GEOPHYSICS, VOL. 62, NO. 6 (NOVEMBER-DECEMBER 1997); P. 1855–1866, 6 FIGS.

Jack Cohen passed away last year before he could revise his originally submitted manuscript. His colleagues at the Colorado School of Mines were kind enough to take on this responsibility. Jack's own paper now honors his memory.

—Sven Treitel
Editor

Analytic study of the effective parameters for determination of the NMO velocity function in transversely isotropic media

Jack K. Cohen†

ABSTRACT

In their studies of transversely isotropic media with a vertical symmetry axis (VTI media), Alkhalifah and Tsvankin observed that, to a high numerical accuracy, the normal moveout (NMO) velocity for dipping reflectors as a function of ray parameter p depends mainly on just two parameters, each of which can be determined from surface P-wave observations. They substantiated this result by using the weak-anisotropy approximation and exploited it to develop a time-domain processing sequence that takes into account vertical transverse isotropy.

In this study, the two-parameter Alkhalifah-Tsvankin result was further examined analytically. It was found that although there is (as these authors already observed) some dependence on the remaining parameters of the problem, this dependence is weak, especially in the practically important regimes of weak to moderately strong transverse isotropy and small ray parameter. In each of these regimes, an analytic solution is derived for the anisotropy parameter η required for time-domain P-wave imaging in VTI media.

In the case of elliptical anisotropy ($\eta = 0$), NMO velocity expressed through p is fully controlled just by the zero-dip NMO velocity—one of the Alkhalifah-Tsvankin parameters. The two-parameter representation of NMO velocity also was shown to be exact in another limit—that of the zero shear-wave vertical velocity.

The analytic results derived here are based on new representations for both the P-wave phase velocity and normal moveout velocity in terms of the ray parameter, with explicit expressions given for the cases of vanishing on-axis shear speed, weak to moderate transverse isotropy, and small to moderate ray parameter. Using these formulas, I have rederived and, in some cases, extended in a uniform manner various results of Tsvankin, Alkhalifah, and others. Examples include second-order expansions in the anisotropy parameters for both the P-wave phase-velocity function and NMO-velocity function, as well as expansions in powers of the ray parameter for both of these functions. I have checked these expansions against the corresponding exact functions for several choices of the anisotropy parameters.

INTRODUCTION

Transversely isotropic models are often characterized by five parameters introduced in Thomsen (1986): the on-axis P- and S-phase velocities (V_{P0} and V_{S0}) and the anisotropy coefficients γ, δ, and ϵ. Focusing exclusively on P-wave data removes γ from consideration and reduces the number of relevant parameters to four. Remarkably, Alkhalifah and Tsvankin (1995) showed that with only two well-selected parameters and the ray parameter (a quantity that can be obtained from zero-offset seismic sections) as a running variable, it is possible to determine the P-wave NMO-velocity function in transversely isotropic media with a vertical symmetry axis (VTI) media to good numerical accuracy. Moreover, the two parameters selected can be determined from P-wave surface data alone. With the moveout function determined, they developed dip-moveout (DMO) and time-migration algorithms, which have performed well on synthetic and field data exhibiting transverse isotropy (Alkhalifah et al., 1996).

The two parameters selected by Alkhalifah and Tsvankin (1995) are the zero-dip moveout velocity $V_{\text{nmo}}(0)$ and a combination of the Thomsen anisotropy parameters denoted by

Manuscript received by the Editor December 14, 1995; revised manuscript received January 13, 1997.
†(deceased) Center for Wave Phenomena, Department of Mathematical and Computer Sciences, Colorado School of Mines, Golden, Colorado 80401-1887.

η. These authors discuss how to determine these parameters from P-wave surface data. The definitions of these quantities in terms of the Thomsen parameters are given below.

When the Alkhalifah-Tsvankin principal parameters $V_{\text{nmo}}(0)$ and η are adopted, the remaining two independent parameters can be taken to be the V_{S0}/V_{P0} ratio and one of the Thomsen P-wave anisotropy parameters, e.g., δ. Alkhalifah and Tsvankin observed that, in general, the moveout function, expressed as a function of the ray parameter, does exhibit some residual dependence on the two nonprincipal parameters (hereafter referred to as "residual parameters"). However, by doing extensive numerical tests, they found that just choosing reasonable values for these nonobservable (from P-wave reflection traveltimes) parameters provides a numerical NMO function that is highly accurate even when measured against moveout functions constructed with significantly different values for the residual parameters. It should be noted that the dependence of P-wave NMO velocity on reflector dip (or ray parameter) in nonelliptical VTI media strongly deviates from the corresponding isotropic function (Tsvankin, 1995; Levin, 1990).

There are special cases in which the two principal parameters suffice to determine the moveout function exactly. For example, in the elliptical limit ($\epsilon/\delta \to 1$), the moveout function expressed in terms of the ray parameter depends on only $V_{\text{nmo}}(0)$. This result, discussed in Alkhalifah and Tsvankin (1995), can be obtained from the NMO equations for elliptical anisotropy given in Byun (1982) and Uren et al. (1990). Although the elliptical case is a common analytic simplification, it is of limited practical application. Here, another such theoretical limit is introduced: $V_{S0}/V_{P0} \to 0$. It is once again shown that the resulting moveout function depends only on the two principal parameters.

More important for treating field data is the weak-anisotropy limit. This limit is characterized by dropping all quadratic and higher-order dependencies on δ and ϵ. For this limit, Alkhalifah and Tsvankin (1995) show that the moveout function expressed in terms of the ray parameter again depends on only the two principal parameters. I extend this expansion here to the case in which quadratic anisotropy parameters are retained but cubic and higher-order terms are dropped. This regime is designated moderate transverse isotropy and, of course, subsumes the regime of weak transverse isotropy. In the moderate regime, I show that there is dependence on the residual parameters. Thus, the numerical evidence of some residual dependence is quantified analytically. However, the critical result is that the residual dependence is confined to terms that (1) are relatively small and (2) vary smoothly with changes in the residual parameters. Thus, the two-parameter Alkhalifah-Tsvankin methodology is given further analytic support in the most common physical situations. That is, these results support the notion that supplying crude reasonable values for the two parameters that cannot be determined from surface P-wave data is sufficient to allow accurate determination of the moveout function and to justify the subsequent use of DMO and time-migration algorithms based on it.

This paper also gives analytic support for the two-parameter methodology for strong anisotropy when the phase angle—or, equivalently—the ray parameter is small. Again, the residual-parameter dependence is shown to be weak and smoothly varying, thus justifying the use of nominal values.

Finally, in both the moderate-anisotropy and the small-angle regimes, I derive analytic approximations for the anisotropy parameter η.

The analytic results derived here are based on obtaining explicit formulas for both P-wave phase velocity and NMO velocity in terms of the ray parameter. Using these formulas, various results of Tsvankin, Alkhalifah, and others are rederived and, in some cases, extended in a uniform manner. In particular, I derive second-order expansions in the anisotropy parameters for both the P-wave phase-velocity function and NMO-velocity function. Likewise, expansions in the powers of the ray parameter are derived for both of these functions. These expansions are checked against the corresponding exact functions for several choices of the anisotropy parameters.

Throughout this study, *Mathematica* was used extensively to derive and check results. In particular, the *Mathematica* package Thomsen.m (Cohen, 1995) was used to obtain various results in the limit of weak transverse isotropy. Similarly, the use of *Mathematica* facilitated computing symbolic derivatives, series expansions, etc.

PHASE VELOCITY AS A FUNCTION OF RAY PARAMETER

Begin with the formula for P-wave phase velocity in a homogeneous, transversely isotropic medium in terms of the phase angle with the symmetry axis θ (e.g., White, 1983),

$$\begin{aligned} 2\rho V^2(\theta) = {} & (C_{11} + C_{44})\sin^2\theta + (C_{33} + C_{44})\cos^2\theta \\ & + \left\{\left[(C_{11} - C_{44})\sin^2\theta - (C_{33} - C_{44})\cos^2\theta\right]^2 \right. \\ & \left. + 4(C_{13} + C_{44})^2 \sin^2\theta\cos^2\theta\right\}^{1/2}. \end{aligned} \tag{1}$$

Introduce horizontal slowness (ray parameter) p and vertical slowness m,

$$\begin{aligned} p &= \sin\theta/V(\theta), \\ m &= \cos\theta/V(\theta), \end{aligned} \tag{2}$$

to rewrite equation (1) as an equation for the slowness surface:

$$\begin{aligned} 2\rho = {} & (C_{11} + C_{44})p^2 + (C_{33} + C_{44})m^2 + \left\{\left[(C_{11} - C_{44})p^2 \right.\right. \\ & \left.\left. - (C_{33} - C_{44})m^2\right]^2 + 4(C_{13} + C_{44})^2 p^2 m^2\right\}^{1/2}. \end{aligned} \tag{3}$$

To obtain a formula for $V(p)$, follow the instructions given in Alkhalifah and Tsvankin (1995, Appendix A). Convert equation (3) to Thomsen notation and introduce the on-axis P and S velocities V_{P0} and V_{S0}. As a convenient parameter to characterize the V_{S0}/V_{P0} ratio, follow Tsvankin (1996) to define

$$f \equiv 1 - \frac{V_{S0}^2}{V_{P0}^2}. \tag{4}$$

Then the Thomsen-notation form of the slowness surface is

$$\begin{aligned} \frac{2}{V_{P0}^2} = {} & (2 - f)m^2 + (2 + 2\epsilon - f)p^2 \\ & + \sqrt{4f(2\delta + f)m^2p^2 + [f(p^2 - m^2) + 2\epsilon p^2]^2}. \end{aligned} \tag{5}$$

Next, solve equation (5) for m^2 and use equation (2) to write

$$V(p) = \frac{1}{\sqrt{p^2 + m^2(p)}}. \tag{6}$$

This program produces $V^2(p)$ in the form

$$V^2(p) = V_{P0}^2 \frac{A + \sqrt{B}}{2C}, \tag{7}$$

where

$$\begin{aligned} A &= 2 - f - 2(\epsilon - f\delta)z, \\ B &= f^2 - 4fk[\epsilon - (2 - f)\delta]z + 4[2f(1 - f)(\epsilon - \delta) \\ &\quad + (\epsilon - f\delta)^2]z^2, \\ C &= 1 - 2\epsilon z - 2f(\epsilon - \delta)z^2. \end{aligned} \tag{8}$$

The phase velocity is expressed in terms of the dimensionless parameter z as

$$z = V_{P0}^2 p^2. \tag{9}$$

Before turning to some important special cases, observe the following consequences of the definitions in equation (2):

$$\begin{aligned} \sin\theta &= pV(p), \\ \cos\theta &= mV(p) = \sqrt{1 - (pV)^2}, \\ p^2 + m^2 &= \frac{1}{V^2}. \end{aligned} \tag{10}$$

These equations are used to translate between representations in the phase angle and representations in the ray parameter.

On-axis velocity

Observe that for $p = 0\,(z = 0)$, the constants A, B, and C reduce to

$$\begin{aligned} A &= 2 - f, \\ B &= f^2, \\ C &= 1, \end{aligned} \tag{11}$$

so that

$$V^2(0) = V_{P0}^2 \frac{2 - f + f}{2} = V_{P0}^2, \tag{12}$$

leading to the expected result

$$V(0) = V_{P0}. \tag{13}$$

Elliptical anisotropy

In the elliptical case, $\delta = \epsilon$ and the constants A, B, and C reduce to

$$\begin{aligned} A &= 2 - f - 2(1 - f)\delta z, \\ B &= [f + 2(1 - f)\delta z]^2, \\ C &= 1 - 2\delta z, \end{aligned} \tag{14}$$

so that

$$V^2(p) = V_{P0}^2 \frac{2}{2(1 - 2\delta z)} = \frac{V_{P0}^2}{1 - 2\delta z}. \tag{15}$$

The well-known equivalent elliptical-limit value of V^2 in terms of the phase angle θ is given as

$$V^2(\theta) = V_0^2 \cos^2\theta + V_{90}^2 \sin^2\theta, \tag{16}$$

where $V_0 = V(0)$ and $V_{90} = V(\pi/2)$ in the phase-angle form of V given by equation (1). Likewise, equation (15) is equivalent to

$$V(\theta) = V_{P0}\sqrt{1 + 2\delta \sin^2\theta} \qquad (\epsilon = \delta), \tag{17}$$

which is given, for example, in Alkhalifah and Tsvankin (1995).

Zero shear-wave velocity

Another case in which the P-wave phase velocity takes a simple form is in the limit $f = 1\,(V_{S0} = 0)$. Expression (8) reduces to

$$\begin{aligned} A &= 1 - 2(\epsilon - \delta)z, \\ B &= [1 - 2(\epsilon - \delta)z]^2, \\ C &= 1 - 2\epsilon z - 2(\epsilon - \delta)z^2, \end{aligned} \tag{18}$$

so that

$$V^2(p) = V_{P0}^2 \frac{1 - 2(\epsilon - \delta)z}{1 - 2\epsilon z - 2(\epsilon - \delta)z^2}, \qquad (f = 1). \tag{19}$$

Weak and moderate transverse isotropy

In its strict interpretation, the limit of weak transverse isotropy implies retaining only linear terms in δ and ϵ (Thomsen, 1986). In this case, equation (7) gives

$$V(p) \approx V_{P0}[1 + \delta z + (\epsilon - \delta)z^2]. \tag{20}$$

Thomsen's (1986) equivalent expression in terms of phase angle is

$$V(\theta) \approx V_{P0}[1 + (\delta \cos^2\theta + \epsilon \sin^2\theta)\sin^2\theta]. \tag{21}$$

Retaining up to quadratic terms in δ and ϵ in equation (7) yields the moderate transverse-isotropy expansion

$$\begin{aligned} V(p) \approx V_{P0}[&1 + \delta z + ((\epsilon - \delta)(1 + 2\delta/f) + 3\delta^2/2)z^2 \\ &+ (\epsilon - \delta)(5\delta + 2(\epsilon - 2\delta)/f)z^3 \\ &+ (\epsilon - \delta)^2(7/2 - 2/f)z^4]. \end{aligned} \tag{22}$$

The moderate transverse-isotropy regime, in which this quadratically augmented expansion is valid, subsumes the ordinary weak transverse-isotropy regime.

The left panels of Figure 1 compare the exact phase velocity with its weak and moderate approximations for the specific values $f = 0.75$ and $V_{P0} = 1.0$ and the values of the anisotropy parameters shown on the individual panels. Observe that although the weak and moderate approximations are nominally valid on the basis of the sizes of δ and ϵ, once these parameters are specified, the approximations deteriorate as z (or, equivalently, the ray parameter or the phase angle) increases. The complicated relationship $z = V_{P0}^2 p^2 = V_{P0}^2 \sin^2\theta / V^2(z)$ between z and phase angle θ precludes solving for z explicitly, so the corresponding phase angles have been shown as functions of z in the right panels of Figure 1. Figure 1 shows that for moderately strong to strong anisotropy and, with an error tolerance of 2%, the validity of the weak case typically extends to at least 30° and that of the moderate case extends to at least 45°.

The moderate-anisotropy expansion for the square of the phase velocity given in equation (22) is

$$V^2(p) \approx V_{P0}^2[1 + 2\delta z + 2((\epsilon - \delta)(1 + 2\delta/f) + 2\delta^2)\, z^2 + 4(\epsilon - \delta)(3\delta + (\epsilon - 2\delta)/f)z^3 + 4(\epsilon - \delta)^2(2 - 1/f)z^4]. \quad (23)$$

The equivalent expression in phase angle was given in Tsvankin (1996), and an argument based on it was made for the weak dependence on f (or V_{S0}) of kinematic P-wave signatures in homogeneous, transversely isotropic media.

Small ray parameter

Equation (7) yields the small-p expansion

$$V(p) = V_{P0}[1 + \delta z + ((\epsilon - \delta)(1 + 2\delta/f) + 3\delta^2/2)z^2 + ((\epsilon - \delta)(1 + 2\delta/f)(5\delta + 2(\epsilon - 2\delta)/f) + 5\delta^3/2)z^3 + ((\epsilon - \delta)(1 + 2\delta/f) \times (7(\epsilon - \delta + 5\delta^2)/2 - (2(\epsilon - \delta) - 7\delta(3\epsilon - 5\delta))/f - 4(5\delta(\epsilon - \delta) - \epsilon^2)/f^2) + 35\delta^4/8)z^4 + O(z^5)]. \quad (24)$$

This expansion shows that the correct interpretation of small p is that $z = (V_{P0}\,p)^2 \ll 1$. Since $V \approx V_{P0}$ for $z \ll 1$, equation (10) shows that this inequality for z can be written as $\sin^2\theta \ll 1$.

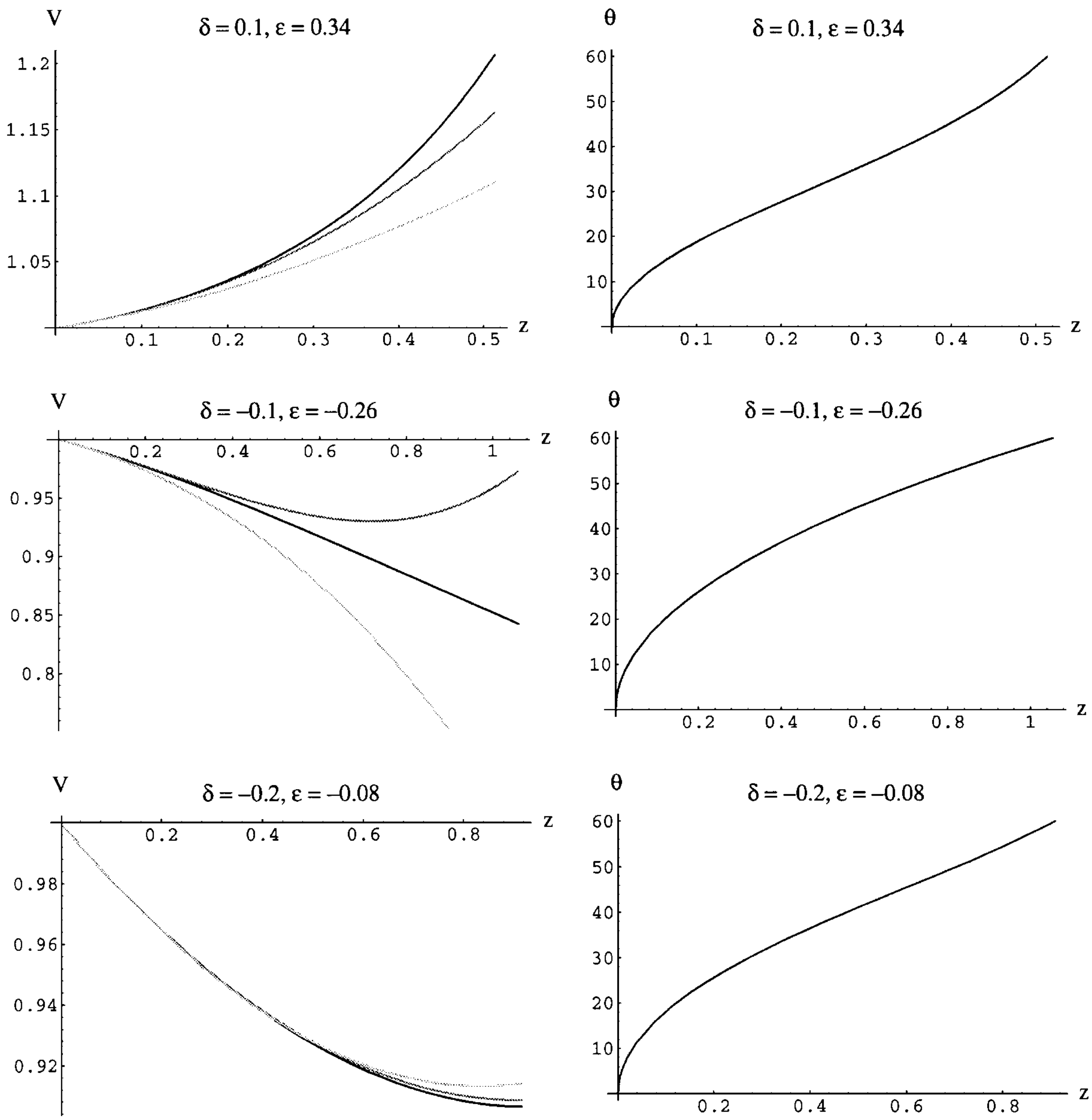

FIG. 1. For the specific values $f = 0.75$ and $V_{P0} = 1.0$ and the anisotropy parameter values shown, the left panels are plots of the exact phase velocity (dark line), its moderate approximation (medium line), and its weak approximation (light line) as functions of z. The right panels show the corresponding phase angles as functions of z.

Thus, in dimensionless terms, small p represents small phase angle θ.

Figure 2 compares the exact phase velocity to the successive small-z approximations of orders 1, 2, 3, and 4 for the same parameter values as in the previous section. Again, with an error tolerance of 2%, the validity of the four-term expansion in equation (24) typically extends to 45° for moderate to strong anisotropy.

NMO VELOCITY AS A FUNCTION OF RAY PARAMETER

The derivation of the NMO velocity as a function of ray parameter follows from the corresponding derivation of its expression in terms of the phase angle given in Tsvankin (1995).

First, use equation (10) to find the following relationship for $dp/d\theta$:

$$mV = \cos\theta = \frac{d\sin\theta}{d\theta} = (pV)'\frac{dp}{d\theta}, \tag{25}$$

where the prime notation is used for p differentiation. Thus,

$$\frac{dV}{d\theta} = V'\frac{dp}{d\theta} = \frac{mVV'}{(pV)'} \tag{26}$$

and

$$\frac{d^2V}{d\theta^2} = \frac{mV}{(pV)'}\left(\frac{mVV'}{(pV)'}\right)'. \tag{27}$$

Next, we use the main result of Tsvankin's (1995) paper—equation (9) for the NMO velocity of any pure mode in the dip plane of the reflector:

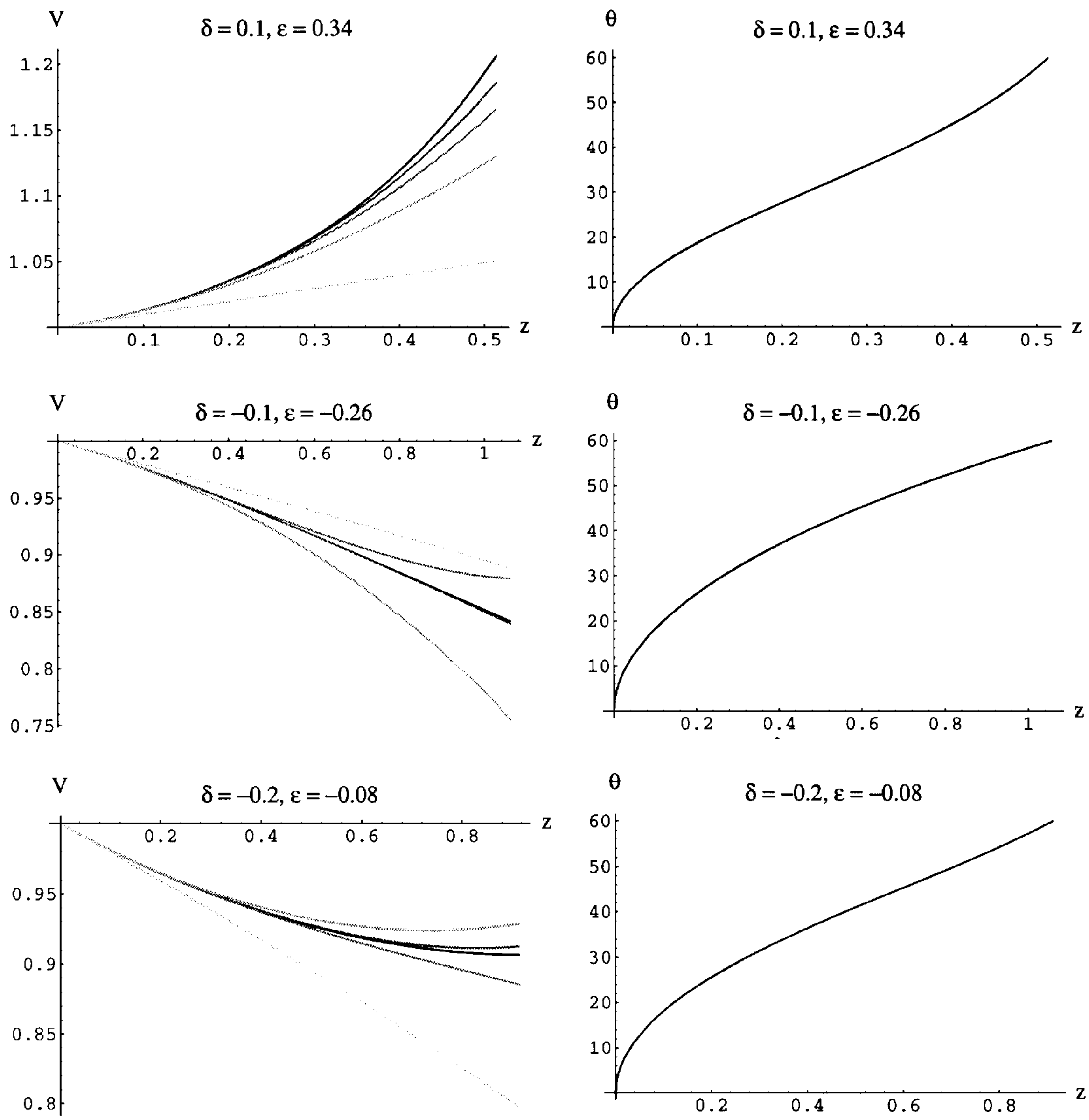

FIG. 2. For the specific values $f = 0.75$ and $V_{P0} = 1.0$ and the anisotropy parameter values shown, the left panels are plots of the exact phase velocity (dark line) and its 4-, 3-, 2-, and 1-term approximations in z [see equation (24) (successively lighter lines) as functions of z. The right panels show the corresponding phase angles as functions of z.

$$V_{\text{nmo}}(\phi) = \frac{V(\phi)}{\cos\phi} \frac{\sqrt{1 + \frac{1}{V(\phi)} \frac{d^2V}{d\theta^2}\Big|_{\theta=\phi}}}{1 - \frac{\tan\phi}{V(\phi)} \frac{dV}{d\theta}\Big|_{\theta=\phi}}, \qquad (28)$$

where ϕ is the reflector dip.

Substituting equations (26) and (27) into equation (28) and simplifying using equation (10) yields an explicit formula for the NMO velocity as a function of ray parameter, written as

$$V^2_{\text{nmo}}(p) = \frac{(1 - p^2V^2)VV'' + (3p^2V^2 - 2)V'^2 + 2pV^3V' + V^4}{(1 - p^2V^2)V(pV)'}. \qquad (29)$$

In the *Mathematica* implementation of this result, a layer of square roots was avoided by writing V^2_{nmo} in terms of $W \equiv V^2$. Also, the dimensionless variable z, defined in equation (9), was used, resulting in

$$V^2_{\text{nmo}}(z) = V^2_{P0} \frac{2z\left(V^2_{P0} - zW\right)W\ddot{W} + \left(V^2_{P0} + zW\right)W\dot{W} - \left(3V^2_{P0} - 4zW\right)\dot{W}^2 + W^3}{\left(V^2_{P0} - zW\right)W(W + z\dot{W})}. \qquad (30)$$

In equation (30), the dot denotes z differentiation.

The Alkhalifah-Tsvankin methodology

Before examining the simplifications of $V^2_{\text{nmo}}(p)$ corresponding to the special cases for V discussed earlier, it is helpful to describe briefly the methodology of Alkhalifah and Tsvankin (1995). These authors introduce the parameter

$$\eta = \frac{\epsilon - \delta}{1 + 2\delta} \qquad (31)$$

and establish, both from the linearized weak-anisotropy approximation and numerically, that $V_{\text{nmo}}(p)$ depends mainly on the principal parameters η and $V_{\text{nmo}}(0)$. They exploit this reduction to surface-observable parameters to develop a time-domain seismic processing sequence that includes, in particular, DMO and time migration and that takes vertical transverse isotropy into account. Finally, they confirm their conclusions by the successful processing of field seismic data (Alkhalifah et al., 1996).

The only other parameters that $V^2_{\text{nmo}}(p)$ can depend on are $f = 1 - (V^2_{S0}/V^2_{P0})$ and δ (or an equivalent pair, such as the V_{S0}/V_{P0} ratio and δ). Thus, the Alkhalifah-Tsvankin methodology is tantamount to the assertion that $V^2_{\text{nmo}}(p)$ depends only weakly on the residual parameters f and δ. Because of this, nominal values can be used for these parameters with little degradation of the results. Typical choices are $f = 3/4$ (i.e., $V_{P0}/V_{S0} = 2$) and $\delta = 0$.

Horizontal reflector

The two leading terms of equation (24) yield the approximations

$$\begin{aligned} V &= V_{P0} + V^3_{P0}\delta p^2 + O(p^4), \\ V' &= 2V^3_{P0}\delta p + O(p^3), \\ V'' &= 2V^3_{P0}\delta + O(p^2). \end{aligned} \qquad (32)$$

Inserting these results into the general NMO equation (29) yields

$$V^2_{\text{nmo}}(p) = V^2_{P0}(1 + 2\delta) + O(p^2). \qquad (33)$$

Setting $p = 0$ establishes

$$V_{\text{nmo}}(0) = V_{P0}\sqrt{1 + 2\delta}. \qquad (34)$$

Along with equation (31) for η, this expression completes the explicit definition of the two principal parameters in the Alkhalifah-Tsvankin methodology discussed above.

Trivially, in this first-order small-p limit, $V_{\text{nmo}}(p) = V_{\text{nmo}}(0) + O(p^2)$, so for this order, $V_{\text{nmo}}(p)$ depends on only (one of) the Alkhalifah-Tsvankin principal parameters.

Elliptical anisotropy

Inserting the elliptical P-wave phase velocity as a function of ray parameter given in equation (15) into equation (29) results in

$$V^2_{\text{nmo}}(p) = \frac{V^2_{P0}(1 + 2\delta)}{1 - (1 + 2\delta)V^2_{P0}p^2}. \qquad (35)$$

In terms of $V_{\text{nmo}}(0)$ as given in equation (34), the elliptical moveout result is expressed by

$$V_{\text{nmo}}(p) = \frac{V_{\text{nmo}}(0)}{\sqrt{1 - p^2V^2_{\text{nmo}}(0)}} \qquad (\delta = \epsilon), \qquad (36)$$

in agreement with Alkhalifah and Tsvankin (1995). Note that in the elliptical case, the Alkhalifah-Tsvankin methodology introduces no error, since $V_{\text{nmo}}(p)$ depends on only (one of) the Alkhalifah-Tsvankin principal parameters.

In expressions for the moveout function, it is convenient to introduce the dimensionless quantity

$$y = V^2_{\text{nmo}}(0)p^2 \qquad (37)$$

and the notation

$$V_{\text{ell}}(y) \equiv \frac{V_{\text{nmo}}(0)}{\sqrt{1 - p^2V^2_{\text{nmo}}(0)}} = \frac{V_{\text{nmo}}(0)}{\sqrt{1 - y}} \qquad (38)$$

for the elliptical result.

Zero shear-wave velocity

Inserting equation (19) into equation (30) gives

$$V^2_{\text{nmo}}(p) = \frac{V^2_{P0}(1 + 2\delta)(1 + 4(\epsilon - \delta)z - 6(\epsilon - \delta)(1 + 2\epsilon)z^2)}{(1 - (1 + 2\epsilon)z)(1 - 2(\epsilon - \delta)z)(1 - 4(\epsilon - \delta)z + 2(\epsilon - \delta)(1 + 2\epsilon)z^2)}. \qquad (39)$$

To express the result in terms of the Alkhalifah-Tsvankin parameters, use equations (9), (31), and (37) to eliminate z, ϵ, and V_{P0} via the equations

$$z = \frac{y}{1+2\delta},$$
$$\epsilon = \delta + \eta(1+2\delta), \quad (40)$$
$$V_{P0} = \frac{V_{\text{nmo}}(0)}{\sqrt{1+2\delta}},$$

obtaining

$$V_{\text{nmo}}^2(p) = V_{\text{nmo}}^2(0) \times \frac{1+2y(2-3y)\eta - 12y^2\eta^2}{(1-2y\eta)(1-y-2y\eta)(1-2y(2-y)\eta+4y^2\eta^2)}. \quad (41)$$

Thus, the Alkhalifah-Tsvankin methodology is exactly verified in the special case of $f = 1$. This case does have some practical significance because the NMO velocity changes only slightly as the V_{S0}/V_{P0} ratio is reduced from typical values (e.g., 0.5) to 0. In the next section, however, we turn from theoretically interesting limits to an approximation of practical interest.

Weak and moderate transverse isotropy

Using equation (20) in equation (30) gives

$$V_{\text{nmo}}^2(p) = \frac{V_{P0}^2}{1-z}\left(1 + \frac{2(\delta + (\epsilon-\delta)z(6-9z+4z^2))}{1-z}\right). \quad (42)$$

Use equation (40) to introduce the Alkhalifah-Tsvankin principal parameters, converting equation (42) into

$$V_{\text{nmo}}^2(p) = V_{\text{ell}}^2(y)(1+2\eta F(y)), \quad (43)$$

with

$$F(y) = \frac{y(6-9y+4y^2)}{1-y}, \quad (44)$$

in agreement with Alkhalifah and Tsvankin (1995). Equation (38) has been used to point out that the elliptical result $V_{\text{ell}}^2(y)$ is a factor of the weak-limit result. Since by its definition y [equation (37)] involves only a principal parameter and the ray parameter, equation (43) shows that there is no residual-parameter dependence at all in the weak limit.

Note that the weak-anisotropy expansion is singular at $y = 1$. Indeed, this singularity is already present in the elliptical moveout function (38) (and so is already present even in the isotropic limit). For elliptical anisotropy, $y = 1$ leads to the infinite value of the moveout velocity for reflections from vertical boundaries. For general VTI media, the exact NMO velocity from a vertical reflector still goes to infinity, but this infinite value no longer corresponds to $y = 1$. Mathematically, successive terms in the power-series expansion of the anisotropy parameters are successively more singular at this critical value, thus invalidating this expansion when y approaches 1. Thus, applications of the weak-anisotropy expansion must be limited by taking into account both the magnitudes of the anisotropy parameters and the size of y (which can be related to the size of the phase angle for given anisotropy parameters).

Equation (43) may be used to obtain the weak-limit estimate

$$\eta \approx \eta_0 = \frac{U}{2F(y)},$$
$$U = \frac{V_{\text{nmo}}^2(p)}{V_{\text{ell}}^2(y)} - 1. \quad (45)$$

At the next order for the anisotropy parameters (i.e., in the moderate-transverse-isotropy case),

$$V_{\text{nmo}}^2(p) = V_{\text{ell}}^2(y)\left[1 + 2\eta F(y) + 24\left(\frac{1}{f}-1\right)\eta\delta P_1(y) + \frac{12}{f}\eta^2 P_2(y) + 4\eta^2 Q(y)\right], \quad (46)$$

where F is defined in equation (44) and

$$P_1(y) = y(1-2y)(1-y),$$
$$P_2(y) = y^2(5-8y)(1-y), \quad (47)$$
$$Q(y) = \frac{y^2(26-68y+63y^2-20y^3)}{(1-y)^2}.$$

The left panels of Figure 3 compare the exact moveout function with its weak and moderate approximations for the specific values $f = 0.75$ and $V_{P0} = 1.0$. Just as in the case of the phase velocity, the weak and moderate approximations of the moveout function deteriorate as the phase angle (here monitored indirectly by y) increases and are valid for about the same phase angle ranges, as can be verified from the right panels of Figure 3.

Observe that in the elliptical limit, all terms in equation (46) except for the first vanish because of the factor of η multiplying them. Hence, consistency with the elliptical case is immediate. Thus, the elliptical limit and the isotropic limit are the same except for the implicit δ in $V_{\text{nmo}}(0)$.

The higher-order terms introduce dependence on the residual parameters δ and f. However, in addition to the fact that these terms are often small because they are quadratic in the anisotropy parameters, their importance is further mitigated both by their small size relative to the other terms and by their smooth dependence on the residual parameters. Since this is important analytic support for the Alkhalifah-Tsvankin methodology, these matters are now described in detail.

There are two quadratic-order terms involving the residual parameters. In the Alkhalifah-Tsvankin methodology, since these parameters cannot be determined from the P-wave surface seismic data, they are just set to reasonable values. To see why this is justified, note first that the two terms depending on the residual parameters are small relative to the other terms. Indeed, the $\eta\delta$ term is multiplied by $1/f - 1$, which lies in the interval [1/4, 2/3] when f is in the typical interval [0.6, 0.8]. The polynomial $P_1(y)$, which also multiplies this term, has an absolute maximum value less than 0.1 in the interval $0 \le y \le 1$. Thus, for practical parameter values, the coefficient of the $\eta\delta$ term is less than $1/15 \approx 0.067$ in absolute value. Similarly, the higher-order η^2 term that is multiplied by $1/f$ is also multiplied by the polynomial $P_2(y)$, which is less than 0.2 in the interval $0 \le y \le 1$, so the overall coefficient of this term is less than 1/3 in absolute value.

Furthermore, when the anisotropy is strong enough that the quadratic-order terms do influence the data, both of the terms just considered will be dominated by the final η^2 contribution, whose rational function coefficient $Q(y)$ grows rapidly as y increases from 0. Note that this last, and dominant, higher-order contribution depends on only the Alkhalifah-Tsvankin principal parameters $V_{\rm nmo}(0)$ and η.

Figure 4 illustrates the preceding comments. Note that for small y values, all three terms mentioned above coincide with the linear approximation, whereas for large y values, the residual quadratic terms are swamped by the rapid growth of the principal-parameter term (however, as previously noted, the whole expansion breaks down when y gets too close to 1). In the middle range of y values, for larger values of η there are, indeed, visible deviations because of the residual dependencies, but these deviations are small for the practical values of the parameters.

Thus, even if these terms were neglected totally, their contribution would often be negligible. However, in the Alkhalifah-Tsvankin methodology, they are not ignored but rather are implemented by using reasonable values for the unknown residual parameters. If the errors made in these parameters are denoted by $\Delta\delta$ and Δf, then the absolute error is $V_{\rm ell}^2(y)$ times

$$24\left(\frac{1}{f}-1\right)\eta P_1(y)\Delta\delta - \frac{24}{f^2}\eta\delta P_1(y)\Delta f - \frac{12}{f^2}\eta^2 P_2(y)\Delta f.$$

With the typical choices, $f = 3/4$ and $\delta = 0$, and noting the bounds on P_1 and P_2, this error factor is less than

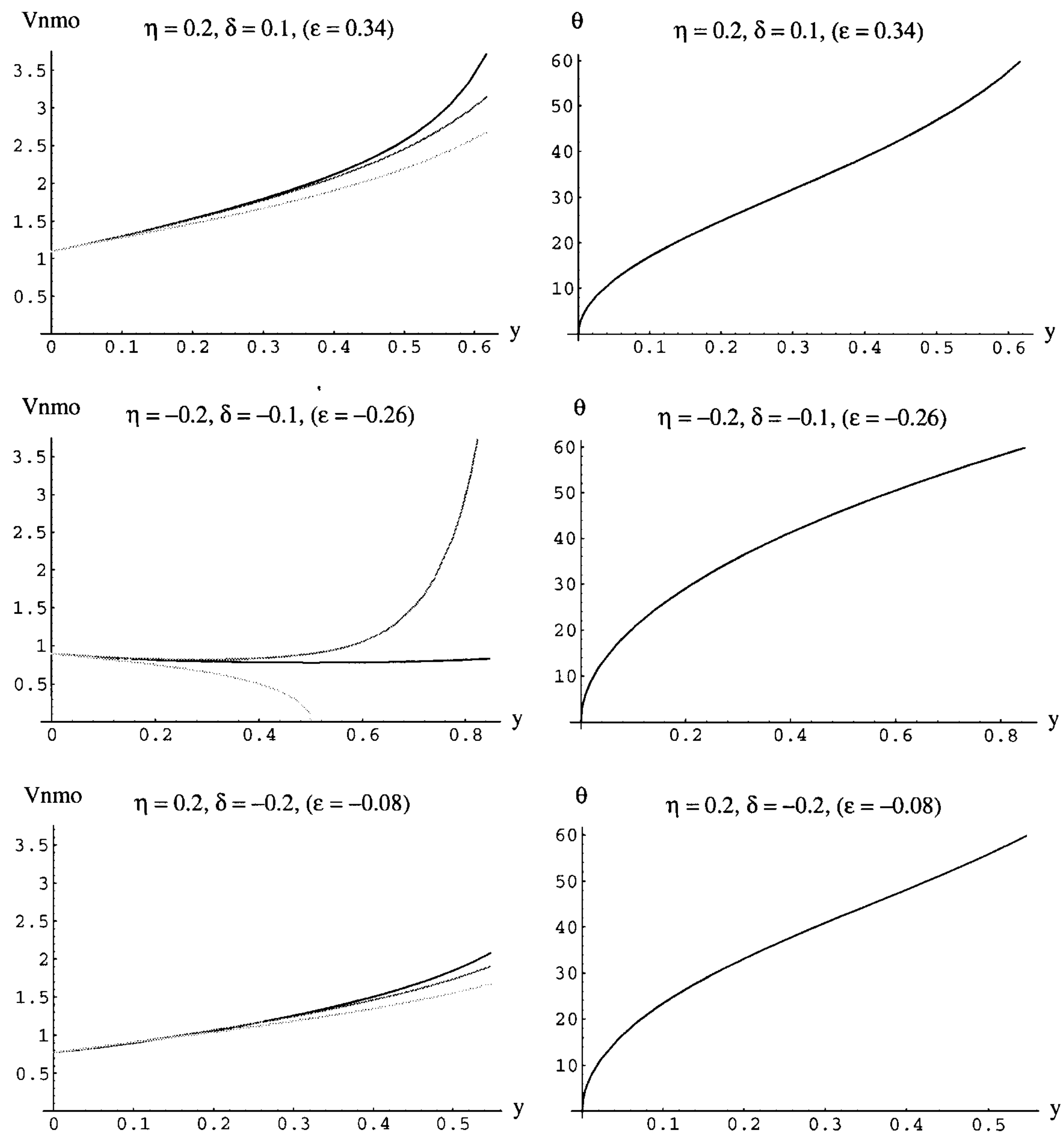

FIG. 3. For the specific values $f = 0.75$ and $V_{P0} = 1.0$ and the anisotropy parameter values shown, the left panels are plots of the exact moveout velocity (dark line), its moderate approximation (medium line) [see equation (46)], and its weak approximation (light line) [see equation (43)] as functions of z. The right panels show the corresponding phase angles as functions of z.

$$0.8\eta|\Delta\delta| + 17.1\eta^2|\Delta f|$$

in absolute value. Considering that typically $|\Delta\delta| < 0.2$ and $|\Delta f| < 0.1$, it seems that the error resulting from setting the residual parameters rarely will be noticed.

In summary, the higher-order expansion supports the Alkhalifah-Tsvankin approach of using "best-guess" values for the unobservable (from P-wave reflection traveltimes) residual parameters δ and f in the important regime of moderate anisotropy.

On the theoretical side, observe that if f is set to one, then the δ contributions drop out completely, along with the f contributions, and the residual-parameter dependence is entirely removed. This is in agreement with the general result in equation (41), and expanding that result to the second order in the anisotropy parameters gives the $f = 1$ limit of the present result.

The quadratic equation (46) can be used to improve the estimate for $\eta = \eta_0$ given in equation (45). We seek the solution of this quadratic equation that is close to η_0. Using only the terms free of dependence on the residual parameters, this solution is given by

$$\eta = \frac{2\eta_0}{1+\sqrt{1-4\nu}} \approx \eta_0(1+\nu), \qquad \nu = -\frac{UQ}{F^2}, \tag{48}$$

where the final approximation is valid under the assumption that $\nu \ll 1$ and U is defined in equation (45). In most cases, it is of some numerical benefit to include the P_2 term defined in equation (47). To do this requires only replacing Q with $Q + (3/f)P_2$ in equation (48) and making a nominal choice of f. Similarly, inclusion of P_1 is accomplished by replacing F with

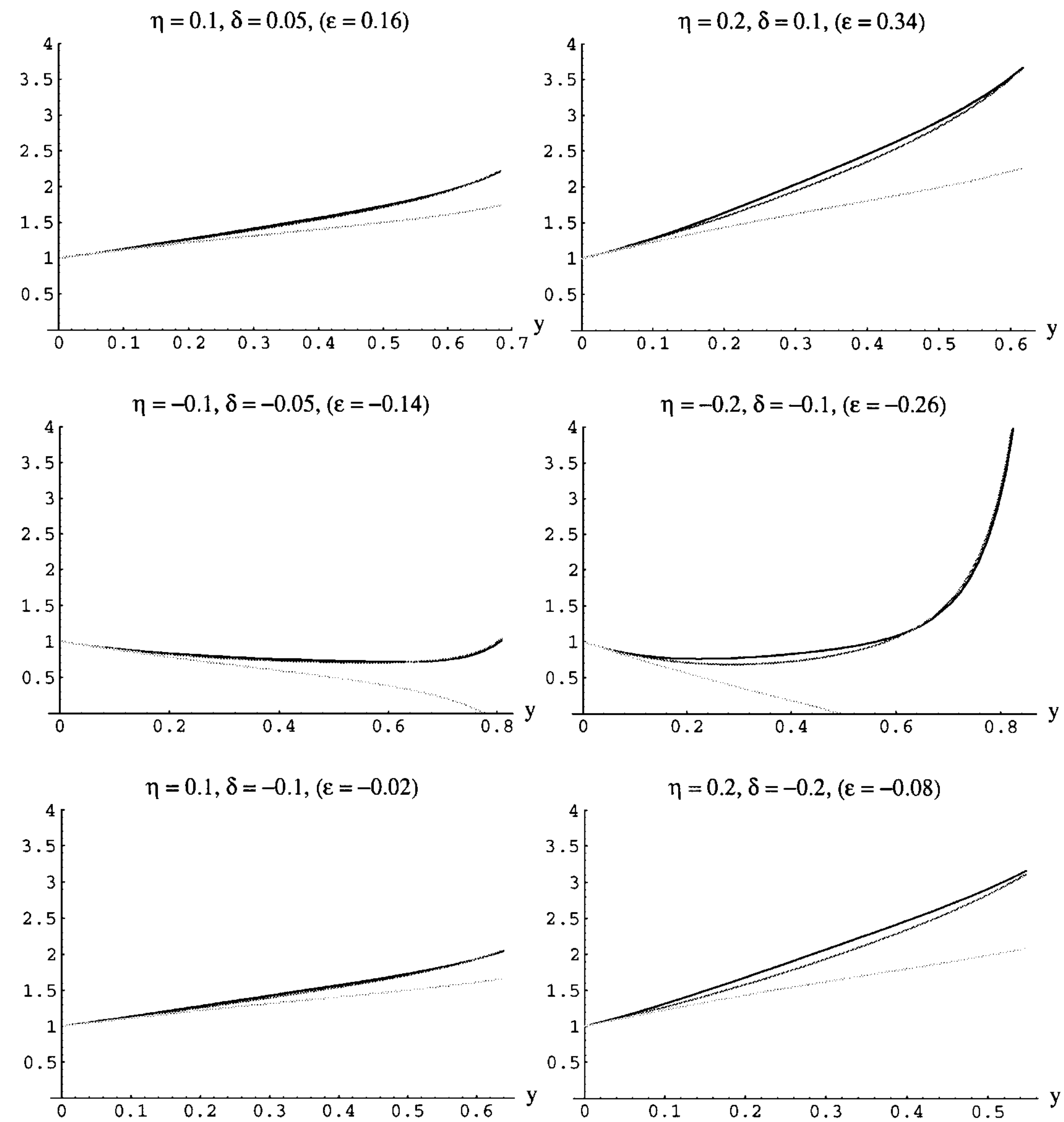

FIG. 4. With the typical parameter value $f = 0.75$ and the values of η and δ and the computed value of ϵ shown, each panel depicts $V^2_{\mathrm{nmo}}(p)$ normalized by $V^2_{\mathrm{ell}}(y)$ plotted against $y = V^2_{\mathrm{nmo}}(0)p^2$ for the linear term alone (light line), the linear term plus the portion of the quadratic term that depends only on the Alkhalifah-Tsvankin principal parameters (medium line), and the linear term plus the full quadratic term (dark line). The y range in each plot corresponds to phase angles up to 60°.

$F + (12\delta/f)P_1$ in both equations (45) and (48). However, the P_1 term is almost always negligible (and is altogether absent if one makes the usual nominal choice $\delta = 0$). Numerical experiments indicate that this second-order approximation for η affords an accurate starting value for obtaining a high-precision answer with only a few Newton iterations.

Small ray parameter

In the small-p limit, it is possible to obtain analytic results even for strong anisotropy. At first, use just the first three terms of equation (24) in equation (29) to obtain

$$V^2_{\rm nmo}(p) = V^2_{P0}\left[1 + 2\delta + (-24\delta^2 + 24\delta\epsilon + f - 8\delta f + 4\delta^2 f + 12\epsilon f)\frac{(V_{P0}p)^2}{f} + O(V_{P0}p)^4\right]. \tag{49}$$

Expressing the result in terms of the Alkhalifah-Tsvankin principal parameters gives

$$V^2_{\rm nmo}(p) = V^2_{\rm nmo}(0)\left[\left(1 + \left(1 + 12\eta\frac{1+2\delta/f}{1+2\delta}\right)y\right) + O(y^2)\right]. \tag{50}$$

Numerical tests show that using this small-p series for a dip of 15° incurs about a 2% error in estimating $V^2_{\rm nmo}(p)$ for typical values of f and δ.

Equation (50) gives analytic support to the Alkhalifah-Tsvankin methodology, since the only dependence on the residual parameters occurs in the ratio

$$g = \frac{1+2\delta/f}{1+2\delta} \tag{51}$$

that multiplies η. Figure 5 shows a plot of this function over the ranges of f and δ that are relevant in practice. Observe that the function $g(\delta, f)$ varies slowly over these ranges, justifying the use of reasonable values of f and δ in place of the true values. In particular, the choice $\delta = 0$ leads to $g = 1$; therefore, with this choice, the choice of f is irrelevant in the small-p regime.

As in the moderate transverse-isotropy regime, the analytic small-p result allows the estimation of η from surface observations. Indeed, if one has observations of $V_{\rm nmo}(p)$ at $p = 0$ and some other (not too large) value $p = p_1$, the solution for η implied by equation (50) is

$$\eta \approx \frac{1}{12g}\left(\frac{V^2_{\rm nmo}(p_1) - V^2_{\rm nmo}(0)}{p_1^2 V^4_{\rm nmo}(0)} - 1\right), \tag{52}$$

where, once again, nominal values of δ and f can be used in factor g without incurring much error. Of course, for the estimation of η to be sufficiently stable, the value of the ray parameter should not be too small either.

Finally, note that the full form of the series for $V^2_{\rm nmo}(p)$ is given by

$$V^2_{\rm nmo}(p) = V^2_{\rm nmo}(0)\left[1 + c_2 y + c_4 y^2 + c_6 y^3 + \cdots\right]. \tag{53}$$

Using equation (24), which keeps eighth-order terms in the small-p expansion of $V(p)$, yields

$$\begin{aligned} c_2 &= 1 + 12g\eta, \\ c_4 &= 1 + 6g(6-5g)\eta + \frac{60g}{f}\eta^2, \\ c_6 &= 1 + 8g(9 - 15g + 7g^2)\eta \\ &\quad + 8g\left(13g + \frac{6(5-7g)}{f}\right)\eta^2 + \frac{224g}{f^2}\eta^3. \end{aligned} \tag{54}$$

The left panels of Figure 6 compare the exact moveout function to the successive small-y approximations of orders 1, 2, 3, and 4 for the same parameter values as those used in Figure 1. The corresponding right panels show the phase angles as functions of y.

Aside from the dependence on g that was discussed above, the higher-order terms in the small-p (small-y) expansion of the moveout velocity function have explicit dependencies on f. The first of these occurs in the final term in c_4. However, in the present small-p expansion, this term is multiplied not only by p^4 but also by η^2, which ameliorates its influence. Similarly, c_6 contains terms of order $\eta^2 p^6$ and $\eta^3 p^6$, which also give rise to the same type of mild divergence from strict dependence on the two Alkhalifah-Tsvankin principal parameters.

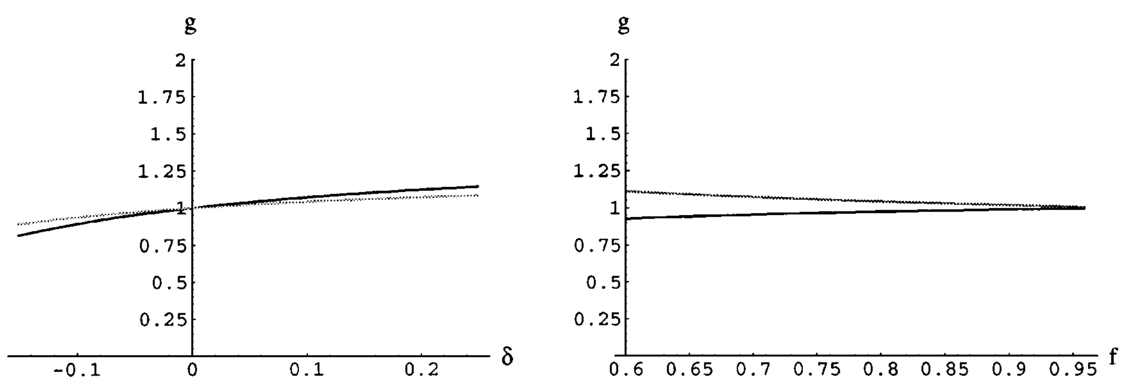

FIG. 5. Plot of the factor $g = (1 + 2\delta/f)/(1 + 2\delta)$ in the left panel as a function of δ for $f = 0.7$ (dark line) and $f = 0.8$ (light line) and in the right panel as a function of f for $\delta = -0.05$ (dark line) and $\delta = +0.1$ (light line).

If f is set to unity, the residual-parameter dependence is removed entirely, since it enters only through g (which equals one when $f = 1$) and f itself; that is, there is no δ dependence other than that contained in g. Again, this result is in agreement with the general result in equation (41). Indeed, expanding that result for small-y approximations yields the $f = 1$ limit of the present result.

As a further check for consistency, observe that for the small-p (small-y) expansion, the elliptical result in equation (36) expands to

$$V_{\text{ell}}^2(y) = \frac{V_{\text{nmo}}^2(0)}{1-y} = V_{\text{nmo}}^2(0)[1 + y + y^2 + \cdots]. \quad (55)$$

This agrees with the small-p expansion given above, since for the elliptical case, $\eta = 0$ and hence c_2, c_4, and c_6 in equation (54) all reduce to unity. Therefore, equation (53) reduces to equation (55). Thus, just as in the case of moderate transverse isotropy, to the order computed, the small-p expansion in the elliptical limit is identical to the small-p expansion in the isotropic limit, except for the δ implicit in $V_{\text{nmo}}(0)$.

CONCLUSIONS

I derived (in some cases rederived) formulas for the P-wave phase and NMO velocities as functions of ray parameter, with explicit results given for the cases of vanishing on-axis shear-wave velocity, weak to moderate transverse isotropy, and small to moderate ray parameter. The various expressions for the phase velocity, nominally expansions in the ray parameter p, turn out to be expansions in the dimensionless parameter

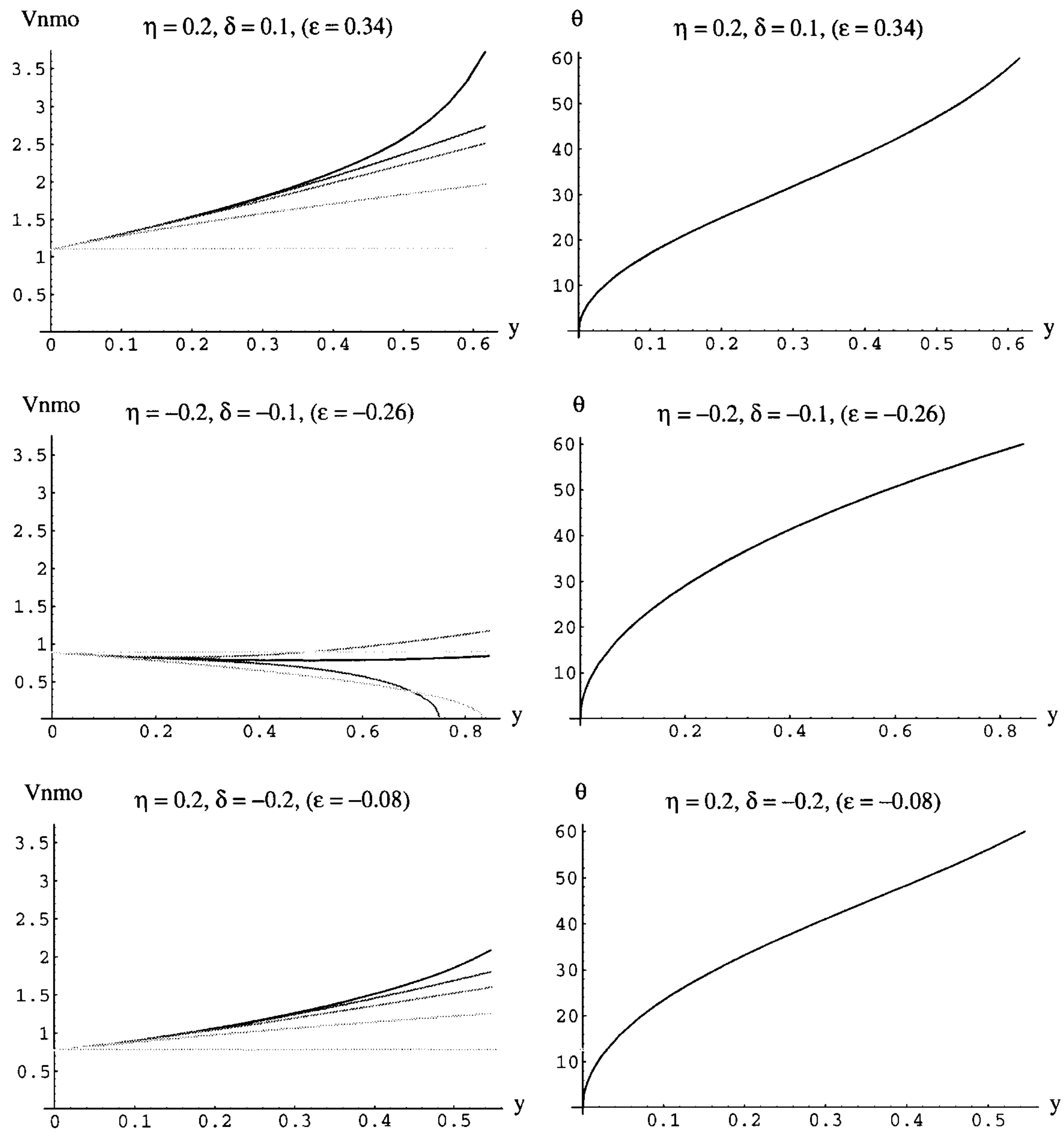

FIG. 6. For the specific values $f = 0.75$ and $V_{P0} = 1.0$ and the anisotropy parameter values shown, the left panels are plots of the exact moveout velocity (dark line) and its 4-, 3-, 2-, and 1-term approximations in y [see equation (53)] (successively lighter lines) as functions of y. The right panels show the corresponding phase angles as functions of y.

$z = V_{P0}^2 p^2$. Similarly, the NMO function expansions turn out to be expansions in $y = V_{\mathrm{nmo}}^2(0)p^2$. For some selected choices of the anisotropy parameters, both the phase velocity and moveout function expansions were compared with the corresponding exact functions.

Further analytic explanation is provided here for the observation in Alkhalifah and Tsvankin (1995) that, to a high numerical accuracy, the NMO velocity as a function of ray parameter depends mainly on its zero-dip value and on the parameter η and only weakly on the residual parameters f and δ. In particular, an expansion in the anisotropy parameters explained the weak residual dependence in the important physical case of weak to moderately strong transverse isotropy, and an expansion in the ray parameter established the result even for strong anisotropy with small to moderate dip angles. In establishing the weak residual dependences, it is crucial that the Alkhalifah-Tsvankin principal parameters be introduced in a consistent and uniform manner. In this study, equation (40) provides this transformation. The residual-parameter dependence vanishes completely in several special cases: $p = 0$ (i.e., horizontal reflector), $\delta = \epsilon$ (elliptical case), and when $f = 1$ ($V_{S0} = 0$).

ACKNOWLEDGMENTS

I thank Tariq Alkhalifah for performing some preliminary numerical tests and for his helpful suggestions, Ken Larner for his encouragement and careful critiques of several drafts of the text, and especially Ilya Tsvankin for sharing his deep understanding of anisotropic issues at several crucial junctures in this research. Support for this work was provided by the members of the Consortium Project on Seismic Inverse Methods for Complex Structures at the Center for Wave Phenomena, Department of Mathematical and Computer Sciences, Colorado School of Mines, and by the United States Department of Energy (project "Velocity Analysis, Parameter Estimation, and Constraints on Lithology for Transversely Isotropic Sediments" within the framework of the Advanced Computational Technology Initiative).

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