Finite-difference solution of linearized eikonal equation for transversely isotropic media

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ABSTRACT

The eikonal equation can be used to compute first-arrival or diffraction traveltimes, and its linearized form is suitable for modeling of traveltime perturbations as well as inversion gradients in seismic tomography. Here, we propose to solve the linearized eikonal equation for transversely isotropic media with a vertical symmetry axis (VTI) using an efficient and robust finite-difference (FD) methodology. The traveltimes for the background (unperturbed) model are computed with a fast-marching algorithm that uses a second-order FD scheme. The same FD scheme in combination with the background traveltimes is employed to solve the linearized eikonal equation for the perturbed medium. To check the accuracy of the proposed method, first we compute the traveltime perturbations caused by Gaussian parameter anomalies embedded in a homogeneous VTI background. Then the algorithm is tested on the perturbed structurally complex VTI Marmousi model. Comparison of the approximate traveltime perturbations with the values obtained directly from the eikonal equation confirms that our method is suitable for computing the traveltime gradients in anisotropic tomography.

1 INTRODUCTION

Traveltime tomography is often used in building near-surface isotropic and anisotropic velocity models from surface seismic data (e.g., Tromp et al., 2005). This method has also been applied to data acquired in vertical seismic profiling (VSP) and crosshole surveys (Bregman et al., 1989). Our primary interest is in applying traveltime tomography to traveltimes of diffracted waves with the goal of building initial velocity fields for transversely isotropic (TI) media.

The key steps of traveltime tomography are modeling of traveltimes and computation of their gradient with respect to the model parameters. Traveltime modeling can be carried out by solving the eikonal equation using ray-tracing methods (Cerveny, 2005). The traveltime gradients with respect to the pertinent parameters of anisotropic media can also be obtained from ray theory (Chapman and Pratt, 1992). However, the major drawback of these methods is a limited ray coverage in the presence of strong spatial velocity variations (e.g., near salt bodies). Ray tracing also involves the cumbersome task of re-computing traveltimes and the corresponding gradients from the ray coordinates to regular grids.

A number of publications are devoted to numerical solutions of the eikonal equation for isotropic and anisotropic media using finite-difference approximations (Vidale, 1990; Van Trier and Symes, 1991; Qin and Schuster, 1993; Cao and Greenhalgh, 1994; Sethian and Popovici, 1999). The Fast Marching (FM) method proposed by Sethian (1996) and the Fast Sweeping (FS) method presented by Zhao (2005) are among the most robust and efficient FD techniques for traveltime computation. Fomel (2004) applies the FM method to model P-wave traveltimes for VTI (TI with vertical symmetry axis) media, whereas Waheed et al. (2015a,b) employ the FS method to traveltime modeling in 2D TI media and 3D tilted orthorhombic media. Traveltime tomography can be performed using adjoint-state methods where gradients of the objective function are obtained implicitly using the FS method (Huang and Bellefleur, 2012; Waheed et al., 2016). Alternatively, the gradients can be found explicitly by solving a linearized eikonal equation in which the intermediate step of computing the adjoint-state variables is eliminated. Then traveltime tomography can be performed using the Gauss-Newton approximation (Li et al., 2013; Treister and Haber, 2016). This approach should be particularly useful in tomographic inversion for multiple parameters of anisotropic media.

Here, we solve the linearized eikonal equation for VTI media using a second-order finite-difference approximation (Rickett and Fomel, 1999; Franklin and Harris, 2001), with the background traveltimes modeled by the FM method. The accuracy of our methodology is verified by computing traveltime per-
2 Yogesh Arora and Ilya Tsvankin

turbations caused by Gaussian anomalies in the vertical ($V_{p0}$) and horizontal ($V_{h0}$) P-wave velocities and in the anellipticity parameter $\eta$. We also present a numerical example for a perturbed VTI Marmousi model (Alkhalifah, 1997). In both tests, the computed traveltime perturbations are compared with the differences between the traveltimes for the perturbed and background velocity models. The main goal of this work is to build a framework for computation of the inversion gradients in anisotropic traveltome of diffracted waves.

2 THEORY

The objective function in traveltime tomography is generally chosen as the $L_2$-norm of the difference between the observed ($T^{obs}$) and calculated ($T^{cal}$) traveltimes:

$$\mathcal{J}(\textbf{m}) = \frac{1}{2} ||T^{obs} - \textbf{P} T^{cal}(\textbf{m})||^2,$$

(1)

where $\mathbf{m}$ is the velocity model, and $\mathbf{P}$ is an operator that projects the simulated traveltimes onto the receiver locations. The gradient of the objective function with respect to the model parameters is:

$$\frac{\partial \mathcal{J}(\textbf{m})}{\partial \textbf{m}} = \textbf{P}^T \left[ T^{obs} - \textbf{P} T^{cal}(\textbf{m}) \right] \frac{\partial T^{cal}(\textbf{m})}{\partial \textbf{m}},$$

(2)

where $\textbf{P}^T$ is the transpose of $\textbf{P}$. It is evident from equation 2 that tomographic inversion requires computation of both the traveltimes and their gradients. To model P-wave traveltimes for VTI media, we solve the eikonal equation:

$$|\nabla T|^2 = \frac{1}{V^2(\theta)},$$

or

$$\left( \frac{\partial T}{\partial x} \right)^2 + \left( \frac{\partial T}{\partial z} \right)^2 = \frac{1}{V^2(\theta)},$$

(3)

where $(x, z)$ is the source location, $V(\theta)$ is the P-wave phase velocity as a function of the phase angle $\theta$ with the symmetry axis. We use the following approximation for the phase velocity presented by Alkhalifah (1998):

$$2V^2(\theta) \approx V_{h0}^2 \sin^2 \theta + V_{p0}^2 \cos^2 \theta + \sqrt{V_{h0}^2 \sin^2 \theta + V_{p0}^2 \cos^2 \theta} \cdot \sqrt{V_{h0}^2 - V_{p0}^2} \cdot \sin^2 2\theta,$$

(4)

where $V_{p0}, V_{h0} = V_{p0} \sqrt{1 + 2\epsilon}$, and $V_{h0} = V_{p0} \sqrt{1 + 2\delta}$ are the P-wave vertical, horizontal, and normal-moveout velocities, respectively; $\epsilon$ and $\delta$ are Thomsen anisotropy coefficients. To compute the traveltime gradient, we solve the linearized eikonal equation for the phase velocity $V(\theta)$:

$$\nabla T \cdot \nabla \tau = -\nabla^2 V(\theta),$$

or

$$\frac{\partial T}{\partial x} \frac{\partial \tau}{\partial x} + \frac{\partial T}{\partial z} \frac{\partial \tau}{\partial z} = -\nabla^2 V(\theta),$$

(5)

where $\tau$ and $\nu$ are the gradients (derivatives) of the traveltime ($\partial T/\partial m$) and phase velocity ($\partial V/\partial m$), respectively. Also, equation 5 can be used to obtain traveltime perturbations ($\tau = \partial T/\partial \lambda$) as a function of perturbations in the phase velocity ($\nu = \partial V/\partial \lambda$). We use the following first-order approximation for the phase-velocity perturbation:

$$\delta V \approx \frac{\partial V}{\partial \lambda} \delta \lambda,$$

(6)

where $\lambda$ is a medium parameter (e.g., $V_{p0}, V_{h0}, \eta$) and $\delta \lambda$ is its perturbation. If necessary, equation 6 can be replaced with a higher-order approximation in a straightforward way.

3 FAST MARCHING METHOD

To solve equation 3 on regular grids, one can use finite-difference approximations which produce a smooth traveltime distribution even in the presence of strong velocity variations. Here, we follow the Fast Marching method in applying an upwind FD scheme starting from the source location. At each grid point $(i, j)$, the traveltimes are computed sequentially similarly to Dijkstra’s shortest-path algorithm with the following approximation for the gradient operator in equation 3 ( Sethian and Popovici, 1999):

$$|\nabla T|^2 \approx \max(D_{ij}^{+x} T, 0)^2 + \min(D_{ij}^{-x} T, 0)^2 + \max(D_{ij}^{+z} T, 0)^2 + \min(D_{ij}^{-z} T, 0)^2,$$

(7)

where the operators $D_{ij}^{+x}$ and $D_{ij}^{+z}$ are obtained using a second-order finite-difference approximation ( Rickett and Fomel, 1999; Franklin and Harris, 2001):

$$D_{ij}^{+x} T = \frac{\pm 3T_{i,j} + 4T_{i+1,j} + T_{i+2,j}}{2\Delta x},$$

$$D_{ij}^{+z} T = \frac{\pm 3T_{i,j} + 4T_{i,j+1} + T_{i,j+2}}{2\Delta z}.$$

(8)

Likewise, we approximate the gradient operators in equation 5 as follows:

$$\nabla T \cdot \nabla \tau \approx D_{ij}^{+x} T \left[ \max(D_{ij}^{-x} \tau, 0) + \min(D_{ij}^{+x} \tau, 0) \right] + D_{ij}^{+z} T \left[ \max(D_{ij}^{-z} \tau, 0) + \min(D_{ij}^{+z} \tau, 0) \right].$$

(9)

Fomel (2004) implements equations 7 and 8 for VTI media using the phase velocity from equation 4 in Madagascar (Fomel et al., 2013) program sfektionalvt. To solve the linearized eikonal equation, first we compute the traveltimes ($T$) for the background model using that program and obtain the slownesses $D_{ij}^{-x} T$ and $D_{ij}^{-z} T$. These slownesses determine the phase direction for the wavefront propagation. Then equation 9 is solved for the traveltime perturbations ($\delta T$) in the same sequence as the one employed to calculate the background traveltimes. Because the method is based on a first-order linearization, the perturbations in the medium parameters should be small enough to ensure that the phase direction remains close to that for the background model. The VTI medium is parameterized by the velocities $V_{p0}$ and $V_{h0}$ and the anellipticity parameter $\eta$ (Tsivankin, 2012),

$$\eta = \frac{\epsilon - \delta}{1 + 2\delta}.$$  

(10)
Figure 1. Traveltimes \( T_b \) computed for a source located at \( x = z = 0 \) km in a homogeneous background VTI medium with \( V_{P0} = 3.1 \) km/s, \( V_{hor} = 3.80 \) km/s, and \( \eta = 0.34 \) (based on the model of Greenhorn shale).

Figure 2. (a) Perturbation in the parameter \( V_{P0} \) for the model in Figure 1. (b) The traveltime perturbations for the source at \( x = z = 0 \) km obtained from the linearized eikonal equation. (c) The traveltime differences between the perturbed and background models computed with the FM method.

4 NUMERICAL EXAMPLES

To verify the accuracy of the proposed technique, we first use a background VTI model with the parameters of Greenhorn shale (Fomel, 2004). Then we introduce a Gaussian anomaly in one of the VTI parameters \( (V_{P0}, V_{hor}, \eta) \) and compute corresponding traveltime perturbations \( \delta T \) from equation 5. The background traveltimes \( T_b \) are obtained from equation 3 using the FM method (Figure 1). For comparison, we also calculate the traveltimes \( T \) for the perturbed model with the FM method and subtract the background values to find the exact differences \( (T - T_b) \). The traveltime perturbations produced by our method are almost identical to these actual values (see Figure 2 for the perturbation in \( V_{P0} \)), and the sum \( T_b + \delta T \) practically coincides with the traveltimes for the perturbed model. Next, the structurally complex VTI Marmousi model is used as the background medium (Figures 3 and 4). We perturb the medium parameters and compute the corresponding traveltime perturbations using the algorithm described above (Figures 5–7). As in the previous test, the traveltime perturbations are close to the actual traveltime differences computed with the FM method. Clearly, the proposed numerical scheme for solving the linearized eikonal equation is sufficiently accurate even for pronounced parameter perturbations in the presence of substantial heterogeneity.

5 CONCLUSIONS

Solving the eikonal equation is an important step in implementing traveltome tomography, which is often used to build
Yogesh Arora and Ilya Tsvankin

Figure 5. (a) Perturbation in the parameter $V_{P0}$ for the background model in Figure 3. (b) The traveltime perturbations obtained from the linearized eikonal equation. (c) The traveltime difference between the perturbed and background models computed with the FM method.

Figure 6. (a) Perturbation in the parameter $V_{\text{hor}}$ for the background model in Figure 3. (b) The traveltime perturbations obtained from the linearized eikonal equation. (c) The traveltime difference between the perturbed and background models computed with the FM method.

Figure 7. (a) Perturbation in the parameter $\eta$ for the background model in Figure 3. (b) The traveltime perturbations obtained from the linearized eikonal equation. (c) The traveltime difference between the perturbed and background models computed with the FM method.

Initial models for migration velocity analysis in seismic imaging. We propose a methodology based on second-order finite-differences to solve the linearized eikonal equation for VTI media. Application of this algorithm permits robust and efficient traveltime computation in the presence of strong spatial velocity variations. The algorithm was first employed to model the traveltime perturbations caused by Gaussian anomalies in the parameters $V_{P0}$, $V_{\text{hor}}$, and $\eta$ responsible for P-wave kinematics. The traveltimes for both the background and perturbed models were also obtained by solving the eikonal equation with the Fast Marching method. The results of both computations are almost identical, which confirms the accuracy of the proposed algorithm. Then the test was successfully repeated for the structurally complex VTI Marmousi model. The developed numerical scheme can also be used to compute the traveltime gradients from the linearized eikonal equation. Ongoing work involves incorporating these gradients with respect to the TI parameters into anisotropic traveltime tomography of diffracted waves.

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