A mimetic imaging technique for the acoustic inverse scattering problem

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ABSTRACT
We introduce an imaging procedure based on solving an integral equation that mimics the well-known Lippmann-Schwinger equation for the scattered pressure field of an acoustic medium. The imaging problem is formulated by creating a regularly sampled grid over a desired imaging domain and relating the impulse response of the background medium computed from each grid point to the scattered pressure field through the discretized mimetic integral equation. This discretization leads to a linear system of equations that can be solved for the unknown contrast source function that generates the scattered field without using any prior knowledge or weak scattering approximations of the contrast source. An image is constructed over the specified imaging domain by computing the energy of the obtained solution at each grid point. We apply this technique to imaging morphologically random inclusions and compare the results with those obtained using the linear sampling method.

Key words: inverse scattering, mimetic imaging, linear sampling method

1 INTRODUCTION
Imaging is a qualitative inverse scattering problem that seeks to localize the boundaries or interfaces between media of different physical properties. In this paper, we restrict our attention to imaging acoustic media, in which scalar pressure fields propagate and the scattering is determined by variations in the bulk modulus and mass density of the media. Imaging the boundaries of such variations is challenging due to imprecise (or altogether absent) knowledge of the medium. If the contrast in physical properties is sufficiently strong, much of the wave energy that is sent into a medium can become localized amongst the heterogeneities, where it is scattered and redirected multiple times, before finally being recorded at a receiver. Such strong multiple scattering makes it difficult to interpret where the observed scattered fields originated.

To construct accurate images, it is necessary for imaging algorithms to properly account for the multiple scattering undertaken by a wave field as it propagates through a medium. The linear sampling method [Colton and Kirsch (1996); Colton et al. (1997); Chen et al. (2010); Guo et al. (2013); Haddar et al. (2014); Prunty and Snieder (2019)] is one such imaging technique that has garnered considerable attention. To demarcate the boundary of the scatterer, the method relies on a characteristic blowup behavior of the solution to a boundary integral equation, called the near-field equation, which relates the scattered field recorded on a surface to the observed impulse response from a specified point in the known background medium. In particular, the norm of the solution to the near-field equation becomes arbitrarily large when the specified point lies outside the support of the scatterer. By solving this boundary integral equation for each point in a regularly sampled grid that covers the unknown scatterer, it is possible to image the scatterer by noting where the norm of the solutions become arbitrarily large.

Conversely, the Lippmann-Schwinger equation [e.g., Lechleiter and Monk (2015)] provides an exact representation of the scattered field as a volume integral of the impulse response over the region of the background medium containing the scatterer. Thus, an imaging procedure based on solving the Lippmann-Schwinger equation requires the same information as the linear sampling method: namely, the observed scattered field and the impulse response of the background medium. In principle, one could use the same sampling domain from the linear sampling method to discretize the vol-
ume integral in the Lippmann-Schwinger equation. As the Lippmann-Schwinger equation is not a boundary integral representation of the scattered field, an imaging procedure based on its solution is expected to require less surface data than boundary integral methods to obtain a satisfactory image.

While the Lippmann-Schwinger equation has primarily been investigated for quantitative inversion (e.g., [van den Berg and Kleinman (1997); van den Berg et al. (1999); Giorgi et al. (2013)]), its use for imaging purposes is not new. Previous attempts at using the Lippmann-Schwinger equation for imaging have often relied on a weak scattering approximation [e.g., Caorsi et al. (1999); Giorgi et al. (2013)], its use for imaging more than a matrix-vector product to define the support of a scatterer from measurements of the scattered field, and mass density and refractive index $n$, the velocity contrast $m$, and the density ratio $q$ as

$$n := \frac{c_0^2}{c^2}, \quad m := 1 - n, \quad q := \frac{\rho_0}{\rho}.$$  (3)

We can decouple the acoustic wave equation (2) to separate the influence of the background medium from that of the scatterer. It follows that if the unperturbed wave $p_0$ satisfies

$$\rho_0 \nabla \cdot \left( \frac{1}{\rho_0} \nabla p_0 \right) - \frac{1}{c_0^2} \frac{\partial^2 p_0}{\partial t^2} = -\delta(x - x_0) \zeta(t), \quad x \in \mathbb{R}^3, \quad t \geq 0,$$  (4a)

$$p_0(x, t; x_0) = 0, \quad \frac{\partial p_0(x, t; x_0)}{\partial t} = 0, \quad x \in \mathbb{R}^3, \quad t < 0,$$  (4b)

then the scattered pressure wave $p_s$ satisfies

$$\rho_0 \nabla \cdot \left( \frac{1}{\rho_0} \nabla p_s \right) - \frac{1}{c_0^2} \frac{\partial^2 p_s}{\partial t^2} = \chi(x, t; x_0), \quad x \in \mathbb{R}^3, \quad t \geq 0,$$  (5a)

$$p_s(x, t; x_0) = 0, \quad \frac{\partial p_s(x, t; x_0)}{\partial t} = 0, \quad x \in \mathbb{R}^3, \quad t < 0,$$  (5b)

where $\chi$ is the contrast source function given by

$$\chi := - \left[ \frac{1}{q} \nabla q \cdot \nabla p + \frac{m}{c_0^2} \frac{\partial^2 p}{\partial t^2} \right].$$  (6)

In expression (6), $p$ denotes the total pressure field at the scatterer. Note that the density ratio $q$ is constant for all $x \in \mathbb{R}^3 \setminus D$, so that its gradient $\nabla q$ is nonzero only for $x \in D$. As seen from definition (3), the velocity contrast $m$ clearly has compact support $D$. It follows that the contrast source function $\chi$ has compact support $D$, and is a combination of both monopole sources (due to velocity variations) and dipole sources (due to density variations).
Let $G_0$ denote the unperturbed Green function of the background medium satisfying

$$
\rho_0 \nabla \cdot \left( \frac{1}{\rho_0} \nabla G_0 \right) - \frac{1}{\rho_0} \frac{\partial^2 G_0}{\partial t^2} = -\delta(x - \eta)\delta(t),
$$

$x \in \mathbb{R}^3, \ t \geq 0,$

$$
G_0(x, t; \eta) = 0, \quad \frac{\partial G_0(x, t; \eta)}{\partial t} = 0, \quad x \in \mathbb{R}^3, \ t < 0.
$$

By linearity, the solutions to equations (4) and (5) are given by

$$p_0(x, t; x_s) = \int_{\mathbb{R}} G_0(x, t - \tau; x_s)\zeta(\tau) d\tau,$n

and

$$p_s(x, t; x_s) = \int_D \int_{\mathbb{R}} G_0(x, t - \tau; \eta)\chi(\eta, \tau; x_s)\, d\eta\, d\tau,$n

respectively. Equation (9) is the Lippmann-Schwinger equation for scattered pressure field $p_s$. If the contrast source function (6) is known, we could use equation (9) to exactly compute the scattered pressure field observed at any point $x$ and at any time $t$. In practice, however, the contrast source function (6) is unknown since we do not have precise knowledge of the acoustic medium or the total pressure field at the scatterer.

On the other hand, suppose we know (or can estimate) the background medium without the scatterer so that we can compute the unperturbed Green function $G_0$. Then, provided we can separate the scattered pressure field $p_s$ from the recorded total field $p$, equation (9) provides a means to solve for the unknown contrast source function $\chi$ without any prior knowledge or weak scattering approximations of the contrast source.

3 A MIMETIC SAMPLING METHOD

In an imaging experiment, we record the total pressure field $p$ at receiver locations $x_r$, which are typically restricted to an acquisition surface we denote by $\Gamma_r$. Similarly, the sources used to generate the pressure field are restricted to points $x_s$ of an acquisition surface we denote by $\Gamma_s$ (possibly equal to $\Gamma_r$). Both surfaces $\Gamma_r$ and $\Gamma_s$ are assumed to be disjoint from $\overline{D}$, where $\overline{D}$ denotes the closure of $D$. By evaluating equation (9) on the receiver surface $\Gamma_r$, we can relate the recorded data to the unknown contrast source function $\chi$.

As it stands, equation (9) is difficult to solve for a number of reasons. For one, the unperturbed Green function satisfying equation (7) is not a known analytic expression, since the background medium may be arbitrarily heterogeneous. In general, numerical methods are needed to approximate the Green function. Such methods are numerically unstable when trying to model the impulse response of a system, but fair better when modeling a function of finite bandwidth. As a result, what is often obtained from a numerical approximation is not the Green function itself, but an approximation of the Green function convolved with some band-limited time function. Second, to properly account for the spatial distribution of the contrast source function $\chi$, it is necessary to discretize the volume integration in equation (9) using a quadrature rule. This may prove challenging and computationally expensive when attempting to accommodate the arbitrary structure of the background medium.

To the first issue, let $S \in C^2(\mathbb{R})$ be a time-dependent function possessing the same frequency band as the scattered pressure field $p_s$. For an arbitrary point $z \in \mathbb{R}^3$, we define the test function

$$\Psi(x_r, t; z) := \int_{\mathbb{R}} G_0(x_r, t - \tau; z)S(\tau)\, d\tau.$n

The test function (10) represents a band-limited impulse response of the background acoustic medium that can be effectively modeled using numerical methods. The choice of the time function $S$ need only be made in consideration of the frequency band of the data, and does not necessarily need to equal to the generating pulse function $\zeta$ (which may be unknown). We will use expression (10) as a proxy for the unperturbed Green function $G_0$ in equation (9). Let $\Omega$ denote a subset of the background medium that contains the scatterer. Our imaging procedure is then based on solving the mimetic equation

$$p_s(x_r, t; x_s) = \int_{\mathbb{R}} \int_{\Omega} \Psi(x_r, t - \tau; z)\chi(z, \tau; x_s)\, dz\, d\tau.$n

Due to the compact support $D \subset \Omega$ of the contrast source function, we expect $\chi = 0$ for all $z \in \Omega \setminus \overline{D}$ and for all times $\tau \in \mathbb{R}$. Conversely, we expect $\chi \neq 0$ for $z \in D$ and for some $\tau \in \mathbb{R}$. That is, we have

$$\|\chi(z, \cdot; x_s)\|_{L^2(\mathbb{R})} = 0, \quad z \in \Omega \setminus \overline{D}$$

and

$$\|\chi(z, \cdot; x_s)\|_{L^2(\mathbb{R})} \neq 0, \quad z \in D.$$

It follows that an image of the scatterer can be obtained by plotting the energy of the solution to equation (11) over the domain $\Omega$.

We numerically implement our proposed imaging technique as follows. Given a model of the background medium, we discretize the imaging domain $\Omega$ into $N_x$ distinct points using a regularly sampled grid that covers the region of interest. For each grid point $z_n, n =$
1, ..., N_s, we discretize the test function observed at the receiver locations as
\[ \Psi(i, k, n) := \Psi(x_i, k\Delta t; z_n) \]
where \( i = 1, ..., N_r \) is the number of receivers, \( k = 0, ..., N_t - 1 \) is the number of time samples, and \( \Delta t \) is the sampling interval. For a general heterogeneous background medium, numerical methods are needed to compute these test functions. In Section 5, we consider the special case of a constant background medium for which the test functions assume a simple, analytic expression. Note that the test functions can be efficiently computed using source-receiver reciprocity whenever the number of grid points is expected to be greater than the number of receivers.

Similarly, the scattered pressure field is discretized as
\[ p(i, k, j) := p_s(x_i, k\Delta t; y_j), \]
where \( j = 1, ..., N_s \) is the number of sources. Figure 1 shows a schematic in which we construct a regularly sampled grid that covers an unknown scatterer.

Next, we discretize the mimetic integral equation (11) into a sum over the grid points \( z_n \). Since equation (11) is convolutional in time, the problem is efficiently solved in the frequency domain. Here, we use a circumflex \( \hat{\cdot} \) to denote frequency-domain quantities and \( \omega \) to denote a vector of angular frequencies. For each source \( j = 1, ..., N_s \), we take the scattered pressure field \( \hat{p}(\cdot, j) \) and solve the discretized mimetic equation
\[ \sum_{n=1}^{N_s} \hat{\Psi}(i, \omega, n) \hat{x}(n, \omega, j) = \hat{p}(i, \omega, j) \] (12)
in a least-squares sense. An image is obtained for each source as
\[ I_j(z_n) := \| \hat{x}(n, \cdot, j) \|_2, \] (13)
where the 2-norm is computed over all angular frequencies \( \omega \). The final image \( I_{\text{MSM}}(z_n) \) is defined as the root-mean-square of the normalized images \( I_j \):
\[ I_{\text{MSM}}(z_n) := \left( \frac{1}{N_s} \sum_{j=1}^{N_s} f_j(z_n)^2 \right)^{1/2}, \] (14a)
\[ f_j(z_n) := \frac{I_j(z_n) - \min I_j}{\max I_j - \min I_j}. \] (14b)
It follows that the final image satisfies \( 0 \leq I_{\text{MSM}}(z_n) \leq 1 \) for all points \( z_n \) in the imaging domain \( \Omega \), where values close to one indicate a likely location of the scatterer and values close to zero indicate an unlikely location of the scatterer.

4 COMPARISON WITH THE LINEAR SAMPLING METHOD

The linear sampling method is an inverse scattering technique that is based on solving the near-field equation
\[ \Psi(x_r, t; z) = \int_{\mathbb{R}^+} \int_{\Gamma_s} p_s(x_r, t - \tau; x_s) \varphi(x_s, \tau; z) ds(x_s) d\tau, \] (15)
where \( \varphi \) is an unknown density function. The discretization of the near-field equation leads to the linear system
\[ \sum_{j=1}^{N_s} \hat{p}(i, \omega, j) \hat{\varphi}(j, \omega, n) = \hat{\Psi}(i, \omega, n) \] (16)
Due to the blowup behavior of the solution \( \hat{\varphi} \), we obtain an image as
\[ I_{\text{LSM}}(z_n) := \frac{g(n) - \min g}{\max g - \min g}, \] (17a)
\[ g(n) := \frac{1}{\| \hat{\varphi}(\cdot, n) \|_2}. \] (17b)
It follows that the LSM image also satisfies \( 0 \leq I_{\text{LSM}}(z_n) \leq 1 \) for all points \( z_n \) in the imaging domain.

By comparing equations (11) and (15), we see that the mimetic sampling method is reciprocal to the linear sampling method in the sense that the roles of the scattered pressure field \( p_s \) and the test function \( \Psi \) have been interchanged. In equation (11), we take a linear combination of the test functions over the imaging domain to match the observed scattered field at the receiver surface. In equation (15), we take a linear combination of the observed scattered fields to match a prescribed test function which radiates from a single point in the imaging domain. Thus, the integral operators in equations (11) and (15) perform the same actions but using opposite kernels.

An important difference between the mimetic sampling method and the linear sampling method is that the contrast source function \( \hat{x} \) is a global solution over the imaging domain \( \Omega \), while the density function \( \varphi \) is local to the specific imaging point \( z \in \Omega \). Thus, the mimetic sampling image \( I_{\text{MSM}}(z) \) at a given point \( z \) is dependent on every other point in the imaging domain, but the linear sampling image \( I_{\text{LSM}}(z) \) is independent of every other point in the imaging domain. Another important difference between the two methods is that the success of the mimetic sampling method is entirely dependent on the imaging domain containing the scatterer. Should the chosen imaging domain not contain the scattering target \( D \not\subset \Omega \), the mimetic sampling method would fail since the least-squares solution of equation (12) forces a nonzero solu-
tion \( \chi \). In contrast, the image obtained using the linear sampling method would remain unaffected since it relies on the blowup behavior of the solution \( \varphi_s \) to localize the scattering target.

5 NUMERICAL EXAMPLES

We test our imaging algorithm using a two-dimensional, homogeneous acoustic medium in which we embed morphologically random inclusions (Figure 2). The background medium is characterized by a constant velocity \( c_0 = 2 \text{ m/s} \) and a constant density \( \rho_0 = 1 \text{ kg/m}^3 \). The random inclusions represent perturbations in the ground medium characterized by a constant velocity \( c \) and density \( \rho \) as the data. We set the signal-to-noise ratio (SNR) to 0.3, as defined by

\[
\text{SNR} = \frac{P_{\text{signal}}}{P_{\text{noise}}},
\]

where \( P \) denotes the average power (i.e., the mean-square amplitude). In this case, \( P_{\text{signal}} \) denotes the average signal power taken over 24 traces per shot gather. A common shot gather of the scattered pressure field with noise is shown in Figure 4.

Figure 5 shows the resulting images of the random inclusions obtained for both the mimetic sampling method as well as the linear sampling method. In both cases, we observe a dilation in the size and location of the random inclusions due to the erroneous background velocity. The imprint of the noisy signal is evident in the nonuniform amplitudes of the inclusions as compared with Figure 3. In general, we again see that the quality of the reconstructions using the linear sampling method is worse than that for the mimetic sampling method.

Finally, it is worth pointing out a limitation of the current formulation of our imaging method. As can be seen from expression (6), the contrast source function depends on the direction of the impinging...
Figure 2. (a) The acoustic velocity and (b) density models for the random inclusions. Sources are indicated by blue dots and receivers by green dots. The dashed line indicates the imaging domain. The dominant wavelength of the Ricker pulse is shown for scale.

Figure 3. Reconstructions of the random inclusions using (a) the mimetic sampling method and (b) the linear sampling method.

6 CONCLUSIONS & FUTURE WORK

We have presented an imaging procedure based on solving an integral equation over a specified imaging domain that mimics the Lippmann-Schwinger equation (11) alone. In particular, the mimetic sampling method (like the linear sampling method) treats each grid point as a monopole source. Thus, these imaging methods will fail in the limiting case in which only density varies in the acoustic medium. To see this, suppose we generate the scattered pressure field \( p_s \) using only the density model shown in Figure 2(b), while the acoustic velocity is held fixed at \( c_0 = 2 \) m/s everywhere in the model. Figure 6 shows a typical shot gather corresponding to this limiting case.

Figure 7 shows the corresponding images of the random inclusions obtained using the mimetic sampling method and the linear sampling method. In both cases we see that the methods have failed to identify any coherent structure in the random inclusions, as expected.
constructing morphologically complex shapes when the background medium is known and homogeneous. Compared to the linear sampling method, our approach gives qualitatively cleaner images of the inclusions given the same scattered field data and test functions. Even in the presence of noise and an erroneous background model, our imaging technique gives reasonable reconstructions of the inclusions.

The current formulation of our imaging algorithm is unable to reconstruct directionally dependent sources, as evident in our numerical example of a pure density perturbation. The same is true for the linear sampling method. A possible remedy for this shortcoming would
be to consider how the solutions vary with respect to source locations; that is, considering the gradients $\nabla_x \chi$ and $\nabla_z \varphi$.

While the mimetic sampling method demonstrates favorable imaging capabilities in the numerical examples we have presented (excluding the pure density perturbation example), much research remains to be done on testing its performance when applied to exploration-type applications. In particular, it is worth exploring to what extent the mimetic sampling method can succeed with limited aperture data, as well as higher noise levels, acquisition errors, and errors in the assumed background model, particularly when the background medium is known to be heterogeneous.
Finally, it is worth pointing out that our imaging procedure has effectively ignored any quantitative information about the density and velocity variations contained in our solution to the mimetic integral equation. If we approximate the total pressure field \( p \) in the imaging domain (say, using a truncated scattering series), we could use expression (6) to invert for the density and velocity variations of the medium.

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**REFERENCES**


