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Adaptive absorbing boundaries for finite-difference modeling of acoustic waves

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ABSTRACT

In finite-difference modeling, acoustic waves incident on model boundaries may be absorbed by well known modifications to the finite-difference equations at those boundaries. The simplest such modification attenuates waves incident at a particular angle (usual chosen to be zero), but reflects waves incident at other angles with an angle-dependent reflection coefficient. More complicated modifications to the boundary equations (also well known) enable the boundaries to absorb waves for an extended range of incident angles.

The effectiveness of the simplest absorbing boundary equation can be improved by letting its design parameter, the parameter that determines which incident angle to absorb, be determined from the incident wavefield. This improved absorbing boundary equation is non-linear, because the coefficient in the equation is determined from the waves to which it is applied. Tests demonstrate that this adaptive absorbing boundary effectively attenuates boundary reflections.

INTRODUCTION

I recently wrote a simple finite-difference acoustic modeling program in support of other work at CWP. While implementing the finite-difference equations for the boundaries, I recalled an idea (Toldi and Hale, 1982) for attenuating boundary reflections in migration. The idea was to use the simplest absorbing boundary equation described by Clayton and Engquist (1980), but to let the coefficient in that equation be determined from the seismic wavefield. In other words, *instead of using complicated fixed-coefficient boundary equations, use simple adaptive-coefficient boundary equations.*

Thinking that this idea may also be useful in finite-difference modeling of acoustic waves, I found my copy of Clayton and Engquist's (1977) classic paper on absorbing boundaries for wave equations, and developed an adaptive version of their simplest absorbing boundary equation.

I have not conducted sufficient tests to enable comparison of this simple adaptive boundary equation with more complicated (but well known and widely used) boundary equations. Nevertheless, the simple adaptive absorbing boundaries derived and illustrated below have served me well.

ADAPTIVE ABSORBING BOUNDARIES

Suppose that the wave incident on the lower boundary of a computational grid is a plane wave:

$$p(t,x,z) = f\left(\frac{\cos\theta}{v}z + \frac{\sin\theta}{v}x - t\right),$$

where t denotes time, x and z are horizontal and vertical spatial coordinates, v denotes velocity, and θ denotes the angle of propagation with respect to the vertical z axis. Differentiating with respect to t, x, and z, we obtain:

$$\frac{\partial p}{\partial t} = -f' \tag{1}$$

$$\frac{\partial p}{\partial x} = \frac{\sin\theta}{v}f' \tag{2}$$

$$\frac{\partial p}{\partial z} = \frac{\cos\theta}{v}f'. \tag{3}$$

Combine equations (1) and (2) to obtain

$$\frac{\partial p}{\partial x} + \frac{\sin\theta}{v}\frac{\partial p}{\partial t} = 0, \tag{4}$$

and combine equations (1) and (3) to obtain

$$\frac{\partial p}{\partial z} + \frac{\cos\theta}{v}\frac{\partial p}{\partial t} = 0. \tag{5}$$

Equation (5) is a one-way equation suitable for finite-difference representation on the lower boundary of a finite-difference modeling grid (Clayton and Engquist, 1977). To absorb plane waves normally incident on the boundary, set $\theta = 0$ in equation (5). More generally, waves incident at any angle $\pm\theta$ may be absorbed by a finite-difference implementation of equation (5) for that particular θ.

One problem with equation (5) is that only one angle of incidence θ may be chosen, so that not all waves arriving from multiple directions at some point on the boundary can be absorbed. A second problem is that, even if the waves arrive from only one direction, the incident angle θ is usually not known *a priori*.

As suggested by Clayton and Engquist (1977), such a priori knowledge of the incident angle might be used to "tune" the absorbing boundary equation (5). In practice, however, this angle is unknown except for simple models, for which finite-difference modeling is unlikely to be necessary. Therefore, when implementing an absorbing boundary via equation (5), one typically chooses the angle $\theta = 0$ or, equivalently, $\cos\theta = 1$.

Rather than assuming $\cos\theta = 1$, I instead use equation (4) to compute:

$$\begin{aligned} \cos\theta &= \left(1 - \sin^2\theta\right)^{1/2} \\ &= \left[1 - v^2 \left(\frac{\partial p}{\partial x}\right)^2 \left(\frac{\partial p}{\partial t}\right)^{-2}\right]^{1/2}. \end{aligned} \tag{6}$$

For each time t and location x along the lower boundary of the computational grid, I compute the derivatives $\partial p/\partial t$ and $\partial p/\partial x$ via centered second-order finite-difference approximations:

$$\begin{aligned} \frac{\partial p}{\partial t} &\approx \frac{p(t+\Delta t, x, z) - p(t-\Delta t, x, z)}{2\Delta t} \\ \frac{\partial p}{\partial x} &\approx \frac{p(t, x+\Delta x, z) - p(t, x-\Delta x, z)}{2\Delta x}. \end{aligned}$$

When the argument to the square-root in equation (6) is negative, I simply set $\cos\theta = 0$.

Let z denote the lower computational boundary where, for all x, we must compute $p(t+\Delta t, x, z)$, given $p(t, x, z)$ and $p(t, x, z-\Delta z)$ from a previous time step, and $p(t+\Delta t, x, z-\Delta z)$ computed using the interior finite-difference scheme. A simple second-order finite-difference approximation to equation (5) is (Thomée, 1962; Richtmyer and Morton, 1967, p. 137)

$$p(t+\Delta t, x, z) = \gamma\left[p(t+\Delta t, x, z-\Delta z) - p(t, x, z)\right] + p(t, x, z-\Delta z), \tag{7}$$

where γ is defined by

$$\gamma \equiv \frac{1 - \frac{\cos\theta\Delta z}{v\Delta t}}{1 + \frac{\cos\theta\Delta z}{v\Delta t}}, \tag{8}$$

At each time t and for all x along the lower boundary, the difference coefficient γ is determined adaptively by computing $\cos\theta$ according to equation (6), using the wavefield at depth $z - \Delta z$ adjacent to the boundary. Note that, because this coefficient depends upon previously computed values of the wavefield, the adaptive absorbing boundary equation is non-linear.

For definiteness, I assumed in the derivation above that the boundary of interest is the lower boundary. Similar absorbing boundary equations can be obtained for the upper, left, and right boundaries by simply rotating the x and z coordinates.

The derivation above also begins with the supposition that the wave incident on the boundary is a single plane wave. The effectiveness of adaptive absorbing boundaries depends on how well the incident wave *locally* approximates a single plane wave at each point along the boundary. The following section illustrates the validity of this approximation for a point source.

A SIMPLE TEST

The absorbing boundary equation derived in the preceding section was incorporated into a second-order (in both time and space) finite-difference approximation to the acoustic wave equation. A snapshot of an expanding wavefront corresponding to a point source is illustrated in Figure 1. Almost no reflected energy is apparent, which suggests that the single-plane-wave assumption made in the preceding section is valid for this example.

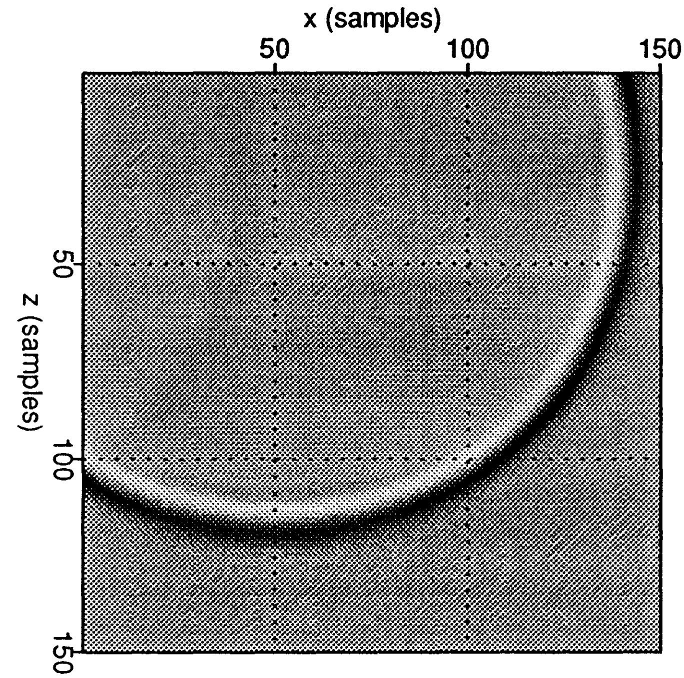

FIG. 1. Absorbing boundaries with an adaptive-coefficient boundary equation. Almost no reflected energy is apparent. Compare with Figure 2. This figure is a snapshot of a wave propagating from a point source located at $x = 50$ and $z = 25$.

For comparison, Figure 2 illustrates the result of using the same finite-difference scheme, but with the coefficient γ of equation (8) fixed by setting $\cos\theta = 1$. This choice corresponds to the simplest absorbing boundary equation proposed by Clayton and Engquist (1977). Because this boundary equation absorbs best for small angles of incidence, no normally reflected energy is evident in Figure 2. However, a significant amount of energy is apparently reflected at oblique angles of incidence.

Either one of the adaptive- or fixed-coefficient boundary equations is considerably more absorbing than the simple zero-value boundary equation. Simply setting the wavefield on the boundaries to zero yields the perfect reflections illustrated in Figure 3.

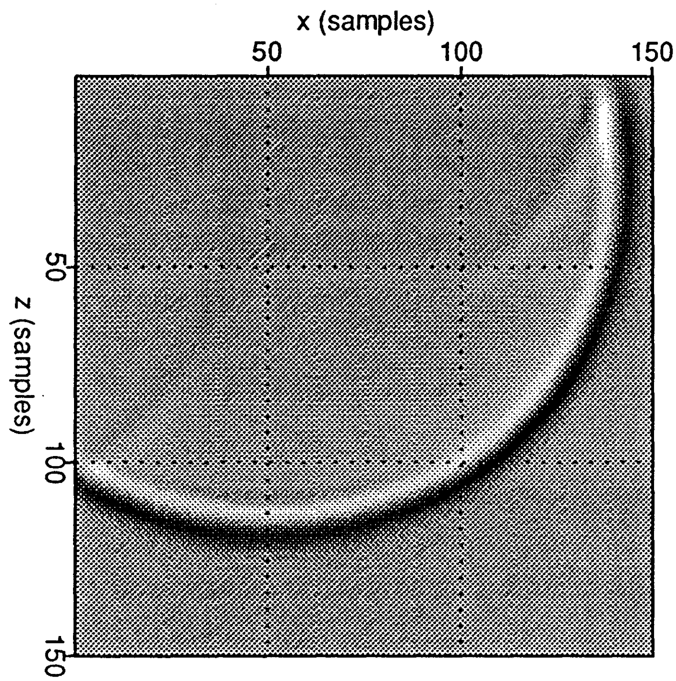

FIG. 2. Absorbing boundaries with a fixed-coefficient boundary equation designed to attenuate waves normally incident on the boundary. Note the reflections off the boundaries for oblique incidence angles. Compare with Figure 1. This figure is a snapshot of a wave propagating from a point source located at $x = 50$ and $z = 25$.

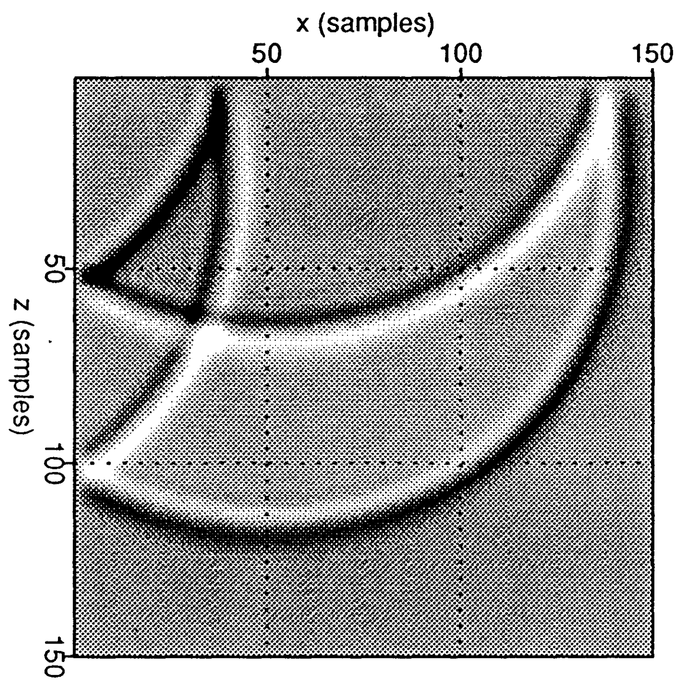

FIG. 3. Perfectly reflecting (zero-value) boundary condition. This figure is a snapshot of a wave propagating from a point source located at $x = 50$ and $z = 25$.

CONCLUSIONS

The simplest absorbing boundary equation may be improved by letting its coefficient adapt according to values of the wavefield near the boundary. Improvement is limited by the extent to which a wave incident on the boundary locally approximates a single plane wave. The simple test illustrated here (and my experience with other more complicated models not shown) suggests that this approximation is reasonable.

Although not demonstrated in this paper, more complicated boundary equations with fixed coefficients are capable of absorbing an extended range of incident angles. Further testing is necessary to determine how the performance of these equations compares with that of the adaptive absorbing boundaries described here.

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