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Mathematica^{TM} and the Method of Steepest Descents: Part I

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ABSTRACT

Mathematica is a symbolic manipulator with graphical capabilities. During the fall 1990 semester, I used Mathematica on my NeXT workstation to create graphics for teaching the method of steepest descents. This required level curve plots and surface plots for the real or imaginary part of functions of a complex variable. By the end of this topic in the course, I was doing near-research-level analysis of complex valued functions. The response of my class was also very positive. In this paper, I describe some of the more elementary examples discussed in the class. In Part II, I will describe the steepest descent analysis for the Lamb problem as a representative of research level application and utility of Mathematica for this type of analysis. This is the easiest example demonstrating how refracted and evanescent waves in a layered medium are treated by the method of steepest descents with the additional enhancement of Mathematica.

***.**—Key words. Mathematica, method of steepest descents, critical points, saddle points, branch points, branch cuts, ContourPlot, ContourLevels, PlotPoints, Plot3d, ViewPoint, PlotRange, Ticks, BoxRatios.

***.**—AMS(MOS) subject classifications. 30 (Functions of a complex variable), 98 (Mathematical Education, Collegiate)

INTRODUCTION

If the truth be known, my first love in mathematics is the application of classical complex function theory to the analysis of problems in wave propagation. Consequently, both of my books [Bleistein, 1984; Bleistein and Handelsman, 1986] have long chapters on the method of steepest descents. At the time that I was writing the first of these references, I had just acquired a microcomputer with graphics capability. Of course, I learned how to use its graphics by writing a program to plot steepest descent directions as small arrows covering the complex plane. (See pages 218, 221, Bleistein, 1984).

Ten years later I was preparing to teach a special topics course in complex variables in the fall 1990 semester. The title of the course was "Multi-valued functions and their applications." One of the major topics of the course was the method of steepest descents [Copson, 1965; Bleistein, 1984; Bleistein and Handelsman, 1986].

At the beginning of the summer, I acquired a NeXT workstation with a version of Mathematica configured to exploit the NeXT's graphics capability. In addition, my colleague Jack K. Cohen, enhanced the Mathematica on our system by installing the suite of programs, complex.m, available in the first chapter of Maeder [1990], as well as some enhancements of his own.

I could not resist the opportunity to apply my new computer capability to my first mathematical love. The results were a great success! Playing with conformal mappings and examining nuances of paths of steepest descents was as much an education for me as it was for my students. I could show details of regions near singular points in great enough detail to expose a new richness in mappings not easily accessible with hand calculations and freehand drawing. Of course, all of this was done interactively, with feedback in a few minutes, at most, for even the hardest examples. The graphics of complex valued functions evolved from an expository tool to a research tool as I will describe below.

When I started out, I would provide my students with copies of my files along with the graphics output. Eventually, I gave them a graphic assignment of their own. Two of the students actually thanked me for getting them started in Mathematica. Thereafter, graphic output became a regular part of homework submissions, whether or not I requested them as part of the assignment.

In this two part paper, I describe a series of examples of computer graphics taken from my course. The applications became more sophisticated as I learned and as the material became more difficult. Please note that I am the least facile computer user in our group. So, the reader should not expect to see particularly stylistic programming. The real point of these papers is to show what is possible with the use of Mathematica with graphics in a particular area of applied analysis. Part I shows standard examples as one might find in a text discussion of the method of steepest descents; Part II describes somewhat more sophisticated research level examples, where the graphics actually becomes a research tool in analyzing functions by for application of the method of steepest descents.

I have only "doctored" the Mathematica output by using a graphics program to produce larger numbers and tick marks on the axes. Thus, I invoked Mathematica graphics with the option, `Ticks -> None`; for classroom use, I would use Mathematica's own typescript numbers which are admittedly inferior and not changeable to other fonts, as far as I know.

I should also point out that some of my research was carried out on a more primitive graphics terminal than the NeXT monitor; I used Mathematica in terminal mode and included an appropriate file describing the graphics profile of the terminal with the Mathematica include (<<) command. Typically, I would set up a sparser grid of curves than I would have on the NeXT terminal because of the 2400 baud rate of my telephone communication. The result was lesser quality graphics than on my NeXT, but adequate for many purposes. There is one distinct advantage to the graphics at 2400 baud. The curves evolve on the screen at a slow enough rate to actually see the development and, at times, insight is gained from the evolution.

This is not a tutorial on Mathematica. Therefore, Mathematica constructs—including complex.m augmentation—will not be fully explained. However, the user of Mathematica should be able to follow the command sequence and, hopefully, the non-user will become hooked by seeing the examples.

I also used the Mathematica graphics in teaching the Cagniard-de Hoop method [de Hoop, 1960; Cagniard, 1962; Aki & Richards, 1980] in this course. In this method, straightforward conformal mappings become useful, as well as the analysis of the position of singularities in the mapped plane. Some of these applications will appear in a forthcoming paper [Bleistein and Cohen, 1991].

BACKGROUND: ON THE METHOD OF STEEPEST DESCENTS

The method of steepest descents is concerned with integrals of the form

$$I(\lambda) = \int_C g(z) \exp\{\lambda w(z)\} dz. \tag{1}$$

Here, we use the notations, $z = x + iy$ and $w(z) = u(x,y) + iv(x,y)$, with x, y, u, v, all being real. In this integral, C is a contour in the complex z plane. The method is concerned primarily with asymptotic approximations of this integral for large positive values of λ, although the method is also relevant to numerical approximations for any choice of λ.

The objective of the method is to use Cauchy's theorem to justify replacing the given contour of integration by one or a sum of contours that have been chosen for particularly rapid convergence of the resulting integrals. The rapid convergence is achieved by choosing these contours in such a manner that $u(x,y) = \text{Re}\, w(z)$ decreases most rapidly from its reference value at a so-called *critical point.* Candidate critical points include points where w and/or g fails to be analytic and points where the first and, possibly, higher derivatives of w vanish. These latter are called saddle points for reasons that will be explained below.

Paths along which $u(x,y)$ decreases (increases) are called paths of descent (ascent). Paths along which this decrease (increase) is most rapid compared to neighboring paths are called paths of steepest descent (ascent). At each point on these paths, the tangents are called directions of steepest descent (ascent).

Using complex function theory one can show the following.

1. At points of analyticity, there are always one or more directions of steepest descent (ascent), with the number of such directions determined by the number of vanishing derivatives at that point: first derivative nonzero, one of each; first derivative, zero and second derivative nonzero, two of each; first $n-1$ derivatives zero and nth derivative nonzero, n directions of each type.

2. The paths of steepest descent and ascent are the curves of constant $v(x,y) = \text{Im}\, w(z)$. At points of analyticity, this is just a consequence of Cauchy's theorem; at points where analyticity fails, it remains true, as well.

3. The functions $|w(z)|$, $u(x,y)$, $v(x,y)$, cannot have local maxima in the interior of the domain of analyticity of $w(z)$. Similarly, except for $|w(z)|$, which can assume the minimum of zero, these functions cannot have local minima in the

interior of the domain of analyticity of $w(z)$. In particular, then, from the prespective of any point in the domain of analyticity, the surface $u(x, y)$ is made up of *hills* and *valleys* rolling off towards infinity or towards finite points where $w(z)$ fails to be analytic.

As a consequence of these observations, we can conclude that in order for the integral in (1) to converge when C is infinite, the path of integration must end in a valley or, at worst, have the *boundary* between hill and valley—where $u(x, y)$ remains finite—as asymptote. In the deformation of a contour, then, the integral will be replaced by a sum of integrals, each being of one of the following three types. The first type of integral has a finite endpoint at an endpoint of integration of the original integral or at some sort of singularity. Its other endpoint is at infinity in a valley of $u(x, y)$. The second type of integral has two endpoints at infinity in the same valley of $u(x, y)$ with the contour "looping around" a singularity in the finite plane (yielding merely a residue, if the singularity is a pole). The third type is an integral with one endpoint at a saddle point and the other at infinity in a valley of $u(x, y)$. The role of contours through the saddle points, typically, is to connect different valleys at infinity for integrals both of whose endpoints are at infinity.

For the purposes of this paper, the point of all of this is to convince the reader that we need techniques to facilitate the location of saddle points and the paths of steepest descent and ascent of the function, $w(z)$. The most direct way to find these paths is to find the level curves of $v(x, y)$. As noted above, at a saddle point, there are multiple directions of descent and ascent. Therefore, we should expect to see crossing level curves at such points. Hence, this depiction will let us pick out the saddle points, as well. Alternatively, we can draw perspective plots of the surface, $u(x, y)$. These will aid in our conceptual understanding of the hills and valleys of $u(x, y)$. They provide important expositional tools. In addition, to these two depictions, one can depict the actual steepest descent and ascent paths with Mathematica, however, somewhat imperfectly in any nontrivial example. In combination with the former two depictions, we obtain the information we need for deformation of contours in the complex plane.

Clearly, for simple examples, this can all be done by hand. However, even in those cases, I can assure the reader from my classroom experience that exposition is greatly improved by the computer graphics such as the ones presented below. Furthermore, hands-on experience by the student becomes a valuable learning tool. Finally, I will describe a research-level problem, where the computer implementation helped me to see a saddle point that is the solution of a fourth order equation in z^2. This was a case in which I already knew that the saddle point was out there somewhere. However, I would suggest that a similar analysis is possible where the presence of such saddle points is only suspected.

GRAPHICAL EXAMPLES

Below is a series of graphic examples from my lectures on the method of steepest descents. I will start with the simplest of functions with saddle points and proceed through a hierarchy of progressively more difficult examples, ending in Part II with the analysis of a function $w(z)$ that arises in studying refracted and evanescent waves.

Prototype Saddle Points

Here we describe two of the simplest functions with a saddle point at the origin. These prototype examples help to develop our ability to identify saddle points in more complicated situations.

*.—Simple Saddle

We begin with the function

$$w(z) = z^2,\ u(x,y) = x^2 - y^2,\ v(x,y) = 2xy, \tag{2}$$

with derivatives

$$w'(z) = 2z,\ w''(z) = 2. \tag{3}$$

We show only two derivatives here because this function only has one saddle point at $z = 0$, with the second derivative, nonvanishing there. As noted above, this is a case in which there will be two directions of steepest descent and ascent at $z = 0$ and only one direction of each type everywhere else, where the first derivative is nonzero. Also, this function is entire—analytic everywhere.

This example is basic enough that we can read off the directions of steepest descent and ascent from (2). The level curves of v are the hyperbolas $2xy =$ constant, with the level curves through the origin being $x = 0,\ y = 0$.

We can check from the equation for u in (2) that the paths of steepest descent from the origin are the lines $x = 0$, while the paths of steepest ascent from the origin are the lines $y = 0$. At any other point, we simply find the hyperbola $v(x,y) =$ constant through it; one direction through the hyperbola is the direction of steepest descent and the other is the direction of steepest ascent. For example, on the hyperbolas in the first quadrant, passing from lower right to upper left is the direction of steepest descent; passing from upper left to lower right is the direction of steepest ascent.

Now we will use Mathematica to show these results. First, we depict the level curves of $v(x,y)$ by using the following command file.

```
ContourPlot[Im[(x + I y)^2 ],{x,-4,4},{y,-4,4},
ContourLevels -> 41,
Ticks -> None,
PlotPoints -> 81]
```

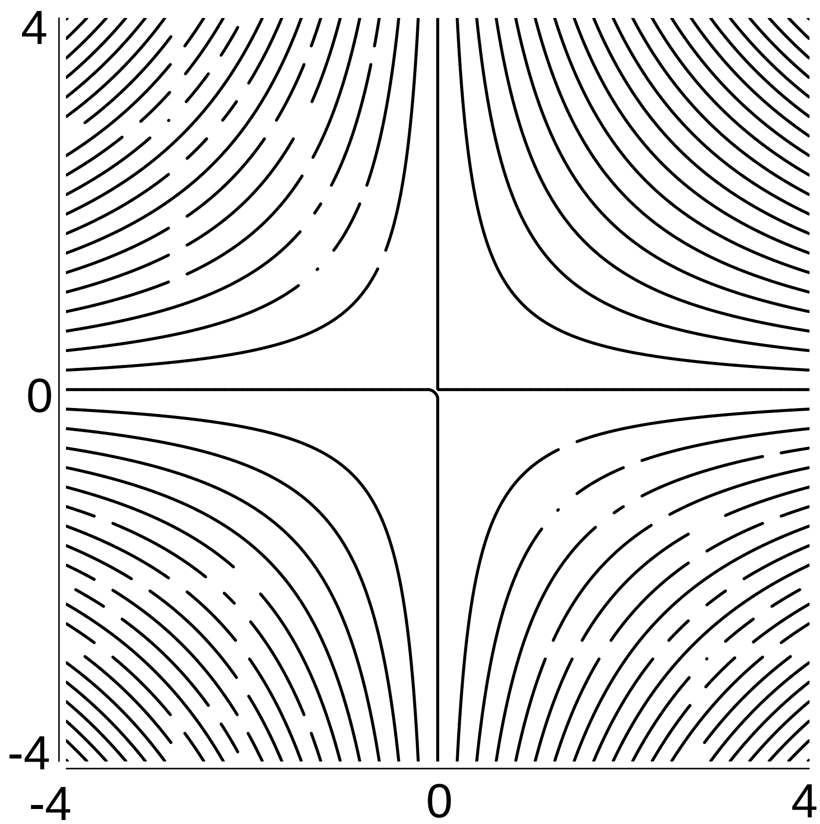

FIG. 1. Level curves of Im z^2.

This leads to Figure 1. Note that the range in x and y is (-4,4) as invoked in the first line of the command file; although I requested 41 contour levels in the second line, above, only 39 contour levels appear in the figure (including almost imperceptible dots at the four corners of the figure); each curve is made up of straight lines connecting 81 points at which `Im[(x + Iy)^2 ]` takes on the value for that particular contour. One can see here the various curves described analytically, above, in the second equation of (3), including the paths of steepest descent and ascent through the saddle point. In more complicated examples, we might not be fortunate enough to see these curves through the saddle point. Therefore, it is important to note from this figure the telltale signature of a saddle point from the nearby level curves of $v(x, y)$, namely, that these curves come nearby and then move away from the saddle, each being confined to one sector of the level curves through the saddle. Each of these curves is the projection of a horizontal slice through the surface $v(x, y)$. Of course, the surface $v(x, y)$ has the same type of saddle as the surface $u(x, y)$, except that it is rotated through 45° compared to the latter saddle.

We can see the surface $u(x, y)$ with a Mathematica command such as the following.

```
Plot3D[Re[(x + I y)^2],{x,-4,4},{y,-4,4},
Ticks -> None,
ViewPoint -> {1000,500,500}]
```

The result is shown in Figure 2. Here, again, the range of x and y is (-4,4) and the curve is viewed by looking back towards the origin from the `ViewPoint` (1000,500,500). (I use large numbers like this when I want to experiment with small incremental changes in viewpoint.) The surface is viewed from a point in the first octant with the positive y axis moving to the right, the positive x axis moving to the lower left and coming out of the paper. It can be seen from this figure that the surface $u(x, y)$ does, indeed, dip downward in the directions $y \to \pm\infty$ and bends upward in the directions $x \to \pm\infty$. Furthermore, locally, near $(0, 0, 0)$ the surface does look like a saddle. Because only the first derivative vanishes for this function, we refer to this as a *simple saddle.*

*****.—Monkey Saddle

Now we set

$$w(z) = z^3,\ u(x, y) = x(x^2 - 3y^2),\ v(x, y) = y(3x^2 - y^2). \tag{4}$$

For this function,

$$w'(z) = 3z^2,\ w''(z) = 6z,\ w'''(z) = 6. \tag{5}$$

We see that, again, the function has a saddle point at $z = 0$—of second order, in this case— and is entire with nonvanishing first derivative everywhere else. Figure 3 shows the curves of constant v while Figure 4 shows the surface u. In this latter figure, the viewpoint is (1000,-500,500); that is, the figure is being viewed from the

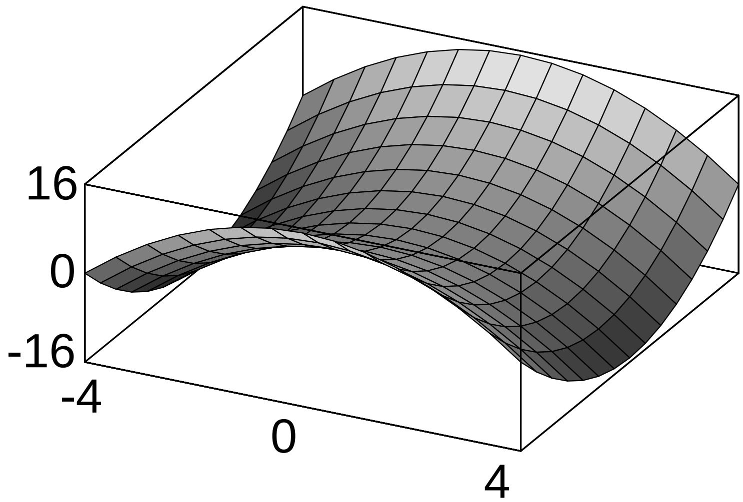

FIG. 2. Surface of Re z^2.

fourth quadrant in x and y and from above the plane, $z = 0$. From the former, we see six rays intersecting at $z = 0$. In the outward direction, three of them represent paths of steepest descent and three of them represent paths of steepest ascent. These alternate as one encircles the origin. By checking $u(x, y)$ in (4), one can verify that the rays at angles $\pm\pi/3$, π, are the paths of steepest descent while the rays at angles, 0, $\pm 2\pi/3$, are the paths of steepest ascent. Again, we can see the signature of the saddle point in the nearby level curves, all confined to sectors between these rays moving inward along one ray and outward along the adjacent ray.

From Figure 4 we see the three-lobe structure of the surface $u(x, y)$ and can confirm that the level curves of the previous figure really are the projections of the paths of steepest descent and ascent on the surface $u(x, y)$.

Airy Function Exponent

The function

$$w(z) = z - z^3/3, \ u(x, y) = x(1 + y^2 - x^2/3), \ v(x, y) = y(1 + y^2/3 - x^2), \quad (6)$$

arises in the analysis of the integral representation of the Airy function [Bleistein, 1984]. For this function,

$$w'(z) = 1 - z^2, \ w''(z) = -2z. \quad (7)$$

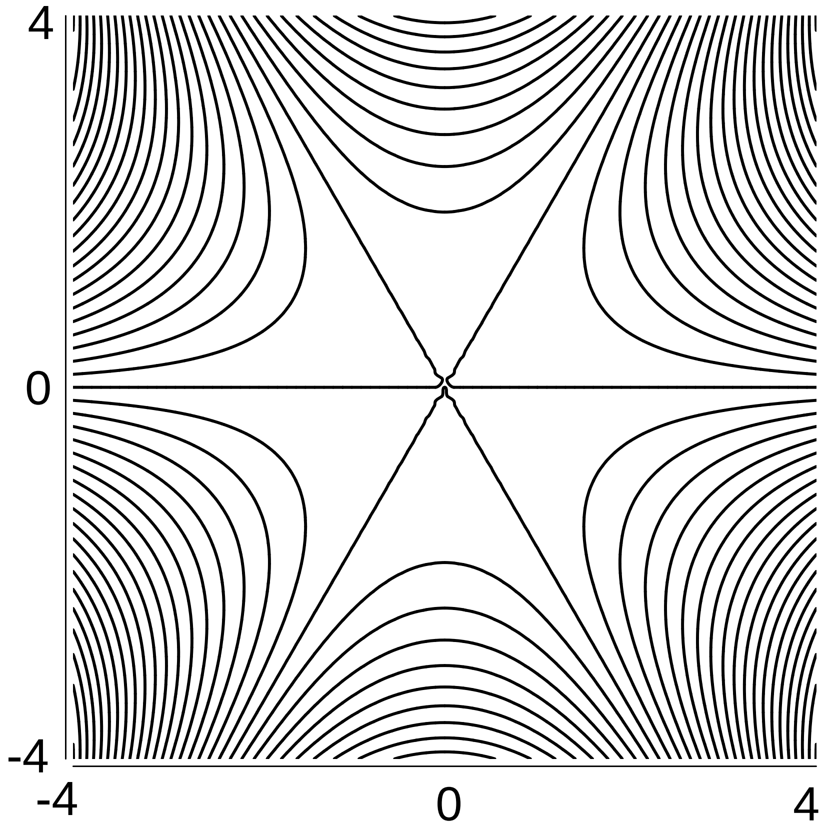

FIG. 3. Level curves of Im z^3.

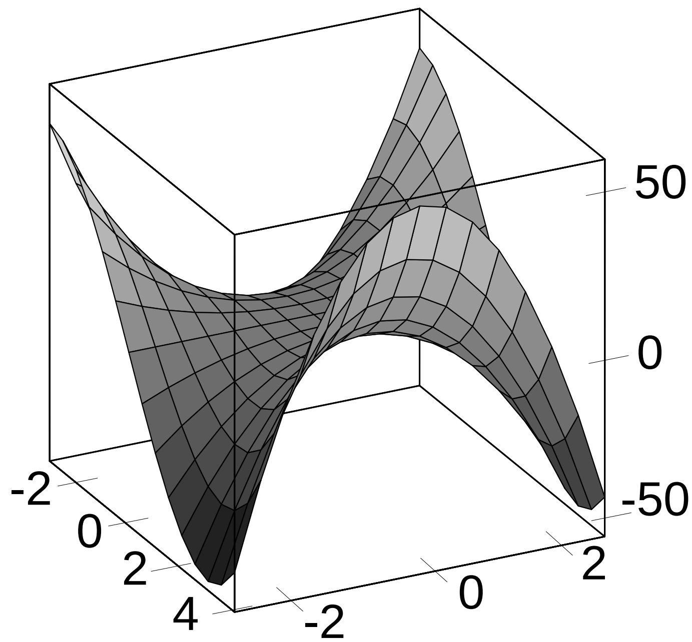

FIG. 4. Surface of Re z^3.

This is an entire function with two saddle points at $z = \pm 1$ and

$$w(\pm 1) = \pm 2/3, \;\; w''(\pm 1) = \mp 2 \neq 0. \tag{8}$$

From this evaluation, we see that $v(\pm 1, 0) = 0$, so that the level curves through the saddle points satisfy

$$y(1 + y^2/3 - x^2) = 0, \;\Rightarrow\; y = 0, \text{ or } x^2 - y^2/3 = 1. \tag{9}$$

The first of these curves, $y = 0$, is the x axis; the second curve, $x^2 - y^2/3 = 1$, is a hyperbola. By examining $u(x, y)$ one can verify that the segment of the axis connecting -1 to 1 is a path of steepest ascent from the saddle at -1 and correspondingly, it is a path of steepest descent from the saddle point at +1. Outside that range, the x axis is a path of steepest ascent on the left and a path of steepest descent on the right.

For the left branch of the hyperbola, each half, viewed as a curve directed away from -1, is a path of steepest descent, while for the the branch through +1, each half is a path of steepest ascent.

To verify that we have the curves right, we examine the plot of the level curves of $v(x, y)$ in Figure 5. To confirm that we have the descent and ascent directions right, we examine the surface $u(x, y)$ in Figure 6. The viewpoint in this figure is (-7,10,4). Note here that positive x is to the back left and positive y comes forward to the

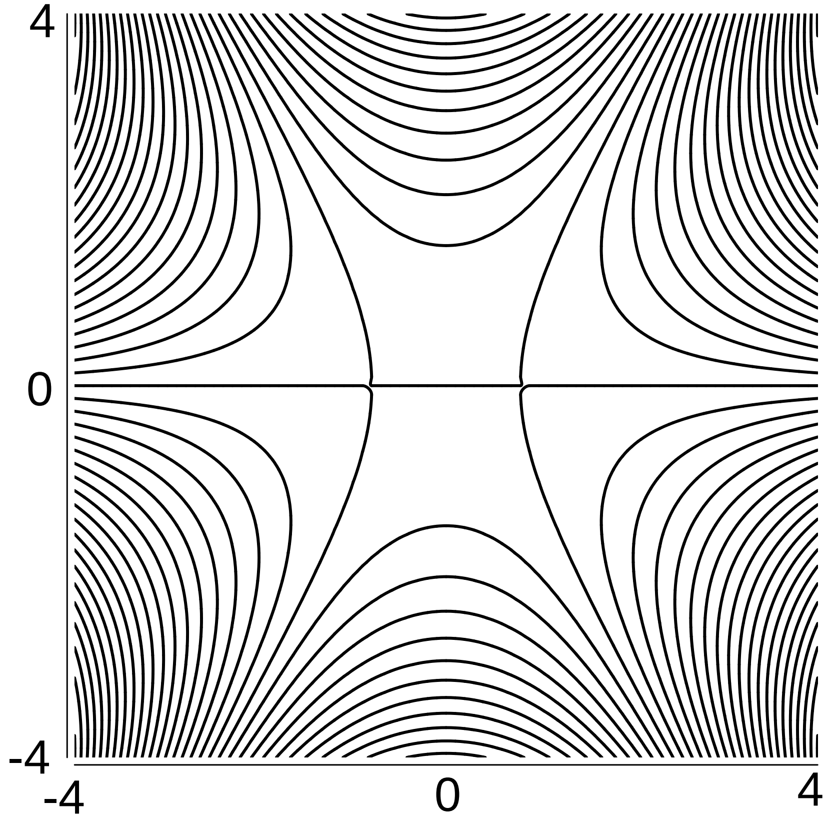

FIG. 5. Level curves of Im $\{z - z^3/3\}$.

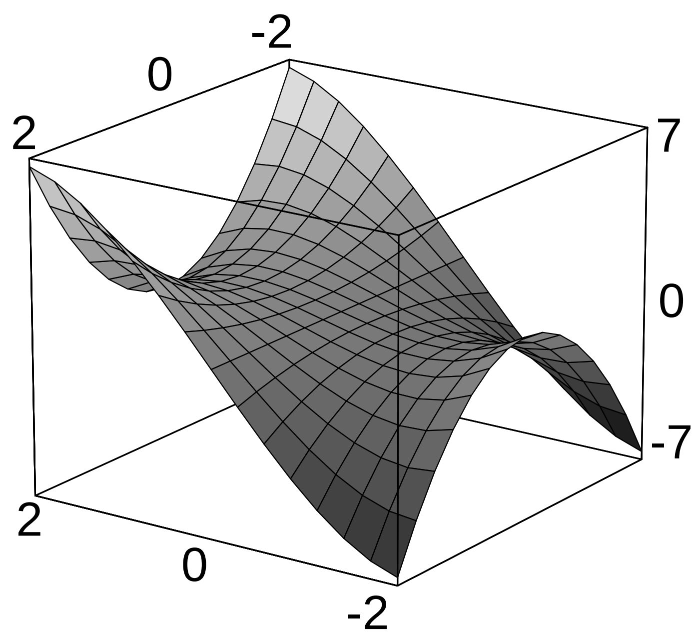

FIG. 6. Surface of Re $\{z - z^3/3\}$.

left. We can see the surface "descending" as we pass from $(1, 0, 2/3)$ to $(-1, 0, -2/3)$. Qualitatively we can also see here that the projections of the curves of steepest descent and ascent are as described above and suggested by the Figure 5.

The last two figures were produced with the following commands, respectively.

```
f[x_,y_] := y + y^3 /3 - x^2 y;
     ContourPlot[f[x,y],{x,-5,5},{y,-5,5},
     ContourLevels -> 41,
     Ticks -> None,
     Plotpoints -> 81]
```

```
fr[x_,y_] := x - x^3 /3 + x y^2 ;
     Plot3D[fr[x,y],{x,-2,2},{y,-2,2},
     ViewPoint -> {-7,10,4},
     Ticks -> None,
     BoxRatios -> {5,5,4}]
```

The effect of *BoxRatios* here is to slightly "squash" the figure, with visual increments in z being only four-fifths of the increments in x and y. This is done to produce a figure that emphasizes the positions of the saddles through shading.

For this more interesting example, it is worthwhile to produce the actual steepest descent and ascent paths. The command line I used to do this is as follows.

```
f[x_,y_] := y + y^3 /3 - x^2 y;
     ContourPlot[f[x,y],{x,-5,5},{y,-5,5},
     ContourLevels -> 1,
     PlotPoints -> 81,
     Ticks -> None,
     PlotRange -> {-.0001,1}]
```

The result is shown in Figure 7. By choosing only one contour level here, starting from a lower limit of -.0001, I have assured that only the curve, $f(x, y) = -.0001$, will be plotted. I found that $f = 0$ would miss pieces of the curves in Figure 7, apparently due to roundoff errors.

This last figure verifies our claims for the previous figures. We present the figures in this order because this type of figure will tend to degrade more than the others for more complicated examples. Thus, in my view, the ability to develop insight based on the previous pair of figures is desirable.

Bessel Function Exponent

Here we consider the exponent

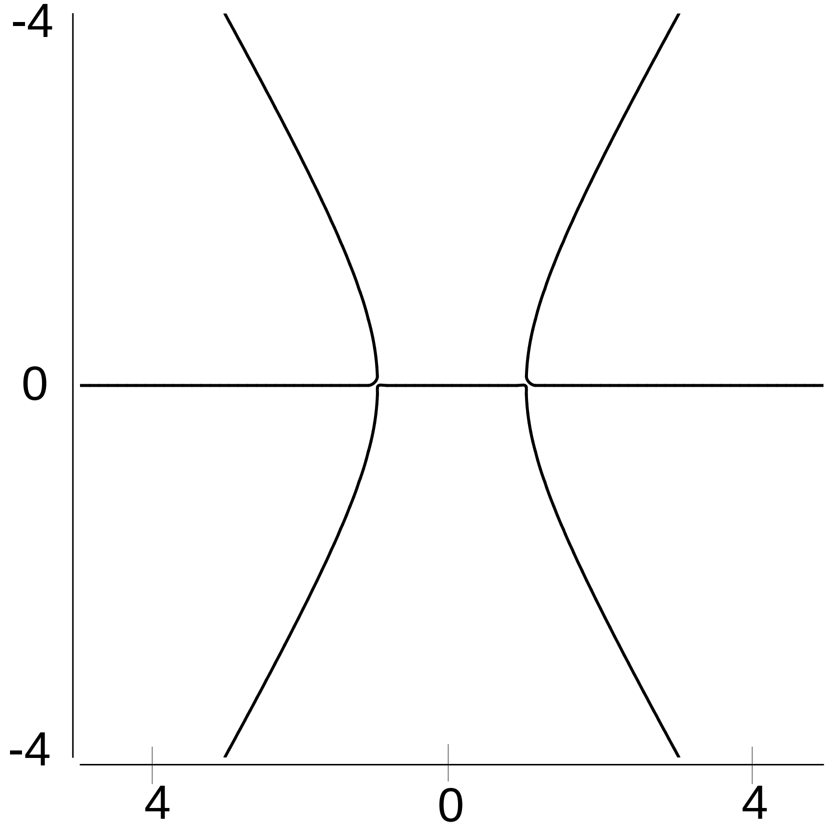

FIG. 7. Steepest descent and ascent paths for the Airy function exponent

$$w(z) = i[\cos z + .5(z - \pi/2)],$$
$$u(x,y) = \sin x \sinh y - .5y, \;\; v(x,y) = \cos x \cosh y + .5(x - \pi/2). \tag{10}$$

This exponent arises in the analysis of the integral representation of the Bessel functions when the ratio of order to argument is .5. For this function,

$$w'(z) = i[-\sin z + .5], \quad w''(z) = -i\cos z. \tag{11}$$

We will focus our attention, here, on the strip $-\pi/2 < x \leq 3\pi/2$.

This exponent has two saddle points, symmetrically located about $\pi/2$ with $\sin z = .5$; that is, $z = \pi/6, \; 5\pi/6$ and

$$w(\pi/6) = i[\sqrt{3}/2 - \pi/6], \quad w''(\pi/6) = -i\sqrt{3}/2,$$
$$w(5\pi/6) = -i[\sqrt{3}/2 - \pi/6], \quad w''(\pi/6) = +i\sqrt{3}/2. \tag{12}$$

For this function, the analysis of the steepest descent paths and the location of hills and valleys is somewhat more complicated than for those above, since determination of the level curves requires the solution of the mixed algebraic/transcendental equation $v(x,y) =$ constant. We content ourselves here with resorting to computer graphics to show us where the paths of steepest descent and ascent are. The level curves of $v(x,y)$ are shown in Figure 8. Here the level curves through the saddle point do not appear. This is a consequence of the manner in which Mathematica picks out the values at which to draw level curves. Nonetheless, we infer the presence of the saddle points from the telltale signature of nearby level curves bending into and away from two points on the Re z axis. The reader should compare the level curves near the saddle points in this figure with the level curves near the saddle point in Figure 1. Also, we can see here that the paths of steepest descent and ascent through the saddle points have the vertical lines at $x = \pm\pi/2, \; 3\pi/2$, as asymptotes. In fact, *all* of the steepest paths through any points in this strip have these lines as asymptotes.

The surface $u(x,y)$ is shown in Figure 9. Note that we are viewing the surface from a point above the third quadrant of the (x,y) plane, namely, (-1000,-500,500). It can be seen here that the x axis is a level curve of this surface; that is, $w(z)$ is purely imaginary on axis and $u(0,y) = 0$. Furthermore, we can see that the paths of steepest descent have $-i\pi/2$, $3\pi i/2$, as asymptotes as $y \to +\infty$ and have the single line at $i\pi/2$ as asymptote as $y \to -\infty$. One can show this analytically, but the pictures surely help!

The first figure here was produced with the following command line.

```
ContourPlot[Im[I ( Cos[x + I y] +.5 (x + I y - Pi/2))],
```

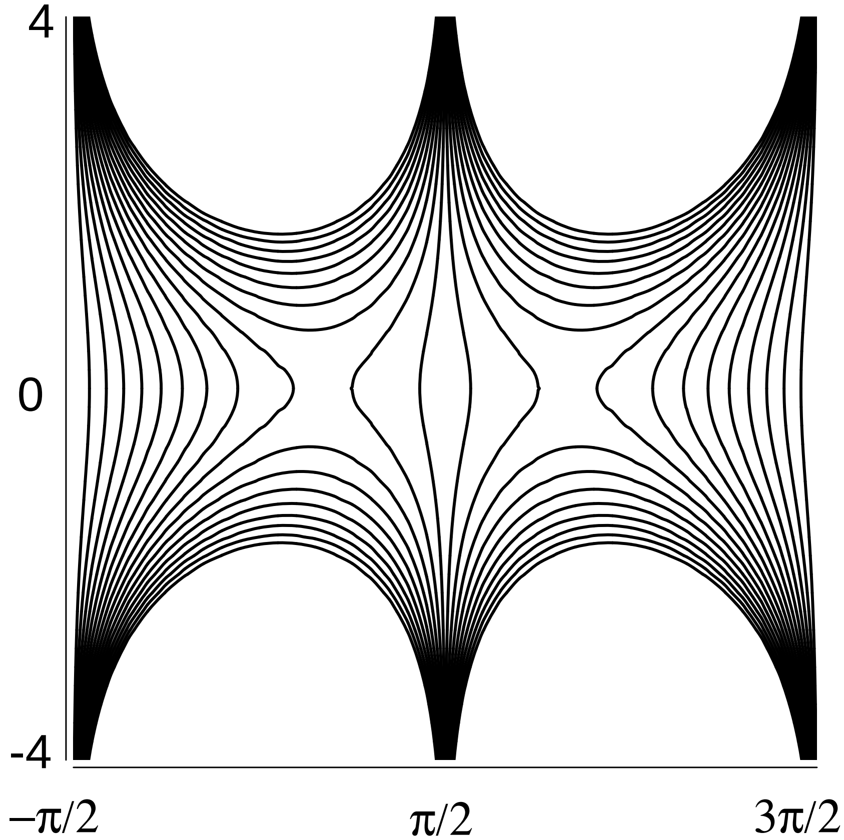

FIG. 8. Level curves of Im $\{i[\cos z + .5(z - \pi/2)]\}$.

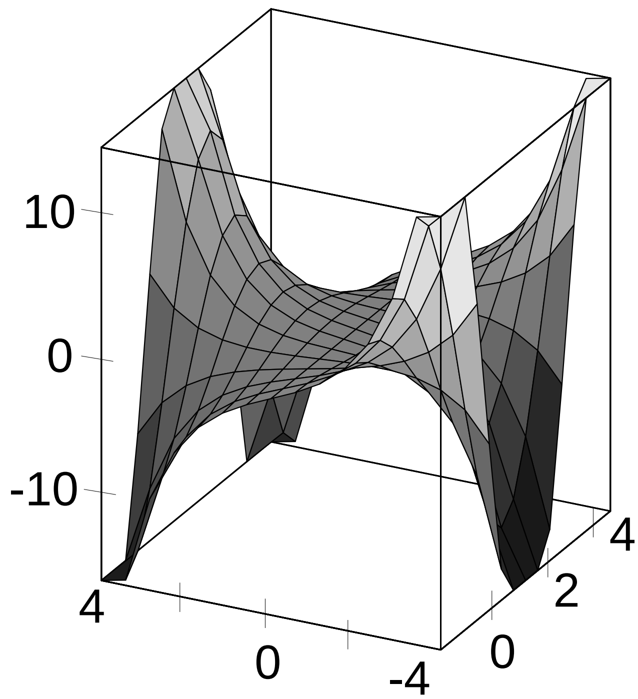

FIG. 9. Surface of Re $\{i[\cos z + .5(z - \pi/2)]\}$.

```
    {x, - Pi /2 , 3 Pi /2}, {y, -4,4},
    PlotPoints -> 31,
    PlotRange -> {-2,2},
    Ticks -> None,
    ContourLevels -> 20]
```

For this function, we have let Mathematica compute the imaginary part. Also, we have some problem here because of the exponential character of the complex trigonometric functions and more plot points than the default were necessary to adequately cover the range near the real axis; without this, not enough samples are taken there. The constrained PlotRange also limits the exponential growth of v in the plot. The number of ContourLevels, was also chosen by experimentation to provide an informative picture.

The second figure was produced with the command line

```
Plot3D[Re[I ( Cos[x + I y] +.5 (x + I y - Pi/2))],
      {x, - Pi /2 , 3 Pi /2}, {y, -4,4},
      ViewPoint -> {-1000,-500,500},
      Ticks -> None,
      BoxRatios -> {4,4,5}]
```

Again, the effect of the BoxRatios command here is to slightly enhance the dip and rise of the surface $u(x, y)$.

Figure 10 shows the steepest descent and ascent paths through the saddle point at $\pi/6$. Note that another curve in the neighborhood of $x = 3\pi/2$ also appears in the figure. On this curve, $v(x, y)$ must take on the same value as it does at the saddle point. However, there are no crossing curves here, hence, no saddle point. A similar figure could be generated for the saddlepoint at $5\pi/6$. The command lines for this figure were

```
ContourPlot[Im[I ( Cos[x + I y] +.5 (x + I y - Pi/2))],
     {x, - Pi /2 , 3 Pi /2}, {y, -4,4},
     PlotRange -> {Cos[Pi/6] +.5 (Pi/6 - Pi/2),
         Cos[Pi/6] +.5 (Pi/6 - Pi/2)},
     ContourLevels -> 1,
     Ticks -> None,
     PlotPoints -> 61]
```

Note that here we have allowed Mathematica to evaluate Im $w(z)$ at the saddle point in order to fix the value in the PlotRange option. This seemed a better means of function evaluation than if we were to choose the accuracy of this irrational number approximation.

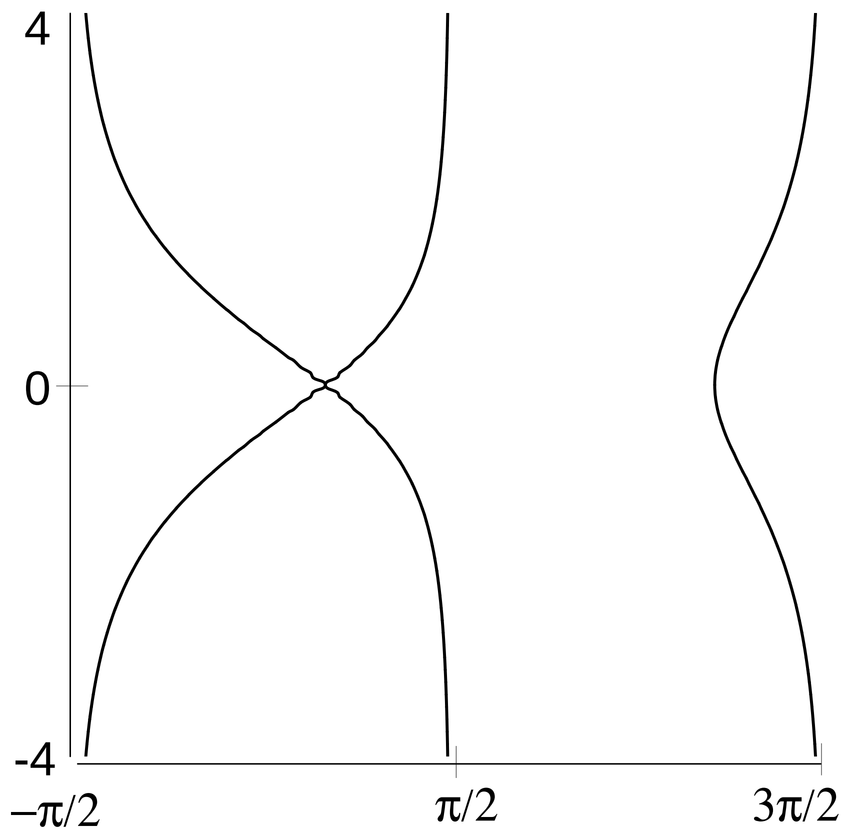

FIG. 10. Steepest descent paths for the Bessel function exponent with saddle point at $\pi/6$.

CONCLUSIONS

Through a hierarchy of progressively more difficult examples, not unlike the order presented in my course, I have described the application of Mathematica to the analysis techniques needed for the method of steepest descents. I found the use of Mathematica to be an excellent expository aid in this course. In addition, Mathematica was used in this course in the discussion of conformal mappings and the Cagniard-de Hoop method. The use of Mathematica was straightforward enough that my students learned with only a few examples to guide them, and then they used this tool routinely in homework assignments. It is clear from my experience that Mathematica can be used as a research tool in these applications.

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