writing the seminal textbook "Quantitative Seismology" with Paul Richards, but also for his prolific and groundbreaking research in seismology, and for his caring and creative frame of mind. Aki is one of the pioneers in the analysis of coda waves; he developed and implemented the analysis of the decay rate of these waves as a tool to monitor time-lapse changes in the Earth. He carried out numerous studies to apply this to monitor fault zones and volcanoes with the ultimate goal to develop better tools for hazard assessment.

Aki's use of coda waves has focused on the amplitude of these waves. At the Colorado School of Mines we developed a new technique, coda wave interferometry, that takes the phase information of these waves into account as well. This provides a tool to monitor time-lapse changes in the subsurface based on changes in the waveforms in the coda. We have applied this to measure velocity changes in rocks due to changes in the temperature or stress, to monitor stress changes in a mining environment, to determine the distance between earthquakes, and for volcano monitoring. We look forward to collaborate with our sponsors on the application of coda wave interferometry to reservoir monitoring.

The following paper gives the theory of coda wave interferometry. It will appear in the special issue of Pure and Applied Geophysics titled "Advances in Studies of Heterogeneities in the Earth Lithosphere: The Keiiti Aki Volume II." The treatment of this paper is general and can be used to a variety of different applications. The work on coda wave interferometry has been largely financed completely with financial support from the National Science Foundation.

1 INTRODUCTION

The seismic coda constitutes the tail of strongly scattered waves in a seismogram. Aki was one of pioneers in using the seismic coda [Aki and Chouet, 1975] . He used the temporal decay of the seismic coda as measure of the scattering in the earth, and proposed to use changes of coda Q to monitor changes in the stress in the subsurface [Aki, 1985; Jin and Aki, 1986]. This approach considers the amplitude of the coda waves, but does not use the phase information in the coda.

Here I present the theory of coda wave interferometry, a technique to monitor time-lapse changes based on the phase and amplitude information of coda waves.

In a strongly-scattering medium, the waves repeatedly sample the same region in space. Such a medium therefore works as a natural interferometer. Just as in a man-made interferometer [Lauterborn et al., 1995] the multiply-scattered waves are extremely sensitive to minute changes in the medium. In coda wave interferometry we exploit this sensitivity to measure small changes in the medium.

This idea is not new, the sensitivity of coda waves has been used to estimate velocity changes in fault zones [Poupinet et al., 1984], in volcanoes [Ratdomopurbo and Poupinet, 1995; Matsumoto et al., 2001] , in a mining environment [Grêt et al., 2004] , and in ultrasound experiments [Roberts et al, 1992; Snieder et al., 2002; Grêt et al., 2005] . Temporal changes in the coda waves within a a couple of days have been observed in a volcano . In the physics community a related technique called diffusing wave spectroscopy [Weitz and Pine, 1993; Cowan et al., 2002] has been used to monitor fluidized suspensions [Cowan et al., 2000; Page et al., 2000].

In this work I present the theory of coda wave interferometry. Because of the generality of the derivation, it can be applied to a a number of different applications of coda wave interferometry. The theory is based on the path summation (section 2). This is a formulation of scattering that states that the total wave field is the superposition of the waves that propagate along all possible scattering paths. In section 3 I show how the changes in the coda waves can be characterized with the time-shifted correlation coefficient. This quantity is related to the distribution of the travel time perturbation when averaged over all scattering paths (section 4). In principle, the distribution of the travel time perturbation can be obtained from the time-shifted correlation coefficient, but I show in section 5 how in practical application the mean and variance of the travel time perturbation can be obtained from the time-shifted correlation coefficients. In section 6 I apply the theory to three examples; a change in the velocity, uncorrelated perturbations of the scatterer locations, and a change in the source position.

2 THE PATH SUMMATION AND THE CHANGE IN THE WAVES

The theory of coda wave interferometry is based on the path summation where the wavefield at a given location is written as a sum of the waves that propagate along all possible paths [Snieder, 1999]:

$$u(t) = \sum_P S_P(t) \ . \tag{1}$$

The path summation is valid when the wavefield can be written as a discrete sum over all possible scattering paths, and expression (1) simply states that the total wavefield is the sum of the waves that have propagated along all possible scattering paths. I use a scalar notation, but expression (1) is also valid for elastic waves; in that case equation (1) can be used for each of the components of the wave motion. For elastic waves conversions between P and S-waves occur at the scatterers; in that case the path summation includes a sum P and S waves for each segment of the scattering path, so that $\sum_P$ includes all the possible wave conversions along each path as well.

A perturbation of the medium leads to a perturbation of the waves. I assume that the scattering properties of the scatterers in the medium do not change, but that either the propagation velocity, the position of the scatterers, or the source position is weakly perturbed. These perturbations correspond to a change in the phase of the wave propagation changes, the geometrical spreading, and the scattering angle for every scattering event.

Let us consider a wave that propagates over a distance l between scatterers. The average distance between the scatterers, as seen by the waves, is given by the scattering mean free path [Lagendijk and van Tiggelen, 1996; van Rossem and Nieuwenhuizen, 1999]. In the frequency domain the propagation over this distance corresponds to a phase shift $\exp(ikl)$. Let us first consider a change δl in the path length. This change corresponds to a phase change $\exp(ik\delta l) = 1 + i\delta kl$, so that a change in the path length gives the following change in the wavefield:

$$|\delta u|^{phase} \sim |k\delta l u| \ . \tag{2}$$

In three dimensions the geometrical spreading for each segment of propagation varies as $1/l$, hence the change in the wavefield due to the geometrical spreading associated with a change in path length is given by

$$|\delta u|^{spreading} \sim \left| \frac{\delta l}{l} u \right| \ . \tag{3}$$

A change δl in the scatterer position leads to a change in the scattering angle that is of the order $\delta\theta \sim \delta l/l$. When the scattering amplitude, or source radiation pattern, varies as $\cos m\theta$ or $\sin m\theta$, the change in the wavefield due to this change in scattering angle is of the order

$$|\delta u|^{angle} \sim |m\delta\theta| \, u = \left| \frac{m\delta l}{l} u \right| \ . \tag{4}$$

The change is the phase of the wavefield dominates when the right hand side expression (2) is larger than both (3) and (4). Since in general $|m| > 1$ this conditions imply that the change in the phase is dominant when

$$\frac{l}{\lambda} > \frac{|m|}{2\pi} \ , \tag{5}$$

where the wavelength is given by $\lambda = 2\pi/k$. This means that the change in the phase dominates the change of the wavefield when the scattering mean free path is much larger than a wavelength. If this condition is not satisfied the waves are localized [van Tiggelen, 1999] and the theory of this paper does not hold.

This means that when the distance between the scatterers is changed, the change in the phase is the most important change. It follows from expression (2) that the change in the phase varies linearly with frequency, because $k = \omega/c$. Such a change in the phase corresponds in the time-domain a change in the phase corresponds to a change in the arrival time of the wave. A change in the velocity of propagation also gives a change in the arrival time of the wave. I denote the change in the travel time of the wave that propagates along path P by τ_P. This means that the perturbed wavefield is given by

$$\tilde{u}(t) = \sum_P S_P(t - \tau_P) \ . \tag{6}$$

This expression should be compared with equation (1) for the unperturbed wave.

3 MEASURING THE CHANGE IN THE WAVEFIELD

The unperturbed and perturbed waves can be compared using the time-shifted correlation coefficient defined as

$$R(t_s) = \frac{\int_{t-T}^{t+T} u(t')\tilde{u}(t'+t_s)dt'}{\sqrt{\int_{t-T}^{t+T} u^2(t')dt' \int_{t-T}^{t+T} \tilde{u}^2(t')dt'}} \ . \tag{7}$$

In this expression t_s is the time shift of the unperturbed and perturbed waves in the correlation. The correlation is computed of a finite time-window with center-time t and window length $2T$. Let us first analyze the numerator of this expression that is given by

$$N(t_s) = \int_{t-T}^{t+T} u(t')\tilde{u}(t'+t_s)dt' \ . \tag{8}$$

Inserting the expressions (1) and (6) in this equation gives a double sum $\sum_{PP'}$over paths. Such a double sum can be divided into diagonal terms, for which $P = P'$, and cross-terms $P \neq P'$:

$$\sum_{PP'} (\cdots) = \sum_{P=P'} (\cdots) + \sum_{P\neq P'} (\cdots) \ , \tag{9}$$

with equations (1) and (6) this gives:

$$\begin{aligned} N(t_s) \quad &= \textstyle\sum_P \int_{t-T}^{t+T} S_P(t')S_P(t'+t_s-\tau_P)dt' \\ &+ \textstyle\sum_{P\neq P'} \int_{t-T}^{t+T} S_P(t')S_{P'}(t'+t_s-\tau_{P'})dt' \ . \end{aligned} \tag{10}$$

The cross-terms are uncorrelated, this means that on average the cross terms integrate to zero. In a single realization this is not necessarily the case, and the second term in expression (10) may not be zero. Snieder [2004] estimated the magnitude of the cross terms $\sum_{P\neq P'}$ to the diagonal terms $\sum_{P=P'}$, and showed that when the DC-component of the signal vanishes that in a single realization

$$\frac{|\text{cross terms}|}{|\text{diagonal terms}|} \sim \sqrt{\frac{T_{corr}}{2T}} \ , \tag{11}$$

where T_{corr} is the width of the autocorrelation of the signal in the time domain. This width is proportional to $1/\Delta f$, with Δf the bandwidth of the signal. This means that the ratio in expression (11) can also be written as

$$\frac{|\text{cross terms}|}{|\text{diagonal terms}|} \sim \sqrt{\frac{1}{\Delta f \ 2T}} \ . \tag{12}$$

The bandwidth times the window length is the number of degrees of freedom in the signal [Bucci and Franceschetti, 1989; Landau, 1967] , hence the ratio of the cross-terms to the diagonal terms is for a single realization equal to $1/\sqrt{N}$, with N the number of degrees of freedom in the data. The important point of expression (11) is that the cross-terms decrease when the window length is increased. Note that this argument does not hold for monochromatic data, because in that case $\Delta f = 0$ and the right hand side of expression (12) cannot be reduced by increasing the window length. This means that in order to ignore the cross-terms both the window length and the bandwidth must be sufficiently large.

In the following I assume that this indeed the case and that the cross-terms can be ignored, in this approximation

$$N(t_s) = \sum_P \int_{t-T}^{t+T} S_P(t')S_P(t'+t_s-\tau_P)dt' \ . \tag{13}$$

This integral can be written as a sum of the cross-correlations of the waves that have propagated along the individual paths that is defined as

$$C_P(t_0) \equiv \int_{t-T}^{t+T} S_P(t')S_P(t'+t_0)dt' \ . \tag{14}$$

With this definition $I(t_s)$ can be written as

$$N(t_s) = \sum_P C_P(t_s - \tau_P) \ . \tag{15}$$

A similar treatment can be applied to the terms in the denominator of equation (7), this gives

$$\int_{t-T}^{t+T} u^2(t')dt' = \int_{t-T}^{t+T} \tilde{u}^2(t')dt' = \sum_P C_P(0) \ . \tag{16}$$

In deriving this result the same approximations are used as in the derivation of expression (15), specifically the cross-terms $\sum_{P\neq P'}$ are ignored. Inserting the expressions (15) and (16) into the definition (7) of the correlation coefficient gives

$$R(t_s) = \frac{\sum_P C_P(t_s - \tau_P)}{\sum_P C_P(0)} \ . \tag{17}$$

Before we analyze this correlation coefficient I introduce another approximation. In the frequency domain the definition (14) for the correlation of S_P corresponds to

$$C_P(\omega) = S_P^*(\omega)S_P(\omega)\exp(i\omega t_0) = |S_P(\omega)|^2 \exp(i\omega t_0) \ . \tag{18}$$

I assume that the waves that propagate along the different trajectories have a power spectrum with the same shape, but that the amplitude of each of these waves may be different. Note that this does not imply that the waves $S_P(t)$ are the same, because the phase spectrum may be different. In fact, the phase spectrum will be different because these waves have different arrival times. Since the autocorrelation is the Fourier transform of the power spectrum, the assumption that shape of the the power spectrum of the waves is the same implies that up to a constant the autocorrelation also is the same, so that

$$C_P(t_0) = I_P C(t_0) \qquad \text{with} \qquad C(0) = 1 \; . \tag{19}$$

In this expression $C(t_0)$ is the autocorrelation of the $S_P(t)$ normalized at its maximum for $t_0 = 0$, while I_P measures the intensity of the wave that has propagated along path P. Using these results the correlation coefficient is given by

$$R(t_s) = \frac{\sum_P I_P C(t_s - \tau_P)}{\sum_P I_P} \; . \tag{20}$$

4 THE PROBABILITY DENSITY FUNCTION OF THE TRAVEL TIME PERTURBATION

The time-shifted cross correlation coefficient can be determined from the recorded waves using expression (7). The goal is to infer properties of the travel time perturbation from this measurement. One way to achieve this is to define the normalized energy of the arrivals in the employed time window with a travel time shift between τ and $\tau + d\tau$ as $P(\tau)d\tau$. This definition implies that

$$P(\tau)d\tau = \frac{\sum_{P \text{ such that } \tau < \tau_P < \tau + d\tau} I_P}{\sum_{all\ P} I_P} \; . \tag{21}$$

The sum in the numerator is all paths that have a travel time change between τ and $\tau + d\tau$. Since the denominator contains a sum over all paths, and hence all relevant values of τ, the function $P(\tau)$ is normalized:

$$\int_{-\infty}^{\infty} P(\tau)d\tau = 1 \; . \tag{22}$$

The definition (21) consists of a ratio of positive numbers, therefore $P(\tau)$ is positive and normalized. For this reason it has the same properties as a probability density function. Using statistical jargon, we use the following definition of an expectation value:

$$\langle f(\tau) \rangle \equiv \int_{-\infty}^{\infty} P(\tau) f(\tau) d\tau \; . \tag{23}$$

Using expressions (21) this can also be written in the path summation as

$$\langle f(\tau) \rangle \equiv \frac{\sum_P I_P f(\tau_P)}{\sum_P I_P} \; . \tag{24}$$

With the definitions (21) and (24), equation (20) can be written as

$$R(t_s) = \langle C(t_s - \tau) \rangle = \int P(\tau) C(t_s - \tau) d\tau \; . \tag{25}$$

The first identity shows that coda wave interferometry leads to a weighted average of a function of the travel time perturbation. The coda waves travel along all possible paths, and the contribution of the travel time perturbation is averaged over all possible paths with a weight function given by the intensity of the waves for each path.

The second identity of expression (25) states that the time-shifted correlation coefficient is given by the convolution of $P(\tau)$ with $C(\tau)$. The time-shifted correlation coefficient $R(t_s)$ follows from the recorded waves. The function $C(t)$ follows from the power spectrum of the recorded waves, and is known as well. In principle $P(\tau)$ can the be obtained from expression (25) by deconvolution. This provides direct information on the distribution of the travel time perturbations over the paths that have arrivals within the employed time window. Note that in this approach we do not need to assume that the travel time perturbation is small.

In practice this deconvolution approach may not work well for the retrieval of $P(\tau)$. In the frequency domain, the right hand side of expression (25) corresponds to the multiplication of the frequency spectra of P and C. The frequency spectrum of $C(\tau)$ is the power spectrum of the data. In general, there is no guarantee that the frequency spectrum of $P(\tau)$ overlaps with the power spectrum of the data. This would only be the case when the distribution of the travel time perturbation peaks near the dominant period of the waves. Since we cannot be sure that this condition is satisfied in a given experiment we take a different approach and extract the first and second moments of the travel time perturbation from expression (25).

5 EXTRACTING THE MOMENTS OF THE TRAVEL TIME PERTURBATION FROM THE CORRELATION

In this section we assume that the travel time perturbations do not change vary much among all the different paths with arrivals within the employed time window. Specifically, we assume that we can use a second-order Taylor expansion of $C(t)$. The autocorrelation is an even function, with $C(0) = 1$, the second-order Taylor expansion is given by

$$C(t) = 1 - \frac{1}{2} \ddot{C}(t = 0) t^2 \; , \tag{26}$$

where the dots denote the second time derivative. In the following I consider the situation before the perturbation. In that case $\tau_P = 0$, and expression (20) gives

$$C(t) = R(t) \; , \tag{27}$$

where $R(t)$ is the correlation coefficient defined in expression (7) with the unperturbed state equal to the perturbed state: $u = \tilde{u}$. Using this in expression (27) gives

$$C(t) = \frac{\int u(t')u(t'+t)dt'}{\int u^2(t')dt'} , \tag{28}$$

where the integration is over the time-window under consideration. Note that it follows from this expression that $C(0) = 1$, as required in equation (19).

Differentiating expression (28) twice with respect to time gives

$$\ddot{C}(t) = \frac{\int u(t')\dfrac{d^2u(t'+t)}{dt^2}dt'}{\int u^2(t')dt'} = \frac{\int u(t')\dfrac{d^2u(t'+t)}{dt'^2}dt'}{\int u^2(t')dt'} , \tag{29}$$

where second derivative in the last term is with respect to t'. Setting $t = 0$, and using an integration by parts gives under the assumption that u vanishes at the end of the integration interval:

$$\ddot{C}(0) = -\frac{\int \dot{u}^2(t')dt'}{\int u^2(t')dt'} . \tag{30}$$

In practice, one tapers the integrand in the correlation coefficient (7) in order to suppress truncation artifacts. This taper ensures that the integrand of expression (7) indeed vanishes at the endpoint of the interval. The right hand side of expression (30) has the physical dimension $frequency^2$; for this reason I introduce the following definition of the mean squared angular frequency:

$$\overline{\omega^2} \equiv \frac{\int \dot{u}^2(t')dt'}{\int u^2(t')dt'} . \tag{31}$$

Note that this quantity can directly be computed from the recorded data. Using this result in the expressions (26) and (30) gives the following second-order Taylor expansion:

$$C(t) = 1 - \frac{1}{2}\overline{\omega^2}t^2 . \tag{32}$$

Inserting this result in expression (20) gives

$$R(t_s) = 1 - \frac{1}{2}\overline{\omega^2}\frac{\sum_P I_P(t_s - \tau_P)^2}{\sum_P I_P} . \tag{33}$$

It follows by differentiation that the correlation coefficient attains its maximum when

$$0 = \frac{dR(t_s)}{dt_s} = -\overline{\omega^2}\frac{\sum_P I_P(t_s - \tau_P)}{\sum_P I_P} . \tag{34}$$

This maximum is reached for

$$t_s = t_{\max} = \frac{\sum_P I_P\tau_P}{\sum_P I_P} . \tag{35}$$

This equation states that the correlation coefficient attains its maximum for a time shift that is equal to the intensity-weighted travel time perturbation. According to the notation of expression (24) this result can also be written as

$$t_{\max} = \langle\tau\rangle . \tag{36}$$

The value $R_{\max}$ of the correlation coefficient at its maximum follows by replacing t_s in expression (33) by the average $\langle\tau\rangle$, thus

$$R_{\max} = 1 - \frac{1}{2}\overline{\omega^2}\frac{\sum_P I_P(\tau_P - \langle\tau\rangle)^2}{\sum_P I_P} . \tag{37}$$

Using expression (24) the ratio in the last term satisfies

$$\frac{\sum_P I_P(\tau_P - \langle\tau\rangle)^2}{\sum_P I_P} = \left\langle(\tau - \langle\tau\rangle)^2\right\rangle = \sigma_\tau^2 , \tag{38}$$

where σ_τ^2 is the variance of the travel time perturbation. Combining this result with expression (37) gives

$$R_{\max} = 1 - \frac{1}{2}\overline{\omega^2}\sigma_\tau^2 . \tag{39}$$

The time-shifted correlation coefficient can for every employed time window be computed from expression (7) given the data before and after the perturbation, and according to expression (36) the time-shifted correlation coefficient attains its maximum for a shift time that is equal to the mean travel time perturbation. According to equation (39) the value of the time-shifted correlation coefficient gives the variance of the travel time perturbation. This means that if the unperturbed and perturbed waveforms are known, the mean and variance of the travel time perturbation can be computed. Computing the two lowest moments of the distribution of the travel time perturbation is a less ambitious goal than determining the distribution $P(\tau)$ of the travel time perturbations from equation (25), but as we will see in the examples of the next section the mean and variance of the travel time perturbation are useful in a number of practical applications.

The data may be contaminated with noise. The noise has two effects on the time-shifted cross correlation coefficient. First, the noise introduces fluctuations in the estimated coefficient. Without knowing the noise there is no way to eliminate these fluctuations other than using non-overlapping time windows of the coda to obtain independent estimates of the cross-correlation. Second, noise leads to a bias because noise lowers the value of the cross-correlation. This bias can be estimated given the energy in the noise ($\langle n^2\rangle$) and the energy of the noise-contaminated data ($\langle u^2\rangle$). Douma and Snieder [2005] show that the bias in the correlation coefficient can be accounted for by using the corrected correlation coefficient that is related to the uncorrected coefficient by the following relation

$$R_{corr} = R/\sqrt{1 - \frac{\langle n^2\rangle}{\langle u^2\rangle}}\sqrt{1 - \frac{\langle\tilde{n}^2\rangle}{\langle\tilde{u}^2\rangle}}. \tag{40}$$

The noise levels $\langle n^2\rangle$ and $\langle\tilde{n}^2\rangle$ before and after the perturbation can be estimated from the data recorded before the first-arriving waves.

6 EXAMPLES

In this section I consider three different perturbations that are of relevance for practical applications; a constant velocity perturbation, random displacement of scatterers, and a perturbation in the source position.

6.1 A constant velocity perturbation

Suppose that in a medium the velocity is perturbed with a perturbation δv, and that the relative velocity perturbation $\delta v/v$ is the same at every location in space. The unperturbed travel time is given by

$$t = \int_P \frac{1}{v} ds \; , \tag{41}$$

where the integration is along path P. The perturbed travel time is to first order in the velocity perturbation given by

$$t + \tau_P = \int_P \frac{1}{v + \delta v} ds = \int_P \left(\frac{1}{v} - \frac{\delta v}{v^2} \right) ds \; . \tag{42}$$

With expression (41) this gives

$$\tau_P = - \int_P \frac{1}{v} \frac{\delta v}{v} ds = - \left(\frac{\delta v}{v} \right) \int_P \frac{1}{v} ds \; , \tag{43}$$

where I used in the last identity that the relative velocity perturbation is assumed to be constant. With expression (41) the travel time perturbation can be written as

$$\tau_P = - \left(\frac{\delta v}{v} \right) t \; . \tag{44}$$

Note that the velocity v is not necessarily constant. According to expression (44), the travel time perturbation depends on the arrival time of the wave only, but is independent of the particular path. This means that in a small time window the mean travel time perturbation is also given by

$$t_{max} = \langle \tau \rangle = - \left(\frac{\delta v}{v} \right) t \; . \tag{45}$$

Since the travel time perturbation is the same for all trajectories, the variance of the travel time perturbation vanishes

$$\sigma_\tau^2 = 0 \; . \tag{46}$$

As shown in expression (36), the mean travel time perturbation is equal to the shift time that gives the maximum of the time-shifted correlation coefficient, this quantity can easily be retrieved from the data. According to expression (45) the relative velocity change then follows from

$$\left(\frac{\delta v}{v} \right) = - \frac{t_{max}}{t} \; . \tag{47}$$

This has been applied by Snieder et al. [2002] to measure the velocity change in a granite sample with temperature. In their experiment coda wave interferometry is sufficiently sensitive to detect a velocity change of about 0.1%. This change in the velocity can be inferred from different non-overlapping time windows in the coda. This redundancy serves as a consistency check on the method, and can be used for error estimation. In these measurements the estimated error in the velocity change was about 0.02% [Sniedet et al., 2002] . Grêt et al. [Grêt et al., 2005] applied this technique also to measure the velocity change in rocks due to changes in the stress state in laboratory conditions, and in a mining environment [Grêt et al., 2004].

For elastic waves there are two wave velocitites. It follows from the intensity-averaged travel time perturbation (35) that for elastic waves the inferred velocity change is a weighted average of the change in the P-wave and S-wave velocities [Snieder, 2002]:

$$\frac{\delta v}{v} = \frac{\beta^3}{2\alpha^3 + \beta^3} \frac{\delta \alpha}{\alpha} + \frac{2\alpha^3}{2\alpha^3 + \beta^3} \frac{\delta \beta}{\beta} \; . \tag{48}$$

where α and β are the velocities of P-waves and S-waves, respectively. Since $\beta < \alpha$ the perturbation in the shear velocity dominates the perturbation in the P-wave velocity. For example, for a a Poisson medium ($\alpha = \sqrt{3}\beta$):

$$\frac{\delta v}{v} = 0.09 \frac{\delta \alpha}{\alpha} + 0.91 \frac{\delta \beta}{\beta} \; . \tag{49}$$

The theory of this section is valid when the relative velocity change is independent of location. In realistic situations this is not necessarily the case. The theory can be extended for situations where the relative velocity change depends on position. In that case the mean travel time change is linearly related to the velocity change:

$$\langle \tau \rangle = \int K(\mathbf{r}, t) \delta v(\mathbf{r}) dV \; , \tag{50}$$

where the kernel $K(\mathbf{r}, t)$ depends on the intensity-weighted average over all scattering paths. This expression can be used as the basis of a standard linear inversion for the velocity change $\delta v(\mathbf{r})$ given the mean travel time change observed for different source-receiver pairs and different time windows. The kernel $K(\mathbf{r}, t)$ has been derived both for single-scattered waves [Pacheco and Snieder, 2005a] as well as for strongly scattered waves [Pacheco and Snieder, 2005b].

6.2 Random displacement of scatterers

As a second example we consider a perturbation that consists of uncorrelated movement of the scatterers. This is of relevance for studying the motion of particles in colloidal suspensions [Heckmeier and Maret, 1997] and of bubbles in a turbulent fluid [Cowan et al., 2000; Page et al., 2000]. The theory presented here is equivalent to diffusing wave spectroscopy [Cowan at al., 2002;

Weitz and Pine, 1993], although the derivation is different.

Let us consider scatterers that move independently in three dimensions and that have a root mean square displacement δ between the two measurements of the waves that are used to study the motion of the scatterers. On average the path length for each scattered wave does not change, hence the mean perturbation of the travel time vanishes:

$$\langle \tau \rangle = 0 \ . \tag{51}$$

Some scattering paths are longer, while other scattering paths are shorter, therefore the variance of the travel time perturbation is nonzero.

We compute the variance of the travel time by using that for a wave that has scattered n times, the variance in the path length is given by [Snieder and Scales, 1998]:

$$\sigma_L^2 = 2n\left(1 - \overline{\cos\theta}\right)\delta^2 \ . \tag{52}$$

In this expression $\overline{\cos\theta}$ is the average of the cosine of the scattering angle over all paths in the employed time window. The number of scatterers encountered is related to the travel time by $n = vt/l$, where v is the wave velocity and l the scattering mean free path. Using this in expression (52) gives

$$\sigma_L^2 = \frac{2vt\delta^2}{l_*} \ , \tag{53}$$

where l_* is the transport mean free path. This is the distance of propagation over which the scattered wave has lost all information about its direction of propagation [Lagendijk and van Tiggelen, 1996; van Rossum and Nieuwenhuizen, 1999]. For a constant velocity the variance in the path travel time is related to the variance in the path length by $\sigma_\tau = \sigma_L/v$, so that

$$\sigma_\tau^2 = \frac{2t\delta^2}{vl_*} \ , \tag{54}$$

With expression (39) this means that the measured maximum of the cross-correlation is related to the mean displacement of the scatterers by

$$R_{max} = 1 - \frac{\overline{\omega^2}\delta^2 t}{vl_*} \ . \tag{55}$$

Given a measured value of the maximum of the cross-correlation, one can infer the mean scatterer displacement δ from this expression if the wave velocity and the transport mean free path is known. Snieder et al. [2002] show in a numerical experiment that the mean scatterer displacement can correctly be retrieved from the coda waves. The scatterer displacement can be computed from several non-overlapping time windows of the coda, and they show how this redundancy can be used to compute error bounds on the scatterer displacement inferred from the coda waves.

6.3 A displaced source position

The relative distance between earthquakes can be found from the absolute location of the events, but in this approach errors in the employed velocity model may lead to large errors in the distance between the events [Pavlis, 1992] . An alternative approach is the double-difference method where the relative event location is computed from the differential travel time of the direct arrivals [Shearer, 1997; Astiz and Shearer, 2000; Walhauser and Ellsworth, 2002] . Coda wave interferometry can also be used to determine the relative distance between earthquakes, provided the source mechanisms of the events are identical. This provides additional information on the relative position between events that can be used in addition to the constraints obtained from the double-difference method.

Let us consider two nearby seismic sources with the same source mechanism. When the source position is perturbed over the distance $\mathbf{r}$, the distance from the source to the first scatterer along every scatterer path is perturbed. The distance between the scatterers is unperturbed, so we only need to account for the change in the distance to the first scatterer along every scattering path. For this reason it does not matter if the wave travels from the first scatterer along every path directly to the receiver, or whether the wave visits many other scatterers first. The theory of this section therefore is applicable both for single scattering as well as for multiple scattering.

For a scattering path P with take-off direction $\hat{\mathbf{t}}_P$ at the source, the perturbation in the travel time is given by

$$\tau_P = -\frac{\left(\hat{\mathbf{t}}_P \cdot \mathbf{r}\right)}{v} \ . \tag{56}$$

For a scattering path where the wave leaves the source as a P-wave, v is the P-wave velocity at the source location, whereas for a path where the wave leaves the source as an S-wave, v denotes the shear velocity. When the scatterers are distributed homogeneously, some paths are longer when the source location is perturbed, while others are shorter. The resulting mean travel time perturbation vanishes [Snieder and Vrijlandt, 2005] . This can be shown by integrating the travel time perturbation (56) over all take-off directions.

The variance of the travel time perturbation, however, is nonzero, because some paths are longer when the source location is perturbed while others are shorter. The variance of the travel time perturbation depends on the type of source (explosion, point force, double couple), as well as on the orientation of the perturbation of the source location relative to the source mechanism [Snieder and Vrijlandt, 2005]. An important application is the location of aftershocks. In that case all events are located in the plane of the fault-plane of the main shock. Snieder and Vrijlandt [2005] show that in this case the variance of the travel time is given by

$$\sigma_\tau^2 = \frac{\left(\frac{6}{\alpha^8} + \frac{7}{\beta^8}\right)}{7\left(\frac{2}{\alpha^6} + \frac{3}{\beta^6}\right)} r^2 \; . \tag{57}$$

In this expression α is the P-wave velocity, and β denotes the shear-wave velocity. This expression depends on both velocities, because the path summation includes waves that leave the source as a P-wave, as well as waves that leave the source as an S-wave. The P- and S-wave velocities in expression (57) are raised to high powers. Since $\beta < \alpha$, the terms with the shear velocity dominate. Note that expression (57) depends on the distance r between the events, but not on their relative orientation in the fault plane. The reason for this is that the averaging (38) in this application involves an integration over all take-off directions. In this directional averaging information on the direction of the event separation is lost.

Expression (57) can be used to estimate the source separation in the following way. For several time windows in the coda the time-shifted correlation coefficient can be computed from expression (7). Using expression (39), the maximum of this function can be equated to the variance of the travel time perturbation given by equation (57). The resulting expression can then be solved for the event separation r. In the presence of significant noise, the correction factor of equation (40) can be used to eliminate the bias in the cross-correlation due to noise.

Snieder and Vrijlandt [2005] applied this technique to events on the Hayward Fault, California, and showed that the event separation obtained from the coda waves agrees with the event separation of the same events determined by Waldhauser and Ellsworth [2002] with the double-difference method.

Note that the estimation of the event separation can be carried out with a single station. An error estimate of the event separation can be obtained by comparing the event separation inferred from different non-overlapping time windows in the coda.

7 DISCUSSION

The main result of the theory of this paper is that the time-shifted correlation coefficient of expression (7) can be related to the mean and variance of the travel time perturbation. According to expression (36), the mean travel time perturbation follows from the shift time that gives the maximum of the time-shifted correlation coefficient, and equation (39) relates the maximum of the cross-correlation coefficient to the variance of the travel time perturbation. This means that the mean and variance of the travel time perturbation can be obtained from a comparison of the coda waves before and after the perturbation. Additive random noise leads to a bias in the maximum of the correlation coefficient, this bias can be removed using the correction factor of expression (40).

The present theory can be applied to a constant change in the velocity, to uncorrelated perturbations in the locations of the scatterers, and to changes in the source position. For a change in the velocity (section 6.1), the means travel time perturbation is nonzero (equation (45)), but the variance of the travel time perturbation is zero (expression (46)). For random perturbations in the scatterer location the mean travel time perturbation vanishes (expression (51)), but according to equation (52) the variance of the travel time is nonzero and depends linearly on time. This contrasts the case of a perturbation in the source position where the mean travel time perturbation also vanishes and where according to expression (57) the variance is nonzero and independent of time. In general, it may not be obvious how a medium is perturbed. As shown above, the three different perturbation leave a different imprint on the mean and the variance of the travel time perturbation. Since these quantities can be estimated for several independent windows of the coda waves using the time-shifted correlation coefficient, it is possible to discriminate between these different perturbations using the recorded coda waves.

REFERENCES

K. Aki. Theory of earthquake prediction with special reference to monitoring of the quality factor of lithosphere by the coda method. *Earthquake Res. Bull.*, 3:219–230, 1985.

K. Aki and L.B. Chouet. Origin of coda waves: source, attenuation, and scattering effects. *J. Geophys. Res.*, 80:3322–3342, 1975.

L. Astiz and P.M. Shearer. Earthquake locations in the inner Continental Borderland, offshore southern California. *Bull. Seism. Soc. Am.*, 90:425–449, 2000.

O.M. Bucci and G Franceschetti. On the degrees of freedom of scattered fields. *IEEE Trans. on Antennas and Propagation*, 37:918–926, 1989.

M.L. Cowan, I.P. Jones, J.H. Page, and D.A. Weitz. Diffusing acoustic wave spectroscopy. *Phys. Rev. E*, 65:066605–1/11, 2002.

M.L. Cowan, J.H. Page, and D.A. Weitz. Velocity fluctuations in fluidized suspensions probed by ultrasonic correlation spectroscopy. *Phys. Rev. Lett.*, 85:453–456, 2000.

H. Douma and R. Snieder. Correcting for bias due to noise in coda wave interferometry. *Submitted to Geophys. J. Int.*, 2005.

A. Grêt, R. Snieder, R.C. Aster, and P.R. Kyle. Monitoring rapid temporal changes in a volcano with coda wave interferometry. *Geophys. Res. Lett., in press*, 2005.

A. Grêt, R. Snieder, and J. Scales. Time-lapse monitoring of rock properties with coda wave interferometry. *Submitted to J. Geophys. Res.*, 2005.

R. Grêt, A.and Snieder and U. Ozbay. Monitoring stress change in an undergraound mining environment with coda wave interferometry. *Submitted to the J. of Mining Applications*, 2004.

M. Heckmeier and G. Maret. Dark speckle imaging of colloidal suspensions in multiple light scattering media. *Progr. Colloid. Polym. Sci.*, 104:12–16, 1997.

A. Jin and K. Aki. Temporal changes in the coda Q before the Tangshan earthquake of 1976 and the Haicheng earthquake of 1975. *J. Geophys. Res.*, 91:665–673, 1986.

A. Lagendijk and B.A. van Tiggelen. Resonant multiple scattering of light. *Phys. Rep.*, 270:143–215, 1996.

H.J. Landau. Necessary density conditions for sampling and interpolation of certain entire functions. *Acta Mathematica*, 117:37–52, 1967.

W. Lauterborn, T. Kurz, and M. Wiesenfeldt. *Coherent Optics, Fundamentals and Applications*. Springer-Verlag, Berlin, 1995.

S. Matsumoto, K. Obra, K. Yoshimoto, T. Saito, A. Ito, and A. Hasegawa. Temporal changes in P-wave scatterer distribution associated with the M6.1 earthquake near Iwate volcano, northeastern Japan. *Geophys. J. Int.*, 145:48–58, 2001.

C. Pacheco and R. Snieder. Localizing time-lapse changes with multiply scattered waves. *Submitted to J. Acoust. Soc. Am.*, 2005.

C. Pacheco and R. Snieder. Localizing time-lapse changes with single scattered waves. *Submitted to Geophys. J. Int.*, 2005.

J.H. Page, M.L. Cowan, and D.A. Weitz. Diffusing acoustic wave spectroscopy of fluidized suspensions. *Physica B*, 279:130–133, 2000.

G. Pavlis. Appraising relative eartquake location errors. *Bull. Seism. Soc. Am.*, 82:836–859, 1992.

G. Poupinet, W.L. Ellsworth, and J. Fréchet. Monitoring velocity variations in the crust using earthquake doublets: an application to the Calaveras Fault, California. *J. Geophys. Res.*, 89:5719–5731, 1984.

A. Ratdomopurbo and G. Poupinet. Monitoring a temporal change of seismic velocity in a volcano: Application to the 1992 eruption of Mt. Merapi (Indonesia). *Geophys. Res. Lett.*, 22:775–778, 1995.

P.M. Roberts, W. Scott Phillips, and M.C. Fehler. Development of the active doublet method for measuring small velocity and attenuation changes in solids. *J. Acoust. Soc. Am.*, 91:3291–3302, 1992.

P.M. Shearer. Improving local earthquake locations using the L_1-norm and waveform cross-correlation: Application to the Whittier Narrows California aftershock sequence. *J. Geophys. Res.*, 102:8269–8283, 1997.

M. Snieder and M. Vrijlandt. Constraining relative source locations with coda wave interferometry: Theory and application to earthquake doublets in the Hayward Fault, California. *J. Geophys. Res., in press*, 2005.

R. Snieder. Imaging and averaging in complex media. In J.P. Fouque, editor, *Diffuse waves in complex media*, pages 405–454. Kluwer, Dordrecht, 1999.

R. Snieder. Coda wave interferometry and the equilibration of energy in elastic media. *Phys. Rev. E*, 66:046615–1,8, 2002.

R. Snieder. Extracting the Green's function from the correlation of coda waves: A derivation based on stationary phase. *Phys. Rev. E*, 69:046610, 2004.

R. Snieder, A. Grêt, H. Douma, and J. Scales. Coda wave interferometry for estimating nonlinear behavior in seismic velocity. *Science*, 295:2253–2255, 2002.

R. Snieder and J.A. Scales. Time reversed imaging as a diagnostic of wave and particle chaos. *Phys. Rev. E*, 58:5668–5675, 1998.

M.C.W. van Rossum and Th.M. Nieuwenhuizen. Multiple scattering of classical waves: microscopy, mesoscopy and diffusion. *Rev. Mod. Phys.*, 71:313–371, 1999.

B.A. van Tiggelen. Localization of waves. In J.P. Fouque, editor, *Diffuse waves in complex media*, pages 1–60. Kluwer, Dordrecht, 1999.

F. Waldhauser and W.L. Ellsworth. Fault structure and mechanics of the Hayward Fault, California, from double-difference earthquake locations. *J. Geophys. Res.*, 107:10.1029/2000JB000084, 2002.

D.A. Weitz and D.J. Pine. Diffusing wave spectroscopy. In W. Brown, editor, *Dynamic light scattering, The method and some applications*, pages 652–720. Clarendon Press, Oxford, 1993.