The coherent backscattering effect for moving scatterers

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ABSTRACT

The constructive interference of backscattered waves that propagate along the same path in opposite directions doubles the intensity of these waves. When the scatterers move during wave propagation, the enhancement factor is reduced from 2 to a lower value. I derive the enhancement factor for coherent backscattering under the assumption that the scatterers move independently with a constant velocity. The resulting enhancement value depends exponentially on $\langle v^2 \rangle^{1/2} t/\lambda$, which is the ratio of the root-mean-square displacement of the scatterers during the wave propagation to the wavelength, and on the number of scatterers encountered.

Key words: multiple scattering, coherent backscattering effect

1 INTRODUCTION

The constructive interference of backscattered waves that propagate in opposite directions along the same scattering paths leads to an enhancement of the backscattered waves by a factor 2 (e.g., Akkermans, et al., 1986, 1988; Sheng, 1995). [This phenomena is called the *coherent backscattering effect*. This constructive interference is similar to that of waves that travel in opposite directions along loops (Haney and Snieder, 2003). The coherent backscattering effect has been observed for light (Wolf and Maret, 1985; van Albada, et al., 1987; Maret, 1995; Wiersman, et al., 1997), for acoustic waves (Tourin, et al., 1997), and for elastic waves (van Tiggelen, et al., 2001). It has been used to account for the brightness of the moon (Hapke et al., 1993), and to characterize the heterogeneity in human bone (Derode, et al., 2005).

The enhancement factor of 2 occurs only when the scatterers do not move as the waves propagate through the scattering medium. Movement of the scatterers leads to a phase change of the backscattered waves that propagate in opposite directions along a scattering path. This decreases the enhancement factor for coherent backscattering from 2 to a lower value. Here I compute the enhancement factor for coherent backscattering when the scatterers move independently for the special case where the scatterers are illuminated with an impulsive wave with a duration that is short compared to the time scale associated with the movement of the scatterers.

The physical problem analyzed here differs from diffusing acoustic wave spectroscopy (Weitz and Pine, 1993; Cowan, et al, 2000; Cowan, et al., 2002) and coda wave interferometry (Snieder, et al., 2002) because in those applications one compares the wave propagation before and after the medium has changed. Here I consider changes in the medium *during* the wave propagation.

2 PHASE PERTURBATION DUE TO MOVING SCATTERERS

Consider the case that the scatterers move as the wave is being scattered, as shown in fig. 1. The solid and dashed lines indicate the scattering paths in the forward and reverse directions, respectively. The scatterers are at different locations along these paths as the wave is being scattered at each scatterer. This perturbs the relative phase accumulated along the forward and reverse scattering paths, which weakens the coherent backscattering effect. In the following treatment I assume that the velocity of the scatterers is much smaller than the wave velocity, so that the Doppler effect can be ignored, and I don't account for resonant scattering. When the

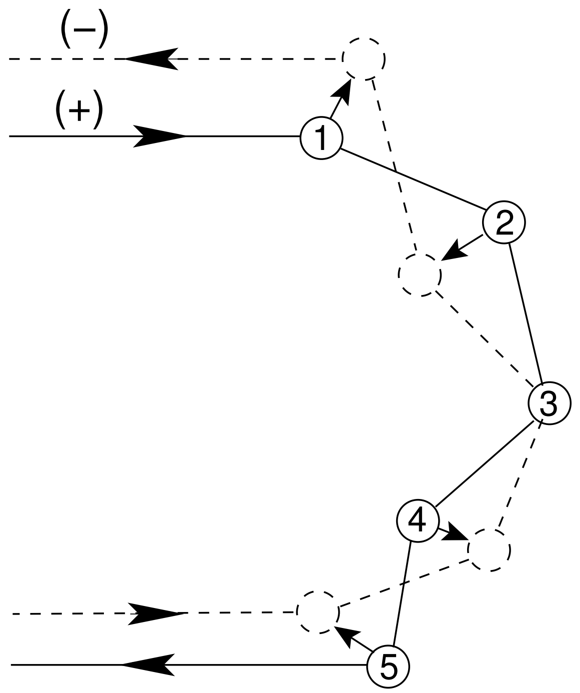

Figure 1. Scattering paths for forward (solid line) and reverse (dashed line) propagation. The motion of the scatterers in the time interval between the visit of the waves along the forward path and reverse path is indicated by arrows.

scatterers move, the scattering amplitude and geometrical spreading change. For a mean-free path larger than a wavelength, the change in the phase due to the change in the path length dominates the changes in scattering amplitude and geometrical spreading (Snieder and Vrijlandt, 2005; Snieder, 2006). For this reason I analyze the change in the length of the scattering paths due to the motion of the scatterers.

In practical situations the scatterers may change their velocity due to Brownian motion, or collisions with other scatterers. I analyze the situation where the propagation time t of the wave is less than the time t_C in which the velocities of the scatterers change. In this case the velocity of the scatterers can assumed to be constant with time. In the following $x_i^{(j)}$ denotes the i-coordinate of scatterer j along a given scattering trajectory, and $\Delta x_i^{(j)}$ the associated perturbation in this quantity due to the movement of the scatterer. (The scatterers are numbered consecutively along the forward scattering path with the index j.) I assume that the velocities of the scatterers are uncorrelated; hence

$$\langle \Delta x_i^{(j)} \Delta x_n^{(m)} \rangle = \delta_{in} \delta_{jm} \langle (\Delta x_i^{(j)})^2 \rangle \ , \tag{1}$$

where the brackets $\langle \cdots \rangle$ denote the average over the motion of the scatterers. During a time Δt_j, scatterer j moves over a distance $v_i^{(j)} \Delta t_j$ in the i-direction, where $v_i^{(j)}$ is the i-component of the velocity of scatterer j. Assuming that the root-mean-square velocity of the scatterers is identical, $\langle (\Delta x_i^{(j)})^2 \rangle = \langle v_i^2 \rangle (\Delta t_j)^2$. When the velocity of the scatterers has an isotropic distribution

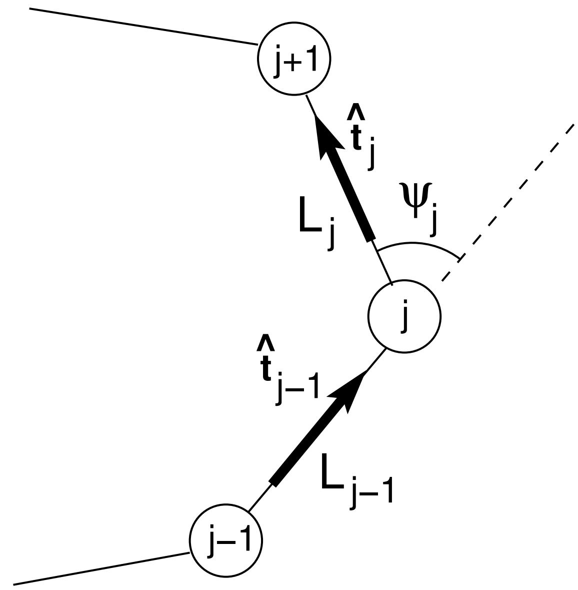

Figure 2. Definition of the length L_j, the unit vector $\hat{\mathbf{t}}^{(j)}$, and the scattering angle ψ_j.

$$\langle v_x^2 \rangle = \langle v_y^2 \rangle = \langle v_z^2 \rangle = \frac{1}{3} \langle v^2 \rangle \ ; \tag{2}$$

hence, for each of the components of the displacement of scatterer j,

$$\langle (\Delta x_i^{(j)})^2 \rangle = \frac{1}{3} \langle v^2 \rangle \, (\Delta t_j)^2 \ . \tag{3}$$

In order to compute the relative phase shift along the solid and dashed trajectories of fig. 1, we need to compute the change in the path length L caused by the motion of scatterers. As shown in fig. 2, the path length L_j measures the distance from scatterer j to the next scatterer along the forward trajectory. I first consider the motion of just scatterer j along a path, which causes only the path lengths L_{j-1} and L_j to change, so that (Snieder and Scales, 1998)

$$\frac{\partial L}{\partial x_i^{(j)}} = \frac{\partial (L_j + L_{j-1})}{\partial x_i^{(j)}} = \hat{t}_i^{(j-1)} - \hat{t}_i^{(j)} \ , \tag{4}$$

where $\hat{\mathbf{t}}^{(j)}$ is the unit vector that points along the scattering path from scatterer j to scatterer $j+1$. These unit vectors define the scattering angle at scatterer j by $(\hat{\mathbf{t}}^{(j)} \cdot \hat{\mathbf{t}}^{(j-1)}) = \cos\psi_j$. Using this relationship, assuming that the scatterers move independently, and summing over the n scatterers along a path gives, with expression (3), the variance in the perturbation in the path length

$$\langle(\Delta L)^2\rangle = \sum_{j=1}^{n}\sum_{i=1}^{3}\left(\frac{\partial L}{\partial x_i^{(j)}}\right)^2 \langle(\Delta x_i^{(j)})^2\rangle = \sum_{j=1}^{n}\frac{2}{3}\left(1-\cos\psi_j\right)\langle v^2\rangle(\Delta t_j)^2 \ , \quad (5)$$

In the sum (5), $\cos\psi_j$ can be replaced by its value $\overline{\cos\psi}$ averaged over all scattering paths; hence,

$$\langle(\Delta L)^2\rangle = \frac{2}{3}\left(1-\overline{\cos\psi}\right)\langle v^2\rangle\sum_{j=1}^{n}(\Delta t_j)^2 \ . \quad (6)$$

On average, the waves encounter a scatterer after propagation over the mean free path l. With the wave velocity c, this gives a mean free time $\tau = l/c$. For the sake of argument I present the case of an odd number of scatterers along a scattering path, but the final result holds for scattering paths with an even number of scatterers as well. The waves on the forward and reverse trajectories visit each scatterer at a different moment in time; for scatterer j this time difference is equal to

$$\Delta t_j = (n-2j+1)\tau \ . \quad (7)$$

For the scatterer in the middle of the trajectory ($j = (n+1)/2$), this time difference is equal to zero because this scatterer is visited at the same moment in time for both the forward and reverse paths, see fig. 1.

Summation of the series in expression (6) gives (Jolley, 1961)

$$\sum_{j=1}^{n}(\Delta t_j)^2 = \frac{1}{3}n\left(n^2-1\right)\tau^2 \approx \frac{1}{3}n^3\tau^2 \ , \quad (8)$$

where I assumed in the above approximation many scatterers along each scattering path ($n \gg 1$). Inserting this in eq. (6) and using that the number of scatterers along the path is related to the time of flight t by $n = ct/l$, gives, with $\tau = l/c$,

$$\langle(\Delta L)^2\rangle = \frac{2}{9}\frac{\langle v^2\rangle ct^3}{l_*} \ , \quad (9)$$

where the transport mean free path is defined by $l_* = l/(1-\overline{\cos\psi})$ (van Rossum and Nieuwenhuizen, 1999). The phase difference φ associated with the difference in the propagation distance along the forward and reverse trajectories thus satisfies

$$\langle\varphi^2\rangle = k^2\langle(\Delta L)^2\rangle = \frac{2}{9}\frac{k^2\langle v^2\rangle ct^3}{l_*} \ , \quad (10)$$

with k the dominant wavenumber.

Let us compare this expression with the corresponding result in diffusing wave spectroscopy, where one studies the change in the medium between two measurements of the multiply scattered waves. Equation (16.22) of Weitz and Pine (1993) is in the notation of this work given by $\langle\varphi^2\rangle_{DWS} = 2k^2\langle(\Delta r)^2\rangle ct/3l_*$. If the time interval t_{int} between the two measurements of the wave propagation is smaller than t_C, then $\langle(\Delta r)^2\rangle = \langle v^2\rangle t_{int}^2$, and $\langle\varphi^2\rangle_{DWS,t_C} = 2k^2\langle v^2\rangle t_{int}^2 ct/3l_*$. When t_{int} is larger than the time t_B required for the motion of the scatterers to be diffusive with diffusion constant D, then $\langle(\Delta r)^2\rangle = Dt_{int}/3$, and $\langle\varphi^2\rangle_{DWS,t_B} = 2k^2Dt_{int}ct/9l_*$. In both cases, $\langle\varphi^2\rangle_{DWS}$ varies linearly with the propagation time t. This contrasts the t^3-dependence on time in expression (10). That expression is applicable to changes in the positions of the scatterers *during* the wave propagation.

3 THE COHERENT BACKSCATTERING EFFECT

The intensity, normalized by the average intensity, for an incoming wave with wavenumber $\mathbf{k}_{in}$ and outgoing wave with wavenumber $\mathbf{k}_{out}$ is, after averaging over multiple realizations, equal to (Sheng,1995)

$$E = \sum_P\langle 1+\cos(\mathbf{k}_{in}+\mathbf{k}_{out})\cdot(\mathbf{r}_{P,in}-\mathbf{r}_{P,out}) +\varphi_P\rangle/\sum_P(1) \ , \quad (11)$$

where P labels the scattering trajectories, $\mathbf{r}_{P,in}$ and $\mathbf{r}_{P,out}$ are the positions of the first and last scatterers along the forward trajectory P, and φ_P is the phase difference for the associated forward and reverse trajectories. In this sum, forward and backward propagation along the same trajectory is counted once (Sheng, 1995). For backscattering, $\mathbf{k}_{in}+\mathbf{k}_{out} = 0$, and the enhancement factor for coherent backscattering is equal to

$$E = \sum_P\langle 1+\cos\varphi_P\rangle/\sum_P(1) = 1+\langle\cos\varphi\rangle \ . \quad (12)$$

The phase perturbation φ has zero mean, and is the sum of the independent motion of many scatterers along a path. Because of the central limit theorem, φ has a Gaussian distribution. For a Gaussian distribution with zero mean $\langle\cos\varphi\rangle = \exp\left(-\langle\varphi^2\rangle/2\right)$, so that, using eq. (10)

$$E = 1+\exp\left(-\frac{k^2\langle v^2\rangle ct^3}{9l_*}\right) \ . \quad (13)$$

This enhancement factor is shown in fig. 3 as a function of t/t^*, where

$$t^* \equiv \left(\frac{9l_*}{k^2\langle v^2\rangle c}\right)^{1/3} \ . \quad (14)$$

The enhancement factor for backscattered waves decreases with time over the characteristic time t^*. During this time the scatterers have moved so far that the constructive interference of the waves that propagate along the forward and reverse scattering paths is destroyed.

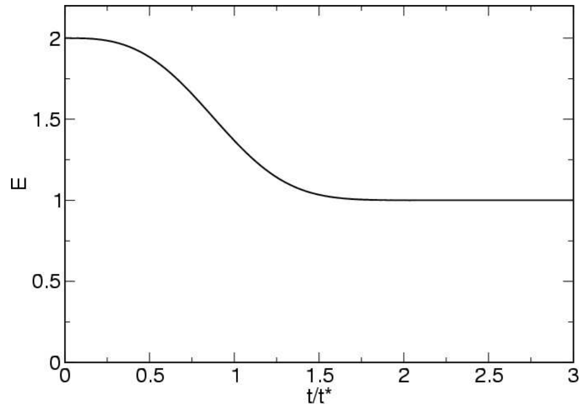

Figure 3. The enhancement factor from eq. (13) as a function of t/t^*.

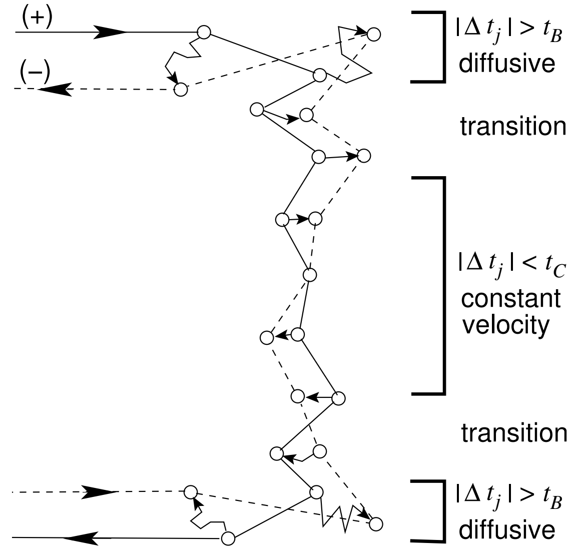

Figure 4. The paths for the wave propagating along the forward (solid line) and reverse (dashed line) trajectories for a long path, for the special case $t > t_B$. The motion of the scatterers is indicated with small arrows. For the scatterers near the endpoints of the path diffusive motion is appropriate, while for scatterers near the middle of the path the velocity is constant between the visits of the waves propagating in opposite directions. For intermediate scatterers one needs to account for the transition from a constant velocity to diffusive motion.

4 DISCUSSION

The enhancement factor in eq. (13) accounts for the movement of scatterers as the waves propagate through the medium. The treatment is valid for an impulsive illumination of the scattering medium, and the resulting enhancement factor is time-dependent. One might think that for a monochromatic illumination, one needs to average the enhancement factor over all scattering paths using the intensity of the waves as a weight factor, e.g., (Akkerman, et al., 1988; Weitz and Pine, 1993). This is, however, not the case because a monochromatic wave has an infinite duration, hence the condition $t < t_C$ cannot be satisfied.

The derivation is valid when the propagation time t of the scattered wave is smaller than the time t_C over which the velocity of the scatterers changes. The latter time has been monitored experimentally for bubbles that scatter acoustic waves (Cowan, et al., 2002; Page, et al., 2000). For propagation times larger than t_C the assumption of a constant velocity of the scatterers must be modified. In Akkermans, et al. (1988) a time t_B is defined as the time after which the motion of the scatterers is given by Brownian motion. In general, $t_B > t_C$. If the motion of the scatterers over *all* time intervals Δt_j would be diffusive, and $\langle(\Delta\mathbf{x}^{(j)})^2\rangle = D\Delta t_j/3$, with D the diffusion constant of the Brownian motion. Using the reasoning of this work this would give an enhancement factor

$$E_{diffusive} = 1 + \exp\left(-\frac{Dk^2ct^2}{3l_*}\right) , \qquad (15)$$

which should be compared with expression (13). This result is, however, incorrect. As shown in fig. 4, the middle scatterer along every trajectory is visited at the same time by the waves propagating in the forward and reverse directions. For sufficiently long scattering paths, and for scatterers near the endpoints of the trajectories, the time interval $|\Delta t_j|$ between the visits to scatterer j of the waves propagating in opposite directions can indeed be sufficiently large for the corresponding motion of the scatterers to be diffusive ($|\Delta t_j| > t_B$). Near the middle of the trajectory, the time difference $|\Delta t_j|$ does to zero. This means that near the middle of the trajectory $|\Delta t_j| < t_C$, and the velocity of each scatterers must assumed to be constant. For intermediate scatterers $t_C < |\Delta t_j| < t_B$, hence the motion of the scatterers is in transition from a constant velocity to diffusive motion. This implies that diffusive motion for *all* scatterers along the trajectory is not a correct dynamic model, hence expression (15) must be modified to account for the transition of a constant velocity of the scatterers to diffusive motion.

The theory presented here is valid when $t < t_C$. According to eq. (13) the enhancement factor depends on $\langle v^2\rangle$. Measurements of the enhancement factor as a function of time can thus be used to infer the root-mean-square velocity of the scatterers. Expressed in the dominant wavelength λ, the enhancement factor is given by

$$E = 1 + \exp\left(-\frac{4\pi^2}{9}\left(\frac{\langle v^2\rangle t^2}{\lambda^2}\right)\frac{ct}{l_*}\right) . \qquad (16)$$

The enhancement factor thus depends on the average motion of the scatterers measured in wavelengths (vt/λ), and on the ratio ct/l_* that measures the number of scatterers encountered. This means that measurements of the enhancement factor may resolve the average movements of the scatterers on a length scale much smaller than the wavelength of the employed waves.

According to expression (16) the coherent backscattering associated with the movement of scatterers is observable when $4\pi^2\langle v^2\rangle ct^3/9\lambda^2 l_* = O(1)$. For a path length $L = ct$ the coherent backscattering effect thus is observable when $L^3 \approx 9\lambda^2 l_* c^2/4\pi^2\langle v^2\rangle$. As an example, consider a sound wave with a frequency of 10 kHz propagating through water (c= 1500 m/s) that is being scattered by bubbles moving with a velocity $\langle v^2\rangle^{1/2} =$ 0.01 m/s. For a transport mean free path $l_* = 1$ m, the coherent backscattering effect due to the motion of the bubbles is observable for path lengths of about 200 m. Such an estimation can be used to design experiments based on the theory presented here.

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