GEOPHYSICS, VOL. 62, NO. 3 (MAY-JUNE 1997); P. 713–722, 7 FIGS., 1 TABLE.

P-wave reflection coefficients for transversely isotropic models with vertical and horizontal axis of symmetry

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ABSTRACT

The study of P-wave reflection coefficients in anisotropic media is important for amplitude variation with offset (AVO) analysis. While numerical evaluation of the reflection coefficient is straightforward, numerical solutions do not provide analytic insight into the influence of anisotropy on the AVO signature.

To overcome this difficulty, I present an improved approximation for P-wave reflection coefficients at a horizontal boundary in transversely isotropic media with vertical axis of symmetry (VTI media). This solution has the same AVO-gradient term describing the low-order angular variation of the reflection coefficient as the equations published previously, but is more accurate for large incidence angles. The refined approximation is then extended to transverse isotropy with a horizontal axis of symmetry (HTI), which is caused typically by a system of vertical cracks. Comparison of the approximate reflection coefficients for P-waves incident in the two vertical symmetry planes of HTI media indicates that the azimuthal variation of the AVO gradient is a function of the shear-wave splitting parameter γ, and the anisotropy parameter describing P-wave anisotropy for nearvertical propagation in the vertical plane containing the symmetry axis.

INTRODUCTION

Reflection and transmission of plane waves at a plane boundary between two isotropic media is one of the most fundamental problems in wave propagation. Zoeppritz (1919) was among the first to publish the solution to this problem invoking continuity of stress and displacement at the reflecting horizon.

One important practical application of the reflection coefficient studies is the analysis of amplitude variations with offset (AVO). The variation of P-wave amplitudes with incidence angle provides one of the few direct hydrocarbon indicators (Ostrander, 1984) and has drawn substantial attention within the geophysical community.

Conventional AVO analysis is based on analytic expressions for P-wave reflection coefficients in isotropic media, and it needs to be modified if anisotropy is present on either side of the reflecting boundary. Reflection and transmission of plane waves at an interface between two anisotropic media have been discussed, for example, in Musgrave (1970), Henneke (1972), Keith and Crampin (1977) and Daley and Hron (1977). The solution for reflection coefficients at interfaces of anisotropic media is very involved algebraically, and the general solution requires a numerical inversion of 6×6 matrices. The complexity of the problem obscures any physical insight into the AVO signature. Empirical and analytical studies, however, show that the presence of anisotropy can significantly distort conventional AVO analysis (Wright, 1987; Banik, 1987; Kim et al., 1993).

In the first part of this paper, I discuss the quasi P-wave[1] reflection coefficient at a boundary between two transversely isotropic media with vertical axes of symmetry (VTI media). The reflection coefficients are evaluated for the displacement vector and the positive polarization direction is chosen according to the sign convention in Aki and Richards (1980). Following Thomsen (1993), I derive a refined approximation for the P-P reflection coefficient assuming weak anisotropy and a boundary with small discontinuities in elastic properties. The new approximation has the same gradient term as the solution presented in Banik (1987) and Thomsen (1993), but is more accurate at larger angles.

The second part of the paper is devoted to media with azimuthally varying properties. One plausible explanation for the

[1]The qualifier in "quasi-P-wave" refers to the fact that, for anisotropic media, the longitudinal wave is not polarized in either the slowness or ray directions; similarly, the SV-wave in anisotropic media is not polarized normal to the slowness and ray directions. In the following, I will omit the qualifier "quasi" in "P-wave" and "SV-wave."

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azimuthal anisotropy observed in seismic field data (e.g., Lynn et al., 1995) is natural fracturing of rocks, often aligned in accordance with the dominant stress direction (Crampin, 1984, 1985). This observation can be of crucial importance to hydrocarbon exploration since fracture networks often determine direction and amount of fluid flow through reservoir rocks. Other physical reasons for azimuthally isotropic media include dipping shale sequences and intrinsic anisotropy.

Probably the most simple azimuthally anisotropic medium is the transversely isotropic model with a horizontal axis of rotational symmetry (HTI media) that is used frequently to describe vertically aligned penny-shaped cracks in an isotropic matrix (e.g., Thomsen, 1995). Because of its importance for reservoir characterization, much attention has been devoted to this model, mainly through analysis of shear-wave birefringence (e.g., Martin and Davis, 1987; Mueller, 1991). Based on synthetic modeling studies, Mallick and Frazer (1991) suggest to alternatively determine fracture orientation from P-wave AVO. More recently, Lefeuvre (1994) discussed the possibility of using P-wave reflections to statistically infer fracture parameters; however, he did not provide analytic insight into the relation between P-wave signatures and anisotropy parameters.

Here, I discuss the approximate solutions for P-P reflection coefficients for a horizontal interface between two HTI media. The solutions are based on a limited analogy between P-wave propagation in VTI and HTI models. This analogy is the key step not only in deriving the reflection coefficient, but also for the derivation of kinematic properties and polarizations in HTI media. Additionally, this analogy implies that the known exact solutions of VTI reflection coefficients can be used to evaluate the reflection coefficient for waves incident in the vertical plane containing the symmetry axis of HTI models.

The original goal of AVO analysis was to extract shear-wave information from the variations of the reflection coefficient with incidence angle. Based on the studies shown in this paper, Rüger and Tsvankin (1995) additionally have proposed to use the azimuthal differences of P-wave AVO gradients to invert for fracture parameters.

P-P REFLECTIONS IN VTI MEDIA

The first step away from simple, isotropic models is to consider a transversely isotropic medium with a vertical axis of symmetry (VTI). One can illustrate this symmetry type with a horizontal stack of layers such as shown in Figure 1. It is obvious that the AVO response in VTI media is azimuthally independent, i.e., for a given angle of incidence, the reflection amplitude does not vary with azimuth (measured with respect to the x_1-axis). This is true for both isotropic and VTI-overburden. Furthermore, one has to consider only the coupling of P- and SV-waves at the interface. The third wave type, the SH-wave, is polarized perpendicular to the incidence plane and is not excited by incident P- and SV-waves. Thus, the continuity requirements at a plane horizontal interface lead to a system of four boundary conditions that can be solved for the transmission and reflection coefficients of the P- and SV-waves.

Approximate solution

Exact algebraic solutions to this problem have been developed, for example, by Daley and Hron (1977) and Graebner (1992). The inverse problem of estimating medium parameters from the angular changes of the reflection response represents a much more difficult problem. Koefoed (1955) went through the laborious exercise of numerically investigating reflection coefficients for many different sets of elastic, isotropic parameters. Extending this approach to anisotropic media is not feasible. Instead, it is more helpful to derive approximate scattering coefficients to learn about the influence of the anisotropy parameters on the reflection response.

To examine the contribution of the jumps in velocities, densities and anisotropy on the reflection amplitudes, I derive an approximate solution to the P-P-reflection coefficient similar to the solution derived in Thomsen (1993). The main assumptions [that are also intrinsic to other classical approximations of the isotropic reflection coefficients (e.g., Aki and Richards, 1980; Shuey, 1985)] are that of small discontinuities in elastic properties across the boundary and precritical angles of incidence; additionally the geologically reasonable assumption of weak anisotropy is used. The vertical P-wave velocity of the overburden VTI layer can be described as

$$V_{P0_1} = \bar{V}_{P0}\left(1 - \frac{1}{2}\frac{\Delta V_{P0}}{\bar{V}_{P0}}\right), \qquad (1)$$

while the vertical P-wave velocity of the lower VTI medium can be represented as

$$V_{P0_2} = \bar{V}_{P0}\left(1 + \frac{1}{2}\frac{\Delta V_{P0}}{\bar{V}_{P0}}\right). \qquad (2)$$

Corresponding expressions are valid for V_{S0_1}, V_{S0_2}, ρ_1, and ρ_2. V_{S0} and ρ denote the S-wave vertical velocity and density, respectively. Subscripts 1 and 2 refer to the media above and below the reflector. The "bar" terms indicate average quantities and Δ denotes the differences between lower and upper medium parameters. The anisotropy in the two layers is described using anisotropy coefficients ϵ, δ and γ, as suggested by Thomsen (1986). The assumptions of small discontinuities in elastic parameters and weak anisotropy hence translate into

$$\left|\frac{\Delta V_{P0}}{\bar{V}_{P0}}\right|, \left|\frac{\Delta V_{S0}}{\bar{V}_{S0}}\right|, \left|\frac{\Delta\rho}{\bar{\rho}}\right|, |\delta|, |\epsilon|, |\gamma| \ll 1; \qquad (3)$$

in practice, the approximation is acceptable if the above small terms are smaller than 0.2 (see Thomsen, 1986, 1993).

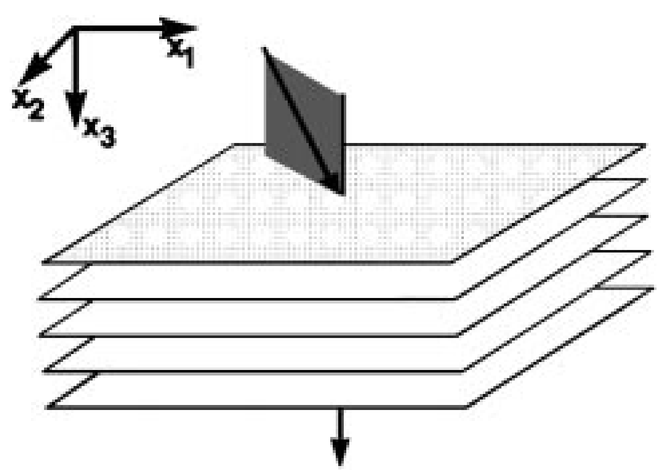

FIG. 1. Sketch of a VTI model. Note that there is no azimuthal variation in AVO response at interfaces between VTI media.

I used the symbolic calculation software *Mathematica* to derive the following solution for the approximate P-P reflection coefficient (Appendix):

$$R_P^{VTI}(\theta) = \frac{1}{2}\frac{\Delta Z}{\bar{Z}} + \frac{1}{2}\left\{\frac{\Delta V_{P0}}{\bar{V}_{P0}} - \left(\frac{2\bar{V}_{S0}}{\bar{V}_{P0}}\right)^2 \frac{\Delta G}{\bar{G}} + \Delta\delta\right\}\sin^2\theta + \frac{1}{2}\left\{\frac{\Delta V_{P0}}{\bar{V}_{P0}} + \Delta\epsilon\right\}\sin^2\theta\tan^2\theta, \quad (4)$$

where θ denotes the incident phase angle, $Z = \rho V_{P0}$ is the vertical P-wave impedance, and $G = \rho V_{S0}^2$ denotes the vertical shear modulus. The differences in anisotropy across the boundary are written as $\Delta\delta = (\delta_2 - \delta_1)$, $\Delta\epsilon = (\epsilon_2 - \epsilon_1)$.

If all anisotropic parameters (ϵ_j and δ_j, $j = 1, 2$) are identically zero, equation (4) reduces to an expression equivalent to Shuey's approximation for the isotropic reflection coefficient (Shuey, 1985). Equation (4) differs from Thomsen's result (Thomsen, 1993) in that here, the difference in anisotropy parameter $\Delta\delta$ does not appear in the $\sin^2\theta\tan^2\theta$-term. $\Delta\delta$ enters the $\sin^2\theta$ term and hence describes the influence of anisotropy on the small-angle reflection coefficient and the AVO slope, while $\Delta\epsilon$ is responsible for the $\sin^2\theta\tan^2\theta$ term and hence is more dominant at larger incidence angles. This is a manifestation of the well-known fact that δ controls the influence of anisotropy on near-vertically traveling P-waves, while ϵ dominates near-horizontal wave propagation.

Equation (4) was discussed in detail in Thomsen (1993) and Blangy (1994), but with the wrong $\sin^2\theta\tan^2\theta$-term. Note that the presence of $\Delta\delta$ in Thomsen's result may cause inaccuracies at large angles; specifically, for nonzero values of $\Delta\delta$, Thomsen's approximation does not work well for angles larger than 20° and will break down for angles greater than 45° where the third term dominates over the second one.

Figure 2 illustrates the accuracy of equation (4) by comparing the approximate reflection coefficient with the exact VTI solution and the corresponding exact isotropic reflection coefficient (assuming that $\epsilon_2 = \delta_2 = 0$). As throughout this paper, velocities and densities are hereby given in units of km/s and g/cm^3, respectively. This example shows that the presence of anisotropy can severely distort or even reverse the variation of the reflection coefficient with incidence angle.

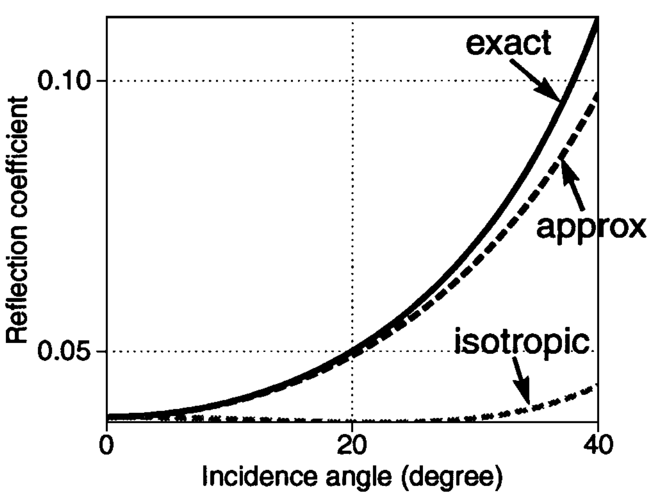

FIG. 2. Reflection coefficient for VTI media. Shown are the exact VTI solution (solid line), the approximation based on equation (4), and the corresponding isotropic ($\delta_2 = 0$, $\epsilon_2 = 0$) reflection coefficient. Model parameters are $V_{P0_1} = 2.9$, $V_{S0_1} = 1.8$, $\rho_1 = 2.18$, $\epsilon_1 = 0$, $\delta_1 = 0$, $V_{P0_2} = 3.1$, $V_{S0_2} = 1.85$, $\rho_2 = 2.2$, $\epsilon_2 = 0.1$, and $\delta_2 = 0.2$.

P-*P* REFLECTIONS IN HTI MEDIA

Most upper-crustal media and certain reservoir rocks of interest to hydrocarbon exploration show azimuthal anisotropy of various types and strength (Crampin, 1985), implying that the angular dependence of reflection coefficients varies with azimuth. The "first-order" model for azimuthal anisotropy is transverse isotropy with a horizontal axis of symmetry (HTI). Common physical reasons for a medium of HTI symmetry are a sequence of dipping shale sequences and/or a system of parallel vertical cracks embedded in an isotropic matrix, similar to the medium shown in Figure 3.

Let us assume that the symmetry axis of the HTI model is parallel to the x_1 direction. The $[x_1, x_3]$-plane that contains the symmetry axis is hereafter referred to as the "symmetry-axis plane." Waves confined to the plane normal to the symmetry axis do not experience any anisotropy; hence, this plane is the so-called "isotropy plane" or "fracture plane." Shear waves propagating in the isotropy plane can travel with two different velocities, depending on whether their polarization is confined to the isotropy plane or perpendicular to it.

Analytic expressions for reflection coefficients, either exact or approximate, are restricted mostly to VTI models (Daley and Hron, 1977; Banik, 1987, Graebner, 1992; Thomsen, 1993; Blangy, 1994). Studies of P-wave AVO signatures in HTI media have been of a purely empirical nature without an analytic study of the AVO response (e.g., Lefeuvre, 1994). The present work fills in this gap by complementing exact numerical results with concise analytic expressions. First, I show numerically evaluated, exact reflection coefficients for various azimuths. I then discuss the angular dependence of P-P reflection coefficients in the isotropy plane and, subsequently, exploit analogies between VTI and HTI media to extend the analysis for VTI media to AVO responses in the HTI symmetry-axis plane.

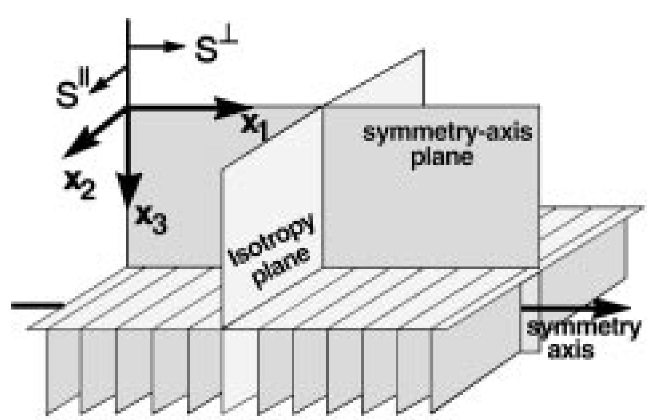

FIG. 3. Sketch of an HTI model. Reflections of P-waves confined to the two vertical symmetry planes, here called the *symmetry-axis* plane and the *isotropy* plane, are discussed in the text. Shear waves polarized parallel and normal to the isotropy plane have different vertical velocities.

Exact solution for HTI media

A plane P-wave incident on a HTI medium outside the symmetry-axis and isotropy plane generates three plane waves with mutually orthogonal polarization directions in the lower medium:

1) A P-wave polarized approximately along its propagation direction.
2) A shear wave polarized within the isotropy plane, referred to as $S^{\parallel}$.
3) A shear wave polarized in the plane formed by the slowness vector and the symmetry axis, here called the $S^{\perp}$-wave (the slow mode in the vertical direction).

Boundary conditions have to be applied to solve for the reflection and transmission coefficients. These conditions are the continuity of particle displacement and of shear and normal traction. After evaluating vertical slownesses, polarizations, and phase velocities for all generated wave types, the boundary conditions yield a system of six linear equations for the reflection/transmission coefficients.

Based on this approach, I implemented an algorithm to compute the exact reflection and transmission coefficients for interfaces between two HTI media with the same direction of the symmetry axis. In Figure 4, reflection-coefficient curves are shown as a function of incidence angle, for azimuths of 0°, 30°, 60°, and 90° measured with respect to the x_1-axis. In this experiment, the upper medium is purely isotropic. V_{P0_2} and V_{S0_2} are symmetry-direction velocities, and anisotropy parameters δ_2, ϵ_2 and γ_2 are (generic) Thomsen parameters defined with respect to the horizontal symmetry axis of the second medium. The change between the absolute values, as well as the change of slope of the reflection coefficient, is significant among the individual azimuths. Only for normal incidence do the curves coincide.

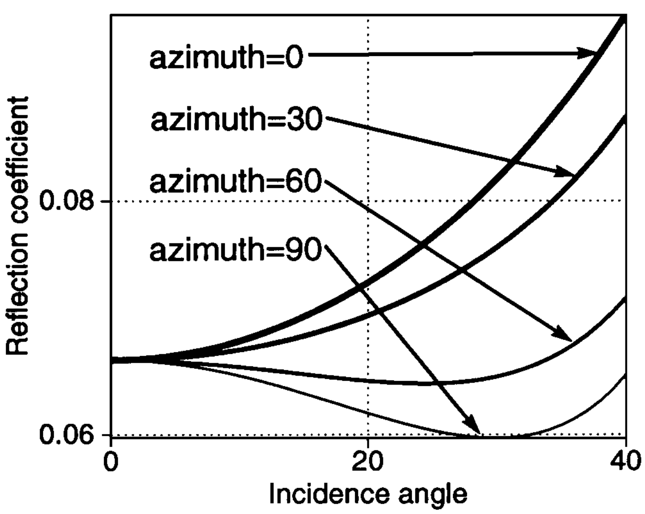

FIG. 4. Reflection coefficients as a function of incidence and azimuthal angle for a boundary between an isotropic overburden and an HTI medium. $V_{P0_1} = 2.26$, $V_{S0_1} = 1.428$, $\rho_1 = 2.6$, $\epsilon_1 = 0$, $\delta_1 = 0$, $\gamma_1 = 0$, $V_{P0_2} = 2.37$, $V_{S0_2} = 1.36$, $\rho_2 = 2.7$, $\epsilon_2 = 0.05$, $\delta_2 = 0.02$, and $\gamma_2 = 0.1$. V_{P0} and V_{S0} are symmetry-direction velocities, and anisotropy parameters δ_2, ϵ_2, and γ_2 are generic Thomsen parameters defined with respect to the horizontal symmetry axis of the second medium.

Clearly, evaluating reflection coefficients for HTI media is more complicated than for VTI media. However, as indicated in the above example (Figure 4), ignoring the presence of anisotropy in HTI media has the potential of severely distorting the conventional AVO analysis. On the other hand, a careful study of the reflection response in HTI media can provide additional information for extraction of physical properties of the medium.

In general, exact evaluation of the HTI reflection coefficient at arbitrary azimuth requires a numerical calculation. The next section describes how to obtain an analytic solution and useful approximations to reflection coefficients for incident P-waves confined to the isotropy and symmetry-axis planes. Comparison of these coefficients can then help in estimating the magnitude of azimuthal change and relating it to the anisotropy of the subsurface.

Exact and approximate solution for isotropy-plane reflections

Two different shear waves propagate in HTI media. An incident P-wave confined to the isotropy plane excites the $S^{\parallel}$-mode polarized within the fractures; i.e., this shear wave travels with a different speed than $S^{\perp}$-waves that would be generated by an incident P-wave confined to the symmetry-axis plane.

It is straightforward to compute exact P-P reflection coefficients in the isotropy plane using existing solutions for isotropic media. For example, one can use the exact analytic representation of the P-P reflection coefficient given in Aki and Richards (1980), by simply inserting the proper (fracture-plane) velocities.

In the following, it is important to note that because of the different velocities of $S^{\parallel}$ and $S^{\perp}$ waves, several different parameterizations are available to represent approximate solutions to the reflection problem in HTI media. To avoid any confusion, I explicitly state the meaning of the parameters used in the remainder of this paper by relating them to the stiffness elements [in Voigt notation, see for example, Musgrave (1970)], defined for the coordinate frame shown in Figure 3, e.g., the symmetry axis is pointing in the x_1-direction:

$$\begin{aligned} \beta &= \sqrt{\frac{c_{44}}{\rho}}, \\ \alpha &= \sqrt{\frac{c_{33}}{\rho}}, \\ G &= \rho\beta^2, \\ Z &= \rho\alpha. \end{aligned} \tag{5}$$

Here, β and α are the isotropy-plane velocities of the $S^{\parallel}$-wave and the compressional wave, respectively. G denotes shear modulus and Z denotes P-wave impedance. Note that β in equation (5) refers to the fast vertical shear-wave velocity and, in general, is different from V_{S0}, the velocity of shear waves propagating in the symmetry-axis direction.

Now we can state the approximate solution for the P-P reflection coefficient in the isotropy plane. Using the parameterization in equation (5), we find

$$R_P^{iso}(\theta) = \frac{1}{2}\frac{\Delta Z}{\bar{Z}} + \frac{1}{2}\left\{\frac{\Delta\alpha}{\bar{\alpha}} - \left(\frac{2\bar{\beta}}{\bar{\alpha}}\right)^2 \frac{\Delta G}{\bar{G}}\right\}\sin^2\theta + \frac{1}{2}\left\{\frac{\Delta\alpha}{\bar{\alpha}}\right\}\sin^2\theta\tan^2\theta. \tag{6}$$

In essence, equation (6) is just the classical approximate reflection coefficient for interfaces between isotropic media with the faster S-wave vertical velocity.

Analogy between VTI media and the symmetry-axis plane of HTI models

Consider the VTI and HTI models shown in Figure 5. As before, the symmetry axis of the VTI model points in the x_3-direction, whereas the symmetry axis of the HTI model coincides with the x_1-axis. In both cases, five independent components of elastic stiffness exist, and if the physical cause of anisotropy is known, some of the components may be related. For the given coordinate system with the x_3-axis pointing downwards, we can represent the elastic stiffnesses in terms of symmetric 6×6 matrices $\underset{\sim}{\mathbf{C}}_{VTI}$ and $\underset{\sim}{\mathbf{C}}_{HTI}$ (Musgrave, 1970):

$$\underset{\sim}{\mathbf{C}}_{VTI} = \begin{pmatrix} c_{11} & (c_{11}-2c_{66}) & c_{13} & & & \\ (c_{11}-2c_{66}) & c_{11} & c_{13} & & & \\ c_{13} & c_{13} & c_{33} & & & \\ & & & c_{55} & & \\ & & & & c_{55} & \\ & & & & & c_{66} \end{pmatrix} \tag{7}$$

$$\underset{\sim}{\mathbf{C}}_{HTI} = \begin{pmatrix} c_{11} & c_{13} & c_{13} & & & \\ c_{13} & c_{33} & (c_{33}-2c_{44}) & & & \\ c_{13} & (c_{33}-2c_{44}) & c_{33} & & & \\ & & & c_{44} & & \\ & & & & c_{55} & \\ & & & & & c_{55} \end{pmatrix}. \tag{8}$$

Although both models are of the same symmetry class, the $\underset{\sim}{\mathbf{C}}_{VTI}$ and $\underset{\sim}{\mathbf{C}}_{HTI}$ stiffness matrices differ because of the different directions of the symmetry axes. For example, c_{55} is identical to c_{44} in VTI models while c_{55} equals c_{66} in HTI models.

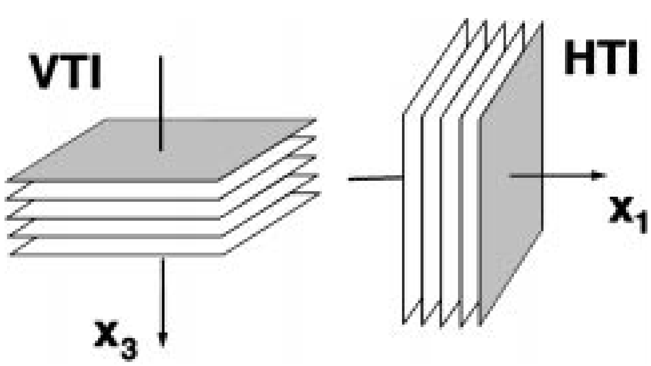

FIG. 5. The analogy between VTI and HTI models helps to extend solutions for VTI reflection coefficients to HTI media.

Propagation of body waves in elastic media is governed by the Christoffel equation, here shown in $c_{ijk\ell}$-tensor notation with slowness components p_i as

$$\det|c_{ijk\ell}\, p_j\, p_\ell - \rho| = 0, \tag{9}$$

where the Einstein summation rule of repeated indices applies. Inserting $\underset{\sim}{\mathbf{C}}_{VTI}$ and $\underset{\sim}{\mathbf{C}}_{HTI}$ into equation (9) generally yields two different solutions for waves propagating in VTI and HTI media. However, the key observation for this study is that equation (9), applied to propagation in the [x_1, x_3]-plane, is identical for both VTI and HTI models. Specifically, using the Voigt recipe to compact the indices of the fourth-order tensor $c_{ijk\ell}$ (Musgrave, 1970), equation (9) then yields

$$\det \begin{vmatrix} c_{11}p_1^2 + c_{55}p_3^2 - \rho & (c_{13}+c_{55})p_1p_3 \\ (c_{13}+c_{55})p_1p_3 & c_{33}p_3^2 + c_{55}p_1^2 - \rho \end{vmatrix} = 0 \tag{10}$$

for P- and SV-wave propagation in media of either VTI or HTI symmetry.

Equation (10) provides the solution for the velocities and the polarization direction of P- and $S^{\perp}$-wave propagation in the [x_1, x_3]-plane. Thus, all equations describing velocities, traveltime, polarization, and stresses for waves propagating in the [x_1, x_3]-plane are identical for media with the symmetry axis pointing in either the x_1 (HTI) or the x_3 (VTI) direction!

In other words, for any HTI model there exists an "equivalent" VTI model that has the same kinematic properties and polarizations of P- and $S^{\perp}$-waves in the [x_1, x_3]-plane. Thus, P- and $S^{\perp}$-wave propagation in the [x_1, x_3]-plane (i.e., the symmetry-axis plane) of HTI media can be described by the known VTI equations using the elastic stiffness components c_{ij} or, alternatively, using Thomsen's parameters $\epsilon^{(V)}$ and $\delta^{(V)}$ defined with respect to vertical; i.e., they are different from the generic coefficients defined with respect to the horizontal symmetry axis:

$$\begin{aligned} \epsilon^{(V)} &\equiv \frac{c_{11}-c_{33}}{2c_{33}} \\ \delta^{(V)} &\equiv \frac{(c_{13}+c_{55})^2-(c_{33}-c_{55})^2}{2c_{33}(c_{33}-c_{55})}. \end{aligned} \tag{11}$$

The coefficients of the equivalent VTI model $\epsilon^{(V)}$ and $\delta^{(V)}$ can be expressed through the generic Thomsen parameters ϵ and δ as defined with respect to the horizontal symmetry axis:

$$\begin{aligned} \epsilon^{(V)} &= -\frac{\epsilon}{1+2\epsilon} \\ \delta^{(V)} &= \frac{\delta - 2\epsilon[1+(\epsilon/f)]}{(1+2\epsilon)[1+(2\epsilon/f)]} \end{aligned} \tag{12}$$

where

$$f \equiv 1 - (V_{S0}/V_{P0})^2,$$

as introduced in Tsvankin (1996). Both V_{S0} and V_{P0} are measured along the horizontal symmetry axis.

For completeness, it should be mentioned that the discussion above extends naturally to $S^{\parallel}$-wave propagation in the

$[x_1, x_3]$-plane of HTI media. Thomsen's parameter $\gamma^{(V)}$ of the equivalent VTI model is defined as

$$\gamma^{(V)} = \frac{c_{66} - c_{44}}{2c_{44}} \tag{13}$$

and is related to the generic parameter γ defined with respect to the horizontal symmetry axis as

$$\gamma^{(V)} = -\frac{\gamma}{1 + 2\gamma}. \tag{14}$$

Exact solution for HTI symmetry-plane reflections

As stated above, solutions for propagation properties of P- and $S^{\perp}$-waves in the $[x_1, x_3]$-plane are valid for both VTI and HTI media. Therefore, all information required to determine P-P reflection coefficients is available without any extra derivation. The exact solution for the VTI reflection problem is valid for reflections in HTI media, as long as the incident P-wave is confined to the symmetry-axis plane. Furthermore, this solution is also valid for interfaces between VTI and HTI media, no matter whether the HTI medium is above or below the boundary. Exact numerical algorithms designed to evaluate reflection and transmission coefficients in VTI media can be used without any modification to compute the reflection response of waves propagating in the symmetry-axis plane of HTI models. For example, algorithms based on Graebner's (1992) exact VTI solution for the reflection/transmission coefficients (stated in terms of density normalized stiffness components) can be related to the symmetry-axis plane reflection by simply inserting the correct stiffness matrices.

One note of caution. The analogy between VTI and HTI media as stated in this work does *not* imply that the value of the reflection coefficients, phase velocities, or polarization is identical for VTI and HTI-symmetry-axis plane propagation if the HTI medium is obtained by a 90° rotation of the VTI model. The "equivalent" VTI model introduced above does *not* result from a rotation of the HTI medium but simply denotes that this VTI medium has the identical propagation properties for waves traveling in the $[x_1, x_3]$-plane as the corresponding HTI model.

Approximate solutions for symmetry-axis-plane reflections

The principle of relating properties in the symmetry-axis plane of HTI media with those of VTI media can be applied to equation (4), the approximate solution for VTI reflection coefficients. The approximate P-wave reflection coefficient for waves incident in the symmetry-axis plane is given by

$$\begin{aligned} R_P^{\text{sym}}(\theta) &= \frac{1}{2}\frac{\Delta Z}{\bar{Z}} \\ &\quad + \frac{1}{2}\left\{\frac{\Delta\alpha}{\bar{\alpha}} - \left(\frac{2\bar{\beta}^{(V)}}{\bar{\alpha}}\right)^2 \frac{\Delta G^{(V)}}{\bar{G}^{(V)}} + \Delta\delta^{(V)}\right\}\sin^2\theta \\ &\quad + \frac{1}{2}\left\{\frac{\Delta\alpha}{\bar{\alpha}} + \Delta\epsilon^{(V)}\right\}\sin^2\theta\tan^2\theta. \end{aligned} \tag{15}$$

$\delta^{(V)}$ and $\epsilon^{(V)}$ are introduced in equation (11) and the superscripts in $\beta^{(V)}$ and $G^{(V)}$ indicate that these quantities correspond to the shear wave confined to the symmetry-axis plane (the SV-wave in the equivalent VTI model).

The main reason for this analytic study, however, is an improved understanding of the azimuthal changes in the reflection response. In that case, it is important to relate the shear modulus $G^{(V)}$ and the shear velocity $\beta^{(V)}$ of the equivalent VTI model to the properties in the isotropy plane [equation (5)]. As mentioned above, this is necessary because in the symmetry-axis plane, the P-wave is coupled with the $S^{\perp}$-wave which has a velocity that differs from the velocity of the shear wave confined to the isotropy plane. The difference between $S^{\parallel}$- and $S^{\perp}$-wave vertical speeds can be expressed using parameter $\gamma^{(V)}$ or, equivalently, the shear-wave splitting parameter γ. Here, I choose the latter because of its importance in shear-wave birefringence studies. It then follows that

$$\begin{aligned} \beta^{(V)} &= \beta(1 - \gamma) \\ G^{(V)} &= G(1 - 2\gamma). \end{aligned} \tag{16}$$

Using this set of parameters, equation (4) can be rewritten for the symmetry-axis plane of an HTI medium as

$$\begin{aligned} R_P^{\text{sym}}(\theta) &= \frac{1}{2}\frac{\Delta Z}{\bar{Z}} \\ &\quad + \frac{1}{2}\left\{\frac{\Delta\alpha}{\bar{\alpha}} - \left(\frac{2\bar{\beta}}{\bar{\alpha}}\right)^2\left(\frac{\Delta G}{\bar{G}} - 2\Delta\gamma\right) + \Delta\delta^{(V)}\right\} \\ &\quad \times \sin^2\theta + \frac{1}{2}\left\{\frac{\Delta\alpha}{\bar{\alpha}} + \Delta\epsilon^{(V)}\right\}\sin^2\theta\tan^2\theta, \end{aligned} \tag{17}$$

where θ again denotes the incident phase angle. This equation is valid for general HTI media. Specifically, the derivation of equation (17) did not assume that the anisotropy has been caused by vertically aligned cracks. [Thomsen (1995) has shown that the anisotropy coefficients $\delta^{(V)}$, $\epsilon^{(V)}$, and γ are not independent in the case of fracture-induced anisotropy.]

The most important message of equation (17) is that the contrasts in the anisotropy parameters $\delta^{(V)}$ and γ have a non-negligible, first-order influence on the angular dependence of the reflection coefficient. In fact, if we approximate $\bar{\beta}/\bar{\alpha} \approx 1/2$, the contrast in γ is twice as important as the P-wave contrast $\Delta\alpha/\bar{\alpha}$ in the gradient ($\sin^2\theta$)-term. This raises hopes that a proper inversion procedure may be able to extract these parameters from the AVO-response in the symmetry-axis plane or from the azimuthal variation in the reflection coefficient.

Equation (17) is valid for precritical incidence on an interface between two weakly anisotropic HTI media with the same symmetry-axis direction and small jumps in the elastic properties across the boundary. Before estimating the accuracy of equation (17), a question to be answered is what values of $\epsilon^{(V)}$, $\delta^{(V)}$, and γ are indeed physically reasonable. In field and laboratory experiments, γ has been positive and generally is much smaller than one ($1 \gg \gamma > 0$). In rocks with negligible equant porosity and very thin cracks $\epsilon^{(V)} = 0$; otherwise, $\epsilon^{(V)}$ is small and negative ($-1 \ll \epsilon^{(V)} < 0$) (Tsvankin, 1995). On the other hand, $\delta^{(V)}$ can be either positive or negative.

Equation (17) is linearized in nine small quantities $\Delta\alpha/\bar{\alpha}$, $\Delta Z/\bar{Z}$, etc. A total of 45 unknown quadratic terms are dropped in the derivation, and it is not clear how the accuracy of the approximation depends on the medium parameters. In particular, it is of great interest to study the accuracy of equation (17) for different values of anisotropy parameters $\epsilon^{(V)}$,

$\delta^{(V)}$, and γ. Figures 6 and 7 show the reflection coefficients evaluated at boundaries between isotropic and HTI media for incidence angles up to 40°. Approximations are shown for both symmetry-axis-plane and isotropy-plane reflections, together with the exact solutions. The model parameters are listed in Table 1. The same vertical velocities $\alpha_2 = 2.5$ km/s, $\beta_2 = 1.5$ km/s [equation (5)], and a density of 2.7 g/cm^3 in the lower media are used in these examples. In Figure 6, the small elastic parameters $\Delta\alpha/\bar{\alpha}$, $\Delta Z/\bar{Z}$, and $\Delta G/\bar{G}$ are positive; in Figure 7, they are all negative. To perform a representative test of equation (17) and to study the sensitivity of the approximation with respect to the individual parameters, several of the examples shown use only one nonzero anisotropic parameter. This is done to test the approximations for general HTI media; in the particular case of fracture-induced anisotropy, the anisotropy coefficients $\epsilon^{(V)}$, $\delta^{(V)}$, and γ are not independent.

Recall that the isotropy-plane approximation is identical to the conventionally used approximation for P-wave reflections

Table 1. Models used to test the accuracy of equations (17) and (6) and to study the sensitivity of the approximate reflection coefficients. The upper medium is isotropic, the lower medium has the following parameters: α_2 = 2.5, β_2 = 1.5, and ρ_2 = 2.7. α_2 denotes the compressional vertical velocity, β_2 the faster shear-wave velocity, and ρ_2 the density.

	$\Delta\alpha/\bar{\alpha}$	$\Delta Z/\bar{Z}$	$\Delta G/G$	$\delta^{(V)}$	$\epsilon^{(V)}$	γ
Model 1	0.1	0.1	0.1	−0.1	0	0
Model 2	0.1	0.1	0.1	0.1	0	0
Model 3	0.1	0.1	0.1	0	−0.1	0
Model 4	0.1	0.1	0.1	0	0	0.1
Model 5	0.1	0.1	0.1	0.1	−0.1	0.1
Model 6	0.1	0.1	0.1	−0.1	−0.1	0.1
Model 7	−0.1	−0.1	−0.1	−0.1	0	0
Model 8	−0.1	−0.1	−0.1	0.1	0	0
Model 9	−0.1	−0.1	−0.1	0	−0.1	0
Model 10	−0.1	−0.1	−0.1	0	0	0.1
Model 11	−0.1	−0.1	−0.1	−0.1	−0.1	0.1
Model 12	−0.1	−0.1	−0.1	0.1	−0.1	0.1

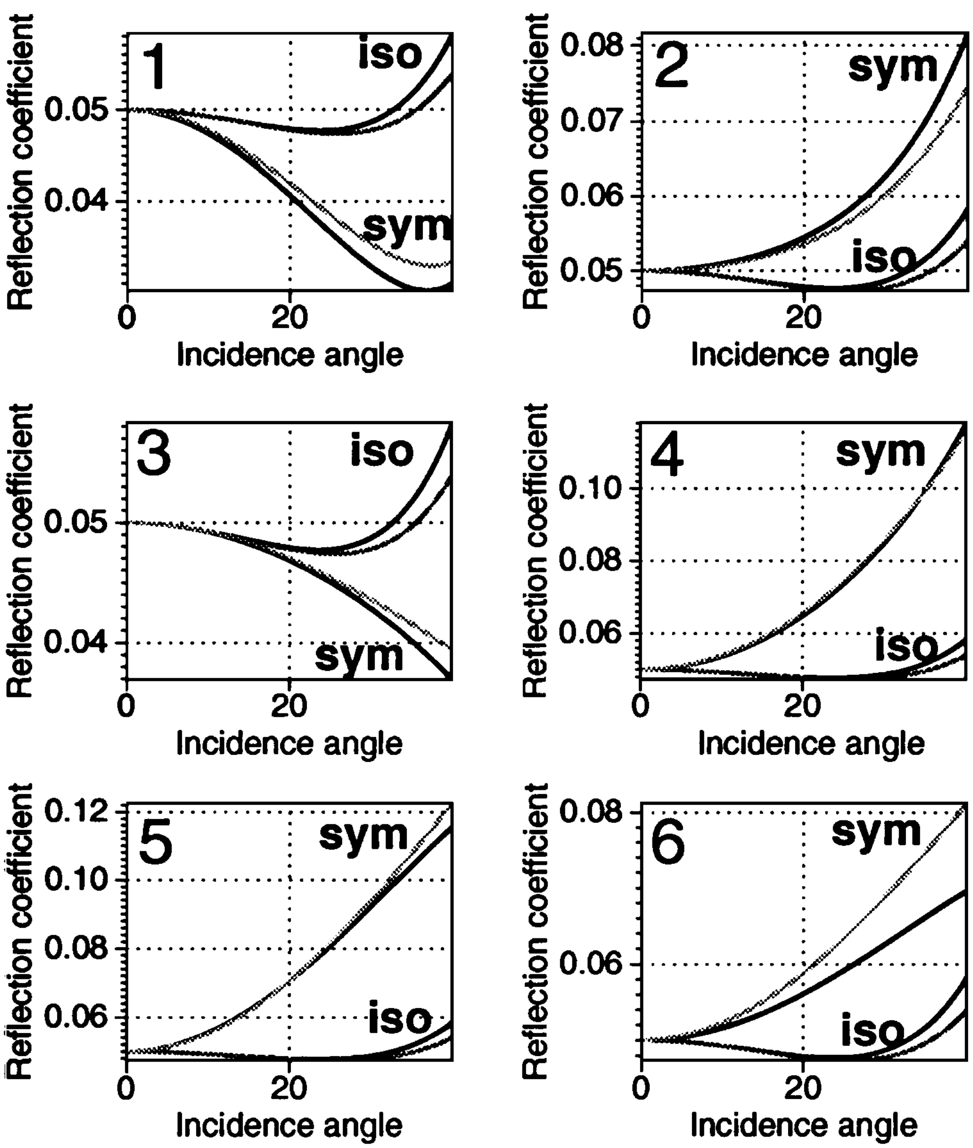

FIG. 6. Reflection coefficient for an isotropic layer overlying an HTI medium. Shown are the exact solutions (black) and the approximations based on equation (17) (gray) for both symmetry-axis-plane (sym) and isotropy-plane (iso) reflections. Table 1 lists the model parameters. $\Delta\alpha/\bar{\alpha}$, $\Delta Z/\bar{Z}$, $\Delta G/\bar{G}$ are positive for all these tests.

in isotropic media. Note that the accuracy for the symmetry-axis-plane approximation (17) is generally lower than that for the isotropy plane. This is not a physical phenomenon, but is caused by the design of the approximation; specifically, the $S^{\perp}$-wave velocity is approximated as a function of β and γ. If accuracy is the main concern in the study, one should use equation (15) or (preferably) compute the exact reflection coefficient as discussed above. The main contribution of the approximations shown above is to establish a physical foundation for (exact) numerical inversion algorithms and highlight simple dependencies that interpreters can use to quickly evaluate the importance of anisotropy in a particular play.

Further analysis of reflections and transmission coefficients

I performed similar investigations for the reflection and transmission coefficients for all pure and converted modes in VTI media and symmetry planes of HTI and orthorhombic models. A collection of explicit approximate expressions is beyond the scope of this paper and is given in Rüger (submitted for publication). Another question of interest concerns the approximate P-wave coefficients for arbitrary azimuth with respect to the symmetry-axis. The approximation for both reflection and transmission coefficients as a function of incidence phase angle and azimuth have been derived in Rüger (1996) and shows that the AVO gradient varies smoothly between the two vertical symmetry planes. The azimuthal variation of the higher-angle term ($\sin^2\theta \tan^2\theta$-term) can also be described by simple trigonometric equations.

DISCUSSION AND CONCLUSIONS

Linearized approximations are helpful for obtaining simple representations of otherwise incomprehensibly complex reflection coefficients in anisotropic media. The main assumptions

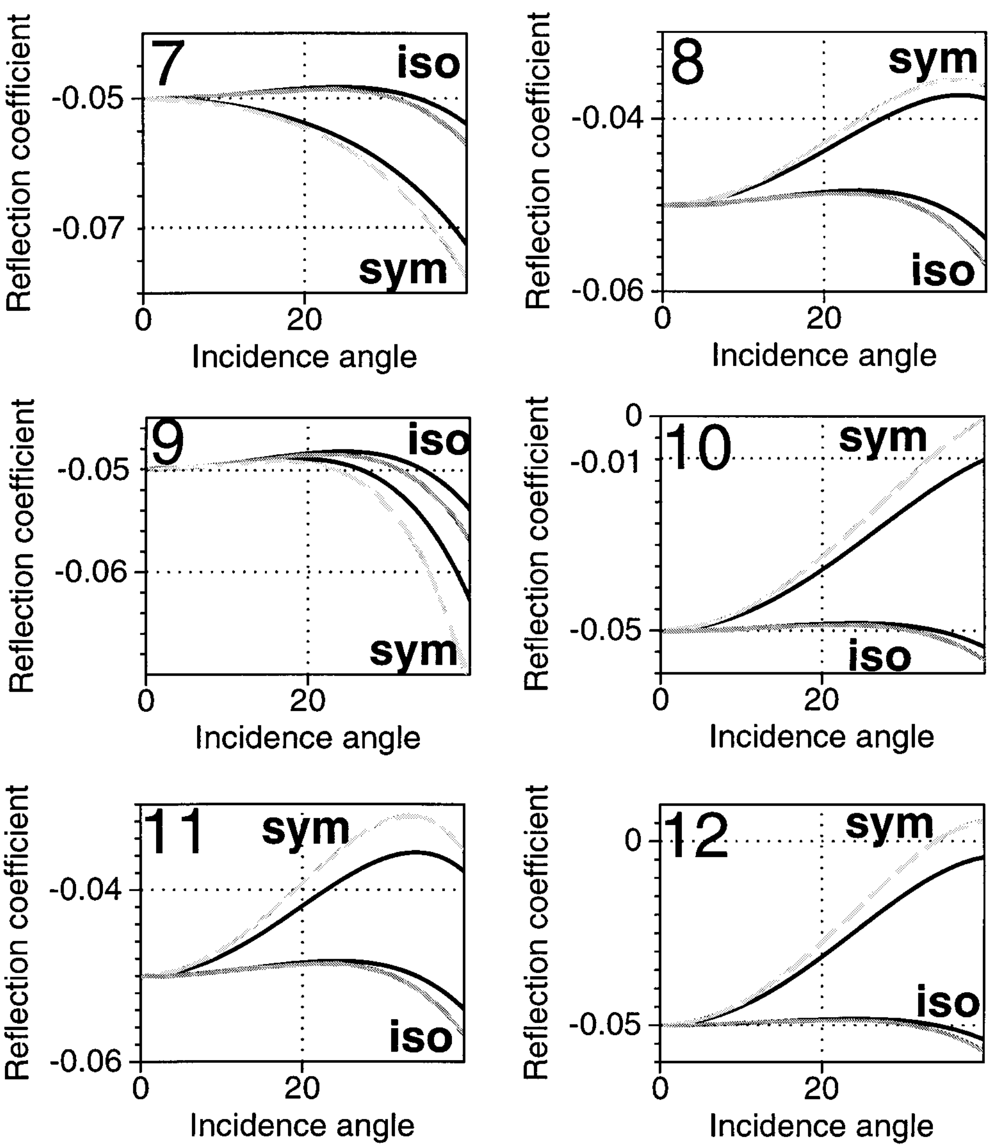

FIG. 7. Same as Figure 6, except that here, the contrast parameters $\Delta\alpha/\bar{\alpha}$, $\Delta Z/\bar{Z}$, $\Delta G/\bar{G}$ are negative. Table 1 lists the model parameters.

in this reflection-coefficient study—small jumps in the elastic parameters across the reflecting interface, precritical incidence and weak anisotropy—are geologically and geophysically reasonable and have proved useful in many exploration contexts.

A refined "Shuey-type" approximate reflection coefficient in VTI media gives a concise expression for the AVO-gradient term and the higher-angle term as a function of Thomsen's (1986) parameters. For small incidence angles, the solution presented here is identical to previously published equations in the literature, but it is more accurate for large angles ($>$20–25°) and a nonzero difference in Thomsen's parameter δ across the reflecting boundary.

The refined approximation for reflection coefficients in VTI media can be related to the approximate solutions for P-P reflection coefficients in transversely isotropic models with a horizontal axis of symmetry (HTI). The derivation is based on a limited analogy between VTI and HTI media that is not only helpful in the evaluation of reflection coefficients, but can also be used to determine kinematic properties and polarizations in HTI media. Furthermore, the relationship between wave propagation in VTI media and the vertical plane containing the symmetry axis (the so-called symmetry-axis plane) of HTI media allows the use of already known exact VTI reflection coefficients for the reflection coefficients in the symmetry-axis plane of HTI media. For example, existing numerical algorithms for VTI media, such as the one proposed in Graebner (1982) can also be used to compute the exact reflection coefficient efficiently in the symmetry-axis plane of HTI models. Probably the most important application of the VTI/HTI analogy is the introduction of new effective "Thomsen-style" anisotropy parameters for HTI media. Tsvankin (1995), for example, shows that the parameter $\delta^{(V)}$ entering the AVO gradient term for HTI symmetry-axis plane reflections also describes the variation of normal moveout velocities in HTI media.

The approximate reflection coefficient expressions derived in this paper relate the AVO response to the anisotropy parameters and provide physical insight into the reflection amplitudes of P-waves at boundaries of media with HTI symmetry. In HTI models, the reflection coefficient becomes dependent on azimuthal direction and the magnitude of the azimuthal change in the AVO gradient is a function of the shear-wave splitting parameter (Thomsen's parameter γ) and the anisotropy parameter describing P-wave anisotropy for near-vertical propagation in the symmetry-axis plane.

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APPENDIX A

DERIVATION OF APPROXIMATE REFLECTION COEFFICIENTS IN VTI MEDIA

Here, I derive an approximate solution to the P-P reflection coefficient similar to the solution given in Thomsen (1993). I assume a weakly anisotropic media with small discontinuities in elastic properties across the reflecting boundary. Specifically, let us introduce seven small quantities:

$$d_1 = \frac{\Delta V_{P0}}{\bar{V}_{P0}}, \qquad d_2 = \frac{\Delta V_{S0}}{\bar{V}_{S0}}, \qquad d_3 = \frac{\Delta \rho}{\bar{\rho}},$$
$$d_4 = \delta_1, \qquad d_5 = \epsilon_1, \qquad d_6 = \delta_2, \qquad d_7 = \epsilon_2,$$

with

$$|d_i| \ll 1\,(i = 1, 2, \ldots, 7),$$

and the operator

$$\boldsymbol{\Delta} = d_j \frac{\partial}{\partial d_j},$$

where the partial derivative is evaluated for $d_j = 0$ ($j = 1, 2, \ldots, 7$). As explained in the main text, the medium is described using small anisotropy coefficients and the average and the difference in elastic parameters. The average and difference in vertical P-wave velocity, for example, is defined as

$$\bar{V}_{P0} = 1/2\big(V_{P0_1} + V_{P0_2}\big),$$
$$\Delta V_{P0} = V_{P0_2} - V_{P0_1},$$

and correspondingly for $\bar{V}_{S0}$, ΔV_{S0}, etc.

Boundary conditions have to be applied to solve for the reflection and transmission coefficients. These conditions are the continuity of particle displacement and of shear and normal traction. After evaluating vertical slownesses, polarizations and phase velocities for all generated wave types, the boundary conditions yield a system of six linear equations for the reflection/transmission coefficients. In general, the system to be solved can be represented in the form

$$\underset{\sim}{\mathbf{A}}\mathbf{R} = \mathbf{b}, \tag{A-1}$$

where $\underset{\sim}{\mathbf{A}}$ is the boundary equation matrix formed by the scattered (reflected and transmitted) wave types, $\mathbf{R}$ is the vector of reflection and transmission coefficients, and $\mathbf{b}$ is composed of the contribution of the incident wave to the boundary conditions (see Aki and Richards, 1980; Thomsen, 1993).

Before trying to solve equation (A-1), let us consider a plane-wave incident at the same incidence angle upon the "average" (or "unperturbed") model, i.e., the model that results by setting all small quantities d_j to zero.

In this case, equation (A-1) can be written in the unperturbed form

$$\underset{\sim}{\mathbf{A}}^u\mathbf{R}^u = \mathbf{b}^u. \tag{A-2}$$

Obviously, $\underset{\sim}{\mathbf{A}}^u$, $\mathbf{R}^u$, and $\mathbf{b}^u$ are much more simple than $\underset{\sim}{\mathbf{A}}$, $\mathbf{R}$, and $\mathbf{b}$; for example, $\mathbf{R}^u$ has only one nonzero component. This suggests rewriting equation (A-1) in the following linearized form:

$$(\underset{\sim}{\mathbf{A}}^u + \boldsymbol{\Delta}\underset{\sim}{\mathbf{A}})(\mathbf{R}^u + \boldsymbol{\Delta}\mathbf{R}) = (\mathbf{b}^u + \boldsymbol{\Delta}\mathbf{b}). \tag{A-3}$$

The quantity of interest is $\boldsymbol{\Delta}\mathbf{R}$ which, to a first approximation, can be written as

$$\boldsymbol{\Delta}\mathbf{R} = (\underset{\sim}{\mathbf{A}}^u)^{-1}(\boldsymbol{\Delta}\mathbf{b} - \boldsymbol{\Delta}\underset{\sim}{\mathbf{A}}\mathbf{R}^u). \tag{A-4}$$

Using Cramer's rule, it is possible to find the inverse of $\underset{\sim}{\mathbf{A}}^u$. With the help of the symbolic calculation software *Mathematica*, I solved equation (A-4) and obtained the following solution for the most important element of $\boldsymbol{\Delta}\mathbf{R}$, the P-P reflection coefficient:

$$R_P^{VTI}(\theta) = \frac{1}{2}\frac{\Delta Z}{\bar{Z}} + \frac{1}{2}\left\{\frac{\Delta V_{P0}}{\bar{V}_{P0}} - \left(\frac{2\bar{V}_{S0}}{\bar{V}_{P0}}\right)^2\frac{\Delta G}{\bar{G}} + \Delta\delta\right\} \times \sin^2\theta + \frac{1}{2}\left\{\frac{\Delta V_{P0}}{\bar{V}_{P0}} + \Delta\epsilon\right\}\sin^2\theta\tan^2\theta, \tag{A-5}$$

where θ denotes the incident phase angle, $Z = \rho V_{P0}$ is the vertical P-wave impedance, and $G = \rho V_{S0}^2$ denotes the vertical shear modulus. The differences in anisotropy across the boundary are written as $\Delta\delta = (\delta_2 - \delta_1)$, $\Delta\epsilon = (\epsilon_2 - \epsilon_1)$. More details on this derivation can be found in Thomsen (1993).