GEOPHYSICS, VOL. 52, NO. 1 (JANUARY 1987); P. 26–36, 5 FIGS.

Two and one-half dimensional Born inversion with an arbitrary reference

Norman Bleistein*, Jack K. Cohen*, and Frank G. Hagin*

ABSTRACT

Multidimensional inversion algorithms are presented for both prestack and poststack data gathered on a single line. These algorithms both image the subsurface (i.e., give a migrated section) and, given relative true amplitude data, estimate reflection strength or impedance on each reflector.

The algorithms are "two and one-half dimensional" (2.5-D) in that they incorporate three-dimensional (3-D) wave propagation in a medium which varies in only two dimensions. The use of 3-D sources does not entail any computational penalty, and it avoids the serious degradation of amplitude incurred by using the 2-D wave equation.

Our methods are based on the linearized inversion theory associated with the "Born inversion." Thus, we assume that the sound speed profile is well approximated by a given background velocity, plus a perturbation. It is this perturbation that we seek to reconstruct. We are able to treat the case of an arbitrary continuous background profile. However, the cost of implementation increases as one seeks to honor, successively, constant background, depth-only dependent background, and, ultimately, fully lateral and depth-dependent background. For depth-only dependent background, the increase in CPU time is quite modest when compared to the constant-background case.

We exploit the high-frequency character of seismic data ab initio. Therefore, we use ray theory and WKBJ Green's functions in deriving our inversion representations. Furthermore, our algorithms reduce to finding quantities by ray tracing with respect to a background medium. In the constant-background case, the ray tracing can be eliminated and an explicit algorithm obtained. In the case of a depth-only dependent background, the ray tracing can be done quite efficiently. Finally, in the general 2.5-D case, the ray-tracing procedure becomes the principal issue. However, the robustness of the inversion allows for a sparse computation of rays and interpolation for intermediary values.

The inversion techniques presented here cover the cases of common-source gather, common-receiver gather, and common-offset gather. Zero offset is a special case of the last of these. For offset data, the reflection coefficient is angle-dependent, so parameter extraction is more difficult than in the zero-offset case. Nonetheless, we are able to determine the unknown angle pointwise and derive parameter estimates at the same time as we produce the image. For each reflector, this estimate of the output is based on the Kirchhoff approximation of the upward-scattered data. Thus, it is constrained to neither small discontinuities in sound speed at the reflector nor to small offset angle as would be the case for a strict "Born approximation" of the reflection process.

The prestack algorithms presented here are inversions of single gathers. The question of how best to composite or "stack" these inversions is analogous to the question for any migration scheme and is not treated here.

INTRODUCTION

We show that the 2.5-D Born integral equation for a sound speed perturbation can be inverted in a general case. We verify this result in three cases of interest for seismic exploration, namely, common source, common receiver, and common offset. In this verification, the estimate of the output of our method is based on the Kirchhoff approximation of the upward scattered data. Thus, it is constrained to neither small discontinuities in sound speed at the reflector nor to small offset angles. We rely heavily on Cohen et al. (1986), which in turn was founded on Beylkin (1985) and Cohen and Hagin (1985).

The notion of "two-and-one-half dimensions" arises from

Manuscript received by the Editor February 24, 1986; revised manuscript received June 27, 1986.
*Center for Wave Phenomena, Department of Mathematics, Colorado School of Mines, Golden, CO 80401.

LIST OF SYMBOLS

$A(\mathbf{x}, \mathbf{x}')$ = Ray theoretic or WKB amplitude of the Green's function for the time harmonic wave equation.

g = Scaling between differential arc length and differential of parameter describing the curve along which data are gathered [equation (21)].

$G(\mathbf{x}, \mathbf{x}', \omega)$ = Green's function for the time harmonic wave equation.

$h(\mathbf{x}, \xi)$ = A fundamental determinant defined by equation (19).

$H(x, z, \xi)$ = A factor of $h(\mathbf{x}, \xi)$ in the 2.5-D specialization [equation (30)].

K = Two-dimensional ray Jacobian [equation (15)].

$\hat{\mathbf{n}}_s, \hat{\mathbf{n}}_r$ = Unit downward normals at the source or receiver points, respectively.

$\mathbf{p} = \nabla\tau$

$\mathbf{p}'$ = Initial value of $\mathbf{p}$ on a ray [equation (7)].

(p, q) = Components of $\nabla\tau$ in two dimensions.

(p_1, p_2, p_3) = Components of $\nabla\tau$ in three dimensions.

$R(\mathbf{x}, \theta)$ = Angular-dependent reflection coefficient at x on reflector.

$\mathbf{x}_s, \mathbf{x}_r$ = Coordinate of the source and receiver points, respectively.

$\mathbf{x}'$ = Initial value of x on a ray [equation (7)].

$\alpha(\mathbf{x})$ = Perturbation [equation (17)].

$\beta(x, z)$ = Output of the inversion algorithm.

I_s', I_r' = 0, 1 factors which define the cases of common-source gather, common-receiver gather [equations (32) and (37)].

$\phi(x, z, \xi)$ = A sum of traveltimes defined by equation (31).

σ = Ray parameter for which $|d\mathbf{x}/d\sigma|^2 = 1/c^2$.

$\tau(\mathbf{x}, \mathbf{x}')$ = Ray theoretic traveltime with respect to the background sound speed.

θ = Angle between incident ray and surface normal.

ξ = Surface parameter; $\mathbf{x}_s$ and $\mathbf{x}_r$ are functions of this variable.

the common exploration situation of a linear (as opposed to planar) distribution of sources and receivers. In this situation, one has no information in the direction orthogonal to the data line and hence can only obtain information about a slice of the subsurface. In order to ignore the missing off-line data, one is forced to proceed as if the subsurface did not vary in the orthogonal direction.

In an attempt to reduce processing efforts, an additional assumption has traditionally been made, namely, that the actual point-like sources were replaced by line sources. This extra assumption allows the replacement of the 3-D wave equation by the 2-D wave equation (Claerbout and Doherty, 1972; Schneider, 1978; Stolt, 1978). Unfortunately, use of the 2-D wave equation causes degradation of amplitude information since this model implies cylindrical rather than spherical wavefronts.

Somewhat surprisingly, retention of the 3-D point-source model over a subsurface with the 2-D symmetry just described does not, in fact, incur a computational penalty in solving the inverse problem (Cohen and Bleistein, 1979a; Bleistein et al., 1985). In a recent paper Bleistein (1986) presented a thorough discussion of the so-called two and one-half dimensional (2.5-D) geometry and its role in modeling. Furthermore, it is shown there that retention of 3-D sources in the direct (modeling) problem can also be carried out efficiently.

Thus it is desirable to retain the 3-D point-source model along with the 2-D subsurface model imposed by the data collection. We refer to this as "two-and-a-half-dimensional" or "2.5-D."

As in our previous inversion results (Cohen and Bleistein, 1979a; Bleistein and Gray, 1985; Cohen and Hagin, 1985; Sullivan and Cohen, 1985; Cohen et al., 1986), we start from a full 3-D inversion; then, by assuming that the data are invariant in one direction, specialize to the 2.5-D situation. With the 2.5-D assumption, we can obtain formal inversions in some cases for which the full 3-D inversion has so far eluded us. Most notable of these cases is inversion of common-offset gathers in the case of fully variable (x, z-dependent) reference velocity; we include the specializations of that result to the case of a z-dependent background velocity. We also present the variable-background results for the case of a common-source gather and a common-receiver gather.

REVIEW OF 2.5-D MODELING RESULTS

Bleistein (1986) gives the derivation of the asymptotic "in-plane" or 2.5-D Green's function. Here we summarize those results, with a slight change in notation. From the WKB approximation

$$G(\mathbf{x}, \mathbf{x}', \omega) \sim A(\mathbf{x}, \mathbf{x}')e^{i\omega\tau(\mathbf{x}, \mathbf{x}')} \tag{1}$$

for the impulse response or Green's function at $\mathbf{x}$ due to a unit point source at $\mathbf{x}'$ and the 3-D reduced wave equation

$$\left[\nabla^2 + \frac{\omega^2}{c^2(\mathbf{x})}\right]G(\mathbf{x}, \mathbf{x}', \omega) = -\delta(\mathbf{x} - \mathbf{x}') \tag{2}$$

we derive the *eikonal equation,*

$$\nabla\tau \cdot \nabla\tau = 1/c^2(\mathbf{x}) \tag{3}$$

for τ and the *transport equation,*

$$2\nabla\tau \cdot \nabla A + A\nabla^2\tau = 0 \tag{4}$$

for A. For 2.5-D wave propagation, we assume that

$$c = c(x_1, x_3). \tag{5}$$

The equations for τ and A can be solved by introducing the slowness vector

$$\mathbf{p} = \nabla\tau, \tag{6}$$

and the *ray equations*,

$$\frac{d\mathbf{x}}{d\sigma} = \mathbf{p}, \qquad \mathbf{x}(0) = \mathbf{x}',$$
$$\frac{d\mathbf{p}}{d\sigma} = \frac{1}{2}\nabla\left[\frac{1}{c^2}\right], \qquad \mathbf{p}(0) = \mathbf{p}' \tag{7}$$

with the indicated initial data. The parameter σ is related to arc length along rays through the first equation above and the eikonal equation (3); i.e., $|d\mathbf{x}/d\sigma| = 1/c(\mathbf{x})$. The initial slowness $\mathbf{p}'$ is constrained by the eikonal equation to satisfy

$$|\mathbf{p}'| = 1/c(\mathbf{x}'). \tag{8}$$

We consider rays which start in-plane, i.e.,

$$x_2' = 0. \tag{9}$$

We then find, from the second components in equation (7), that p_2 is constant and

$$x_2 = p_2\sigma. \tag{10}$$

The necessary and sufficient condition for such a ray to *remain* in-plane is $p_2 = 0$.

We introduce the simpler and more traditional alternative notation

$$(x_1, x_3) = (x, z); \qquad (p_1, p_3) = (p, q) \tag{11}$$

for the in-plane Cartesian coordinates and the horizontal and vertical slownesses. Thus we can describe the in-plane Green's function ($x_2 = x_2' = 0$) by the ray equations

$$\frac{dx}{d\sigma} = p, \qquad x(0) = x',$$
$$\frac{dz}{d\sigma} = q, \qquad z(0) = z',$$
$$\frac{dp}{d\sigma} = \frac{1}{2}\frac{\partial}{\partial x}\left[\frac{1}{c^2(x, z)}\right], \qquad p(0) = p', \tag{12}$$
$$\frac{d\tau}{d\sigma} = \frac{1}{c^2(x, z)}, \qquad \tau(0) = 0$$

and the integrated transport equation

$$A = \frac{1}{4\pi|q'\sigma K|^{1/2}}. \tag{13}$$

The vertical component of slowness and its initial value are determined by

$$q = \pm\sqrt{1/c^2(x, z) - p^2},$$
$$q' = \pm\sqrt{1/c^2(x', z') - p'^2}, \tag{14}$$

and K denotes the in-plane ray Jacobian

$$K = \frac{\partial(x, z)}{\partial(\sigma, p')} = \begin{vmatrix} p & q \\ \dfrac{\partial x}{\partial p'} & \dfrac{\partial z}{\partial p'} \end{vmatrix}. \tag{15}$$

Hence, the asymptotic in-plane Green's function is determined by 2-D ray theory with the extra multiplier of σ in equation (13) accounting for the out-of-plane spreading.

THE 2.5-D INVERSION FORMULA

The Beylkin (1985) inversion formula in 3-D is

$$\alpha(\mathbf{x}) \sim \frac{c^2(\mathbf{x})}{8\pi^3}\iint_{S_0} d^2\xi\, \frac{|h(\mathbf{x}, \xi)|}{A(\mathbf{x}, \mathbf{x}_s)A(\mathbf{x}, \mathbf{x}_r)}$$
$$\times \int d\omega\, F(\omega)\exp\left\{-i\omega\left[\tau(\mathbf{x}, \mathbf{x}_s) + \tau(\mathbf{x}, \mathbf{x}_r)\right]\right\} D(\omega, \xi). \tag{16}$$

Here, $\alpha(\mathbf{x})$ is the perturbation correction to the given reference velocity $c(\mathbf{x})$, with the total velocity $v(\mathbf{x})$ being given by

$$v^{-2}(\mathbf{x}) = c^{-2}(\mathbf{x})\left[1 + \alpha(\mathbf{x})\right]. \tag{17}$$

A and τ are at this point the 3-D amplitude and phase of the WKB Green's function, but with initial points being either the source point $\mathbf{x}_s$ or the receiver point $\mathbf{x}_r$, both of which are parameterized by $\xi = (\xi_1, \xi_2)$ on the datum surface S_0, i.e.,

$$\mathbf{x}_s = \mathbf{x}_s(\xi_1, \xi_2), \qquad \mathbf{x}_r = \mathbf{x}_r(\xi_1, \xi_2) \tag{18}$$

(see Figure 1). Further, $F(\omega)$ is a given high passband filter and $h(\mathbf{x}, \xi)$ is the fundamental determinant introduced by Beylkin (1985),

$$h(\mathbf{x}, \xi) = \begin{vmatrix} \nabla\left[\tau(\mathbf{x}, \mathbf{x}_s) + \tau(\mathbf{x}, \mathbf{x}_r)\right] \\ \dfrac{\partial}{\partial\xi_1}\nabla\left[\tau(\mathbf{x}, \mathbf{x}_s) + \tau(\mathbf{x}, \mathbf{x}_r)\right] \\ \dfrac{\partial}{\partial\xi_2}\nabla\left[\tau(\mathbf{x}, \mathbf{x}_s) + \tau(\mathbf{x}, \mathbf{x}_r)\right] \end{vmatrix} = \begin{vmatrix} \mathbf{p}_s + \mathbf{p}_r \\ \dfrac{\partial}{\partial\xi_1}\left(\mathbf{p}_s + \mathbf{p}_r\right) \\ \dfrac{\partial}{\partial\xi_2}\left(\mathbf{p}_s + \mathbf{p}_r\right) \end{vmatrix}. \tag{19}$$

Finally, $D(\omega, \xi)$ denotes the observed data at $\mathbf{x}_r$ due to the source at $\mathbf{x}_s$. In equation (19) we have introduced the slowness vectors defined as in equation (6) by

$$\mathbf{p}_s = \nabla\tau(\mathbf{x}, \mathbf{x}_s), \qquad \mathbf{p}_r = \nabla\tau(\mathbf{x}, \mathbf{x}_r). \tag{20}$$

We assume h is finite and nonsingular (see Beylkin, 1985, for an interpretation of this condition).

The crucial step in the practical implementation of equation (16) is the efficient computation of h. In Cohen et al. (1986), computation of h in the fully 3-D case was reduced to computation of quantities known from the ray tracing for the following source-receiver configurations: (1) zero-offset (CMP) gather; (2) common-shot gather; and (3) common-receiver gather.

Using the 2.5-D assumption, we obtain an analogous reduction in general. In particular, we can invert another important conventional configuration: (4) the common-offset gather. An alternate reduction to ray tracing is given in Beylkin (1985).

In order to specialize the 3-D formula (16) to the 2.5-D geometry, we assume, as in equation (5), that reference speed c is independent of x_2. We also assume that the datum surface S_0 is cylindrical in the x_2 direction,

$$\mathbf{x}^0(\xi) = \left[x_1^0(\xi_1),\, \xi_2,\, x_3^0(\xi_1) \right] = \left[x^0(\xi),\, \xi_2,\, z^0(\xi) \right]. \tag{21}$$

Here we have adopted the simplified notation (11) and also used $\xi = \xi_1$. Consistent with the 2.5-D hypothesis, we assume that the sources and receivers are along the cylindrical direction

$$x_{2s} = \xi_2 = x_{2s} \quad \text{on} \quad S_0, \tag{22}$$

and we seek α only at $x_2 = 0$. The reduction to 2.5-D is then achieved by a stationary phase calculation in ξ_2 analogous to that performed for the Kirchhoff forward modeling formula in Bleistein (1986). However, as in Cohen et al. (1986), in the inverse problem we use the rays emanating from the field point $\mathbf{x} = (x_1, 0, x_3)$ to construct the asymptotic form of the Green's function $G(\omega, \mathbf{x}, \mathbf{x}_s)$ and $G(\omega, \mathbf{x}, \mathbf{x}_r)$. The stationary phase condition is

$$\left.\frac{\partial \tau(\mathbf{x}, \mathbf{x}_s)}{\partial \xi_2} + \frac{\partial \tau(\mathbf{x}, \mathbf{x}_r)}{\partial \xi_2}\right|_{S_0} = 0. \tag{23}$$

For each of the ray families from $\mathbf{x}$, we have as in equation (10),

$$p_2 = p_2', \qquad y_2 = p_2' \sigma. \tag{24}$$

We denote an arbitrary point on the ray from $\mathbf{x}$ as $\mathbf{y}$ and the running variable on those rays as σ (see Figure 3).

We now evaluate the phase derivative (23) as

$$\begin{aligned} \left.\frac{\partial}{\partial \xi_2}\left[\tau(\mathbf{x}, \mathbf{x}_s) + \tau(\mathbf{x}, \mathbf{x}_r) \right]\right|_{S_0} &= \left. p_{2s} + p_{2r} \right|_{S_0} \\ &= \left.\frac{y_{2s}}{\sigma_s} + \frac{y_{2r}}{\sigma_r}\right|_{S_0} \\ &= \frac{x_{2s}}{\sigma_{s0}} + \frac{x_{2r}}{\sigma_{r0}} \\ &= \left(\frac{1}{\sigma_{s0}} + \frac{1}{\sigma_{r0}} \right) \xi_2 . \end{aligned} \tag{25}$$

Here the first equality is due to definitions (20) and (22); the second is due to the 2.5-D result (24); the third, to the nomenclature for the source and receiver points on the datum surface S_0; the final result is again due to equation (22). We use the superscript 0 on the ray parameters σ_s and σ_r to denote their values on the datum surface S_0.

From the final equality in equations (25) and (23) it is clear that the stationarity condition is $\xi_2 = 0$. To complete the stationary phase calculation, we must compute the second derivative of the phase at $\xi_2 = 0$. From equation (25),

$$\left.\frac{\partial^2}{\partial \xi_2^2}\left[\tau(\mathbf{x}, \mathbf{x}_s) + \tau(\mathbf{x}, \mathbf{x}_r) \right]\right|_{\xi_2 = 0} = \frac{1}{\sigma_{s0}} + \frac{1}{\sigma_{r0}}. \tag{26}$$

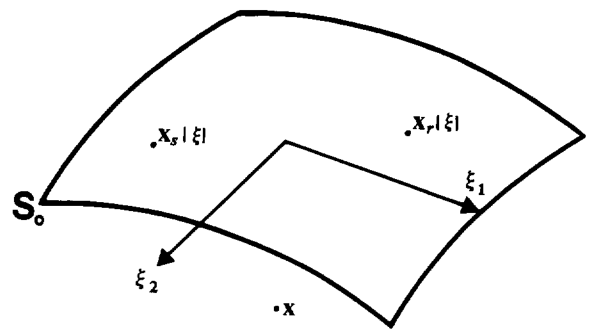

FIG. 1. Source-receiver surface S_0, parameterized by $\xi = (\xi_1, \xi_2)$. Source point $\mathbf{x}_s(\xi)$, receiver point $\mathbf{x}_r(\xi)$.

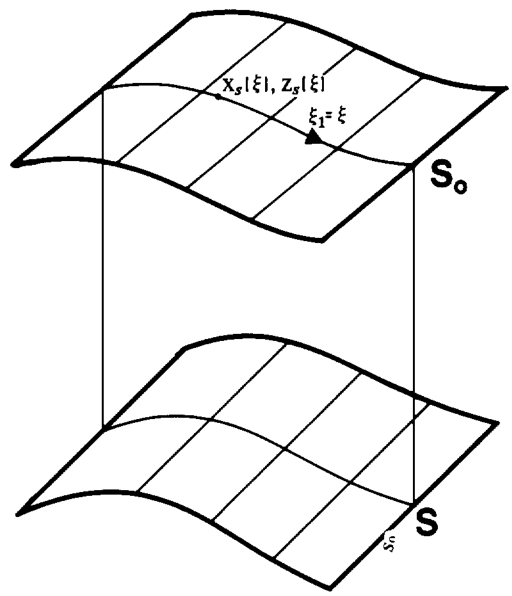

FIG. 2. 2.5-D surface geometry. All surfaces are cylinders generated by curves in the plane $y = 0$.

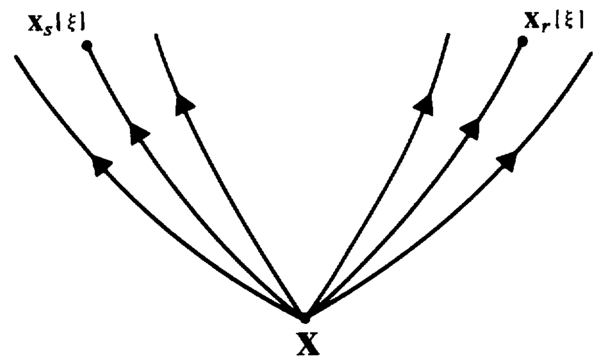

FIG. 3. Ray families from subsurface point $\mathbf{x}$ to the neighborhoods of the source point $\mathbf{x}_s(\xi)$ and the receiver point $\mathbf{x}_r(\xi)$.

These results allow us to state the stationary phase result formally as

$$\alpha(x, z) \sim \frac{c^2(x, z)}{(2\pi)^{5/2}} \int d\xi \frac{|h(\mathbf{x}, \xi)|}{A(\mathbf{x}, \mathbf{x}_s)A(\mathbf{x}, \mathbf{x}_r)} \left(\frac{1}{\sigma_{s0}} + \frac{1}{\sigma_{r0}}\right)^{-1/2} \int \frac{d\omega}{|\omega|^{1/2}} \times \exp\left\{-\frac{i\pi}{4}\operatorname{sgn}(\omega) - i\omega\left[\tau(\mathbf{x}, \mathbf{x}_s) + \tau(\mathbf{x}, \mathbf{x}_r)\right]\right\}\Bigg|_{x_2=\xi_2=0} D(\omega, \xi). \tag{27}$$

To reduce this result to a convenient form for computation, we must evaluate h as defined in equation (19) in terms of quantities computed along rays. First note that at any point on the ray from $\mathbf{x}$ to $\mathbf{x}_s$, the slowness vector $\mathbf{p}_r$ is just the negative of the slowness vector for the ray emanating from $\mathbf{x}_s$. The same is true for $\mathbf{x}_r$ and $\mathbf{p}_r$. Hence, in a computation analogous to equation (25), we find

$$\frac{\partial}{\partial x_2}\left[\tau(\mathbf{x}, \mathbf{x}_s) + \tau(\mathbf{x}, \mathbf{x}_r)\right]\Bigg|_{S_0} = -\left(\frac{1}{\sigma_{s0}} + \frac{1}{\sigma_{r0}}\right)\xi_2. \tag{28}$$

Moreover, from equation (28) it is easy to compute the remaining quantities in the middle column of equation (19). Since we need these results only at the stationary point $\xi_2 = 0$, the first two components of the column vanish and we find on S_0

$$h(\mathbf{x}, \xi)\Bigg|_{\xi_2=0} = \left(\frac{1}{\sigma_{s0}} + \frac{1}{\sigma_{r0}}\right) H(x, z, \xi), \tag{29}$$

where the reduced determinant H is given by

$$H(x, z, \xi) = \begin{vmatrix} \dfrac{\partial \phi}{\partial x} & \dfrac{\partial \phi}{\partial z} \\ \dfrac{\partial^2 \phi}{\partial \xi \partial x} & \dfrac{\partial^2 \phi}{\partial \xi \partial z} \end{vmatrix}, \tag{30}$$

$$\phi \equiv \phi(x, z, \xi) = \tau(x, z, x_s, z_s) + \tau(x, z, x_r, z_r). \tag{31}$$

Since ϕ is a sum of two terms and determinants are linear functions of their rows, the reduced determinant H can be expressed as a sum of four 2×2 determinants. In the special case of a common-source gather where $x_s(\xi)$ is actually a constant, the second row of H is no longer a sum and H reduces to the sum of only two 2×2 determinants. A similar reduction results for the common-receiver gather. Thus we may write

$$H = (H_1 + H_3)I'_s + (H_2 + H_4)I'_r, \tag{32}$$

where

$$H_1 = \begin{vmatrix} p_s & q_s \\ \dfrac{\partial p_s}{\partial \xi} & \dfrac{\partial q_s}{\partial \xi} \end{vmatrix}, \tag{33}$$

$$H_2 = \begin{vmatrix} p_s & q_s \\ \dfrac{\partial p_r}{\partial \xi} & \dfrac{\partial q_r}{\partial \xi} \end{vmatrix}, \tag{34}$$

$$H_3 = \begin{vmatrix} p_r & q_r \\ \dfrac{\partial p_s}{\partial \xi} & \dfrac{\partial q_s}{\partial \xi} \end{vmatrix}, \tag{35}$$

$$H_4 = \begin{vmatrix} p_r & q_r \\ \dfrac{\partial p_r}{\partial \xi} & \dfrac{\partial q_r}{\partial \xi} \end{vmatrix}, \tag{36}$$

$$I'_s = \begin{cases} 0 & \text{common-source gather,} \\ 1 & \text{otherwise,} \end{cases} \qquad I'_r = \begin{cases} 0 & \text{common-receiver gather,} \\ 1 & \text{otherwise.} \end{cases} \tag{37}$$

The evaluation of the H_i ($i = 1, 2, 3, 4$) is carried out just as the special case for h in three dimensions presented in the Appendix of Cohen et al. (1986). Note that in the general 3-D case one must also deal with determinants of the form

$$\begin{vmatrix} \mathbf{p}_s \\ \dfrac{\partial \mathbf{p}_s}{\partial \xi_1} \\ \dfrac{\partial \mathbf{p}_r}{\partial \xi_2} \end{vmatrix}$$

in which the last two rows involve different surface differentiated slowness vectors. Fortunately, this complication does not occur for our reduced determinant. We find in the current setting

$$H = \frac{16\pi^2}{c^2(\mathbf{x})}\left[1 + c^2(\mathbf{x})\mathbf{p}_s \cdot \mathbf{p}_r\right] \times \left[\sqrt{g_s}\,\hat{\mathbf{n}}_s \cdot \mathbf{p}_{s0}\, A^2(\mathbf{x}, \mathbf{x}_s)\sigma_{s0} I'_s + \sqrt{g_r}\,\hat{\mathbf{n}}_r \cdot \mathbf{p}_{r0}\, A^2(\mathbf{x}, \mathbf{x}_r)\sigma_{r0} I'_r\right], \tag{38}$$

where $\mathbf{p}_s$ and $\mathbf{p}_r$ denote the slowness vectors at depth and $\mathbf{p}_{s0}$ and $\mathbf{p}_{r0}$ denote the slowness vectors on the surface S_0. Further, we have introduced the unit normal vectors $\hat{\mathbf{n}}_s$ and $\hat{\mathbf{n}}_r$ and the differential surface elements $\sqrt{g_s}$ and $\sqrt{g_r}$ at the points $\mathbf{x}_s$ and $\mathbf{x}_r$ on S_0. The directions of the normal vectors are determined as follows. Associated with the parameterizations of the surface $\mathbf{x}_s(\xi)$ and $\mathbf{x}_r(\xi)$ are unit tangent vectors obtained by differentiation and normalization. We imbed these vectors in three dimensions and denote them by $\mathbf{t}_s^1 = (t_{s1}, 0, t_{s3})$ and $\mathbf{t}_r^1 = (t_{r1}, 0, t_{r3})$, respectively. Denote by $\mathbf{t}^2$ the out-of-plane unit vector $(0, 1, 0)$. Now $\hat{\mathbf{n}}_s = \mathbf{t}_s^1 \times \mathbf{t}^2$ and $\hat{\mathbf{n}}_r = \mathbf{t}_r^1 \times \mathbf{t}^2$.

Combining equations (16), (29), and (38), we have the general 2.5-D inversion formula

$$\alpha(x, z) \sim 2\sqrt{\frac{2}{\pi}} \int d\xi \left[1 + c^2(x, z)\mathbf{p}_s \cdot \mathbf{p}_r\right][A_s A_r]^{-1} \sqrt{\frac{1}{\sigma_{s0}} + \frac{1}{\sigma_{r0}}} \times \left(\sqrt{g_s}\,\hat{\mathbf{n}}_s \cdot \mathbf{p}_{s0} \frac{A_s}{A_r}\sigma_{s0} I'_s + \sqrt{g_r}\,\hat{\mathbf{n}}_r \cdot \mathbf{p}_{r0} \frac{A_r}{A_s}\sigma_{r0} I'_r\right) \times \int \frac{d\omega}{\sqrt{i\omega}} F(\omega) \exp\left[-i\omega(\tau_s + \tau_r)\right] D(\omega, \xi). \tag{39}$$

Here we adopt the abbreviations

$$\tau_s = \tau(x, z, x_s(\xi), z_s(\xi)) \tag{40}$$

and similarly for τ_r, A_s, and A_r; we also use the shorthand

$$\sqrt{i\omega} = |\omega|^{1/2} e^{i\pi/4 \,\mathrm{sgn}\,(\omega)}. \tag{41}$$

Note that for the case of the common-source gather, $I'_s = 0$ and $I'_r = 1$; for the common-receiver gather, $I'_r = 0$ and $I'_s = 1$. For other gathers, such as common offset, $I'_s = I'_r = 1$. Special cases are discussed later.

2.5-D INVERSION FOR REFLECTOR MAPPING

Our inversion formulas are only valid in a high-frequency limit. As we have shown (Cohen and Bleistein 1979a; 1979b; Bleistein, 1984; Bleistein et al., 1985), under such circumstances it is better to process data for the upward normal derivative $\partial\alpha/\partial n$ at each discontinuity surface of $\alpha(\mathbf{x})$, rather than for $\alpha(\mathbf{x})$ itself. $\partial\alpha/\partial n$ is a sum of weighted *singular functions*. Each singular function is a Dirac delta function which peaks on a surface of discontinuity of $\alpha(\mathbf{x})$, that is, the singular functions peak on the reflectors, thereby providing a map of the subsurface. The weighting factor of each singular function is calculated below.

The references above describe a means of finding $\partial\alpha(\mathbf{x})/\partial n$ given $\alpha(\mathbf{x})$. More precisely, the Fourier transforms of these functions are related in a simple way to leading order at high frequency, namely,

$$\alpha(\mathbf{x}) \leftrightarrow \tilde{\alpha}(\mathbf{k}), \qquad \frac{\partial\alpha(\mathbf{x})}{\partial n} \leftrightarrow ik \,\mathrm{sgn}\,(\omega)\tilde{\alpha}(\mathbf{k}). \tag{42}$$

Implicit in the development in Cohen et al. (1986) is the fact that (39) is indeed an approximate (high-frequency) Fourier integral of $\tilde{\alpha}(\mathbf{k})$, with $\mathbf{k}$ defined by

$$\mathbf{k} = \omega(\mathbf{p}_s + \mathbf{p}_r) \tag{43}$$

or

$$k \,\mathrm{sgn}\,(\omega) = \frac{\omega}{c(\mathbf{x})}\sqrt{2\left[1 + c^2(\mathbf{x})\mathbf{p}_s \cdot \mathbf{p}_r\right]} \tag{44}$$

and

$$\frac{\partial\alpha}{\partial n} \leftrightarrow \frac{i\omega\tilde{\alpha}(\mathbf{k})}{c(\mathbf{x})}\sqrt{2\left[1 + c^2(\mathbf{x})\mathbf{p}_s \cdot \mathbf{p}_r\right]}. \tag{45}$$

We use equation (45) in equation (39) to obtain the following 2.5-D formula for $\partial\alpha/\partial n$:

$$\frac{\partial\alpha}{\partial n}(x, z) \sim \frac{4}{\sqrt{\pi}c(\mathbf{x})}\int d\xi \left[1 + c^2(\mathbf{x})\mathbf{p}_s \cdot \mathbf{p}_r\right]^{3/2} \times [A_s A_r]^{-1}\sqrt{\frac{1}{\sigma_{s0}} + \frac{1}{\sigma_{r0}}} \times \left[\sqrt{g_s}\,\hat{\mathbf{n}}_s \cdot \mathbf{p}_{s0} A_s^2(\mathbf{x}, \mathbf{x}_s)\sigma_{s0} I'_s + \sqrt{g_r}\,\hat{\mathbf{n}}_r \cdot \mathbf{p}_{r0} A_r^2(\mathbf{x}, \mathbf{x}_r)\sigma_{r0} I'_r\right] \times \int d\omega \sqrt{i\omega}\, F(\omega) e^{-i\omega(\tau_s + \tau_r)} D(\omega, \xi). \tag{46}$$

RELATIONSHIP BETWEEN $\partial\alpha/\partial n$, REFLECTION STRENGTH, AND Δc

We now turn to the question of how formula (46) for $\partial\alpha/\partial n$ relates to more familiar quantities, such as reflection strength and the jump in speed Δc. It has been shown that in the zero-offset case, $\partial\alpha/\partial n = 4R\delta(\mathbf{x})$, where R is the normal reflection coefficient. When the offset is nonzero, the situation is more difficult in that both $\partial\alpha/\partial n$ and R depend upon the angle of incidence of a particular pair of rays.

To help interpret $\partial\alpha/\partial n$ correctly [i.e., the result from applying the integral operator in equation (46)], we replace D in equation (46) by Kirchhoff approximate data for a single reflector. The details are in Appendix A where it is shown that if $\hat{\mathbf{n}}(\mathbf{x})$ denotes the unit upward normal to the reflector,

$$\frac{\partial\alpha}{\partial n} \sim 4\cos^2\theta R(\mathbf{x}, \theta)\gamma(\mathbf{x}). \tag{47}$$

$\gamma(\mathbf{x})$ is the singular function of the model surface; $R(\mathbf{x}, \theta)$ is defined for $\mathbf{x}$ on the reflecting surface by

$$R(\mathbf{x}, \theta) = \frac{\cos\theta - \sqrt{c^2(\mathbf{x})/c_+^2(\mathbf{x}) - \sin^2\theta}}{\cos\theta + \sqrt{c^2(\mathbf{x})/c_+^2(\mathbf{x}) - \sin^2\theta}}, \tag{48}$$

with $c_+(\mathbf{x})$ and $c(\mathbf{x})$ being the sound speeds below and above the reflector, respectively. The angle θ is defined by the condition

$$\frac{\cos\theta}{c(\mathbf{x})} = -\hat{\mathbf{n}}(\mathbf{x}) \cdot \mathbf{p}_s = -\hat{\mathbf{n}}(\mathbf{x}) \cdot \mathbf{p}_r, \tag{49}$$

where $\mathbf{p}_s$ and $\mathbf{p}_r$ are defined in (20).

The right term in equation (49) determines a source-receiver pair on the surface for which the point $\mathbf{x}$ on the reflector is the specular reflection point; that is, for which the incident and reflected rays make the same angle θ with the normal to the reflector. In principle, if $\hat{\mathbf{n}}$ and θ were known, in many conventional source-receiver configurations this would uniquely determine $\mathbf{x}_r$ and $\mathbf{x}_s$. However, in practice $\hat{\mathbf{n}}$ and θ are not known.

We now show how to determine $\cos\theta$ and avoid explicit determination of $\hat{\mathbf{n}}$. We introduce another inversion operator, $\beta(x, z)$, with a kernel only slightly modified from that in equation (46):

$$\beta(x, z) \sim \frac{2}{\sqrt{\pi}}\int d\xi \left[1 + c^2(\mathbf{x})\mathbf{p}_s \cdot \mathbf{p}_r\right]^{1/2} [A_s A_r]^{-1}\sqrt{\frac{1}{\sigma_{s0}} + \frac{1}{\sigma_{r0}}} \times \left(\sqrt{g_s}\,\hat{\mathbf{n}}_s \cdot \mathbf{p}_{s0} A_s^2 \sigma_{s0} I'_s + \sqrt{g_r}\,\hat{\mathbf{n}}_r \cdot \mathbf{p}_{r0} A_r^2 \sigma_{r0} I'_r\right) \times \int d\omega \sqrt{i\omega}\, F(\omega) e^{-i\omega(\tau_s + \tau_r)} D(\omega, \xi). \tag{50}$$

The integrand in equation (50) differs from the integrand in equation (46) only in the factor $2(1 + c^2\mathbf{p}_s \cdot \mathbf{p}_r)$, which, under condition (49), becomes $4\cos^2\theta$ when this new inversion operator is applied to Kirchhoff data. Thus, the same analysis which produced equation (47) leads to the result

$$\beta(x, z) \sim R(\mathbf{x}, \theta)\gamma(\mathbf{x}). \tag{51}$$

Thus $\beta(x, z)$ is the quantity previously defined (Bleistein, 1984) as the *reflectivity function*.

When processing for $\partial\alpha/\partial n$, for little additional cost the data may be simultaneously processed to yield $\beta(x, z)$. The ratio of the two at the peak of $\gamma(\mathbf{x})$ yields an estimate of $\cos^2\theta$, that is,

$$\cos^2\theta = \frac{\partial\alpha/\partial n\ (\text{peak})}{4\beta\ (\text{peak})}. \tag{52}$$

With $\cos\theta$ known, we use equations (A-13), (A-17), and (A-18) to calculate the peak value of $\gamma(\mathbf{x})$:

$$\gamma(\mathbf{x})_{\text{peak}} = |\nabla\Phi|_{\text{peak}} \int F(\omega)\,d\omega = \frac{\cos\theta}{c(\mathbf{x})}\int F(\omega)\,d\omega, \tag{53}$$

where Φ is defined in the Appendix.

Since we know the filter, we can now read the value of $R(\mathbf{x}, \theta)$ from the peak value of $\beta(x, z)$ by using expression (51). We then use equation (48) to obtain a robust estimate of $c_{\pm}(\mathbf{x})$, since this is now the only unknown quantity in that equation. We have now determined $c_{+}(\mathbf{x})$ without determining $\hat{\mathbf{n}}(\mathbf{x})$ or the particular source-receiver pair for which $\mathbf{x}$ is the specular point on the reflector.

This verification suggests a recursive application of the method in which the background velocity is updated progressively deeper in the section so that even deeper reflectors may be properly mapped and their attendant velocity increments can be estimated.

This interpretation is based on analysis of the output of our inversion formula when applied to the upward-scattered wave from a single reflector represented by the Kirchhoff approximation. This has certain advantages over an interpretation based on analysis of the output based on Born approximation of the input data. The latter approximation is restricted to small perturbations in sound speed and to small offsets between source and receiver. The Kirchhoff approximation is not subject to either of these constraints. Thus, the method has broader application than its basis in the Born approximation would suggest. On the other hand, multiples from reflectors above the "test reflector" are still neglected. Furthermore, the background sound speed above the test reflector must be close to the true sound speed so that this last reflector can be properly located by the inversion.

COMPUTER IMPLEMENTATION

For equation (46) or equation (50), computer processing proceeds as follows. The function $D(\omega, \xi)$ represents the Fourier transform of the trace, with the source and receiver points given by $\mathbf{x}_r = \mathbf{x}_r(\xi)$ and $\mathbf{x}_s = \mathbf{x}_s(\xi)$. The ω integration represents a filtering and inverse transform of the trace. Given an output point $\mathbf{x} = (x, z)$ and a ξ value, we determine $\mathbf{x}_s$, $\mathbf{x}_r$, and then $\tau(\mathbf{x}, \mathbf{x}_s)$, $\tau(\mathbf{x}, \mathbf{x}_r)$, etc. and add to a weighted discrete sum approximating the integral with respect to ξ.

For a general background velocity $c(x, z)$, the traveltimes and amplitudes can be determined by ray theory. The extreme contrast is the case of a constant background, when these terms may be expressed explicitly in terms of $\mathbf{x}$, $\mathbf{x}_s$, and $\mathbf{x}_r$.

We caution the reader that the results (46) and (50) assume a continuous background velocity $c(x, z)$. We can recover the result for the case in which $c(x, z)$ is discontinuous above the reflector of interest by noting that continuity of c was used only in producing equation (38). The key step in that result is the use of equation (13) to eliminate K. For the case of discontinuous $c(x, z)$, this equation is no longer valid. We can recover the result for discontinuous $c(x, z)$ by replacing the factors of A_s^2, A_r^2 in equations (46) and (50) according to equation (13). The factor $[A_s A_r]^{-1}$ remains unchanged since it did not come from that substitution.

In fact, this alternative representation is the one of choice for computation. We chose the forms (46) and (50) for exposition; the amplitudes of the respective Green's functions are more familiar quantities than the Jacobi determinants K_s and K_r. We present the results in terms of these determinants by using equation (13) in equation (38) to write H in terms of K_{so} and K_{ro}. These are the determinants defined by equation (15) for (p, q) replaced by (p_s, q_s) and (p_r, q_r), respectively, and p' replaced by p_{so} and p_{ro}, respectively. The result is

$$H = \frac{1}{c^2(\mathbf{x})}\left[1 + c^2(\mathbf{x})\mathbf{p}_s\cdot\mathbf{p}_r\right] \times \left[\sqrt{g_s}\,\hat{\mathbf{n}}\cdot\mathbf{p}_{so}\Big/\left(K_{so}q_{so}\right) + \sqrt{g_r}\,\hat{\mathbf{n}}\cdot\mathbf{p}_{ro}\Big/\left(K_{ro}q_{ro}\right)\right]. \tag{54}$$

We now make the same replacements in equations (46) and (50) to obtain

$$\frac{\partial\alpha}{\partial n}(x, z) \sim \frac{1}{4\pi^2\sqrt{\pi c(\mathbf{x})}}\int d\xi\left[1 + c^2(\mathbf{x})\mathbf{p}_s\cdot\mathbf{p}_r\right]^{3/2} \times [A_s A_r]^{-1}\sqrt{\frac{1}{\sigma_{so}} + \frac{1}{\sigma_{ro}}} \times \left[\sqrt{g_s}\,\hat{\mathbf{n}}\cdot\mathbf{p}_{so}\Big/\left(K_{so}q_{so}\right) + \sqrt{g_r}\,\hat{\mathbf{n}}\cdot\mathbf{p}_{ro}\Big/\left(K_{ro}q_{ro}\right)\right] \times \int d\omega\,\sqrt{i\omega}\,F(\omega)e^{-i\omega(\tau_s+\tau_r)}D(\omega, \xi), \tag{55}$$

and

$$\beta(x, z) \sim \frac{1}{2\pi^2\sqrt{\pi c(\mathbf{x})}}\int d\xi\left(1 + c^2(\mathbf{x})\mathbf{p}_s\cdot\mathbf{p}_r\right)^{1/2} \times [A_s A_r]^{-1}\sqrt{\frac{1}{\sigma_{so}} + \frac{1}{\sigma_{ro}}} \times \left[\sqrt{g_s}\,\hat{\mathbf{n}}\cdot\mathbf{p}_{so}\Big/\left(K_{so}q_{so}\right) + \sqrt{g_r}\,\hat{\mathbf{n}}\cdot\mathbf{p}_{ro}\Big/\left(K_{ro}q_{ro}\right)\right] \times \int d\omega\,\sqrt{i\omega}\,F(\omega)e^{-i\omega(\tau_s+\tau_r)}D(\omega, \xi). \tag{56}$$

These results are valid for a fixed source and a line of receivers, for a fixed receiver and a line of sources, or for the common-offset configuration. In all these cases, the imaging predicted will be true for those points on reflectors such that equation (49) is satisfied for some pair $\mathbf{x}_s$, $\mathbf{x}_r$ in the data set. B. L. Sumner developed a computer code consistent with these results for the case of a depth-dependent background and zero offset, as reported in Cohen and Hagin (1985); a code for constant background and constant offset has been developed by Sullivan and Cohen (1985). Computer codes for other source-receiver configurations are presently being developed.

Again, the upper surface of all these cases need not be flat, but could be moderately curved.

SPECIAL CASES

Here we discuss some special cases which can be implemented by minor modification of present techniques. For simplicity we take the upper surface to be flat, so that ξ is a Cartesian variable and

$$
\begin{aligned}
g_s &= g_r = 1,\\
\hat{\mathbf{n}}_s \cdot \mathbf{p}_{0s} &= p_{3s},\\
\hat{\mathbf{n}}_r \cdot \mathbf{p}_{0r} &= p_{3r},\\
z_r &= z_s = 0.
\end{aligned} \tag{57}
$$

To be definite, we write formulas only for $\beta(x, z)$; the corresponding formula for $\partial\alpha/\partial n$ is derived as discussed below equation (50). That is, we must multiply the integrand of the β-formulas below by the factor $2(1 + c^2\mathbf{p}_s \cdot \mathbf{p}_r)$ to obtain the corresponding formula for $\partial\alpha/\partial n$.

Constant background

For c = constant, the solutions of equations (12) and (13) are

$$
\begin{aligned}
\tau(\mathbf{x}, \mathbf{x}_s) &= \frac{|\mathbf{x}_s - \mathbf{x}|}{c},\\
\mathbf{p}_s &= \frac{\mathbf{x}_s - \mathbf{x}}{c|\mathbf{x}_s - \mathbf{x}|},\\
A(\mathbf{x}, \mathbf{x}_s) &= \frac{1}{4\pi|\mathbf{x}_s - \mathbf{x}|},\\
\sigma_{s0} &= c|\mathbf{x}_s - \mathbf{x}|,
\end{aligned} \tag{58}
$$

and similarly for the variables with subscript r replacing subscript s.

Constant background, common-source gather

For the special case of a common-source gather, we set

$$
\begin{aligned}
&I'_s = 0, && I'_r = 1;\\
&x_s = \text{constant}, && x_r = \xi;\\
&\mathbf{x}_s = (x_s, 0), && \mathbf{x}_r = (\xi, 0).
\end{aligned} \tag{59}
$$

Thus, equation (50) becomes

$$
\begin{aligned}
\beta(x, z) \sim{}& \frac{2z|\mathbf{x}_s - \mathbf{x}|}{c^{3/2}\sqrt{\pi}} \int \frac{d\xi}{|\mathbf{x}_r - \mathbf{x}|} \sqrt{1 + \frac{(\mathbf{x}_s - \mathbf{x}) \cdot (\mathbf{x}_r - \mathbf{x})}{|\mathbf{x}_s - \mathbf{x}|\,|\mathbf{x}_r - \mathbf{x}|}}\\
&\times \sqrt{\frac{|\mathbf{x}_s - \mathbf{x}| + |\mathbf{x}_r - \mathbf{x}|}{|\mathbf{x}_s - \mathbf{x}|\,|\mathbf{x}_r - \mathbf{x}|}} \int d\omega \sqrt{i\omega}\, F(\omega)\\
&\times \exp\left[-i\omega\left(|\mathbf{x}_s - \mathbf{x}| + |\mathbf{x}_r - \mathbf{x}|\right)\Big/c\right] D(\omega, \xi).
\end{aligned} \tag{60}
$$

Equation (60) is a 2.5-D common-source, constant-background inversion formula. The common-receiver case is obtained from equation (60) by interchanging $\mathbf{x}_r$ and $\mathbf{x}_s$, with $\mathbf{x}_r$ fixed and

$$
\begin{aligned}
\mathbf{x} &= (\xi, 0),\\
\mathbf{x}_r &= (x_r, 0),
\end{aligned} \tag{61}
$$

and

$$\mathbf{x} = (x, z).$$

Constant-background, common-offset gather

We now set

$$
\begin{aligned}
I'_s &= I'_r = 1,\\
x_s &= \xi - h,\\
x_r &= \xi + h,
\end{aligned} \tag{62}
$$

and again use equations (57) and (58) in equation (50) to obtain

$$
\begin{aligned}
\beta(x, z) \sim{}& \frac{2z}{c^{3/2}\sqrt{\pi}} \int d\xi \sqrt{1 + \frac{(\mathbf{x}_s - \mathbf{x}) \cdot (\mathbf{x}_r - \mathbf{x})}{|\mathbf{x}_s - \mathbf{x}|\,|\mathbf{x}_r - \mathbf{x}|}}\\
&\times \sqrt{\frac{|\mathbf{x}_s - \mathbf{x}| + |\mathbf{x}_r - \mathbf{x}|}{|\mathbf{x}_s - \mathbf{x}|\,|\mathbf{x}_r - \mathbf{x}|}} \left[\frac{|\mathbf{x}_r - \mathbf{x}|}{|\mathbf{x}_s - \mathbf{x}|} + \frac{|\mathbf{x}_s - \mathbf{x}|}{|\mathbf{x}_r - \mathbf{x}|}\right]\\
&\times \int d\omega \sqrt{i\omega}\, F(\omega)\\
&\times \exp\left[-i\omega\left(|\mathbf{x}_s - \mathbf{x}| + |\mathbf{x}_r - \mathbf{x}|\right)\Big/c\right] D(\omega, \xi).
\end{aligned} \tag{63}
$$

This result agrees with Sullivan and Cohen (1985).

Constant-background, zero-offset gather

Since expression (63) provides a means for processing common-offset gathers in isolation, we may, in particular, obtain the zero-offset result by setting $h = 0$:

$$
\begin{aligned}
\beta(x, z) \sim{}& \frac{8z}{c^{3/2}\sqrt{\pi}} \int \frac{d\xi}{\sqrt{\rho}} \int \sqrt{i\omega}\, d\omega\, F(\omega)\\
&\times \exp\left[-2i\omega\left(|\mathbf{x}_s - \mathbf{x}| + |\mathbf{x}_r - \mathbf{x}|\right)\right].
\end{aligned} \tag{64}
$$

This result agrees with Bleistein et al. (1985).

Depth-dependent background

We now consider the case in which c is a continuous function of z alone, that is, $c(\mathbf{x}) = c(z)$. The ray equations, (12) and (13), in this case lead to the following. Note from equation (12) that $p(\sigma) = p'$ = constant; hence, we denote this parameter by p_s. Second, we replace σ by z as the ray parameter [by dividing equations (12) by $dz/d\sigma$]. Hence, the system of equations (12) and (13) leads to

$$
\begin{aligned}
q_s &= \sqrt{\frac{1}{c^2} - p_s^2},\\
|x_s - x| &= p_s \int_0^z \frac{d\zeta}{q_s(\zeta)},\\
\tau(x, x_s) &= \int_0^z \frac{d\zeta}{c^2(\zeta)q_s(\zeta)},\\
\sigma_s &= \int_0^z \frac{d\zeta}{q_s(\zeta)},\\
K_s &= q_s(z)q_s(0) \int_0^z \frac{d\zeta}{c^2(\zeta)q_s(\zeta)},
\end{aligned} \tag{65}
$$

and (66)

$$A(\mathbf{x}, \mathbf{x}_s) = \frac{1}{4\pi\sqrt{q_s(z)\sigma_s(z)K_s(z)}}.$$

Given two points $(x_s, 0)$ and (x, z), we use the equation for $|x_s - x|$ to solve for p_s and then use this value to evaluate τ, σ, and A. The same set of formulas is used with x_s replaced by x_r to obtain the results for p_r, etc., from $|x_r - x|$ and z.

Now we use expression (50) to write the formulas for the special cases just treated, namely, common source, common receiver, or common offset. For a common-source gather, we also must use equations (57) and (59), and for a common-offset gather we use equations (57) and (62).

The value in specializing to $c(z)$ is that the formulas in expression (66) are laterally invariant, thus requiring calculation of ray variables only for $|x_s - x|$ and z $[O(N_x N_z)$ points] rather than for each x_s, x, and z $[O(N_x^2 N_z)]$ points] as in the general $c(x, z)$ case.

CONCLUSIONS

We have derived a 2.5-D velocity inversion formula for reflector mapping and velocity estimation in a fairly general setting. The background velocity can be transversely, as well as vertically, variable; the datum surface may be curved. For a single reflector at depth, we have applied the method to Kirchhoff approximation and have shown that the method does indeed produce the band-limited singular function of the surface multiplied by the angle-dependent reflection coefficient. We have also shown how that angle can be determined at each point, so that the fully nonlinear geometrical optics reflection coefficient can be determined to yield increments of sound speed. We have confirmed the validity of the method for the cases of common-source data, common-receiver data, and common-offset data, with zero offset being a special case of the common-offset case. This confirmation suggests a recursive algorithm in which the output is used to correct the background sound speed progressively deeper, and then further inversion produces a reflector map below the corrected background.

ACKNOWLEDGMENTS

The authors gratefully acknowledge the support of The Office of Naval Research, Mathematics Division, through its Selected Research Opportunities Program, and the Consortium Project on Seismic Inverse Methods For Complex Structures at the Center for Wave Phenomena, Colorado School of Mines. Consortium members included Amoco Production Company; Conoco, Inc.; Digicon, Inc.; Geophysical Exploration Company of Norway A/S; Marathon Oil Company; Mobil Research and Development Corp.; Phillips Petroleum Company; Sun Exploration and Research; Texaco USA; Union Oil Company of California; and Western Geophysical.

REFERENCES

Beylkin, G., 1985, Imaging of discontinuities in the inverse scattering problem by inversion of a casual generalized Radon transform: J. Math. Phys., **26**, 99–108.

Bleistein, N., 1984, Mathematical methods for wave phenomena: Academic Press Inc.

Bleistein, N., Cohen, J. K., and Hagin, F. G., 1985, Computational and asymptotic aspects of velocity inversion: Geophysics, **50**, 1253–1265.

Bleistein, N., and Gray, S. H., 1985, An extension of the Born inversion method to a depth dependent reference profile: Geophys. Prosp., **33**, 999–1022.

Bleistein, N., 1986, Two-and-one-half dimensional in-plane wave propagation: Geophys. Prosp., **34**, May.

Claerbout, J., and Doherty, S., 1972, Downward continuation of moveout-corrected seismograms: Geophysics, **37**, 741–768.

Cohen, J. K., and Bleistein, N. 1979a, Velocity inversion procedure for acoustic waves: Geophysics, **44**, 1077–1085.

——— 1979b, The singular function of a surface and physical optics inverse scattering: Wave Motion, **1**, 153–161.

Cohen, J. K., and Hagin, F. G., 1985, Velocity inversion using a stratified reference: Geophysics, **50**, 1689–1700.

Cohen, J. K., Hagin, F. G., and Bleistein, N., 1986, Three-dimensional Born inversion with an arbitrary reference: Geophysics, **51**, 1552–1558.

Schneider, W. A., 1978, Integral formulation for migration in two and three dimensions: Geophysics, **43**, 49–76.

Stolt, R. H., 1978, Migration by Fourier transform: Geophysics, **43**, 23–48.

Sullivan, M. F., and Cohen, J. K., 1985, Pre-stack Kirchhoff inversion of common offset data: Res. Rep. CWP-027, Center for Wave Phenomena, Colorado School of Mines.

APPENDIX

In this appendix we derive equation (47) by substituting 2.5-D Kirchhoff approximate data for a single reflector into equation (46). These data were given as equation (58) in Bleistein (1984). In the present notation,

$$D(\omega, \xi) \sim \sqrt{2\pi|\omega|}\, e^{3\pi i \operatorname{sgn}\omega/4} \times \int \frac{d\eta\, R(x;\theta')A_s' A_r' e^{i\omega(\tau_s+\tau_r)}\hat{\mathbf{n}}' \cdot [\mathbf{p}_s' + \mathbf{p}_r']}{\sqrt{\dfrac{1}{\sigma_{s0}'} + \dfrac{1}{\sigma_{r0}'}}}. \tag{A-1}$$

In equation (A-1), the primes denote evaluation on the reflector

$$\mathbf{x} = \mathbf{x}'(\eta). \tag{A-2}$$

The variable η is taken to be arc length on the reflector and $\hat{\mathbf{n}}$ is an upward directed unit normal on the reflector. The function $R(\mathbf{x}', \theta')$ is given by equation (48) above with $\mathbf{x}$ replaced by $\mathbf{x}'$ and $\cos\theta'$ defined by the left half of equation (49):

$$\cos\theta' = -c(\mathbf{x}')\hat{\mathbf{n}}' \cdot \mathbf{p}_s. \tag{A-3}$$

We now use equation (A-1) to define D in equation (46). That is, we are applying an inversion generator deduced from Born approximate data to Kirchhoff approximate data. We carry out the ξ and η integrations in equation (46) by the method of stationary phase applied to the total phase:

$$\begin{aligned}\Phi(\mathbf{x}, \mathbf{x}', \mathbf{x}_s, \mathbf{x}_r) &= \tau_s' + \tau_r' - \tau_s - \tau_r \\ &= \tau(\mathbf{x}'(\eta), \mathbf{x}_s(\xi)) + \tau(\mathbf{x}'(\eta), \mathbf{x}_r(\xi)) \\ &\quad - \tau(\mathbf{x}, \mathbf{x}_s(\xi)) - \tau(\mathbf{x}, \mathbf{x}_r(\xi)).\end{aligned} \tag{A-4}$$

The first derivatives of Φ are given by

$$\frac{\partial \Phi}{\partial \eta} = [\nabla' \tau_s' + \nabla' \tau_r'] \cdot \frac{d\mathbf{x}'}{\partial \eta},$$
$$\frac{\partial \Phi}{\partial \xi} = [\nabla_s \tau_s' - \nabla_s \tau_s] \cdot \frac{d\mathbf{x}_s}{d\xi} + [\nabla_r \tau_r' - \nabla_r \tau_r] \cdot \frac{d\mathbf{x}_r}{d\xi}. \tag{A-5}$$

Here

$$\nabla' = \left[\frac{\partial}{\partial x'}, \frac{\partial}{\partial z'}\right],$$
$$\nabla_{s,r} = \left[\frac{\partial}{\partial x_{s,r}}, \frac{\partial}{\partial z_{s,r}}\right]. \tag{A-6}$$

Setting $\partial\Phi/\partial\eta = 0$ gives the law of reflection, namely, the dominant point on the reflector is the point for which the angle of incidence equals the angle of reflection. This conclusion is usually stated in terms of the normal components:

$$\nabla' \tau_r' \cdot \hat{\mathbf{n}}' = \nabla' \tau_s' \cdot \hat{\mathbf{n}}'. \tag{A-7}$$

The condition $\partial\Phi/\partial\xi = 0$ may be written as

$$\nabla_s \tau_s' \cdot \frac{d\mathbf{x}_s}{d\xi} + \nabla_r \tau_r' \cdot \frac{d\mathbf{x}_r}{d\xi} = \nabla_s \tau_s \cdot \frac{d\mathbf{x}_s}{d\xi} + \nabla_r \tau_r \frac{d\mathbf{x}_r}{d\xi}. \tag{A-8}$$

Equation (A-8) may be interpreted geometrically as requiring the equality of the sum of the projections of the gradients of the phase τ and τ' on the observation curve.

If $\mathbf{x}$ is on the reflector, then one solution of equations (A-7) and (A-8) is that $\mathbf{x}' = \mathbf{x}$ and $\mathbf{x}_r$ and $\mathbf{x}_s$ are a pair of points on the upper surface for which the law of reflection (A-7) is satisfied. The former condition determines η and the latter determines ξ. For some cases, e.g., common midpoint, there could be many values of ξ, i.e., many source-receiver pairs which make the phase stationary. However, for the case of common-source, common-receiver, and common-offset gathers, there is at most one ξ, assuming that there are no caustics in the ray family from $\mathbf{x}$ to the upper surface. There is always the possibility that no ξ satisfying equation (A-8) can be found, for example, because the spread length is too short. However, we proceed under the assumption that one ξ does exist.

The question arises as to whether there are other stationary points when $\mathbf{x}$ is on the reflector. To examine this, consider Figure A-1 which shows a reflector C', an observation curve C, an output point $\mathbf{x}$ on C', rays through $\mathbf{x}$ to the $\mathbf{x}_s$ and $\mathbf{x}_r$ which satisfy the stationary phase conditions, and the four regions bounded by these rays.

An additional stationary point on C' would be in regions I or II. Suppose that the stationary point is $(\tilde{\xi}, \tilde{\eta})$ with corresponding spatial coordinates $\tilde{\mathbf{x}}'$, $\tilde{\mathbf{x}}_s$, $\tilde{\mathbf{x}}_r$ with $\tilde{\mathbf{x}}'$ in region I as shown in Figure A-2. According to equation (A-8), we must consider the traveltime functions from $\mathbf{x}$ and $\tilde{\mathbf{x}}$ to $\tilde{\mathbf{x}}_s$ and $\tilde{\mathbf{x}}_r$ and their gradients. The assumption that there are no caustics in the ray families from $\tilde{\mathbf{x}}_s$ and $\tilde{\mathbf{x}}_r$ guarantees that

$$\nabla_s \tau(\tilde{\mathbf{x}}', \tilde{\mathbf{x}}_s) \cdot \frac{d\tilde{\mathbf{x}}_s}{d\xi} > \nabla_s \tau(\mathbf{x}, \tilde{\mathbf{x}}_s) \cdot \frac{d\tilde{\mathbf{x}}_s}{d\xi},$$
$$\nabla_r \tau(\tilde{\mathbf{x}}', \tilde{\mathbf{x}}_r) \cdot \frac{d\tilde{\mathbf{x}}_r}{d\xi} > \nabla_r \tau(\mathbf{x}, \tilde{\mathbf{x}}_r) \cdot \frac{d\tilde{\mathbf{x}}_r}{d\xi}. \tag{A-9}$$

Thus, the sums in equation (A-8) cannot be equal. In region II, the same conclusion holds with reversed inequalities, and hence the only stationary point when $\mathbf{x}$ is on C' is $\mathbf{x}' = \mathbf{x}$.

We now examine the effect of moving $\mathbf{x}$ off of C'. Equations (A-7) and (A-8) will continue to have a unique solution in some neighborhood of C', provided that the Hessian matrix

$$\mathbf{M} = \begin{bmatrix} \dfrac{\partial^2\Phi}{\partial\eta^2} & \dfrac{\partial^2\Phi}{\partial\xi\partial\eta} \\ \dfrac{\partial^2\Phi}{\partial\xi\partial\eta} & \dfrac{\partial^2\Phi}{\partial\xi^2} \end{bmatrix} \tag{A-10}$$

has a nonzero determinant at the initial solution $\mathbf{x}' = \mathbf{x}$. Note that at $\mathbf{x}' = x$, $\Phi = 0$, and so are all of its ξ derivatives, including $\partial^2\Phi/\partial\xi^2$. Thus,

$$\det \mathbf{M}\bigg|_{\mathbf{x}=\mathbf{x}'} = -\left[\frac{\partial^2\Phi}{\partial\xi\partial\eta}\right]^2\bigg|_{\mathbf{x}'=\mathbf{x}}. \tag{A-11}$$

This second derivative is computed below and given by equation (A-26). It is seen to be a sum of nonnegative terms with at least one of I_s' and I_r' being equal to unity. To assure $\partial^2\Phi/\partial\xi\partial\eta$ is positive, one must assume that the ray directions at the upper surface are not tangent to that surface—$\hat{\mathbf{n}} \cdot \mathbf{p}_{s0} > 0$, $\hat{\mathbf{n}} \cdot \mathbf{p}_{r0} > 0$—and that the rays from $\mathbf{x}_s$ and $\mathbf{x}_r$ are not anti-collinear at the point $\mathbf{x} = \mathbf{x}'$, i.e., $\cos\theta \neq 0$. Equivalently, the rays are not tangent to the reflector at $\mathbf{x}'$.

The determinant of $\mathbf{M}$ and the signature of $\mathbf{M}$ are also

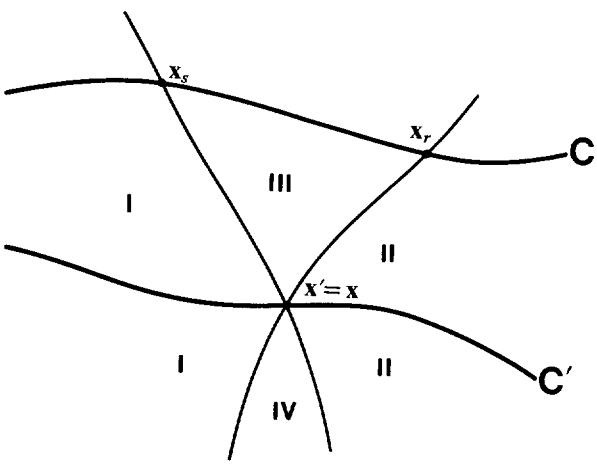

FIG. A-1. Stationary triple $\mathbf{x}'$, $\mathbf{x}_s$, $\mathbf{x}_r$ for $\mathbf{x}$ on the reflector C'. Note $\mathbf{x}' = \mathbf{x}$.

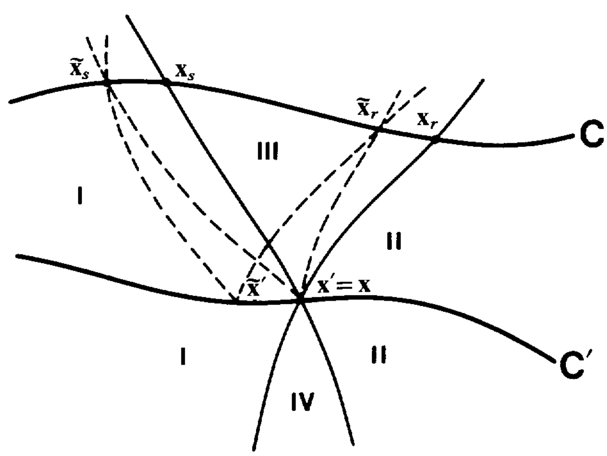

FIG. A-2. Candidate stationary values $\tilde{\mathbf{x}}_s$ and $\tilde{\mathbf{x}}_r$ for stationary point on C' not equal to $\mathbf{x}$ on C'.

required for the stationary-phase formula. Since $\det \mathbf{M} < 0$, the eigenvalues of $\mathbf{M}$ are of opposite sign and $\operatorname{sig} \mathbf{M} = 0$, for x on C' and hence, for x in some neighborhood of C'.

Thus, the result of applying the method of stationary phase to (46) and (A-1) is

$$\frac{\partial\alpha(\mathbf{x})}{\partial n} \sim \frac{8\pi\sqrt{2}}{c(\mathbf{x})}\left[1 + c^2(\mathbf{x})\mathbf{p}_s \cdot \mathbf{p}_r\right]^{3/2} \frac{A(\mathbf{x}', \mathbf{x}_s)A(\mathbf{x}', \mathbf{x}_r)}{A(\mathbf{x}, \mathbf{x}_s)A(\mathbf{x}, \mathbf{x}_r)} R(\mathbf{x}', \theta')$$
$$\times \left[\sqrt{g_s}\,\hat{\mathbf{n}}_s \cdot \mathbf{p}_{so} A^2(\mathbf{x}, \mathbf{x}_s)\sigma_{so} I_s' + \sqrt{g_r}\,\hat{\mathbf{n}}_r \cdot \mathbf{p}_{ro} A^2(\mathbf{x}, \mathbf{x}_r)\sigma_{ro} I_r'\right] \Big/ \left|\frac{\partial^2\Phi}{\partial\xi\partial\eta}\right|$$
$$\times \frac{\sqrt{\dfrac{1}{\sigma_{so}} + \dfrac{1}{\sigma_{ro}}}}{\sqrt{\dfrac{1}{\sigma_{so}'} + \dfrac{1}{\sigma_{ro}'}}} \frac{2\cos\theta'}{c(\mathbf{x}')}$$
$$\times \int d\omega\, F(\omega) \exp\left[i\omega\Phi(\mathbf{x}, \mathbf{x}', \mathbf{x}_s, \mathbf{x}_r)\right]. \tag{A-12}$$

The values of $\mathbf{x}'$, $\mathbf{x}_s$, and $\mathbf{x}_r$ are subject to the stationary phase conditions (A-7) and (A-8). The ω integration can be recognized as a band-limited delta function,

$$2\pi\delta(\Phi(\mathbf{x}, \mathbf{x}', \mathbf{x}_r, \mathbf{x}_s)) = \int d\omega\, F(\omega) \exp(i\omega\Phi(\mathbf{x}, \mathbf{x}', \mathbf{x}_s, \mathbf{x}_r)), \tag{A-13}$$

with the band limiting implicit in the presence of the filter $F(\omega)$ in the integral. Thus, the ω integral is negligible except in some neighborhood of $\Phi = 0$, subject to the stationary phase conditions.

The condition $\Phi = 0$ is satisfied on the curve C' when the stationary condition (A-8) holds, since equation (A-8) implies that $\mathbf{x} = \mathbf{x}'$. We claim that in the neighborhood of C', $\Phi = 0$ only on C'. To see why this is so, examine the total gradient $\nabla_T\Phi$ of Φ, taking into account that the stationary conditions define ξ and η as functions of $\mathbf{x}$. We consider $\mathbf{x}'$ as fixed on C'. This gradient can be written as

$$\nabla_T\Phi = \frac{\partial\Phi}{\partial\xi}\nabla\xi + \frac{\partial\Phi}{\partial\eta}\nabla\eta + \nabla\Phi, \tag{A-14}$$

where

$$\nabla = \left[\frac{\partial}{\partial x'}, \frac{\partial}{\partial z}\right]. \tag{A-15}$$

However, the stationarity conditions are precisely $\partial\Phi/\partial\xi = 0$, $\partial\Phi/\partial\eta = 0$, so that

$$\nabla_T\Phi = \nabla\Phi. \tag{A-16}$$

The right side of equation (A-16) has magnitude

$$|\nabla\Phi|^2 = |\nabla\tau(\mathbf{x}, \mathbf{x}_s) + \nabla\tau(\mathbf{x}, \mathbf{x}_r)|^2 = \frac{2}{c^2(x)}\left[1 + c^2(\mathbf{x})\mathbf{p}_s \cdot \mathbf{p}_r\right]. \tag{A-17}$$

This last expression is zero only if $\mathbf{p}_s$ and $\mathbf{p}_r$ are anticollinear, which we have assumed is not true. Thus, $\nabla_T\Phi \neq \mathbf{0}$ and the only zero of Φ in the neighborhood of C' is C' itself.

We now rewrite our band-limited delta function as a function of normal distance s_n from C',

$$\delta(\Phi) = \frac{\delta(s_n)}{|\nabla\Phi|} = \frac{\gamma(\mathbf{x})}{|\nabla\Phi|}, \tag{A-18}$$

where we have introduced γ as the singular function of C'.

Because the support of the delta function is $\mathbf{C}'$, we need only evaluate the amplitude in equation (A-12) for $\mathbf{x}$ on C', in which case the stationary condition for η guarantees that $\mathbf{x}' = \mathbf{x}$. We obtain

$$\frac{\partial\alpha}{\partial n}(\mathbf{x}) \sim \frac{32\pi^2}{c(\mathbf{x})}(1 + \cos 2\theta)\cos\theta R(\mathbf{x}, \theta)\gamma(\mathbf{x})$$
$$\times \left[\sqrt{g_s}\,\hat{\mathbf{n}}_s \cdot \mathbf{p}_{so} A_s^2 \sigma_{so} I_s' + \sqrt{g_r}\,\hat{\mathbf{n}}_r \cdot \mathbf{p}_{ro} A_r^2 \sigma_{ro} I_r'\right] \Big/ \left|\frac{\partial^2\Phi}{\partial\xi\partial\eta}\right| \tag{A-19}$$

subject to the stationarity conditions. Here we have used relation (49) to simplify the dot product $\mathbf{p}_s \cdot \mathbf{p}_r$.

Finally we evaluate the mixed derivative of Φ at the stationary point. From equation (A-5),

$$\frac{\partial^2\Phi}{\partial\xi\partial\eta} = \frac{\partial}{\partial\xi}[\mathbf{p}_s' + \mathbf{p}_r'] \cdot \frac{\partial\mathbf{x}'}{\partial\eta}. \tag{A-20}$$

From the eikonal equation,

$$\frac{\partial p_{3s}'}{\partial\xi} = -\frac{p_{1s}'}{p_{3s}'}\frac{\partial p_{1s}'}{\partial\xi}, \tag{A-21}$$

and similarly for p_{3r}'. Thus

$$\frac{\partial^2\Phi}{\partial\xi\partial\eta} = -\frac{1}{p_{3s}'}\frac{\partial p_{1s}'}{\partial\xi}\mathbf{p}_s' \cdot \hat{\mathbf{n}}' - \frac{1}{p_{3r}'}\frac{\partial p_{1r}'}{\partial\xi}\mathbf{p}_r' \cdot \hat{\mathbf{n}}', \tag{A-22}$$

where we have introduced the upward normal

$$\hat{\mathbf{n}}' = \left[\frac{\partial z'}{\partial\eta}, -\frac{\partial x'}{\partial\eta}\right]. \tag{A-23}$$

Again, using equation (49) and the stationary condition, $\mathbf{x}' = \mathbf{x}$, we find

$$\frac{\partial^2\Phi}{\partial\xi\partial\eta} = \frac{\cos\theta}{c(\mathbf{x})}\left[\frac{1}{p_{3s}}\frac{\partial p_{1s}}{\partial\xi} + \frac{1}{p_{3r}}\frac{\partial p_{1r}}{\partial\xi}\right]. \tag{A-24}$$

The derivatives of p_{1s} and p_{1r} can be evaluated from the method expounded in Cohen et al. (1986) to yield

$$\frac{1}{p_{3s}}\frac{\partial p_{1s}}{\partial\xi} = 16\pi^2 A^2(\mathbf{x}, \mathbf{x}_s)\sqrt{g_s}\,\hat{\mathbf{n}}_s \cdot \mathbf{p}_{so}\sigma_{so} I_s' \tag{A-25}$$

and a similar result for $\mathbf{p}_r$. Hence

$$\frac{\partial^2\Phi}{\partial\xi\partial\eta} = 16\pi^2\frac{\cos\theta}{c(\mathbf{x})}\left[\sqrt{g_s}\,\hat{\mathbf{n}}_s \cdot \mathbf{p}_{so} A^2(\mathbf{x}, \mathbf{x}_s)\sigma_{so} I_s' + \sqrt{g_r}\,\hat{\mathbf{n}}_r \cdot \mathbf{p}_{ro} A^2(\mathbf{x}, \mathbf{x}_r)\sigma_{ro} I_r'\right], \tag{A-26}$$

and so equation (A-19) becomes

$$\frac{\partial\alpha}{\partial n}(x) \sim 4\cos^2\theta R(\mathbf{x}, \theta)\gamma(\mathbf{x}). \tag{A-27}$$