

Anisotropic parameters and *P*-wave velocity for orthorhombic media

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ABSTRACT

Although orthorhombic (or orthotropic) symmetry is believed to be common for fractured reservoirs, the difficulties in dealing with nine independent elastic constants have precluded this model from being used in seismology. A notation introduced in this work is designed to help make seismic inversion and processing for orthorhombic media more practical by simplifying the description of a wide range of seismic signatures. Taking advantage of the fact that the Christoffel equation has the same form in the symmetry planes of orthorhombic and transversely isotropic (TI) media, we can replace the stiffness coefficients by two vertical (*P* and *S*) velocities and seven dimensionless parameters that represent an extension of Thomsen's anisotropy coefficients to orthorhombic models. By design, this notation provides a uniform description of anisotropic media with both orthorhombic and TI symmetry.

The dimensionless anisotropic parameters introduced here preserve all attractive features of Thomsen notation in treating wave propagation and performing 2-D

processing in the symmetry planes of orthorhombic media. The new notation has proved useful in describing seismic signatures *outside* the symmetry planes as well, especially for *P*-waves. Linearization of *P*-wave phase velocity in the anisotropic coefficients leads to a concise weak-anisotropy approximation that provides good accuracy even for models with pronounced polar and azimuthal velocity variations. This approximation can be used efficiently to build analytic solutions for various seismic signatures.

One of the most important advantages of the new notation is the reduction in the number of parameters responsible for *P*-wave velocities and traveltimes. All kinematic signatures of *P*-waves in orthorhombic media depend on just the vertical velocity V_{P0} and *five* anisotropic parameters, with V_{P0} serving as a scaling coefficient in homogeneous media. This conclusion, which holds even for orthorhombic models with strong velocity anisotropy, provides an analytic basis for application of *P*-wave traveltime inversion and data processing algorithms in orthorhombic media.

INTRODUCTION

The vast majority of existing studies of seismic anisotropy are limited to transversely isotropic (TI) models with different orientation of the symmetry axis. It has been shown in the literature (e.g., Sayers, 1994a) that TI (or hexagonal) symmetry adequately describes the elastic properties of shales, which represent the major source of anisotropy in sedimentary basins. Most shale formations are horizontally layered, yielding an effective transversely isotropic medium with a vertical symmetry axis (the so-called VTI medium). In some cases, shale layers may be dipping (e.g., near flanks of salt domes), which leads to an azimuthally anisotropic medium with a tilted TI symmetry axis [for moveout analysis in such a model, see Tsvankin (1997a)].

However, the transversely isotropic model becomes much more restrictive when applied to the description of cracked

media that contain small (compared to the predominant wavelength) fractures or microcracks. Indeed, only the simplest fractured model with a single system of parallel vertical circular ("penny-shaped") cracks embedded in an isotropic matrix exhibits transverse isotropy with a horizontal symmetry axis (HTI media). Deviations from the circular crack shape, misalignment of the crack planes, the addition of a second crack system, or the presence of anisotropy or thin layering in the matrix lower the symmetry of the effective medium to orthorhombic or less. One of the most common reasons for orthorhombic anisotropy in sedimentary basins is a combination of parallel vertical cracks and vertical transverse isotropy in the background medium (e.g., Wild and Crampin, 1991; Schoenberg and Helbig, 1997) as illustrated by Figure 1. Orthorhombic symmetry can also be caused by two or three mutually orthogonal crack systems or two identical systems of cracks making an arbitrary angle

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with each other. Hence, orthorhombic anisotropy may be the simplest realistic symmetry for many geophysical problems.

Wave propagation in orthorhombic media has been studied in a number of publications including Musgrave (1970), Tsvankin and Chesnokov (1990a, b), Wild and Crampin (1991), Brown et al. (1991), Sayers (1994b) and Schoenberg and Helbig (1997). Such numerical tools as finite-difference or reflectivity methods generalized for anisotropic media make the forward modeling of seismic wavefields in orthorhombic models a relatively straightforward (albeit a costly) procedure. However, extending inversion and processing techniques to orthorhombic media is a much more difficult endeavor.

Seismic methods of fracture detection, extensively developed since the early 1980s, are based on the analysis of the traveltimes and reflection coefficients of split shear waves at near-vertical incidence (Crampin, 1985; Thomsen, 1988). If azimuthal anisotropy is caused by a single system of penny-shaped vertical cracks, the relative time delay or the difference in the normal-incidence reflection amplitude between the shear modes provides a direct estimate of the crack density, while the polarization of the fast *S*-wave¹ determines the crack orientation. These well-known shear-wave splitting methods remain valid in some orthorhombic models such as those in Figure 1. However, vertically traveling shear waves can yield only a single anisotropic parameter (the splitting coefficient), which is not sufficient, for instance, to separate empty and fluid-filled cracks. In more complicated orthorhombic models containing two crack systems or noncircular cracks, interpretation of the shear-wave splitting parameter becomes ambiguous, unless some additional data are available.

Therefore, seismic characterization of orthorhombic media should include, if possible, analysis of azimuthally dependent *P*-wave signatures and shear data at oblique incidence angles. Recently, Lynn et al. (1996a, b) presented field-data examples showing the feasibility of detecting azimuthal anomalies of *P*-wave velocities and amplitudes over typical fractured reservoirs. Interpretation and inversion of such anomalies is impossible without a good understanding of the dependence of

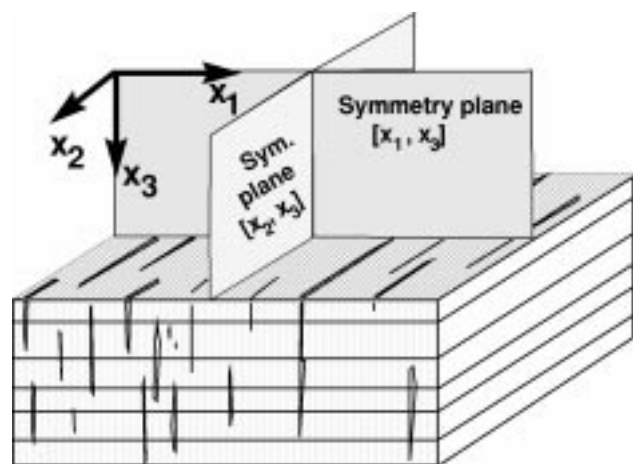


FIG. 1. An orthorhombic model caused by parallel vertical cracks embedded in a medium composed of thin horizontal layers. Orthorhombic media have three mutually orthogonal planes of mirror symmetry.

¹For brevity, the qualifiers in “quasi-*P*-wave” and “quasi-*S*-wave” will be omitted.

seismic signatures on the anisotropy parameters. The complexity of this problem is explained by the fact that general orthorhombic media are described by nine independent stiffness components.

This work is aimed at making seismic treatment of orthorhombic media more practical by identifying the combinations of the stiffness coefficients responsible for a wide range of seismic signatures. The notation that enables us to reach this goal is based on the same principle as that of Thomsen (1986), devised for VTI media. A unified description of vertical transverse isotropy and orthorhombic media provides a convenient bridge between the two models and makes it possible to take full advantage of recent developments in the analytic description of seismic signatures in VTI media (Tsvankin, 1996). This approach has already been successfully applied by R uger (1997) and Tsvankin (1997b) to TI models with a horizontal symmetry axis. For instance, Tsvankin (1997b) showed that the anisotropic parameter that governs azimuthally dependent *P*-wave normal-moveout (NMO) velocity in HTI media represents the *same* combination of the stiffness coefficients as does the Thomsen’s VTI parameter δ . Also, Thomsen-style parameters have been used in physical-modeling studies by Brown et al. (1991) and Cheadle et al. (1991).

Introduction of the new notation is followed by an analytic description of seismic signatures in the symmetry planes of orthorhombic media based on a limited analogy with vertical transverse isotropy. Linearization of the exact phase-velocity equation in the dimensionless anisotropic parameters leads to a simple weak-anisotropy approximation for *P*-wave phase velocity *outside* the symmetry planes. Further analysis shows that *P*-wave kinematic signatures in orthorhombic media are determined by the vertical velocity and only five anisotropic coefficients.

ORTHORHOMBIC STIFFNESS TENSOR

The stiffness tensor c_{ijkl} for orthorhombic media can be represented in a two-index notation (e.g., Musgrave, 1970) often called the “Voigt recipe” as

$$\mathbf{c}_{\text{orthor}} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{pmatrix}. \quad (1)$$

This tensor has the same null components as that for transversely isotropic (TI) media with the symmetry axis aligned with one of the coordinate directions, but the TI tensor has only five independent coefficients. For certain subsets of orthorhombic models (such as orthorhombic media caused by vertical cracks in a VTI background), not all nine stiffnesses in the tensor (1) are independent (Schoenberg and Helbig, 1997), but here we do not restrict ourselves to any specific type of orthorhombic anisotropy.

The phase velocity V and the displacement vector \mathbf{U} of plane waves in arbitrary homogeneous anisotropic media satisfy the Christoffel equation (Musgrave, 1970):

$$[G_{ik} - \rho V^2 \delta_{ik}] U_k = 0, \quad (2)$$

where ρ is the density, δ_{ik} is the Kronecker's symbolic δ , and \mathbf{G} is the symmetric Christoffel matrix,

$$G_{ik} = c_{ijk} n_j n_\ell. \quad (3)$$

Here, \mathbf{n} is the unit vector in the slowness direction; summation over repeated indices is implied. The Christoffel equation (2) describes a standard eigenvalue (ρV^2)-eigenvector (\mathbf{U}) problem for the matrix \mathbf{G} , with the eigenvalues determined by

$$\det [G_{ik} - \rho V^2 \delta_{ik}] = 0. \quad (4)$$

Solving the cubic equation (4) at any specific slowness direction \mathbf{n} (for details, see Appendix A) yields three positive values of

$$\begin{bmatrix} c_{11}n_1^2 + c_{55}n_3^2 - \rho V^2 & 0 & (c_{13} + c_{55})n_1n_3 \\ 0 & c_{66}n_1^2 + c_{44}n_3^2 - \rho V^2 & 0 \\ (c_{13} + c_{55})n_1n_3 & 0 & c_{55}n_1^2 + c_{33}n_3^2 - \rho V^2 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = 0. \quad (11)$$

the squared phase velocity V^2 , which correspond to the P -wave and two S -waves (strictly, the "quasi- P -wave" and "quasi- S -waves"). For certain orientations of the vector \mathbf{n} , the velocities of the split S -waves coincide with each other, which leads to the shear-wave singularities. After the eigenvalues have been determined, the associated displacement (polarization) vectors \mathbf{U} for each mode can be found from equation (2). Since the Christoffel matrix \mathbf{G} is symmetric, the polarization vectors of the three modes are always orthogonal to each other, but none of them is necessarily parallel or perpendicular to \mathbf{n} .

Using equations (1) and (3) and the Voigt recipe for the two-index notation yields the Christoffel matrix for orthorhombic media:

$$G_{11} = c_{11}n_1^2 + c_{66}n_2^2 + c_{55}n_3^2, \quad (5)$$

$$G_{22} = c_{66}n_1^2 + c_{22}n_2^2 + c_{44}n_3^2, \quad (6)$$

$$G_{33} = c_{55}n_1^2 + c_{44}n_2^2 + c_{33}n_3^2, \quad (7)$$

$$G_{12} = (c_{12} + c_{66})n_1n_2, \quad (8)$$

$$G_{13} = (c_{13} + c_{55})n_1n_3, \quad (9)$$

$$\begin{bmatrix} c_{11} \sin^2 \theta + c_{55} \cos^2 \theta - \rho V^2 & (c_{13} + c_{55}) \sin \theta \cos \theta \\ (c_{13} + c_{55}) \sin \theta \cos \theta & c_{55} \sin^2 \theta + c_{33} \cos^2 \theta - \rho V^2 \end{bmatrix} \begin{bmatrix} U_1 \\ U_3 \end{bmatrix} = 0. \quad (12)$$

and

$$G_{23} = (c_{23} + c_{44})n_2n_3. \quad (10)$$

WAVE PROPAGATION IN THE SYMMETRY PLANES

Orthorhombic models have three mutually orthogonal planes of mirror symmetry that coincide with the coordinate planes $[x_1, x_2]$, $[x_1, x_3]$, and $[x_2, x_3]$ (Figure 1). Because of the

relative simplicity of the basic equations, symmetry-plane analysis provides important insights into wave propagation in orthorhombic media. Anticipating future applications in reflection seismology, we assume that the symmetry plane $[x_1, x_2]$ is horizontal. Then information obtained from reflection seismic experiments will be mostly related to phase- and group-velocity variations near the (vertical) x_3 -axis. If none of the symmetry planes is horizontal, the treatment of reflected waves becomes more complicated; however, the equations derived below for a *homogeneous* orthorhombic medium remain entirely valid.

For a wave propagating in the $[x_1, x_3]$ plane, the projection of the slowness vector onto the x_2 -axis vanishes ($n_2 = 0$), the Christoffel matrix \mathbf{G} [equations (5)–(10)] simplifies, and equation (2) takes the following form:

Equation (11) reduces to the corresponding Christoffel equation for the transversely isotropic model with the symmetry axis in the x_3 -direction (VTI medium), if $c_{44} = c_{55}$. (In the orthorhombic model, however, the stiffness components c_{44} and c_{55} are *not* equal to each other, which leads to shear-wave splitting at vertical incidence.) This analogy with VTI media has already been mentioned in the literature (Musgrave, 1970; Cheadle et al., 1991; Schoenberg and Helbig, 1997); here, however, it is used in a systematic fashion to describe wave propagation in the symmetry planes and to introduce new dimensionless anisotropy parameters for orthorhombic media.

The Christoffel equation (11) also reduces to that of an HTI medium with the symmetry axis pointing in the x_1 -direction. The properties of the HTI model and a limited equivalence between vertical and horizontal transverse isotropy were studied in detail by Tsvankin (1997b) and Ruger (1997).

Exactly as in VTI media, equation (11) splits into two independent equations for the in-plane (" P - SV ") motion ($U_2 = 0$) and the pure transverse (" SH ") motion ($U_1 = U_3 = 0$). Expressing the slowness components in equation (11) in terms of the phase angle θ with the vertical (x_3) axis ($n_1 = \sin \theta$; $n_3 = \cos \theta$) yields for the in-plane polarized waves

Equation (12) is identical to the well-known Christoffel equation for the P - SV waves in VTI media that has been thoroughly discussed in the literature (e.g., Payton, 1983). Hence, the phase velocities of the P -wave and the in-plane polarized shear wave (SV -wave) in the $[x_1, x_3]$ plane of orthorhombic media represent the same functions of the stiffness coefficients c_{ij} and the phase angle with vertical as do the P - SV phase velocities in VTI media. As in VTI media, the P -wave phase

(and group) velocity in the vertical (x_3) and horizontal (x_1) directions is determined as $\sqrt{c_{33}/\rho}$ and $\sqrt{c_{11}/\rho}$, respectively, while the vertical and horizontal SV -wave velocity is equal to $\sqrt{c_{55}/\rho}$.

The phase-velocity function of each wave in the $[x_1, x_3]$ symmetry plane is sufficient to obtain the corresponding group (ray) velocity and group angle and, consequently, all other kinematic signatures, such as normal-moveout (NMO) velocity. This means that the kinematics of wave propagation in the $[x_1, x_3]$ plane of orthorhombic media is described fully by known VTI equations. The only exception to this analogy is cuspidal S -wave group-velocity surfaces (wavefronts) formed in the symmetry planes of orthorhombic media near shear-wave point singularities. Some in-plane branches of the cusps may be caused by slowness vectors that lie *outside* of the symmetry planes (Grechka and Obolentseva, 1993); clearly, cuspidal features of this nature cannot exist in VTI media.

Since the displacement (polarization) vectors \mathbf{U} of the in-plane polarized plane waves are determined from the same equation (12), they are also given by the corresponding expressions for VTI media. Furthermore, analysis of the boundary conditions shows that the equivalence with vertical transverse isotropy holds for plane-wave reflection coefficients as well (provided the symmetry planes have the same orientation above and below the boundary).

However, body-wave amplitudes in general do not comply with this “equivalence” principle because they depend on the 3-D shape of the slowness surface, not just in the symmetry plane itself, but also in its vicinity. Out-of-plane velocity variations lead to focusing and defocusing of energy and may have a significant influence on the distribution of energy along the wavefront within the symmetry planes (Tsvankin and Chesnokov, 1990a).

Equation (11) also has a solution corresponding to the shear wave polarized orthogonally to the $[x_1, x_3]$ plane ($U_1 = U_3 = 0, U_2 \neq 0$):

$$c_{66} \sin^2 \theta + c_{44} \cos^2 \theta - \rho V^2 = 0. \quad (13)$$

Equation (13) describes a “pure” shear mode with an elliptical slowness curve and an elliptical wavefront in the $[x_1, x_3]$ plane and is identical to the phase-velocity equation for the SH -wave in VTI media. However, since in orthorhombic media c_{44} is not equal to c_{55} , the vertical velocities of the shear waves are different. In fact, the two shear waves in the $[x_1, x_3]$ plane of an orthorhombic medium are fully decoupled because they depend on different sets of elastic constants [compare equations (12) and (13)].

Similar conclusions can be drawn for wave propagation in the $[x_2, x_3]$ symmetry plane. Substituting $n_1 = 0$ into the Christoffel matrix [equations (5)–(10)] and introducing the in-plane phase angle with vertical ($n_2 = \sin \theta; n_3 = \cos \theta$) yields for P - SV -waves:

Equation (14) becomes identical to the corresponding Christoffel equation (12) for the $[x_1, x_3]$ plane of orthorhombic media if we replace U_2 with U_1 and interchange the indices 1 and 2 in the appropriate components of the stiffness tensor c_{ijkl} , i.e.,

$$c_{22} \rightarrow c_{11}; \quad c_{44} \rightarrow c_{55}; \quad c_{23} \rightarrow c_{13}. \quad (15)$$

Therefore, to obtain the phase velocity of both in-plane polarized modes in the $[x_2, x_3]$ plane as a function of the phase angle with vertical, it is sufficient to make the above substitutions of the elastic constants in the known phase-velocity equations for VTI media (or for the $[x_1, x_3]$ plane of orthorhombic media).

It can be demonstrated exactly in the same fashion that the Christoffel equation in the third ($[x_1, x_2]$) symmetry plane becomes identical to the VTI equations with the appropriate substitution of elastic constants, but the point has already been made.

ANISOTROPIC PARAMETERS FOR ORTHORHOMBIC MEDIA

To take full advantage of the analogy with VTI media, the stiffness coefficients c_{ij} can be replaced with a set of anisotropic parameters (combinations of stiffnesses) that concisely characterize a wide range of seismic signatures for orthorhombic anisotropy. Indeed, since seismic signatures for transversely isotropic media are especially convenient to describe in terms of dimensionless Thomsen (1986) parameters (Tsvankin, 1996), it is natural to extend that notation to the symmetry planes of orthorhombic media. As we will see below, these parameters turn out to be helpful in studying wave propagation outside the symmetry planes as well.

Another, similar version of Thomsen-style notation designed for arbitrary anisotropic media was proposed simultaneously (during the 7th Workshop on Seismic Anisotropy in Miami, 1996) by Gajewski and Pšenčík (1996) and Mensch and Rasolofosaon (1997). Their approach, however, is based on the weak-anisotropy approximation for phase velocity and leads to a different (linearized in c_{ij}) definition of the δ coefficients. One of the main points substantiated below is that the notation introduced here is advantageous for media with any strength of the anisotropy.

Definitions of the parameters

First, following Thomsen’s recipe, we define two vertical velocities (for P - and S -waves) of the reference isotropic model. For orthorhombic media, we can choose either of the two velocities of split shear waves at vertical incidence. Here, the preference is given to the S -wave polarized in the x_1 -direction to make the new notation for the “ P - SV ”-waves in the $[x_1, x_3]$ plane identical to Thomsen’s notation in the VTI case. Thus, the two “isotropic” velocities are defined by

$$\begin{bmatrix} c_{22} \sin^2 \theta + c_{44} \cos^2 \theta - \rho V^2 & (c_{23} + c_{44}) \sin \theta \cos \theta \\ (c_{23} + c_{44}) \sin \theta \cos \theta & c_{44} \sin^2 \theta + c_{33} \cos^2 \theta - \rho V^2 \end{bmatrix} \begin{bmatrix} U_2 \\ U_3 \end{bmatrix} = 0. \quad (14)$$

$$V_{P0} \equiv \sqrt{\frac{c_{33}}{\rho}}, \quad (16)$$

$$V_{S0} \equiv \sqrt{\frac{c_{55}}{\rho}}. \quad (17)$$

Since the Christoffel equation (12) for the waves polarized in the $[x_1, x_3]$ plane is identical to the corresponding equation for vertical transverse isotropy, we introduce the dimensionless coefficients $\epsilon^{(2)}$ and $\delta^{(2)}$ [the superscript (2) refers to the x_2 -axis direction, which defines the orientation of the $[x_1, x_3]$ symmetry plane] through the same equations as those used in Thomsen (1986) for VTI media:

$$\epsilon^{(2)} \equiv \frac{c_{11} - c_{33}}{2c_{33}}, \quad (18)$$

$$\delta^{(2)} \equiv \frac{(c_{13} + c_{55})^2 - (c_{33} - c_{55})^2}{2c_{33}(c_{33} - c_{55})}. \quad (19)$$

Note that in the definition of δ for VTI media, the coefficient c_{44} rather than its equal (c_{55}) has often been used. Since these two parameters differ for orthorhombic media, we should always use c_{55} in equation (19).

As in VTI media, the coefficient $\delta^{(2)}$ from equation (19) provides the *exact* second derivative of P -wave phase velocity at vertical incidence in the $[x_1, x_3]$ plane: $(d^2V/d\theta^2)|_{\theta=0} = 2V_{P0}\delta^{(2)}$. Since this derivative is needed to obtain the NMO velocity (Tsvankin, 1995) and small-angle reflection coefficient, our definition of $\delta^{(2)}$ is particularly suitable for describing reflection seismic signatures.

The analogy with vertical transverse isotropy implies that we can obtain kinematic signatures and polarizations of the P - SV -waves in the $[x_1, x_3]$ plane of orthorhombic media just by substituting V_{P0} , V_{S0} , $\epsilon^{(2)}$, and $\delta^{(2)}$ into the known equations for VTI media expressed in terms of Thomsen parameters V_{P0} , V_{S0} , ϵ , and δ . For instance, using the *exact* phase-velocity equation for VTI media expressed through Thomsen parameters (Tsvankin, 1996) and replacing ϵ and δ with $\epsilon^{(2)}$ and $\delta^{(2)}$, respectively, we find the phase velocities of the P - SV -waves in the $[x_1, x_3]$ plane as

$$\frac{V^2(\theta)}{V_{P0}^2} = 1 + \epsilon^{(2)} \sin^2 \theta - \frac{f}{2} \pm \frac{f}{2} \sqrt{\left(1 + \frac{2\epsilon^{(2)} \sin^2 \theta}{f}\right)^2 - \frac{2(\epsilon^{(2)} - \delta^{(2)}) \sin^2 2\theta}{f}}, \quad (20)$$

$$f \equiv 1 - V_{S0}^2/V_{P0}^2. \quad (21)$$

The plus sign in front of the radical corresponds to the P -wave, while the minus sign corresponds to the SV -wave. In the weak-anisotropy approximation, the P -wave phase velocity in the $[x_1, x_3]$ plane can be linearized in the small parameters $\epsilon^{(2)}$ and $\delta^{(2)}$ to obtain the well-known expression (Thomsen, 1986)

$$V_P(\theta) = V_{P0}(1 + \delta^{(2)} \sin^2 \theta \cos^2 \theta + \epsilon^{(2)} \sin^4 \theta). \quad (22)$$

Phase-velocity equations (20) and (22) demonstrate how we can use the new anisotropic parameters to take advantage of the analogy between the symmetry planes of orthorhombic

media and vertical transverse isotropy (for a more detailed discussion, see the next section).

Next, using the equivalence between equation (13) and the corresponding expression for the SH -wave in VTI media, we introduce the parameter

$$\gamma^{(2)} \equiv \frac{c_{66} - c_{44}}{2c_{44}}. \quad (23)$$

Here, $\gamma^{(2)}$ is identical to Thomsen's coefficient γ for VTI media: it is responsible for the velocity variations of the SH -wave in the $[x_1, x_3]$ plane. The coefficients $\epsilon^{(2)}$, $\delta^{(2)}$, and $\gamma^{(2)}$ coincide with the parameters $\epsilon^{(V)}$, $\delta^{(V)}$, and $\gamma^{(V)}$ (respectively) introduced in Tsvankin (1997b) and Ruger (1997) for transversely isotropic media with the symmetry axis along the x_1 -direction.

In principle, it would be convenient to specify the parameters ϵ , δ , and γ in the same fashion for the other two symmetry planes as well. However, if we did so, some of the anisotropy coefficients would not be independent, and the new notation would suffer from redundancy. In defining the anisotropic parameters, we put emphasis on simplifying seismic signatures in the two vertical planes of symmetry. To describe P - SV -waves in the $[x_2, x_3]$ symmetry plane (the plane normal to the x_1 -axis), we introduce the parameters $\epsilon^{(1)}$ and $\delta^{(1)}$ defined analogously to $\epsilon^{(2)}$ and $\delta^{(2)}$. Using the recipe in equation (15) yields

$$\epsilon^{(1)} \equiv \frac{c_{22} - c_{33}}{2c_{33}}, \quad (24)$$

$$\delta^{(1)} \equiv \frac{(c_{23} + c_{44})^2 - (c_{33} - c_{44})^2}{2c_{33}(c_{33} - c_{44})}, \quad (25)$$

$$\gamma^{(1)} \equiv \frac{c_{66} - c_{55}}{2c_{55}}. \quad (26)$$

Now, for instance, we can obtain the phase velocities of P - SV -waves in the $[x_2, x_3]$ plane from equation (20) by substituting $\epsilon^{(1)}$ for $\epsilon^{(2)}$ and $\delta^{(1)}$ for $\delta^{(2)}$; also, f should be replaced by $f_1 \equiv 1 - V_{S1}^2/V_{P0}^2$, where

$$V_{S1} = \sqrt{c_{44}/\rho} \quad (27)$$

is the vertical velocity of the S -wave polarized in the x_2 -direction.

The two vertical velocities and six anisotropy parameters introduced above can be used instead of eight original stiffness coefficients: c_{11} , c_{22} , c_{33} , c_{44} , c_{55} , c_{66} , c_{23} , and c_{13} . The only remaining stiffness c_{12} can be replaced with a dimensionless anisotropic parameter analogous to the δ coefficients in the vertical planes of symmetry

$$\delta^{(3)} \equiv \frac{(c_{12} + c_{66})^2 - (c_{11} - c_{66})^2}{2c_{11}(c_{11} - c_{66})}. \quad (28)$$

The coefficient $\delta^{(3)}$ plays the role of Thomsen's δ in TI equations written for the $[x_1, x_2]$ symmetry plane, with the x_1 -direction substituted for the symmetry axis. Note that the quantities $\epsilon^{(3)}$ and $\gamma^{(3)}$ would be redundant.

Having introduced the new parameters, it is convenient to list all of them, along with their brief descriptions:

- V_{P0} —the vertical velocity of the P -wave;
- V_{S0} —the vertical velocity of the S -wave polarized in the x_1 -direction;

- $\epsilon^{(2)}$ —the VTI parameter ϵ in the symmetry plane $[x_1, x_3]$ normal to the x_2 -axis (close to the fractional difference between the P -wave velocities in the x_1 - and x_3 -directions);
- $\delta^{(2)}$ —the VTI parameter δ in the $[x_1, x_3]$ plane (responsible for near-vertical P -wave velocity variations, also influences SV -wave velocity anisotropy);
- $\gamma^{(2)}$ —the VTI parameter γ in the $[x_1, x_3]$ plane (close to the fractional difference between the SH -wave velocities in the x_1 - and x_3 -directions);
- $\epsilon^{(1)}$ —the VTI parameter ϵ in the $[x_2, x_3]$ plane;
- $\delta^{(1)}$ —the VTI parameter δ in the $[x_2, x_3]$ plane;
- $\gamma^{(1)}$ —the VTI parameter γ in the $[x_2, x_3]$ plane;
- $\delta^{(3)}$ —the VTI parameter δ in the $[x_1, x_2]$ plane (x_1 is used as the symmetry axis).

While the new parameters are determined uniquely by the nine independent stiffness coefficients of orthorhombic media, the inverse transition is unique only for the coefficients associated with the velocities along the coordinate axes (c_{11} , c_{22} , c_{33} , c_{44} , c_{55} , and c_{66}). To obtain the other three coefficients (c_{12} , c_{13} , c_{23}) from the corresponding δ values, it is necessary to specify the sign of the sums ($c_{13} + c_{55}$) [equation (19)], ($c_{23} + c_{44}$) [equation (25)], and ($c_{12} + c_{66}$) [equation (28)]. As discussed in Tsvankin (1996), exactly the same problem arises with Thomsen parameters in transversely isotropic media. However, since the stability condition (Musgrave, 1970) requires the coefficients c_{55} , c_{44} , and c_{66} to be always positive, the sums under consideration can be negative only for uncommon large negative values of c_{13} , c_{23} , or c_{12} . Therefore, for practical purposes of seismic processing and interpretation, we can assume that ($c_{13} + c_{55}$), ($c_{23} + c_{44}$), and ($c_{12} + c_{66}$) are positive. That would correspond to one of the conditions of so-called “mild anisotropy” as specified in Schoenberg and Helbig (1997) and ensures the absence of anomalous body-wave polarizations in the symmetry planes (Helbig and Schoenberg, 1987). Note that the term “mild anisotropy” does *not* mean that the magnitude of velocity variations is small.

Although the nine parameters introduced above are sufficient to characterize general orthorhombic media, one may need to use different combinations of these coefficients in specific applications. For instance, shear-wave splitting at vertical incidence is described conventionally by the fractional difference between the parameters c_{44} and c_{55} as

$$\gamma^{(s)} \equiv \frac{c_{44} - c_{55}}{2c_{55}} = \frac{\gamma^{(1)} - \gamma^{(2)}}{1 + 2\gamma^{(2)}} \approx \frac{V_{S1} - V_{S0}}{V_{S0}}. \quad (29)$$

The parameter $\gamma^{(s)}$ represents a direct measure of the time delay between two split shear waves at vertical incidence and is identical to the generic Thomsen’s coefficient γ for transversely isotropic media with a *horizontal* symmetry axis that coincides with the x_1 -direction.

Special cases: VTI and HTI media

Both vertical and horizontal transverse isotropy can be considered as degenerate special cases of orthorhombic media. An orthorhombic medium reduces to the VTI model if the properties in all vertical planes are identical, and the velocity of each mode in the $[x_1, x_2]$ plane (the so-called “isotropy plane”) is

constant (although the velocities of the two S -waves generally differ from one another). Hence, the VTI constraints, which reduce the number of independent parameters from nine to five, can be rewritten in terms of the new parameters as

$$\begin{aligned} \epsilon^{(1)} &= \epsilon^{(2)} = \epsilon, \\ \delta^{(1)} &= \delta^{(2)} = \delta, \\ \gamma^{(1)} &= \gamma^{(2)} = \gamma, \end{aligned}$$

and

$$\delta^{(3)} = 0,$$

where ϵ , δ , and γ are Thomsen’s VTI anisotropy coefficients.

Another special case is transverse isotropy with a horizontal axis of symmetry. If the symmetry axis is oriented along the x_1 -direction, then the axes x_2 and x_3 form the isotropy plane, and

$$\begin{aligned} \epsilon^{(1)} &= 0, \\ \delta^{(1)} &= 0, \\ \gamma^{(1)} &= 0. \end{aligned}$$

The parameters $\epsilon^{(2)}$, $\delta^{(2)}$, and $\gamma^{(2)}$ in this case coincide with the coefficients $\epsilon^{(V)}$, $\delta^{(V)}$, and $\gamma^{(V)}$ (respectively) introduced in Tsvankin (1997b) and Ruger (1997). Wave propagation in HTI media is described fully by these three dimensionless anisotropic coefficients and two vertical velocities V_{P0} and V_{S0} . (The last anisotropic parameter, $\delta^{(3)}$, in this model is not independent.)

APPLICATION OF THE NEW NOTATION IN THE SYMMETRY PLANES

By design, the new parameters provide a simple way of describing seismic signatures in the symmetry planes of orthorhombic media using the known equations for VTI media expressed through Thomsen parameters (Thomsen, 1986; Tsvankin, 1996). The coefficients $\epsilon^{(i)}$, $\delta^{(i)}$, and $\gamma^{(i)}$ conveniently quantify the magnitude of velocity anisotropy in orthorhombic media, both within and outside the symmetry planes. The parameters $\epsilon^{(2)}$ and $\epsilon^{(1)}$ are close to the fractional difference between vertical and horizontal P -wave velocities in the planes $[x_1, x_3]$ and $[x_2, x_3]$ (respectively) and, therefore, yield an overall measure of the “ P -wave anisotropy” in these planes. Similarly, the coefficients $\gamma^{(2)}$ and $\gamma^{(1)}$ govern the magnitude of the velocity variation of the elliptical SH -wave in the vertical symmetry planes.

One of the most important advantages of Thomsen notation is the reduction in the number of parameters responsible for P -wave kinematic signatures. The exact P -wave phase-velocity equation for VTI media [analogous to equation (20)] contains four independent parameters (V_{P0} , V_{S0} , ϵ , and δ), but the influence of the shear-wave vertical velocity V_{S0} is practically negligible, even for strong anisotropy (Tsvankin and Thomsen, 1994). As discussed in Tsvankin (1996), the stiffness coefficient that determines the shear-wave vertical velocity ($c_{44} = c_{55}$ in VTI media) does make a contribution to the P -wave velocity equations, but *only* through the parameter δ . The analogy with VTI media implies that P -wave kinematic signatures in each

symmetry plane are determined by just *three* independent coefficients. In the vertical symmetry planes the needed parameters are the vertical P -wave velocity V_{P0} (the scaling factor) and a pair of the anisotropic coefficients introduced above: $\epsilon^{(2)}$ and $\delta^{(2)}$ ($[x_1, x_3]$ plane) or $\epsilon^{(1)}$ and $\delta^{(1)}$ ($[x_2, x_3]$ plane). As in VTI media, the coefficients $\epsilon^{(i)}$ and $\delta^{(i)}$ are responsible for phase and group velocity in different ranges of phase angles, which is extremely convenient for purposes of seismic processing and inversion. Specifically, the coefficients $\delta^{(2)}$ (plane $[x_1, x_3]$) and $\delta^{(1)}$ (plane $[x_2, x_3]$) determine near-vertical P -wave velocity variations [see equation (22)], as well as the anisotropic term in the expression for normal-moveout velocity from horizontal reflectors (discussed in more detail below). Also, the new notation simplifies the elliptical condition in both vertical symmetry planes: for example, the P -wave anisotropy in the $[x_1, x_3]$ plane is elliptical if $\epsilon^{(2)} = \delta^{(2)}$ (the SV -wave velocity in this case is constant).

To obtain any kinematic signature (e.g., phase and group velocity, moveout from horizontal reflectors, etc.), polarization vector, and reflection coefficients of the P - and SV -waves in the $[x_1, x_3]$ plane of orthorhombic media, it is sufficient to substitute V_{P0} , V_{S0} , $\epsilon^{(2)}$, and $\delta^{(2)}$ into VTI equations expressed through V_{P0} , V_{S0} , ϵ , and δ , respectively. The same analogy with vertical transverse isotropy holds for P - SV -waves in the $[x_2, x_3]$ plane, if we use VTI equations with the coefficients V_{P0} , V_{S1} , $\epsilon^{(1)}$, and $\delta^{(1)}$. Adaptation of VTI signatures to the symmetry planes of orthorhombic media using the new anisotropic parameters is discussed briefly below. The reflection coefficients in the symmetry planes of orthorhombic media are described in Ruger (1996).

Phase and group velocity

The exact phase velocity of the P - SV -waves in the $[x_1, x_3]$ plane and the weak-anisotropy approximation for the P -wave phase velocity were expressed through the new parameters in equations (20) and (22), respectively (analogous expressions hold in the $[x_2, x_3]$ plane). In a similar way, we can adapt Thomsen's (1986) weak-anisotropy approximation for the phase velocity of the SV -wave in VTI media. The SV -wave phase velocity in the $[x_1, x_3]$ plane, linearized in the anisotropic parameters, takes the form

$$V_{SV}(\theta)\{[x_1, x_3] \text{ plane}\} = V_{S0}(1 + \sigma^{(2)} \sin^2 \theta \cos^2 \theta), \quad (30)$$

where $\sigma^{(2)}$ was introduced in Tsvankin and Thomsen (1994) as

$$\sigma^{(2)} \equiv \left(\frac{V_{P0}}{V_{S0}} \right)^2 (\epsilon^{(2)} - \delta^{(2)}). \quad (31)$$

The parameter $\sigma^{(2)}$ largely determines the kinematic signatures of the SV -wave in the $[x_1, x_3]$ symmetry plane, although the influence of the individual values of the V_{P0}/V_{S0} ratio, $\epsilon^{(2)}$, and $\delta^{(2)}$ on SV propagation becomes more pronounced with increasing values of anisotropic coefficients. Although the weak-anisotropy approximations for phase velocity become less accurate with increasing velocity anisotropy, they provide a valuable analytic insight into a wide range of seismic signatures (Tsvankin, 1996). It is also possible to improve the accuracy of equations (22) and (30) by adding terms quadratic in the anisotropic parameters. For instance, Tsvankin (1996) presented the quadratic weak-anisotropy approximation for the

P -wave velocity in VTI media that can be directly applied in the symmetry planes of orthorhombic media.

Using the analogy with the SH -wave in VTI media (Thomsen, 1986), we can obtain the exact SH -wave phase velocity in the vertical symmetry planes in the following form:

$$\begin{aligned} V_{SH}(\theta)\{[x_1, x_3] \text{ plane}\} &= V_{S1} \sqrt{1 + 2\gamma^{(2)} \sin^2 \theta} \\ &= V_{S0} \sqrt{\frac{1 + 2\gamma^{(1)}}{1 + 2\gamma^{(2)}}} \sqrt{1 + 2\gamma^{(2)} \sin^2 \theta}; \end{aligned} \quad (32)$$

$$V_{SH}(\theta)\{[x_2, x_3] \text{ plane}\} = V_{S0} \sqrt{1 + 2\gamma^{(1)} \sin^2 \theta}. \quad (33)$$

The group velocity v_{gr} in the symmetry planes of orthorhombic media represents the same function of phase velocity as in VTI media (e.g., Berryman, 1979):

$$v_{gr} = V \sqrt{1 + \left(\frac{1}{V} \frac{dV}{d\theta} \right)^2}, \quad (34)$$

where θ is the phase angle with one of the coordinate axes. [Equation (34), however, will not describe cuspidal branches of S -wave group-velocity surfaces caused by out-of-plane slowness vectors.] Hence, to compute the group velocity in any of the symmetry planes, we just have to substitute the appropriate phase-velocity function into equation (34) (e.g., from equation (20) for the P - SV -waves in the $[x_1, x_3]$ plane). As is the case for TI media, in the linearized weak-anisotropy approximation the group velocity expressed through the phase angle coincides with the phase velocity. However, the group velocity corresponds to the energy propagating at the group angle ψ given in VTI media by (Berryman, 1979)

$$\tan \psi = \frac{\tan \theta + \frac{1}{V} \frac{dV}{d\theta}}{1 - \frac{\tan \theta}{V} \frac{dV}{d\theta}}. \quad (35)$$

Again, equation (35) can be used to calculate the group angle ψ with vertical in the symmetry planes of orthorhombic media by substituting the corresponding phase-velocity functions. [The phase angle θ in equation (35) should be computed with respect to the vertical (x_3) axis]. Also, the weak-anisotropy approximations for the group angles can be adapted easily from the corresponding VTI expressions (Thomsen, 1986). For instance, the P -wave's group angle with vertical in the $[x_1, x_3]$ plane can be represented as

$$\tan \psi = \tan \theta [1 + 2\delta^{(2)} + 4(\epsilon^{(2)} - \delta^{(2)}) \sin^2 \theta]. \quad (36)$$

Note that the difference between the group and phase angles does contain terms linear in the anisotropic coefficients. An obvious modification of equation (36) provides a similar expression in the $[x_2, x_3]$ symmetry plane. For wave propagation along the coordinate axes of orthorhombic media the derivative of phase velocity vanishes, and the group- and phase-velocity vectors for any wave type coincide with each other.

Polarization vector

For a given slowness direction, the polarization vectors of the three *plane* waves (the eigenvectors of the Christoffel equation) are always mutually orthogonal. This orthogonality does not hold, however, for nonplanar wavefronts excited by point sources because the three body waves recorded at any fixed point in space correspond to different slowness directions. Also, the polarization vector of the plane *P*-wave is not necessarily aligned with either phase (slowness) or group (ray) vector. Likewise, in general the split shear waves are not polarized orthogonally to the phase- or group-velocity vector (this explains the qualifier “quasi” usually added to the names of body waves in anisotropic media). In some directions, however, the polarization vector of the *P*-wave is parallel to the slowness vector; Helbig (1993) described such directions as “longitudinal.” In the symmetry planes of orthorhombic media the longitudinal directions include (but are not limited to) the coordinate axes (Schoenberg and Helbig, 1997).

Using the known expression for the polarization vector in transversely isotropic media (e.g., Helbig and Schoenberg, 1987; Tsvankin, 1996), we can write the polarization angle of the plane *P*-wave in the $[x_1, x_3]$ plane of orthorhombic media as

$$\tan \nu \equiv \frac{U_1}{U_3} = \frac{\sin \theta \cos \theta (c_{13} + c_{55})}{\rho V^2 - c_{11} \sin^2 \theta - c_{55} \cos^2 \theta}, \quad (37)$$

where V is the phase velocity of the *P*-wave. Equation (37) is valid for any strength of the anisotropy and determines the *SV*-wave polarization angle as well (the polarization vectors of the *P*- and *SV*-waves are orthogonal). The second shear wave in the $[x_1, x_3]$ plane represents a pure shear (*SH*) mode polarized in the x_2 -direction.

Adapting the results of Rommel (1994) and Tsvankin (1996) obtained for TI media, we find the following weak-anisotropy approximation for equation (37)

$$\tan \nu = \tan \theta \{1 + B[2\delta^{(2)} + 4(\epsilon^{(2)} - \delta^{(2)}) \sin^2 \theta]\}, \quad (38)$$

$$B \equiv \frac{1}{2f} = \frac{1}{2(1 - V_{S0}^2/V_{P0}^2)}.$$

With the appropriate substitutions of the elastic constants [see relations (15)] or anisotropic coefficients, equations (37) and (38) can be used to obtain the polarization angles for *P-SV* waves in the $[x_2, x_3]$ plane.

As discussed in Tsvankin (1996), comparison of the weak-anisotropy approximations for the group angle (36) and the polarization angle (38) shows that for weak and moderate anisotropy, the *P*-wave polarization vector lies closer to the corresponding ray (group) vector than to the phase vector. To obtain body-wave polarizations in the far-field of a point source, it is necessary to find the phase angle θ corresponding to a given group (source-receiver) direction and substitute it into equations (37) or (38). Near-field polarizations, however, depend on the relative amplitudes of several terms of the ray series expansion and, therefore, can be influenced by azimuthal velocity variations in orthorhombic media.

Moveout from horizontal reflectors

Suppose a common-midpoint (CMP) line is parallel to the x_1 -axis of a horizontal orthorhombic layer. Then the phase- and group-velocity vectors of the waves reflected from the bottom of the layer stay in the $[x_1, x_3]$ symmetry plane, and the short-spread normal-moveout (NMO) velocity can be obtained from the known VTI equations (e.g., Thomsen, 1986; Tsvankin, 1996). The substitutions described above yield

$$V_{\text{nmo}}^{(2)} [P\text{-wave}] = V_{P0} \sqrt{1 + 2\delta^{(2)}}, \quad (39)$$

$$V_{\text{nmo}}^{(2)} [SV\text{-wave}] = V_{S0} \sqrt{1 + 2\sigma^{(2)}}, \quad (40)$$

$$V_{\text{nmo}}^{(2)} [SH\text{-wave}] = V_{S0} \sqrt{1 + 2\gamma^{(2)}}, \quad (41)$$

where $\sigma^{(2)}$ was defined in equation (31).

Similarly, the *P*-wave NMO velocity on a line parallel to the x_2 -axis is given by

$$V_{\text{nmo}}^{(1)} [P\text{-wave}] = V_{P0} \sqrt{1 + 2\delta^{(1)}}. \quad (42)$$

Moveout velocity is one of the most important parameters in reflection data processing, and the simplicity of the above expressions is a good illustration of the advantages of the notation introduced here. Note that equations (39)–(42) remain valid for any strength of velocity anisotropy. Evidently, if the symmetry-plane directions and the vertical velocity is known, short-spread *P*-wave moveout in the vertical symmetry planes makes it possible to obtain both δ coefficients.

To find normal-moveout velocity in symmetry planes of *multilayered* laterally homogeneous orthorhombic models, it is sufficient just to apply the conventional Dix equation. This conclusion follows from the general NMO formula for any symmetry plane in anisotropic media given in Alkhalifah and Tsvankin (1995). Of course, for a stack of orthorhombic layers to have a throughgoing vertical symmetry plane, the horizontal coordinate axes in all layers have to be aligned.

For relatively large spreadlengths (exceeding the reflector depth) in anisotropic media, reflection moveout becomes non-hyperbolic and can be described by the following expression developed by Tsvankin and Thomsen (1994) for vertically inhomogeneous VTI media:

$$t^2(x) = t_0^2 + A_2 x^2 + \frac{A_4 x^4}{1 + A x^2}, \quad (43)$$

where t_0 is the zero-offset reflection traveltime, $A_2 = 1/V_{\text{nmo}}^2$, A_4 is the quartic moveout coefficient, and

$$A = \frac{A_4}{\frac{1}{V_{P90}^2} - A_2},$$

where V_{P90} is the horizontal velocity. The normalization of the quartic moveout term makes equation (43) numerically accurate for *P*-wave reflection moveout on long spreads (2–3 times, and more, the reflector depth), even in VTI models with pronounced velocity anisotropy (Tsvankin and Thomsen, 1994). The analogy with vertical transverse isotropy implies that the nonhyperbolic moveout equation (43) can be used without any

modification on CMP lines parallel to one of the horizontal coordinate axes of an orthorhombic layer (or in the symmetry planes of a layered orthorhombic medium). For instance, for a line coinciding with the x_1 -axis we have to make the following substitutions in equation (43) (for the P -wave)

$$A_2 = \frac{1}{V_{P0}^2(1 + 2\delta^{(2)})},$$

$$V_{P90} = V_{P0}\sqrt{1 + 2\epsilon^{(2)}},$$

and

$$A_4 = -\frac{2(\epsilon^{(2)} - \delta^{(2)})}{t_0^2 V_{P0}^4} \frac{1 + 2\delta^{(2)}/f}{(1 + 2\delta^{(2)})^4},$$

with f given by equation (21). While equation (43) as a whole is an approximation for long-spread moveout, the above expression for the quartic coefficient A_4 is exact for any strength of the anisotropy.

Obvious modifications of the above equations provide similar expressions for the $[x_2, x_3]$ symmetry plane. The behavior of the exact normal-moveout velocity outside the symmetry planes is discussed in Grechka and Tsvankin (1996).

Dipping reflectors and 2-D processing

The equivalence between orthorhombic media and vertical transverse isotropy breaks down if the phase- or group-velocity vectors of seismic waves deviate from the vertical symmetry planes. Evidently, 3-D seismic processing in orthorhombic media cannot be carried out by just adapting the known VTI algorithms. However, if the subsurface can be approximated by a 2-D structure with the strike parallel to one of the horizontal coordinate axes of an orthorhombic model, the incident and reflected rays on the dip line (normal to the strike) will be confined to one of the vertical symmetry planes. (For more complicated 3-D models it may be possible to suppress out-of-plane events.) Normal-moveout velocity of in-plane dipping events can be described by the following equation valid for pure modes in any symmetry plane of arbitrary anisotropic media (Tsvankin, 1995):

$$V_{\text{nmo}}(\phi) = \frac{V(\phi)}{\cos \phi} \frac{\sqrt{1 + \frac{1}{V(\phi)} \frac{d^2 V}{d\theta^2} \Big|_{\theta=\phi}}}{1 - \frac{\tan \phi}{V(\phi)} \frac{dV}{d\theta} \Big|_{\theta=\phi}}, \quad (44)$$

where θ is the phase angle with vertical, and ϕ is the dip of the reflector. Equation (44) can be considerably simplified by using the weak-anisotropy approximation. Adapting the result obtained by Tsvankin (1995) for vertical transverse isotropy in the limit of small ϵ and δ , we get the following expression for the P -wave NMO velocity in the $[x_1, x_3]$ plane of orthorhombic media ($|\epsilon^{(2)}| \ll 1$, $|\delta^{(2)}| \ll 1$):

$$\frac{V_{\text{nmo}}^{(2)}(\phi) \cos \phi}{V_{\text{nmo}}^{(2)}(0)} = 1 + \delta^{(2)} \sin^2 \phi + 3(\epsilon^{(2)} - \delta^{(2)}) \sin^2 \phi (2 - \sin^2 \phi). \quad (45)$$

In reflection data processing, NMO velocity is usually treated as a function of the ray parameter $p(\phi)$ that can be determined

from the slope of reflections recorded on zero-offset (or CMP-stacked) seismic sections. For vertical transverse isotropy, P -wave NMO velocity from dipping reflectors [equation (44)] expressed as a function of p is determined by just *two* parameters: the NMO velocity from a horizontal reflector [$V_{\text{nmo}}(0)$] and an anisotropic coefficient defined as $\eta \equiv (\epsilon - \delta)/(1 + 2\delta)$ (Alkhalifah and Tsvankin, 1995). The parameter η can be considered as a measure of “anellipticity” of a particular TI medium. Alkhalifah and Tsvankin (1995) have shown that $V_{\text{nmo}}(0)$ and η control not just the dip-dependent NMO velocity, but also all time-related P -wave processing steps (NMO, DMO, prestack and poststack time migration) in homogeneous or vertically inhomogeneous VTI media.

This conclusion remains entirely valid for P -wave reflections confined to one of the vertical symmetry planes of orthorhombic media. For instance, the kinematics of 2-D time processing in the $[x_1, x_3]$ symmetry plane is governed by the corresponding P -wave NMO velocity from a horizontal reflector [equation (39)] and a parameter $\eta^{(2)}$ given by

$$\eta^{(2)} \equiv \frac{\epsilon^{(2)} - \delta^{(2)}}{1 + 2\delta^{(2)}}. \quad (46)$$

As an example, the nonhyperbolic moveout equation (43) in the $[x_1, x_3]$ plane can be rewritten as a function of just these two effective parameters and the vertical time t_0 (the contribution of V_{S0} to the quartic term A_4 can be ignored):

$$t^2(x) = t_0^2 + \frac{x^2}{[V_{\text{nmo}}^{(2)}(0)]^2} - \frac{2\eta^{(2)}x^4}{[V_{\text{nmo}}^{(2)}(0)]^2 \{t_0^2 [V_{\text{nmo}}^{(2)}(0)]^2 + [1 + 2\eta^{(2)}]x^2\}}. \quad (47)$$

If $\eta^{(2)} = 0$, the P -wave anisotropy in the $[x_1, x_3]$ plane is elliptical and the reflection moveout is purely hyperbolic.

Alkhalifah and Tsvankin (1995) show that both parameters needed for P -wave time processing in VTI media can be obtained from surface P -wave data using NMO velocities from two reflectors with different dips. This algorithm holds for parameter estimation using in-plane dipping events in the vertical symmetry planes of orthorhombic media. For instance, if the strike of the structure is parallel to the x_2 -axis, NMO velocities of reflections from horizontal and dipping interfaces recorded on the dip line (parallel to the x_1 -axis) can be used to recover the zero-dip NMO velocity and the parameter $\eta^{(2)}$ from equation (44). In principle, the coefficient $\eta^{(2)}$ can be estimated using the long-spread (nonhyperbolic) P -wave moveout from horizontal reflectors [see equation (47)], although with lower accuracy (Tsvankin and Thomsen, 1995). Once the two effective parameters have been determined, 2-D time processing in the $[x_1, x_3]$ symmetry plane can be carried out using the methodology developed for vertical transverse isotropy. To perform 2-D depth processing in either of the vertical symmetry planes, it is also necessary to know the P -wave vertical velocity.

In general, all 2-D data-processing steps (NMO, DMO, time and depth migration) in the vertical symmetry planes of orthorhombic media can be performed by VTI algorithms. Body-wave amplitudes, however, will be influenced by azimuthal velocity variations, which may cause distortions in the so-called “true-amplitude” DMO and migration methods.

P-WAVE KINEMATICS OUTSIDE THE SYMMETRY PLANES

While the new notation has obvious advantages in describing symmetry-plane seismic signatures, an open question is whether it can be useful in treating wave propagation outside the symmetry planes. The focus here will be on phase velocity—the most fundamental function that determines all other kinematic signatures for a particular mode. Since Thomsen notation does not provide any tangible simplification of the exact phase-velocity equations for transversely isotropic media (Tsvankin, 1996), our notation can hardly be expected to accomplish this task for orthorhombic models. Indeed, the exact phase velocity in terms of the new coefficients still has to be computed numerically (see Appendix A).

However, as demonstrated by the symmetry-plane analysis in the previous section, the new notation may be helpful in developing concise weak-anisotropy approximations for phase and group velocity, reducing the number of parameters responsible for P -wave kinematics for any strength of the anisotropy, and simplifying the exact analytic expressions for reflection seismic signatures (such as NMO velocity). Hence, our next step here is to transform the phase-velocity equations in the limit of weak anisotropy.

Weak-anisotropy approximation for phase velocity

Since the dimensionless anisotropic parameters introduced above should be small in weakly anisotropic media, the new notation is particularly suitable for developing weak-anisotropy approximations by expanding seismic signatures in powers of $\epsilon^{(i)}$, $\delta^{(i)}$, and $\gamma^{(i)}$ (usually only the linear terms are retained). For the planes of symmetry it is sufficient just to adapt the known expressions for weak transverse isotropy (see the previous sections). Approximate P -wave phase velocity outside the symmetry planes is obtained in Appendix B by linearizing the exact equations (A-8) and (A-9) in the anisotropic coefficients:

$$V_P^2 = V_{P0}^2 [1 + 2n_1^4 \epsilon^{(2)} + 2n_2^4 \epsilon^{(1)} + 2n_1^2 n_3^2 \delta^{(2)} + 2n_2^2 n_3^2 \delta^{(1)} + 2n_1^2 n_2^2 (2\epsilon^{(2)} + \delta^{(3)})]. \quad (48)$$

It is convenient to replace the directional cosines of the slowness (or phase-velocity) vector by the polar (θ) and azimuthal (ϕ) phase angles,

$$n_1 = \sin \theta \cos \phi, \quad n_2 = \sin \theta \sin \phi, \quad n_3 = \cos \theta.$$

Then, taking the square-root of equation (48), we obtain the phase velocity exactly in the same form as in VTI media [equation (22)], but with azimuthally dependent coefficients ϵ and δ :

$$V_P(\theta, \phi) = V_{P0} [1 + \delta(\phi) \sin^2 \theta \cos^2 \theta + \epsilon(\phi) \sin^4 \theta]; \quad (49)$$

$$\delta(\phi) = \delta^{(1)} \sin^2 \phi + \delta^{(2)} \cos^2 \phi,$$

$$\epsilon(\phi) = \epsilon^{(1)} \sin^4 \phi + \epsilon^{(2)} \cos^4 \phi + (2\epsilon^{(2)} + \delta^{(3)}) \sin^2 \phi \cos^2 \phi.$$

In both vertical symmetry planes, equation (49) reduces to Thomsen's (1986) weak-anisotropy approximation for TI media.

The structure of equation (49) is similar to the expansion of P -wave phase velocity in a series of spherical harmonics developed by Sayers (1994b). Here, however, instead of describing the medium by perturbations of the stiffness coefficients (as was done by Sayers), we have obtained a concise approximation in terms of the dimensionless anisotropic parameters $\epsilon^{(i)}$ and $\delta^{(i)}$. Since Sayers (also see Jech and Pšenčík, 1989; Mensch and Rasolofosaon, 1997) neglects terms of order $(\Delta c_{ij}/c_{ij})^2$, whereas we neglect terms of order δ^2 (etc.), this linearization is slightly different. Because of the structure of the Christoffel equation (2) for orthorhombic media, it is convenient to use *nonlinear* combinations of elastic moduli ($\delta^{(i)}$) to characterize the anisotropy. As mentioned above, the δ coefficients defined here yield the *exact* second derivative of P -wave phase velocity at vertical incidence that is needed to describe such reflection seismic signatures as NMO velocity and small-angle reflection coefficient.

Parameters responsible for P -wave velocity

It should be emphasized that equation (49) does not contain any of the three parameters (V_{S0} , $\gamma^{(1)}$, $\gamma^{(2)}$) that describe the shear-wave velocities in the directions of the coordinate axes. Evidently, kinematic signatures of P -waves in *weakly anisotropic* orthorhombic media depend on just *five* anisotropic coefficients ($\epsilon^{(1)}$, $\delta^{(1)}$, $\epsilon^{(2)}$, $\delta^{(2)}$, and $\delta^{(3)}$) and the vertical velocity. Similar conclusions for weakly anisotropic models of various symmetries were drawn by Chapman and Pratt (1992), Sayers (1994b), and Mensch and Rasolofosaon (1997).

The equivalence between the symmetry planes of orthorhombic media and transverse isotropy, discussed in detail above, implies that these six parameters are sufficient to describe P -wave phase velocity in all three symmetry planes, even for *strong* velocity anisotropy. An important question, to be addressed next, is whether or not P -wave phase velocity depends only on the same six parameters *outside* the symmetry planes and in the presence of *significant* velocity variations.

In the numerical examples below, we examine the influence of the parameters V_{S0} and $\gamma^{(i)}$ on the exact phase velocity of P -wave for orthorhombic models with moderate and strong velocity anisotropy. It is sufficient to consider the velocity function in a single octant, for instance, the one corresponding to $0 \leq \theta \leq 90^\circ$, $0 \leq \phi \leq 90^\circ$. Figure 2 shows the dependence of P -wave phase-velocity variations in four vertical planes on the shear-wave vertical velocity V_{S0} (or the V_{P0}/V_{S0} ratio), with all other model parameters being fixed. Clearly, the influence of V_{S0} within a wide range of the V_{P0}/V_{S0} ratios is negligible not only in the $[x_1, x_3]$ symmetry plane ($\phi = 0$), as was expected (Tsvankin, 1996), but outside the symmetry planes as well. This conclusion holds for two models with more pronounced phase-velocity variations shown in Figure 3. Note that even for a medium with uncommonly strong velocity anisotropy (Figure 3b) the difference between the phase-velocity curves corresponding to the two extreme values of the V_{P0}/V_{S0} ratio remains barely visible. The contribution of V_{S0} to the P -wave phase velocity becomes somewhat more noticeable only for uncommon models with close values of the P - and shear-wave velocities in one of the coordinate directions.

It should be emphasized that if the conventional notation is used, the influence of the stiffness coefficients c_{44} , c_{55} , and c_{66} on P -wave velocity cannot be ignored. For instance, c_{55} does make

a contribution to the value of $\delta^{(2)}$ [equation (19)]. As in VTI media, our notation makes it possible to reduce the number of parameters that govern P -wave velocity by combining the stiffness coefficients c_{55} and c_{13} in the single parameter $\delta^{(2)}$; the same holds for the stiffnesses c_{44} and c_{66} as well.

The P -wave phase velocity at two extreme values of the parameter $\gamma^{(1)}$ is shown in Figure 4. Since none of the elastic constants governing P -wave propagation in the $[x_1, x_3]$ plane depends on $\gamma^{(1)}$ (with V_{S0} being fixed), the plot starts at an azimuth of 20° . Although the influence of $\gamma^{(1)}$ increases slightly in the vicinity of the $[x_2, x_3]$ plane ($\phi = 90^\circ$), the contribution of this parameter can be safely ignored at all azimuthal angles. Similarly, P -wave velocity is independent of the coefficient $\gamma^{(2)}$.

Thus, P -wave phase-velocity variations in orthorhombic media are governed by just five anisotropic parameters ($\epsilon^{(1)}$, $\delta^{(1)}$, $\epsilon^{(2)}$, $\delta^{(2)}$, and $\delta^{(3)}$), with the vertical P -wave phase velocity serving as a scaling coefficient (in homogeneous media). The 3-D phase-velocity (or slowness) function fully determines the group-velocity (ray) vector and, therefore, all other kinematic signatures (e.g., reflection traveltime). This means that the five anisotropic parameters listed above and the vertical velocity are sufficient to describe all kinematic properties of P -waves in orthorhombic media.

Furthermore, P -wave reflection traveltimes from mildly dipping reflectors in orthorhombic models with a horizontal symmetry plane should be determined largely by the four

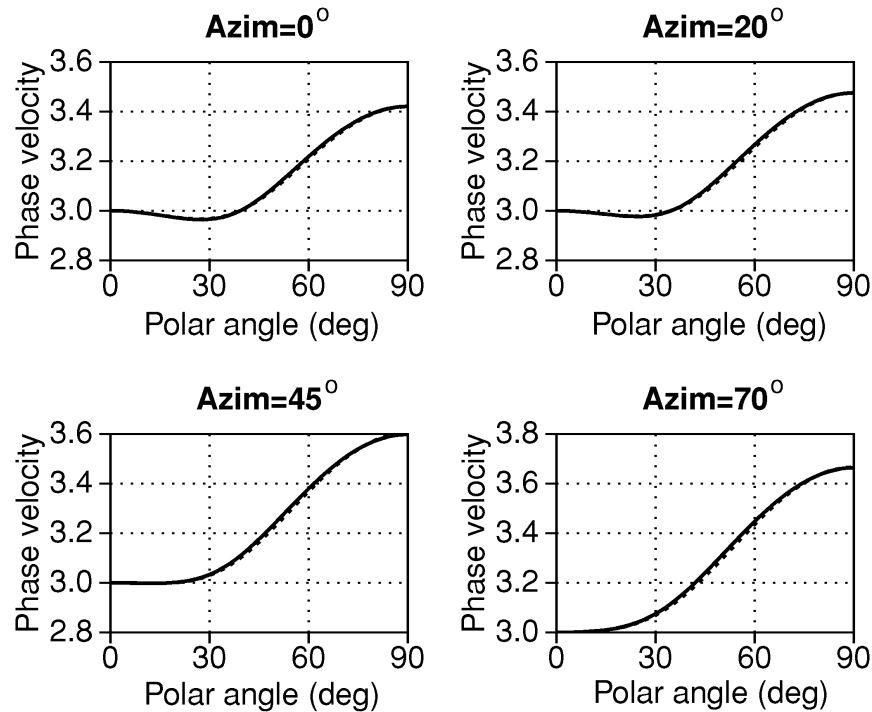


FIG. 2. Influence of the vertical shear-wave velocity V_{S0} on the exact P -wave phase velocity. The velocity is calculated as a function of the phase angle with vertical at the four azimuthal phase angles shown on top of the plots. The solid curve corresponds to $V_{P0}/V_{S0} = 1.5$ ($V_{P0} = 3$ km/s, $V_{S0} = 2$ km/s), the dotted curve to $V_{P0}/V_{S0} = 2.5$ ($V_{P0} = 3$ km/s, $V_{S0} = 1.2$ km/s). The other model parameters are: $\epsilon^{(1)} = 0.25$, $\epsilon^{(2)} = 0.15$, $\delta^{(1)} = 0.05$, $\delta^{(2)} = -0.1$, $\delta^{(3)} = 0.15$, $\gamma^{(1)} = 0.28$, $\gamma^{(2)} = 0.15$.

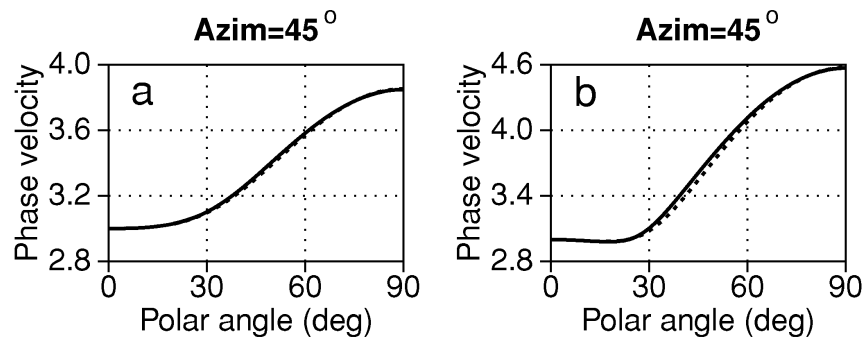


FIG. 3. P -wave phase velocity for different V_{P0}/V_{S0} ratios in media with stronger anisotropy than in Figure 2. The velocity is calculated as a function of the phase angle with vertical at an azimuthal phase angle of 45° . The solid curve corresponds to $V_{P0}/V_{S0} = 1.5$ ($V_{P0} = 3$ km/s, $V_{S0} = 2$ km/s), the dotted curve to $V_{P0}/V_{S0} = 2.5$ ($V_{P0} = 3$ km/s, $V_{S0} = 1.2$ km/s). The other model parameters are: (a) $\epsilon^{(1)} = 0.25$, $\epsilon^{(2)} = 0.45$, $\delta^{(1)} = -0.1$, $\delta^{(2)} = 0.2$, $\delta^{(3)} = -0.15$, $\gamma^{(1)} = 0.28$, $\gamma^{(2)} = 0.15$; (b) $\epsilon^{(1)} = 0.45$, $\epsilon^{(2)} = 0.6$, $\delta^{(1)} = -0.15$, $\delta^{(2)} = -0.1$, $\delta^{(3)} = 0.2$, $\gamma^{(1)} = 0.7$, $\gamma^{(2)} = 0.3$.

anisotropic parameters defined in the vertical planes of symmetry [$\epsilon^{(1)}$, $\delta^{(1)}$, $\epsilon^{(2)}$, and $\delta^{(2)}$] and the symmetry-plane azimuths. Indeed, the influence of the fifth coefficient [$\delta^{(3)}$] is mostly limited to near-horizontal velocity variations outside the vertical symmetry planes [see equation (49)].

Accuracy of the weak-anisotropy approximation

Now that we have shown that the weak-anisotropy approximation (49) contains *all* parameters responsible for *P*-wave velocity in orthorhombic media, we can study the dependence of the error of equation (49) on these anisotropic coefficients. As illustrated by Figure 5, equation (49) provides a good approximation (especially near vertical) for the exact phase velocity in media with moderate velocity anisotropy, both within and outside the symmetry planes. The model in Figure 5, taken from Schoenberg and Helbig (1997), corresponds to an effective orthorhombic medium formed by parallel vertical cracks embedded in a VTI background medium.

Even in the model with uncommonly strong velocity anisotropy shown in Figure 6, equation (49) deviates from the exact solution by no more than 10%. Also, note that for this example the weak-anisotropy approximation does not deteriorate outside the symmetry planes; in fact, it is in the symmetry planes where we observe the maximum error for some models.

Although the accuracy of the weak-anisotropy approximation becomes lower with increasing anisotropic coefficients, the error remains relatively small for polar angles θ up to about 70° . Higher errors of equation (49) near horizontal are not surprising since our notation is designed to provide a better approximation for near-vertical velocity variations. To increase the accuracy of the weak-anisotropy approximation at near-horizontal directions (which may be important in

cross-borehole studies), the “isotropic” parameter can be chosen as one of the horizontal velocities.

Equation (49) can be used to build analytic weak-anisotropy solutions for such signatures as group velocity, polarization angle, point-source radiation pattern, etc. However, since these solutions require multiple additional linearizations and involve the derivatives of phase velocity, their accuracy may be much lower than that of the phase-velocity expression itself.

DISCUSSION AND CONCLUSIONS

Analytic description of wave propagation in orthorhombic media can be significantly simplified by replacing the stiffness coefficients with two “isotropic” vertical velocities and a set of anisotropy parameters similar to the coefficients ϵ , δ , and γ suggested by Thomsen (1986) for vertical transverse isotropy. The definitions of the anisotropy parameters introduced here are based on the analogous form of the Christoffel equation in the symmetry planes of orthorhombic and transversely isotropic media. This analogy implies that all kinematic signatures of body waves, plane-wave polarizations, and reflection coefficients in the symmetry planes of orthorhombic media are given by the same equations (with the appropriate substitutions of c_{ij} 's) as for vertical transverse isotropy. (The only exception is cuspidal branches of shear-wave group-velocity surfaces formed in symmetry planes of orthorhombic media by out-of-plane slowness vectors.) Since Thomsen's notation provides particularly concise expressions for seismic signatures in VTI media (Tsvankin, 1996), the new parameters can be conveniently used to obtain phase, group, and normal-moveout velocity, nonhyperbolic (long-spread) reflection moveout, and plane-wave polarization angle in the symmetry planes of orthorhombic media by adapting the known VTI results. The

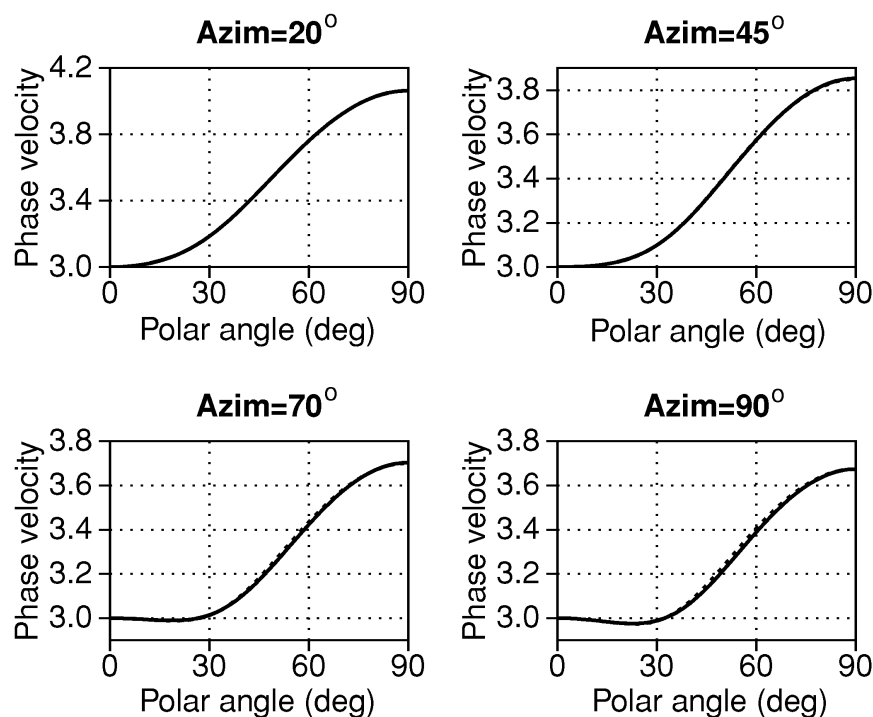


FIG. 4. Influence of the parameter $\gamma^{(1)}$ on the *P*-wave phase velocity. The solid curve corresponds to $\gamma^{(1)} = -0.1$, the dotted curve to $\gamma^{(1)} = 0.65$. The other model parameters are: $V_{P0} = 3$ km/s, $V_{S0} = 1.5$ km/s, $\epsilon^{(1)} = 0.25$, $\epsilon^{(2)} = 0.45$, $\delta^{(1)} = -0.1$, $\delta^{(2)} = 0.2$, $\delta^{(3)} = -0.15$, $\gamma^{(2)} = 0.15$.

equivalence with VTI media can even be used to get analytic expressions for reflection moveout in throughgoing symmetry planes of layered orthorhombic media.

All 2-D seismic processing steps in the vertical symmetry planes can be carried out using the algorithms developed for vertical transverse isotropy, although the amplitude treatment in orthorhombic media should be different. To perform 2-D P -wave time processing (NMO, DMO, time migration) in either symmetry plane, it is necessary to obtain the corresponding zero-dip NMO velocity and the anisotropy coefficient η introduced in Alkhalifah and Tsvankin (1995).

It should be mentioned, however, that the kinematic equivalence between the symmetry planes of orthorhombic media and vertical transverse isotropy *does not* apply to body-wave

amplitudes in general. In particular, point-source radiation patterns in the symmetry planes still depend on azimuthal velocity variations and, therefore, should be studied using analytic and numerical methods developed for azimuthally anisotropic media (Tsvankin and Chesnokov, 1990a; Gajewski, 1993). The influence of out-of-plane velocity variations on body-wave amplitudes also means that near-field polarizations in the symmetry planes of orthorhombic media may be different from those in VTI media. Indeed, polarization in the “nongeometrical” region close to the source is generally nonlinear and depends on the relative amplitudes of at least the first two terms of the ray-series expansion (Tsvankin and Chesnokov, 1990a). In contrast, far-field polarizations are adequately described by the geometrical-seismics (plane-wave)

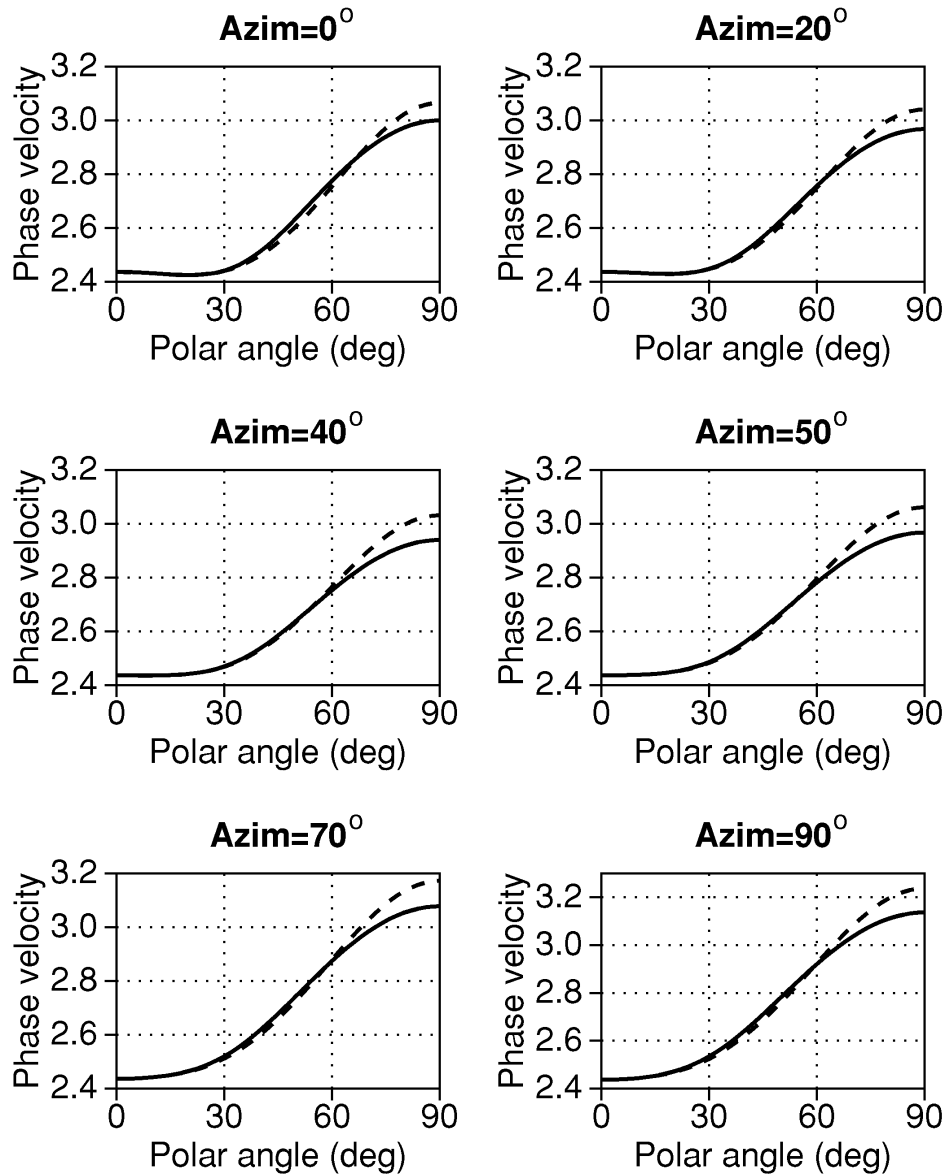


FIG. 5. Comparison between the exact P -wave phase velocity (solid curve) and the weak-anisotropy approximation from equation (49) (dashed curve) for the “standard” orthorhombic model taken from Schoenberg and Helbig (1997). The velocities are calculated at the six azimuthal phase angles shown on top of the plots. The parameters are $V_{P0} = 2.437$ km/s, $\epsilon^{(1)} = 0.329$, $\epsilon^{(2)} = 0.258$, $\delta^{(1)} = 0.083$, $\delta^{(2)} = -0.078$, $\delta^{(3)} = -0.106$ (the other coefficients that do not influence P -wave velocity are $V_{S0} = 1.265$ km/s, $\gamma^{(1)} = 0.182$, $\gamma^{(2)} = 0.0455$).

approximation, which can be obtained by analogy with vertical transverse isotropy.

The dimensionless anisotropic coefficients conveniently characterize the magnitude of anisotropy and represent a natural tool for developing weak-anisotropy approximations *outside* the symmetry planes of orthorhombic media. The linearized weak-anisotropy approximation for P -wave phase velocity in terms of the new parameters has the same form as for vertical transverse isotropy, but with azimuthally dependent coefficients ϵ and δ . This expression provides sufficient accuracy even in models with pronounced velocity anisotropy and can be used to develop weak-anisotropy solutions for the group (ray) angle, polarization vector, etc.

The phase-velocity approximation also shows that the kinematics of P -waves in weakly anisotropic orthorhombic models

is independent of the three parameters responsible for shear-wave velocities along the coordinate axes (the influence of c_{44} , c_{55} , and c_{66} is absorbed by the δ coefficients). Although the error of the weak-anisotropy approximation grows with increasing anisotropy, the exact phase velocity is still governed by only five anisotropic parameters ($\epsilon^{(1)}$, $\delta^{(1)}$, $\epsilon^{(2)}$, $\delta^{(2)}$, and $\delta^{(3)}$) and the vertical P -wave velocity. This means that all kinematic signatures of P -waves in orthorhombic media depend on six parameters (rather than nine in the conventional notation), one of which—the vertical P -wave velocity—represents just a scaling coefficient in homogeneous media. Furthermore, reflection traveltimes from mildly dipping interfaces in orthorhombic media with a horizontal symmetry plane are expected to be only weakly dependent on the coefficient $\delta^{(3)}$. In each symmetry plane, the kinematics of P -waves is determined by three

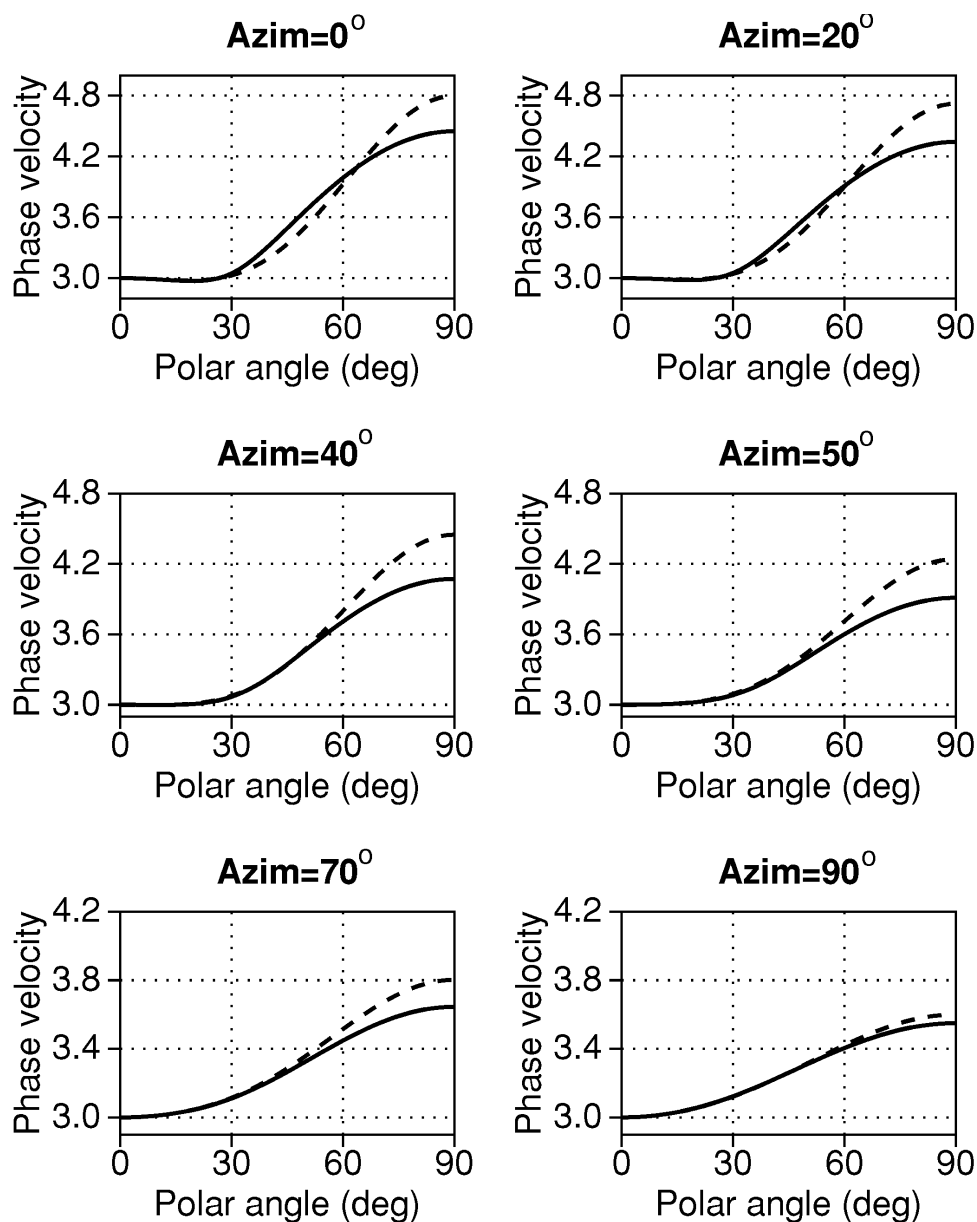


FIG. 6. Comparison between the exact P -wave phase velocity (solid curve) and the weak-anisotropy approximation from equation (49) (dashed curve). The model parameters are $V_{P0} = 3$ km/s, $\epsilon^{(1)} = 0.2$, $\epsilon^{(2)} = 0.6$, $\delta^{(1)} = 0.15$, $\delta^{(2)} = -0.15$, $\delta^{(3)} = -0.2$.

parameters—a pair of the corresponding coefficients ϵ and δ and the P -wave velocity along one of the in-plane coordinate axes. Clearly, the new parameters capture the combinations of the stiffness coefficients responsible for P -wave kinematic signatures in orthorhombic media. Hence, P -wave inversion algorithms for orthorhombic media should target these anisotropic coefficients instead of the stiffness components.

It is important to emphasize that the notation introduced here is convenient to use in orthorhombic media with *any* strength of the anisotropy. The most important advantages of the new anisotropic coefficients, such as the reduction in the number of parameters responsible for P -wave velocity and the simple expressions for seismic signatures in the symmetry planes, remain valid even in strongly anisotropic models. As shown by Grechka and Tsvankin (1996), the *exact* NMO velocity outside the symmetry planes of a horizontal orthorhombic layer is a simple function of the two δ coefficients defined in the vertical symmetry planes. Hence, concise weak-anisotropy approximations can be regarded as just another advantage of this notation, as is the case with Thomsen notation in VTI media (Tsvankin, 1996).

The parameters introduced here are defined with respect to the coordinate system associated with the symmetry planes. In seismic experiments, the orientation of the symmetry planes may be unknown and should generally be determined from the data. The number of unknown coefficients in this case increases from nine to twelve. In many typical fractured reservoirs, however, one of the symmetry planes is horizontal and the only additional (tenth) parameter is the azimuth of one of the vertical symmetry planes.

While the new notation is clearly superior to the stiffness coefficients in the inversion and processing of seismic data, the generic parameter set described above may require some modification for specific acquisition geometries. For example, in the inversion of cross-borehole data it may be convenient to replace $\delta^{(1)}$ and $\delta^{(2)}$ by similar parameters that govern phase-velocity variations near *horizontal*.

The new parameters make up a unified framework for an analytic description of seismic signatures in orthorhombic, VTI, and HTI models. Both vertical and horizontal transverse isotropy can be characterized by specific subsets of the dimensionless anisotropic parameters defined here for the more complicated orthorhombic model. For instance, the parameters $\epsilon^{(V)}$, $\delta^{(V)}$, and $\gamma^{(V)}$ introduced for HTI media by Tsvankin (1997b) and Rüger (1997) are equivalent to the set of the anisotropic coefficients in one of the vertical symmetry planes (depending on the orientation of the symmetry axis). This uniformity of notation simplifies comparative analysis of seismic signatures and transition between different anisotropic models in seismic inversion. On the whole, the new parameters provide an analytic basis for extending inversion and processing algorithms to media with orthorhombic symmetry, which is believed to be typical for realistic fractured reservoirs.

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APPENDIX A

EXACT PHASE VELOCITY FOR ORTHORHOMBIC MEDIA

The goal of this Appendix is to review the known phase-velocity equations for orthorhombic media used to derive the weak-anisotropy approximation for *P*-wave phase velocity in Appendix B. It is also shown how the exact phase velocities of all three waves can be computed using the anisotropic parameters defined in the main text.

Equation (4) of the main text leads to the following cubic equation for the phase velocity valid for any homogeneous anisotropic medium (e.g., Schoenberg and Helbig, 1997):

$$x^3 + ax^2 + bx + c = 0, \quad (\text{A-1})$$

where $x = \rho V^2$ and

$$a = -(G_{11} + G_{22} + G_{33}), \quad (\text{A-2})$$

$$b = G_{11}G_{22} + G_{11}G_{33} + G_{22}G_{33} - G_{12}^2 - G_{13}^2 - G_{23}^2, \quad (\text{A-3})$$

$$c = G_{11}G_{23}^2 + G_{22}G_{13}^2 + G_{33}G_{12}^2 - G_{11}G_{22}G_{33} - 2G_{12}G_{13}G_{23}. \quad (\text{A-4})$$

Through a change of variables ($x = y - a/3$), we can eliminate the quadratic term in equation (A-1) and reduce it to the following form (e.g., Korn and Korn, 1968):

$$y^3 + dy + q = 0, \quad (\text{A-5})$$

with the coefficients

$$d = -\frac{a^2}{3} + b, \quad (\text{A-6})$$

$$q = 2\left(\frac{a}{3}\right)^3 - \frac{ab}{3} + c. \quad (\text{A-7})$$

Using the fact that the matrix \mathbf{G} is symmetric, it can be shown that the coefficient d is negative (Schoenberg and Helbig, 1997). For roots of equation (A-5) to be real, the following combination of d and q should be nonpositive:

$$Q = \left(\frac{d}{3}\right)^3 + \left(\frac{q}{2}\right)^2 \leq 0.$$

Then the three solutions of equation (A-5) can be represented as (Korn and Korn, 1968)

$$y_{1,2,3} = 2\sqrt{\frac{-d}{3}} \cos\left(\frac{\nu}{3} + k\frac{2\pi}{3}\right), \quad (\text{A-8})$$

where $k = 0, 1, 2$, and

$$\cos \nu = -\frac{q}{2\sqrt{(-d/3)^3}}; \quad 0 \leq \nu \leq \pi.$$

The phase velocity is found from

$$\rho V^2 = y - a/3. \quad (\text{A-9})$$

The largest root ($k = 0$) of equation (A-8) yields the phase velocity of the *P*-wave, while the other two roots ($k = 1, 2$) correspond to the split shear waves. For equation (A-5) to have three distinct roots, Q should be negative; if $Q = 0$, two of the roots are identical. In the case of the Christoffel equation, the two smaller roots may coincide with each other, which leads to shear-wave singularities in certain slowness directions. In principle, the two bigger roots may also become identical (then it no longer makes sense to talk about *P*- and *S*-waves), but this is an unusual occurrence that requires a large magnitude of velocity anisotropy.

Equation (A-8) is valid for an arbitrary anisotropic medium provided the appropriate Christoffel matrix is substituted into equations (A-2)–(A-4). Here, we are interested in evaluating the phase velocity in orthorhombic media as a function of the new anisotropic parameters defined in the main text. It can be accomplished by calculating the stiffness coefficients through the new parameters and substituting them into equations (5)–(10) for the components of the Christoffel matrix. Alternatively, we can express the Christoffel matrix explicitly as a function of the anisotropic parameters introduced above. Representing the stiffness coefficients through the new parameters [using equations (16–19), (23), (24–28)] and substituting the results into the expressions for the components of the Christoffel matrix [equations (5)–(10)], we find

$$\frac{G_{11}}{\rho} = n_1^2 V_{P0}^2 (1 + 2\epsilon^{(2)}) + n_2^2 V_{S0}^2 (1 + 2\gamma^{(1)}) + n_3^2 V_{S0}^2, \quad (\text{A-10})$$

$$\begin{aligned} \frac{G_{22}}{\rho} &= n_1^2 V_{S0}^2 (1 + 2\gamma^{(1)}) + n_2^2 V_{P0}^2 (1 + 2\epsilon^{(1)}) \\ &\quad + n_3^2 V_{S0}^2 \frac{1 + 2\gamma^{(1)}}{1 + 2\gamma^{(2)}}, \end{aligned} \quad (\text{A-11})$$

$$\frac{G_{33}}{\rho} = n_1^2 V_{S0}^2 + n_2^2 V_{S0}^2 \frac{1 + 2\gamma^{(1)}}{1 + 2\gamma^{(2)}} + n_3^2 V_{P0}^2, \quad (\text{A-12})$$

$$\frac{G_{12}}{\rho} = n_1 n_2 V_{p0}^2 (1 + 2\epsilon^{(2)}) D \sqrt{1 + 2\delta^{(3)}/D};$$

$$D = 1 - \frac{V_{s0}^2}{V_{p0}^2} \frac{1 + 2\gamma^{(1)}}{1 + 2\epsilon^{(2)}}, \quad (\text{A-13})$$

$$\frac{G_{13}}{\rho} = n_1 n_3 V_{p0}^2 f \sqrt{1 + 2\delta^{(2)}/f}, \quad (\text{A-14})$$

$$\frac{G_{23}}{\rho} = n_2 n_3 V_{p0}^2 E \sqrt{1 + 2\delta^{(1)}/E}; \quad (\text{A-15})$$

$$E = 1 - \frac{V_{s0}^2}{V_{p0}^2} \frac{1 + 2\gamma^{(1)}}{1 + 2\gamma^{(2)}}.$$

In equations (A-13)–(A-15) it is assumed that $c_{12} + c_{66} \geq 0$, $c_{13} + c_{55} \geq 0$, and $c_{23} + c_{44} \geq 0$, which corresponds to the conditions of “mild” anisotropy (Schoenberg and Helbig, 1997) discussed above. After computing the Christoffel matrix at a given slowness direction, we can calculate the coefficients of the cubic equation (A-1) and obtain the phase velocity [equation (A-9)] using the trigonometric solution (A-8).

APPENDIX B

THE WEAK-ANISOTROPY APPROXIMATION FOR P-WAVE PHASE VELOCITY

The exact P -wave phase velocity in orthorhombic media can be obtained from equations (A-8) (with $k = 0$) and (A-9) of Appendix A:

$$\rho V_p^2 = 2\sqrt{\frac{-d}{3}} \cos\left(\frac{\nu}{3}\right) - \frac{a}{3}, \quad (\text{B-1})$$

where

$$\cos \nu = -\frac{q}{2\sqrt{(-d/3)^3}}; \quad 0 \leq \nu \leq \pi;$$

d and q are defined in equations (A-6) and (A-7), a is given by equation (A-2).

Our goal is to obtain the linearized weak-anisotropy approximation for V^2 by expanding p , ν , and a in the anisotropic parameters $\epsilon^{(i)}$, $\delta^{(i)}$, and $\gamma^{(i)}$ and dropping quadratic and higher-order terms. The cosine function in equation (B-1) can be represented as

$$\cos\left(\frac{\nu}{3}\right) \approx 1 - \frac{\nu^2}{18} + \frac{1}{4!} \frac{\nu^4}{81} - \dots$$

Straightforward algebraic transformations show that in the absence of anisotropy $\nu = 0$. This implies that ν , after being expanded in the anisotropic parameters, is composed of linear and higher-order anisotropic terms with no pure “isotropic” contribution. Therefore, ν^2 contains only quadratic or higher-order terms in the anisotropic coefficients, and in the linearized weak-anisotropy approximation $\cos(\nu/3)$ can be replaced with unity. Then the phase-velocity equation (B-1) reduces to

$$\rho V^2 p \approx 2\sqrt{\frac{-d}{3}} - \frac{a}{3}. \quad (\text{B-2})$$

The next step is to obtain the weak-anisotropy approximation for the term $\sqrt{-d/3}$. Using equations (A-2), (A-3), and (A-6), we can express d in the following way through the components of the Christoffel matrix:

$$d = -\frac{1}{3}(G_{11}^2 + G_{22}^2 + G_{33}^2 + 3G_{12}^2 + 3G_{13}^2 + 3G_{23}^2 - G_{11}G_{22} - G_{11}G_{33} - G_{22}G_{33}). \quad (\text{B-3})$$

Substituting the expressions for G_{ik} given in the main text [equations (A-10)–(A-15)] into equation (B-3) and retaining the terms linear in the anisotropic coefficients, we find

$$-3d = n_1^4 A_1 + n_2^4 A_2 + n_3^4 A_3 + n_1^2 n_2^2 A_4 + n_1^2 n_3^2 A_5 + n_2^2 n_3^2 A_6, \quad (\text{B-4})$$

where

$$A_1 = V_{p0}^4 (1 + 4\epsilon^{(2)}) + V_{s0}^4 (1 + 2\gamma^{(1)}) - 2V_{p0}^2 V_{s0}^2 (1 + 2\epsilon^{(2)} + \gamma^{(1)}), \quad (\text{B-5})$$

$$A_2 = V_{p0}^4 (1 + 4\epsilon^{(1)}) + V_{s0}^4 (1 + 2\gamma^{(5)} + 2\gamma^{(1)}) - 2V_{p0}^2 V_{s0}^2 (1 + 2\epsilon^{(1)} + \gamma^{(5)} + \gamma^{(1)}), \quad (\text{B-6})$$

$$A_3 = V_{p0}^4 + V_{s0}^4 (1 + 2\gamma^{(5)}) - 2V_{p0}^2 V_{s0}^2 (1 + \gamma^{(5)}), \quad (\text{B-7})$$

$$A_4 = -V_{p0}^4 (1 + 2\epsilon^{(1)} + 2\epsilon^{(2)}) - V_{s0}^4 (1 - 2\gamma^{(5)} + 8\gamma^{(1)}) + 2V_{p0}^2 V_{s0}^2 (1 + \epsilon^{(1)} + \epsilon^{(2)} - \gamma^{(5)} + 4\gamma^{(1)}) + 3V_{p0}^4 f^2 \left\{ 1 + \frac{2}{f} \left[2\epsilon^{(2)} - \frac{2V_{s0}^2}{V_{p0}^2} \gamma^{(1)} + \delta^{(3)} \right] \right\}, \quad (\text{B-8})$$

$$A_5 = -V_{p0}^4 (1 + 2\epsilon^{(2)}) - V_{s0}^4 (1 - 2\gamma^{(5)} - 2\gamma^{(1)}) + 2V_{p0}^2 V_{s0}^2 (1 + \epsilon^{(2)} - \gamma^{(5)} - \gamma^{(1)}) + 3V_{p0}^4 f^2 \left(1 + \frac{2\delta^{(2)}}{f} \right), \quad (\text{B-9})$$

$$A_6 = -V_{p0}^4 (1 + 2\epsilon^{(1)}) - V_{s0}^4 (1 + 8\gamma^{(5)} - 2\gamma^{(1)}) + 2V_{p0}^2 V_{s0}^2 (1 + \epsilon^{(1)} + 4\gamma^{(5)} - \gamma^{(1)}) + 3V_{p0}^4 f^2 \times \left\{ 1 + \frac{2}{f} \left[-\frac{2V_{s0}^2}{V_{p0}^2} \gamma^{(5)} + \delta^{(1)} \right] \right\}. \quad (\text{B-10})$$

Here, $\gamma^{(5)}$ is the shear-wave splitting parameter defined in equation (29).

It is convenient to transform equation (B-4) by separating the isotropic and anisotropic components:

$$-3d = (V_{p0}^2 - V_{s0}^2)^2 + AN(V_{p0}^2 - V_{s0}^2), \quad (\text{B-11})$$

where AN is an anisotropic term given by

$$AN = n_1^4 B_1 + n_2^4 B_2 + n_3^4 B_3 + n_1^2 n_2^2 B_4 + n_1^2 n_3^2 B_5 + n_2^2 n_3^2 B_6. \quad (\text{B-12})$$

The coefficients B_i are obtained from equations (B-5)–(B-10) as

$$B_1 = 4V_{p0}^2 \epsilon^{(2)} - 2V_{s0}^2 \gamma^{(1)},$$

$$B_2 = 4V_{p0}^2 \epsilon^{(1)} - 2V_{s0}^2 (\gamma^{(S)} + \gamma^{(1)}),$$

$$B_3 = -2V_{s0}^2 \gamma^{(S)},$$

$$B_4 = 2V_{p0}^2 (5\epsilon^{(2)} - \epsilon^{(1)} + 3\delta^{(3)}) - 2V_{s0}^2 (\gamma^{(S)} + 2\gamma^{(1)}),$$

$$B_5 = 2V_{p0}^2 (-\epsilon^{(2)} + 3\delta^{(2)}) - 2V_{s0}^2 (\gamma^{(S)} + \gamma^{(1)}),$$

$$B_6 = 2V_{p0}^2 (-\epsilon^{(1)} + 3\delta^{(1)}) - 2V_{s0}^2 (2\gamma^{(S)} + \gamma^{(1)}).$$

Taking the square root of equation (B-11) and linearizing in the anisotropic coefficients yields

$$\sqrt{-3d} = V_{p0}^2 - V_{s0}^2 + \frac{1}{2}AN. \quad (\text{B-13})$$

The weak-anisotropy approximation for $a = -(G_{11} + G_{22} + G_{33})$ [equation (A-2)] can be found from equations (A-10)–(A-15),

$$\begin{aligned} -a &= V_{p0}^2 + 2V_{s0}^2 + 2n_1^2 (V_{p0}^2 \epsilon^{(2)} + V_{s0}^2 \gamma^{(1)}) \\ &\quad + 2n_2^2 [V_{p0}^2 \epsilon^{(1)} + V_{s0}^2 (\gamma^{(S)} + \gamma^{(1)})] \\ &\quad + 2n_3^2 V_{s0}^2 \gamma^{(S)}. \end{aligned} \quad (\text{B-14})$$

Substituting equations (B-13), (B-12), and (B-14) into the phase-velocity expression (B-2), we obtain the following weak-anisotropy approximation for P -wave phase velocity:

$$\begin{aligned} V_p^2 &= V_{p0}^2 [1 + 2n_1^4 \epsilon^{(2)} + 2n_2^4 \epsilon^{(1)} + 2n_1^2 n_3^2 \delta^{(2)} \\ &\quad + 2n_2^2 n_3^2 \delta^{(1)} + 2n_1^2 n_2^2 (2\epsilon^{(2)} + \delta^{(3)})]. \end{aligned} \quad (\text{B-15})$$