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Three-dimensional Born inversion with an arbitrary reference

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ABSTRACT

Recent work of G. Beylkin helped set the stage for very general seismic inversions. We have combined these broad concepts for inversion with classical high-frequency asymptotics and perturbation methods to bring them closer to practically implementable algorithms. Applications include inversion schemes for both stacked and unstacked seismic data.

Basic assumptions are that the data have relative true amplitude, and that a reasonably accurate background velocity $c(x, y, z)$ is available. The perturbation from this background is then sought. Since high-frequency approximations are used throughout, the resulting algorithms essentially locate discontinuities in velocity.

An expression for a full 3-D velocity inversion can be derived for a general data surface. In this degree of generality the formula does not represent a computationally feasible algorithm, primarily because a key Jacobian determinant is not expressed in practical terms. In several important cases, however, this shortcoming can be overcome and expressions can be obtained that lead to feasible computing schemes. Zero-offsets, common-sources, and common-receivers are examples of such cases.

Implementation of the final algorithms involves, first, processing the data by applying the FFT, making an amplitude adjustment and filtering, and applying an inverse FFT. Then, for each output point, a summation is performed over that portion of the processed data influencing the output point. This last summation involves an amplitude and traveltime along connecting rays. The resulting algorithms are computationally competitive with analogous migration schemes.

INTRODUCTION

A seismic inversion algorithm provides seismic imaging (as in migration) as well as estimation of parameters. Considered here is the subclass of inversion algorithms which assume that the desired parameters are accurately described by unknown, deterministic perturbations from known reference values. The perturbation assumption is often referred to as the "Born approximation." Consideration is further confined to cases adequately described by the acoustic wave equation (with reference speed c and perturbation α) in the high-frequency limit. Such algorithms have been developed for progressively more complex background velocities and source-receiver configurations (Cohen and Bleistein, 1979; Clayton and Stolt, 1981; Bleistein and Gray, 1985; Cohen and Hagin, 1985; Sullivan and Cohen, 1985). In each extension, the crucial issue has been determination of certain properties of a matrix involving derivatives of traveltimes.

Here a recent result, appearing in Beylkin (1985), is discussed and extended along the lines of the references just given. Beylkin (1985) reduces the problem to consideration of a determinant h (defined below), which arises from a transformation of variables. Assuming that h is nonsingular (i.e., finite and nonzero), he establishes a very general inversion result. The singularities (if any) of h are determined by the reference speed and the source-receiver configuration.

The interesting cases of singular h are due to phenomena such as "caustics." However, h is also singular in the case of insufficient data. Thus, for example, such obvious impossibilities as obtaining a three-dimensional (3-D) inversion from a single line of data can be ruled out. In the case of insufficient data, additional assumptions must be made about the subsurface geometry which are consistent with the given data configuration. In the example just mentioned, a necessary assumption would be that the subsurface was independent of the direction orthogonal to the line of observations. Then either the 2-D wave equation (i.e., assume line sources) could be used as in classical migration (Schneider, 1978; Stolt, 1978), or, preferably, the 3-D wave equation could still be used (i.e., use point sources) (Cohen and Bleistein, 1979). The latter approach is known as the $2\frac{1}{2}$-D geometry and is expounded in detail in Bleistein (1986).

Beylkin's (1985) paper uses powerful mathematical tools, such as pseudodifferential operators and generalized Radon transforms. Moreover, Beylkin frames his work in an N-dimensional space. Here, we dispense with much of this mathematical machinery and confine the discussion to the 3-D

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LIST OF SYMBOLS

A = amplitude; equation (8)
a = total amplitude; equation (12)
b = inversion amplitude; equation (13)
c = reference velocity; equation (5)
D = observations; equation (7)
$\mathbf{d}$ = half-offset; equation (4)
$F(\omega)$ = filter function; equation (13)
G = Green's function; equation (7)
g_0, g_r, g_s = square of differential surface area; equations (32), (51)
h = determinant; equation (18)
J = ray Jacobian; equation (A-6)
K = kernel of integral equation; equation (22)
$\mathbf{k} = \omega\nabla\phi$; equation (17)
$\hat{\mathbf{n}}_0, \hat{\mathbf{n}}_r, \hat{\mathbf{n}}_s$ = unit normal at observation point; equations (32), (50)
$\mathbf{p}, \mathbf{p}_0, \mathbf{p}_r, \mathbf{p}_s$ = gradient of traveltime; equations (26), (32), (43), (44)
q_3 = initial value of p_3; equation (40)
R_n = reflection coefficient; equation (35)
$\mathbf{t}_1, \mathbf{t}_2$ = surface tangent vectors; equation (A-5)
u = field; equation (6)
v = speed; equation (5)
$\mathbf{x}, \mathbf{x}'$ = field points; equation (1)
$\mathbf{x}_r$ = receiver location; equation (1)
$\mathbf{x}_s$ = source location; equation (1)
$\mathbf{x}_0$ = zero-offset source point; equation (25)
$\mathbf{y}$ = running point on ray; equation (A-1)
α = perturbation in velocity; equation (5)
β = reflectivity function; equation (35)
γ = singular function; equation (35)
ϕ = total traveltime; equation (12)
σ = parameter along ray; equation (A-1)
σ_f = final value of ray; equation (38)
τ = traveltime; equation (8)
ξ = surface coordinates; equation (1)
ω = circular frequency; equation (7)

case. In the generality of Beylkin's paper the key determinant h is not expressible in computational terms. Here we focus on expressing h; using results presented earlier in Cohen and Hagin (1985), we obtain feasible algorithms in several important cases.

Assuming h is nonsingular, we can evaluate h for the following cases of propagation governed by the acoustic wave equation.

(1) The zero-offset case with a general $c(x, y, z)$ reference velocity (described below).

(2) The common-source gather and also the common-receiver gather with general $c(x, y, z)$ reference velocity (also described below).

(3) The case of common offset with a *constant* reference velocity. An explicit inversion formula was obtained by Sullivan and Cohen (1985).

(4) In the conventional $2\frac{1}{2}$-D geometry, common-source gathers, common-receiver gathers, and common-offset gathers as well as zero-offset data can be inverted for an arbitrary $c(x, z)$ reference velocity. [This case will be addressed in a sequel to this paper.] Note that the 3-D common-source gather treated here does *not* subsume the $2\frac{1}{2}$-D common-source gather. The explanation is that in the 3-D case, the source is on one particular line which precludes the assumption from $2\frac{1}{2}$-D geometry that observations made on parallel lines would be identical.

Actual implementation of a typical algorithm is discussed at the end of the section on zero-offset inversion.

THE GENERAL INVERSION FORMULA

Consider a completely arbitrary source-receiver configuration parameterized by two surface coordinates ξ_1 and ξ_2. Vectors $\mathbf{x}$ and $\mathbf{x}'$ denote arbitrary subsurface field points. Vector $\mathbf{x}_r$ denotes a generic source location, while $\mathbf{x}_s$ denotes a generic receiver location. Any relation between the sources and receivers is defined by the dependencies of $\mathbf{x}_r$ and $\mathbf{x}_s$ on the parameters ξ_1 and ξ_2.

$$\mathbf{x} = (x_1, x_2, x_3),$$
$$\mathbf{x}' = (x_1', x_2', x_3'),$$
$$\xi = (\xi_1, \xi_2), \tag{1}$$
$$\mathbf{x}_s = \mathbf{x}_s(\xi),$$

and

$$\mathbf{x}_r = \mathbf{x}_r(\xi).$$

As indicated, (x_1, x_2, x_3) is used to denote the three Cartesian coordinates of $\mathbf{x}$. The third coordinate x_3 is identical to the depth coordinate z. A typical configuration is

$$\mathbf{x}_r(\xi) = \mathbf{x}_s(\xi), \tag{2}$$

i.e., the offset is zero. The common-source configuration is defined by

$$\mathbf{x}_s(\xi) = \text{constant},$$

and (3)

$$\mathbf{x}_r(\xi) = \text{parameterization of observation surface.}$$

Another example is the common-offset case. If, for simplicity, the data surface is flat, this case can be expressed as

$$\mathbf{x}_r(\xi) = (\xi_1, \xi_2, 0) + 2\mathbf{d}, \tag{4}$$

where $\mathbf{d}$ = constant and $\mathbf{x}_s = (\xi_1, \xi_2, 0)$.

Our formulation of the source-receiver configuration admits the possibility of a curved observation surface, instead of the usual assumption of observations on the flat observation plane $z = x_3 = 0$. We have included this mild generalization because it may aid in the development of prestatics inversion algorithms (May and Covey, 1981; Wiggins, 1984).

Make the following assumptions.

(1) The speed, $v(x_1, x_2, x_3)$, is well-approximated by a known reference speed $c(x_1, x_2, x_3)$, so that

$$\frac{1}{v^2(\mathbf{x})} = \frac{1}{c^2(\mathbf{x})}[1 + \alpha(\mathbf{x})], \tag{5}$$

where $\alpha(x_1, x_2, x_3)$ is a perturbation correction. The inversion algorithms will be designed to compute α (or jumps in α).

(2) The seismic fields are governed with sufficient accuracy by the 3-D acoustic wave equation.

(3) The seismic source can be reduced to an ideal 3-D point source, so that

$$\nabla^2 u(t, \mathbf{x}) - \frac{1}{v^2(\mathbf{x})}\frac{\partial^2}{\partial t^2} u(t, \mathbf{x}) = -\delta(t)\delta(\mathbf{x} - \mathbf{x}_s). \tag{6}$$

These assumptions and application of Green's theorem lead to the linear integral equation relating α to the data:

$$D(\omega, \mathbf{x}_r, \mathbf{x}_s) \approx \omega^2 \iiint d^3x' \frac{G(\omega, \mathbf{x}', \mathbf{x}_s)G(\omega, \mathbf{x}', \mathbf{x}_r)}{c^2(\mathbf{x}')} \alpha(\mathbf{x}'), \tag{7}$$

where D is the Fourier transform ($t \rightarrow \omega$) of the surface data.

Here we use the assumption that α in equation (5) is small, to approximate the field emanating from the sources as a field governed by a wave equation with a velocity function equal to the reference velocity. This assumption also permits linearization of the integral equation (Cohen and Bleistein, 1979). Since our sources are point sources, the functions denoted by G in equation (7) are the Green's functions for the wave equation with velocity $c(x_1, x_2, x_3)$. In equation (7) the unknown is the velocity perturbation $\alpha(x_1, x_2, x_3)$, and data are the observations D at receivers $\mathbf{x}_r$ due to sources $\mathbf{x}_s$.

We now exploit the fact that geophysical data reside in the high-frequency regime (Bleistein, 1984), so that the Green's functions in equation (7) may be replaced by their WKBJ approximations,

$$G(\omega, \mathbf{x}, \mathbf{x}_0) \sim A(\mathbf{x}, \mathbf{x}_0)e^{i\omega\tau(\mathbf{x}, \mathbf{x}_0)}, \tag{8}$$

where $\mathbf{x}_0$ is either $\mathbf{x}_s$ or $\mathbf{x}_r$, with the traveltime phase satisfying the eikonal equation

$$\nabla\tau \cdot \nabla\tau = \frac{1}{c^2(\mathbf{x})}, \tag{9}$$

and the amplitude satisfying the transport equation

$$2\nabla\tau \cdot \nabla A + A\nabla^2\tau = 0. \tag{10}$$

Thus, rewrite equation (7) as

$$D(\omega, \mathbf{x}_r, \mathbf{x}_s) \sim \omega^2 \iiint d^3x' \frac{a(\mathbf{x}', \xi)}{c^2(\mathbf{x}')} e^{i\omega\phi(\mathbf{x}', \xi)}\alpha(\mathbf{x}'), \tag{11}$$

where $\phi(\mathbf{x}, \xi)$ and $a(\mathbf{x}, \xi)$ are given by

$$\phi(\mathbf{x}, \xi) = \tau(\mathbf{x}, \mathbf{x}_s) + \tau(\mathbf{x}, \mathbf{x}_r),$$

and (12)

$$a(\mathbf{x}, \xi) = A(\mathbf{x}, \mathbf{x}_s)A(\mathbf{x}, \mathbf{x}_r).$$

From the basic migration principle of "backward propagation," it is not hard to guess that the inversion operator for expression (11) will have the negative of the phase in expression (11). The correct amplitude is not as easy to guess, and in the inversion equation

$$\alpha(\mathbf{x}) \sim \iint d^2\xi\, b(\mathbf{x}, \xi) \int d\omega\, F(\omega)e^{-i\omega\phi(\mathbf{x}, \xi)}D(\omega, \mathbf{x}_r, \mathbf{x}_s), \tag{13}$$

we merely denote the amplitude by $b(\mathbf{x}, \xi)$ (it is deduced below). In equation (13), $F(\omega)$ denotes a known filter and is included to honor the fact that the data are confined to the high-frequency regime. That is, in implementation, the impulsive source in equation (6) is actually band-limited; hence the resulting reflected data are also band-limited. Note that while, a priori, the unknown amplitude could depend upon ω, we suppress this potential dependence because our prior inversion results suggest that $b(\mathbf{x}, \xi)$ is independent of ω. The results that follow confirm this assumption.

Inserting expression (11) into expression (13) gives an equation which maps $\alpha(\mathbf{x}')$ to $\alpha(\mathbf{x})$. Thus, since this is an integral over all $\mathbf{x}'$ space, the kernel must be the 3-D Dirac delta function $\delta(\mathbf{x} - \mathbf{x}')$. A slight rewriting of this fact gives

$$\iint d^2\xi\, a(\mathbf{x}', \xi)b(\mathbf{x}, \xi) \int d\omega\, \omega^2 F(\omega) \times \exp\left\{i\omega[\phi(\mathbf{x}', \xi) - \phi(\mathbf{x}, \xi)]\right\} \approx c^2(\mathbf{x})\delta(\mathbf{x} - \mathbf{x}'). \tag{14}$$

The approximation in expression (14) is due to the filter $F(\omega)$ which will cause the delta function to be band-limited. Expression (14) must be solved for the unknown amplitude b. Then equation (13) will give the inversion algorithm for α.

Since high-frequency approximations have been used, low-frequency phenomena such as velocity trend information (which are arguably absent from the data) cannot be recovered. Information about the discontinuities (i.e., rapid changes) in the substructure can be expected to be recovered. We therefore assume that critical trend information has been included in the background velocity.

The essence of Beylkin's result is that if attention is confined to the discontinuity structure of α, then in expression (14) only two terms of the Taylor series of $\phi(\mathbf{x}', \xi)$ about $\mathbf{x}' = \mathbf{x}$ must be kept; and similarly only one term of the expansion of $a(\mathbf{x}', \xi)$ must be kept. In order to establish these facts rigorously, Beylkin uses results in the theory of generalized Radon transforms. However, since the result on the left involves a Dirac delta function acting at $\mathbf{x}' = \mathbf{x}$, these approximations are intuitively reasonable. Thus in expression (14), we use the approximations

$$\phi(\mathbf{x}', \xi) \approx \phi(\mathbf{x}, \xi) + \nabla\phi(\mathbf{x}, \xi) \cdot (\mathbf{x}' - \mathbf{x}); \qquad \nabla = \nabla_x,$$

and (15)

$$a(\mathbf{x}', \xi) \approx a(\mathbf{x}, \xi),$$

to obtain

$$\iint d^2\xi\, a(\mathbf{x}, \xi)b(\mathbf{x}, \xi) \int d\omega\, \omega^2 F(\omega) \times \exp\left[i\omega\nabla\phi(\mathbf{x}, \xi) \cdot (\mathbf{x} - \mathbf{x}')\right] \approx c^2(\mathbf{x})\delta(\mathbf{x} - \mathbf{x}'). \tag{16}$$

Following Beylkin (1985), we make the change of variables

$$\mathbf{k} = \omega\nabla\phi(\mathbf{x}, \xi) \tag{17}$$

from (ω, ξ_1, ξ_2) to (k_1, k_2, k_3) (while viewing $\mathbf{x}$ as a "parameter"). Since the ω dependence on the Jacobian of this transformation is just a power, Beylkin (1985) defines the (scaled) Jacobian h by

$$h(\mathbf{x}, \boldsymbol{\xi}) = \omega^{-2} \frac{\partial(\mathbf{k})}{\partial(\omega, \xi_1, \xi_2)} = \begin{vmatrix} \frac{\partial \phi}{\partial x_1} & \frac{\partial \phi}{\partial x_2} & \frac{\partial \phi}{\partial x_3} \\ \frac{\partial^2 \phi}{\partial x_1 \partial \xi_1} & \frac{\partial^2 \phi}{\partial x_2 \partial \xi_1} & \frac{\partial^2 \phi}{\partial x_3 \partial \xi_1} \\ \frac{\partial^2 \phi}{\partial x_1 \partial \xi_2} & \frac{\partial^2 \phi}{\partial x_2 \partial \xi_2} & \frac{\partial^2 \phi}{\partial x_3 \partial \xi_2} \end{vmatrix}. \tag{18}$$

Equation (16) can now be rewritten as

$$\iiint d^3k \frac{a(\mathbf{x}, \boldsymbol{\xi}) b(\mathbf{x}, \boldsymbol{\xi})}{|h(\mathbf{x}, \boldsymbol{\xi})|} F(\omega) e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \approx c^2(\mathbf{x}) \delta(\mathbf{x} - \mathbf{x}'). \tag{19}$$

Once again the filter F will cause the delta function to be band-limited.

From expression (19) and the classical Fourier transform completeness relation, the choice of the inversion amplitude b is now clear, i.e.,

$$b(\mathbf{x}, \boldsymbol{\xi}) = \frac{1}{8\pi^3} \frac{c^2(\mathbf{x}) |h(\mathbf{x}, \boldsymbol{\xi})|}{a(\mathbf{x}, \boldsymbol{\xi})}. \tag{20}$$

Inserting this result into equation (13) gives the inversion result

$$\alpha(\mathbf{x}) \approx \frac{c^2(\mathbf{x})}{8\pi^3} \iint d^2\xi \frac{|h(\mathbf{x}, \boldsymbol{\xi})|}{a(\mathbf{x}, \boldsymbol{\xi})} \int d\omega\, F(\omega) \times e^{-i\omega\phi(\mathbf{x}, \boldsymbol{\xi})} D(\omega, \mathbf{x}_r, \mathbf{x}_s), \tag{21}$$

where ϕ and a are defined in equation (12).

Note that if we define the kernel

$$K(\omega, \mathbf{x}, \mathbf{x}_s, \mathbf{x}_r) = \frac{\omega^2 G(\omega, \mathbf{x}, \mathbf{x}_s) G(\omega, \mathbf{x}, \mathbf{x}_r)}{c^2(\mathbf{x})}, \tag{22}$$

then the integral equation and its inversion (ignoring the filter F) are

$$D(\omega, \mathbf{x}_r, \mathbf{x}_s) \approx \iiint d^3x\, K(\omega, \mathbf{x}, \mathbf{x}_s, \mathbf{x}_r) \alpha(\mathbf{x}) \tag{23}$$

and

$$\alpha(\mathbf{x}) \approx \frac{1}{8\pi^3} \iiint d^3k \frac{1}{K(\omega, \mathbf{x}_s, \mathbf{x}_r, \mathbf{x})} D(\omega, \mathbf{x}_r, \mathbf{x}_s). \tag{24}$$

In equation (24), ω must be regarded as a function of $\mathbf{k}$ as defined by the change of variables [equation (17)].

THE ZERO-OFFSET CONFIGURATION

In the case of zero (source-receiver) offset, a representation of the Jacobian h can be found in terms of the rays used to construct the Green's function. Since the source and receiver points are the same, we denote these points by $\mathbf{x}_0(\boldsymbol{\xi})$ and refer to the set of all such points as the "data surface." In this notation,

$$h(\mathbf{x}, \boldsymbol{\xi}) = \begin{vmatrix} 2\nabla\tau(\mathbf{x}, \mathbf{x}_0) \\ 2\frac{\partial}{\partial \xi_1} \nabla\tau(\mathbf{x}, \mathbf{x}_0) \\ 2\frac{\partial}{\partial \xi_2} \nabla\tau(\mathbf{x}, \mathbf{x}_0) \end{vmatrix} = 8 \begin{vmatrix} \mathbf{p}(\mathbf{x}, \mathbf{x}_0) \\ \frac{\partial}{\partial \xi_1} \mathbf{p}(\mathbf{x}, \mathbf{x}_0) \\ \frac{\partial}{\partial \xi_2} \mathbf{p}(\mathbf{x}, \mathbf{x}_0) \end{vmatrix}, \tag{25}$$

where in the second equality,

$$\mathbf{p}(\mathbf{x}, \mathbf{x}_0) = \nabla\tau(\mathbf{x}, \mathbf{x}_0). \tag{26}$$

Here the $\boldsymbol{\xi}$ dependence in h comes from $\mathbf{x}_0 = \mathbf{x}_0(\boldsymbol{\xi})$ in $\mathbf{p}$. From the eikonal equation (9),

$$\mathbf{p} \cdot \mathbf{p} = \frac{1}{c^2(\mathbf{x})}, \tag{27}$$

and since the speed c is independent of $\boldsymbol{\xi}$, from equation (27) it follows that

$$\mathbf{p} \cdot \frac{\partial}{\partial \xi_i} \mathbf{p} = 0, \quad i = 1, 2. \tag{28}$$

Now multiply the third column of h by p_3 and compensate by putting its reciprocal outside the determinant. Then on multiplying the first two columns by p_1 and p_2, respectively, and adding these to the third column,

$$h(\mathbf{x}, \boldsymbol{\xi}) = \frac{8}{p_3} \begin{vmatrix} p_1 & p_2 & 1/c^2 \\ \frac{\partial p_1}{\partial \xi_1} & \frac{\partial p_2}{\partial \xi_1} & 0 \\ \frac{\partial p_1}{\partial \xi_2} & \frac{\partial p_2}{\partial \xi_2} & 0 \end{vmatrix}. \tag{29}$$

Equation (29) can be written as

$$h(\mathbf{x}, \boldsymbol{\xi}) = \frac{8}{c^2(\mathbf{x}) p_3(\mathbf{x}, \mathbf{x}_0)} \frac{\partial(p_1, p_2)}{\partial(\xi_1, \xi_2)}. \tag{30}$$

Here the Jacobian notation

$$\frac{\partial(p_1, p_2)}{\partial(\xi_1, \xi_2)} = \begin{vmatrix} \frac{\partial p_1}{\partial \xi_1} & \frac{\partial p_2}{\partial \xi_1} \\ \frac{\partial p_1}{\partial \xi_2} & \frac{\partial p_2}{\partial \xi_2} \end{vmatrix} \tag{31}$$

has been used. The Appendix shows that

$$\frac{\partial(p_1, p_2)}{\partial(\xi_1, \xi_2)} = -16\pi^2 A^2(\mathbf{x}, \mathbf{x}_0) p_3(\mathbf{x}, \mathbf{x}_0) \sqrt{g_0}\, \hat{\mathbf{n}}_0 \cdot \mathbf{p}_0, \tag{32}$$

where $\mathbf{p}_0$ is the initial direction of the ray from $\mathbf{x}_0$ to $\mathbf{x}$, $\hat{\mathbf{n}}_0$ is the downward normal, and $\sqrt{g_0}$ is the differential area element [see equation (A-5)] of the data surface $\mathbf{x} = \mathbf{x}_0(\boldsymbol{\xi})$. Thus the absolute value of h is given by

$$|h(\mathbf{x}, \boldsymbol{\xi})| = \frac{8A^2(\mathbf{x}, \mathbf{x}_0)}{c^2(\mathbf{x})} 16\pi^2 \sqrt{g_0}\, \hat{\mathbf{n}}_0 \cdot \mathbf{p}_0, \tag{33}$$

and from equation (21) the zero-offset inversion for general reference velocity can be expressed as

$$\alpha(\mathbf{x}) \sim \frac{16}{\pi} \iint d^2\xi \sqrt{g_0}\, \hat{\mathbf{n}}_0 \cdot \mathbf{p}_0 \int d\omega\, F(\omega) \times e^{-2i\omega\tau(\mathbf{x},\mathbf{x}_0)} D[\omega, \mathbf{x}_0(\xi)]. \quad (34)$$

A few remarks about the implementation of expression (34) are in order. We recommend a discretization of $\xi = (\xi_1, \xi_2)$ corresponding to the geophone spacing. For each discrete value of ξ, the time data are transformed via the fast Fourier transform (FFT) to obtain $D[\omega, \mathbf{x}_0(\xi)]$. Next these data are filtered by $F(\omega)$, and the ω integral is performed by an inverse FFT, again for each ξ value. Finally, for each output point $\mathbf{x}$ of interest, the ξ integration is done by quadrature; the key feature is computation of several items (including traveltime) involving the ray connecting $\mathbf{x}$ to $\mathbf{x}_0(\xi)$. In the important special case of a $c(z)$ reference speed, tables can be set up for amplitude and traveltime as functions of z and offset $[(x - \xi_1)^2 + (y - \xi_2)^2]^{1/2}$. If this idea is efficiently implemented, the computing times for a 3-D inversion in the $c(z)$ case are only marginally more expensive than the constant-reference speed algorithms. In implementation of a full $c(x_1, x_2, x_3)$ reference, the ray computations will be more difficult, but we believe such implementations are feasible for large-scale computers currently available.

The special case in which the reference speed is a function of depth only was derived in Cohen and Hagin (1985). The result they achieved agrees with the present one, aside from a slight change in notation. Of course, the present result also agrees with the constant-reference speed result (Cohen and Bleistein, 1979; Bleistein et al., 1985).

We briefly address two other issues concerning implementation of equation (34). These are specialization to the case of a linear source-receiver array (the $2\frac{1}{2}$-D case mentioned in the Introduction), and the fact that discerning discontinuities is easier if α, which consists of steps, is replaced by the reflectivity function

$$\beta = \sum R_n \gamma_n(s), \quad (35)$$

which consists of delta functions. R_n denotes the reflection coefficient of the nth subsurface reflector, while $\gamma_n(s)$ denotes a delta function in arc length normal to the reflector and peaking on the reflector (Bleistein, 1984).

In the usual case of a data set collected on a linear array instead of a full 2-D surface array, a full 3-D subsurface inversion cannot be obtained. As explained in the Introduction, a model consistent with this restricted set of observations is one in which both the observation surface and the subsurface are assumed to be "cylindrical" with respect to x_2. Thus we assume $c = c(x_1, x_3)$, $\alpha = \alpha(x_1, x_3)$, and the observation surface has the special parameterization:

$$x_1 = x_1(\xi_1), \quad x_2 = \xi_2, \quad (36)$$

and

$$x_3 = x_3(\xi_1).$$

In this $2\frac{1}{2}$-D case, it is possible to eliminate the integral over ξ_2 by the method of stationary phase. Carrying out this step and, for simplicity, further specializing to the case of a planar observation surface:

$$x_1 = \xi_1, \quad x_3 = 0; \quad (37)$$

yields

$$\alpha(x_1, x_3) \sim \frac{16}{\sqrt{\pi}} \int d\xi_1 \sqrt{|\sigma_f|}\, q_3 \int \frac{d\omega}{\sqrt{i\omega}} F(\omega) \times e^{-2i\omega\tau(x_1,x_3,\xi_1,0)} D(\omega, \xi_1). \quad (38)$$

σ_f denotes the final value of the ray parameter σ for the ray from $\mathbf{x}_0$ to $\mathbf{x}$ (or equivalently from $\mathbf{x}$ to $\mathbf{x}_0$) and q_3 is the value of p_3 on the observation array [see equation (40)]. Bleistein (1986) shows that 2-D calculations may be used to compute the rays

$$\frac{dx_1}{d\sigma} = p_1, \quad x_1(0) = \xi_1,$$

$$\frac{dx_3}{d\sigma} = p_3, \quad x_3(0) = 0,$$

$$\frac{dp_1}{d\sigma} = \frac{1}{2} \frac{1}{c^2(x_1, x_3)}, \quad p_1(0) = q_1, \quad (39)$$

$$p_3 = \sqrt{c^2 - p_1^2}$$

and

$$\frac{d\tau}{d\sigma} = \frac{1}{c^2}, \quad \tau(0) = 0.$$

To apply expression (38), the desired field point (x_1, x_3) and a suitable aperture of observation points $(\xi_1, 0)$ on the observation array are fixed. Then trace the ray from $(\xi_1, 0)$ to (x_1, x_3) and compute τ, σ_f, and the ray parameter q_1. Finally, explicitly evaluate

$$q_3 = \sqrt{\frac{1}{c^2(\xi_1, 0)} - q_1^2}. \quad (40)$$

To obtain the reflectivity function β, introduce into expression (38) the factor (Bleistein, 1984),

$$i \operatorname{sgn}(\omega)|k|/4$$

and thereby obtain from expressions (9), (17), and (38),

$$\beta(x_1, x_3) \sim \frac{8}{c(x_1, x_3)\sqrt{\pi}} \int d\xi_1 \sqrt{|\sigma_f|}\, q_3 \times \int d\omega \sqrt{i\omega}\, F(\omega)\, e^{-2i\omega\tau(x_1,x_3,\xi_1,0)} D(\omega, \xi_1). \quad (41)$$

THE COMMON-SOURCE CONFIGURATION

We show here that the computations of the previous section allow us to obtain inversion formulas for the common-source configuration and for the common-receiver configuration. For a common-source gather,

$$\mathbf{x}_s = \text{constant}, \quad (42)$$

so that $\tau(\mathbf{x}, \mathbf{x}_s)$ and

$$\mathbf{p}_s = \nabla\tau(\mathbf{x}, \mathbf{x}_s) \quad (43)$$

are independent of the surface parameters ξ. Denoting the ray direction to the receiver array by $\mathbf{p}_r$,

$$\mathbf{p}_r = \nabla\tau(\mathbf{x}, \mathbf{x}_r), \tag{44}$$

and Beylkin's (1985) determinant becomes

$$h(\mathbf{x}, \xi) = \begin{vmatrix} \mathbf{p}_s + \mathbf{p}_r \\ \dfrac{\partial \mathbf{p}_r}{\partial \xi_1} \\ \dfrac{\partial \mathbf{p}_r}{\partial \xi_2} \end{vmatrix}. \tag{45}$$

Since determinants are linear functions of their rows, h can be written as $h = h_1 + h_2$ where

$$h_1 = \begin{vmatrix} \mathbf{p}_s \\ \dfrac{\partial \mathbf{p}_r}{\partial \xi_1} \\ \dfrac{\partial \mathbf{p}_r}{\partial \xi_2} \end{vmatrix} \tag{46}$$

and

$$h_2 = \begin{vmatrix} \mathbf{p}_r \\ \dfrac{\partial \mathbf{p}_r}{\partial \xi_1} \\ \dfrac{\partial \mathbf{p}_r}{\partial \xi_2} \end{vmatrix}. \tag{47}$$

Since the last two rows of each of these determinants is the same as in the zero-offset case (with $\mathbf{x}_r$, $\mathbf{p}_r$ taking the place of $\mathbf{x}_0$ and $\mathbf{p}$), we can proceed as at the beginning of the last section to obtain

$$h_1 = \frac{\mathbf{p}_s \cdot \mathbf{p}_r}{p_{3r}(\mathbf{x}, \mathbf{x}_r)} \frac{\partial(p_{1r}, \partial p_{2r})}{\partial(\xi_1, \xi_2)} \tag{48}$$

and

$$h_2 = \frac{1}{p_3(\mathbf{x}, \mathbf{x}_r)c^2(\mathbf{x})} \frac{\partial(p_{1r}, p_{2r})}{\partial(\xi_1, \xi_2)}. \tag{49}$$

Using the result of the Appendix,

$$|h(\mathbf{x}, \xi)| = \left[\mathbf{p}_s \cdot \mathbf{p}_r + \frac{1}{c^2(\mathbf{x})}\right] A^2(\mathbf{x}, \mathbf{x}_r) \cdot 16\pi^2 \sqrt{g_r}\, \hat{\mathbf{n}}_r \cdot \mathbf{p}_r, \tag{50}$$

and so, using the general result equation (21),

$$\alpha(\mathbf{x}) = \frac{2}{\pi} \iint d^2\xi \sqrt{g_r} \frac{A(\mathbf{x}, \mathbf{x}_r)}{A(\mathbf{x}, \mathbf{x}_s)} [1 + c^2(\mathbf{x})\mathbf{p}_s \cdot \mathbf{p}_r]\hat{\mathbf{n}}_r \cdot \mathbf{p}_r \times \int d\omega\, F(\omega) \exp\{-i\omega[\tau(\mathbf{x}, \mathbf{x}_s) + \tau(\mathbf{x}, \mathbf{x}_r)]\} D(\omega, \mathbf{x}_r). \tag{51}$$

Implementation of equation (51) is not as simple as in the zero-offset case, because it is necessary to compute the two amplitude factors.

As pointed out in the Introduction, the more practical case of common-source gather inversion from a linear array of receivers rather than from an areal array is not a special case of equation (51); this case will be treated in a sequel devoted to the $2\frac{1}{2}$-D geometry. There we will also address the question of accurate estimation of parameters.

Obviously, the solution for the *common-receiver* configuration can be derived at once from equation (51) by interchanging the subscripts s and r.

CONCLUSIONS

We began from a general setting and obtained an inversion formula which, in principle, covers both prestack and poststack problems and many source-receiver configurations. Unfortunately, in this generality not all quantities are expressed in computationally realizable quantities. Nevertheless, we consider this broad point of view as basically sound and helpful, and we are indebted to the work of G. Beylkin in this regard. In the zero-offset case and in the common-source (or common-receiver) case the formula is expressible in practical terms.

Implementation of the algorithms is as follows. The data are processed by applying the FFT and then, after an amplitude correction and filtering, the inverse FFT is performed. Second, for each output point of interest, a summation is performed over that portion of the processed data influencing the output point. The summation involves computation of an amplitude and a traveltime along the connecting rays, a critical step in the practical implementation of the algorithms. In the case of $c(z)$ background velocity, it is economical to build a table of traveltimes and amplitudes as functions of only depth z and offset. In this case the algorithm is only marginally more expensive to run than the algorithms of constant reference speed. In implementation of a full $c(\mathbf{x})$ reference, the computation of ray information is a major consideration. In spite of this difficulty, we feel that fully 3-D problems, including $c(\mathbf{x})$ reference, are feasible for the large-scale computers currently available.

Note added in proof: G. Beylkin (Pers. comm.) has pointed out alternate methods for evaluating the key Jacobian, h, for several important source-receiver configurations.

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APPENDIX

THE ZERO-OFFSET JACOBIAN

It is natural to think of $\tau(\mathbf{x}, \mathbf{x}_0)$ as the traveltime along a ray-path with initial point $\mathbf{x}_0$ and running point $\mathbf{x}$. In this model, $\mathbf{p}$ is the associated ray direction along the ray from $\mathbf{x}_0$ to $\mathbf{x}$. Furthermore, to solve the eikonal equation in this model by the method of characteristics (or "ray method"), the *family* of rays emanating from the "fixed" point $\mathbf{x}_0$ must be considered and $\mathbf{x}$ must be considered an arbitrary field point. In the present context, in constructing the eikonal τ and transport amplitude A, we can consider $\mathbf{x}_0$ as an arbitrary point on the observation surface; in other words, we repeat the ray method solution for each $\mathbf{x}_0$.

On the other hand, if we think of the field point $\mathbf{x}$ as fixed, it is equally valid to compute the traveltime using a ray with initial point $\mathbf{x}$ and "final" point $\mathbf{x}_0$. In this model, the "upward" ray direction is simply the negative of the ray direction on the "downward" ray at any given point on the ray, and we consider the family of rays emanating from the fixed point $\mathbf{x}$.

In the computation of the 2×2 Jacobian determinant in equation (31), we need to compute derivatives with respect to the second argument of $\mathbf{p}$. Thus we assume $\mathbf{x}$ as fixed. However, in our original description the vector $\mathbf{x}_0$ was bound to the observation surface. Thus for clarity we introduce a new running variable $\mathbf{y}$ governed by the ray equations

$$\frac{d\mathbf{y}}{d\sigma} = \mathbf{p}; \qquad \mathbf{y}(0) = \mathbf{x},$$

$$\frac{d\mathbf{p}}{d\sigma} = \frac{1}{2}\nabla\frac{1}{c^2}; \qquad \mathbf{p}(0)\ \text{free}, \tag{A-1}$$

and

$$\frac{d\tau}{d\sigma} = \frac{1}{c^2}; \qquad \tau(0) = 0.$$

We impose no condition on the starting ray direction in equations (A-1) because, to construct the Green's function, we want the "conoidal" solution of the eikonal equation (Bleistein, 1984), i.e., we consider all possible (upward) directions from the point $\mathbf{x}$. To impose the condition that each ray emanating from $\mathbf{x}$ passes through a point $\mathbf{x}_0$ on the observation surface, we demand

$$\mathbf{y}(\sigma_f) = \mathbf{x}_0(\boldsymbol{\xi}). \tag{A-2}$$

This last condition defines, for each such ray, the final value σ_f of σ as a function of $\boldsymbol{\xi}$. (The question of having a unique ray for each surface point is subsumed in the issue of the nonsingular nature of h.)

We now evaluate our 2×2 Jacobian by introducing the ray parameter σ and then using the chain rule as follows:

$$\begin{aligned}
\frac{\partial(p_1(\mathbf{x}, \mathbf{x}_0), p_2(\mathbf{x}, \mathbf{x}_0))}{\partial(\xi_1, \xi_2)}
&= \left.\frac{\partial[p_1(\mathbf{x}, \mathbf{y}), p_2(\mathbf{x}, \mathbf{y})]}{\partial(\xi_1, \xi_2)}\right|_{\sigma=\sigma_f}, \\
&= \left.\frac{\partial(p_1, p_2, \sigma)}{\partial(\xi_1, \xi_2, \sigma)}\right|_{\sigma=\sigma_f}, \\
&= \left.\frac{\partial(p_1, p_2, \sigma)}{\partial(y_1, y_2, y_3)}\right|_{\sigma=\sigma_f} \left.\frac{\partial(y_1, y_2, y_3)}{\partial(\xi_1, \xi_2, \sigma)}\right|_{\sigma=\sigma_f}
\end{aligned} \tag{A-3}$$

The second of these 3×3 Jacobians can be evaluated as

$$\begin{aligned}
\left.\frac{\partial(y_1, y_2, y_3)}{\partial(\xi_1, \xi_2, \sigma)}\right|_{\sigma=\sigma_f} &= \begin{vmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ -\mathbf{p}_0 \end{vmatrix} \\
&= -\mathbf{p}_0 \cdot \mathbf{t}_1 \times \mathbf{t}_2 \\
&= -\mathbf{p}_0 \cdot \hat{\mathbf{n}}_0 \sqrt{g_0},
\end{aligned} \tag{A-4}$$

where we introduced the surface tangent vectors

$$\mathbf{t}_i = \frac{\partial}{\partial \xi_i}\mathbf{x}_0, \qquad i = 1, 2,$$
$$\sqrt{g_0} = |\mathbf{t}_1 \times \mathbf{t}_2|, \tag{A-5}$$

and where $\mathbf{p}_0$ denotes the initial direction of the ray from $\mathbf{x}_0$ to $\mathbf{x}$ (or the *negative* of the final direction of the ray from $\mathbf{x}$ to $\mathbf{x}_0$). Finally, $\hat{\mathbf{n}}_0$ denotes the unit normal to the data surface at $\mathbf{x}_0$. The remaining Jacobian in equations (A-3) is the reciprocal of the ray Jacobian for rays emanating from $\mathbf{x}$,

$$J(\mathbf{y}) = \frac{\partial(y_1, y_2, y_3)}{\partial(p_1, p_2, \sigma)}. \tag{A-6}$$

From the transport equation, we can derive the relation

$$J(\mathbf{y})A^2(\mathbf{y}, \mathbf{x}) = \text{constant}, \tag{A-7}$$

(see Bleistein, 1984). The constant in equation (A-7) can be evaluated by allowing $\mathbf{y}$ to approach the fixed field point $\mathbf{x}$ and by using the constant c result, with c being $c(\mathbf{x})$. We find, for this limit,

$$\lim_{\mathbf{y}\to\mathbf{x}} JA^2 = \frac{1}{16\pi^2 p_3(\mathbf{x}, \mathbf{x}_0)}, \tag{A-8}$$

so that the ray Jacobian is given by

$$J\big|_{\sigma=\sigma_f} = \frac{1}{16\pi^2 p_3(\mathbf{x}, \mathbf{x}_0)A^2(\mathbf{x}, \mathbf{x}_0)}. \tag{A-9}$$

Here we use reciprocity to switch the arguments of A. Combining equations (A-3), (A-4), and (A-9), we find that the required 2×2 Jacobian is given by equation (32) in the text.