Shortnote:
A Convenient Expression for the NMO Velocity Function in Terms of Ray Parameter

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INTRODUCTION

Tsvankin (1995) derived an expression for the NMO velocity function in terms of the phase velocity expressed as a function of phase angle that is valid in symmetry planes of anisotropic media. However, the phase angle is not a seismic observable, while the ray parameter is. Indeed, Alkhalifah and Tsvankin (1995) pointed out the importance of representing the NMO velocity function as a function of ray parameter and obtained a number of significant results via an indirect use of this variable. Subsequently, Cohen (1996) directly transformed the Tsvankin expression for the moveout velocity obtaining an expression involving the phase velocity expressed as a function of ray parameter. By obtaining an explicit expression in ray parameter for the P-wave phase velocity in a vertically transverse isotropic medium, he was able to get various approximations for the ray parameter form of the moveout function in a vertically transverse isotropic medium and thus give analytic insight into the success of the velocity analysis methodology of Alkhalifah and Tsvankin.

Here, I show that for determining the moveout function as a function of ray parameter, there is actually a simpler expression based on using the slowness surface instead of the phase velocity. This simpler expression follows directly from an expression of the moveout function by Hale, et al. (1992) that also served as the starting point for Tsvankin's derivation. The key step in the derivation of the new moveout formula is inverting the expression for the slowness surface for one component—typically the vertical component. Moreover, inverting the slowness surface for $q = q(p)$ is the first step in getting practical results from the previous form of the moveout function as well, so this step is needed anyway.

The new formula was motivated by the observation that the Hale, et al. expression for the moveout function is expressed in terms of ray parameter. It is natural to pursue a direct derivation in terms of ray parameter instead of first obtaining an expression in terms of phase angle and then transforming back to ray parameter. The WKBJ or "ray" theory provides such a ray-parameter-based framework.

The new expression for the moveout as a function of ray parameter is shown to imply the previously derived ray parameter expression. Thus all the results derived before can be derived (more directly) with the new expression and, so, it suffices to illustrate techniques with a simple example, and the pure shear wave mode of a transversely isotropic medium is chosen for this purpose.

The derivation of the new formula is obtained from the Hale et al. expression by applying the WKBJ or high frequency assumption to the elastic equations, thus obtaining Christoffel's matrix equation. The eigenvalues of the Christoffel matrix imply an eikonal or "dispersion" equation for the travel time appropriate to a specific propagation mode. Applying the "method of characteristics" to the eikonal equation yields the factors required to explicitly evaluate the quantities in the Hale et al. formula for the moveout velocity.

The focus here, is deriving an analog of the moveout functions of Tsvankin (1995) and Cohen (1996) for the case of wave propagation in a symmetry plane of a homogeneous medium. However, the methodology exposed holds promise for deriving moveout functions for inhomogeneous media and for the non-symmetry plane case.

THE WBKJ APPROXIMATION

The Hale, et al. (1992) formula for the moveout function can be written:

$$V_{\mathrm{nmo}}^2(p) = \frac{1}{\tau} \lim_{x\to 0} \frac{dx}{dp}, \tag{1}$$

where τ is the one-way travel time along the zero-offset specular ray to a dipping reflector, x is the horizontal spatial variable, and p is the horizontal ray parameter. The derivation of equation (1) assumes that the rays stay in a plane, which implies, for an anisotropic medium, that the incidence plane is also a symmetry plane. It is interesting to note that equation (1) is closely related to equation (5) in the classic paper of Dix (1955).

The appearance of the ray parameter in the above expression suggests that we directly determine τ and dx/dp from the ray or WKBJ approximation to the elasticity equations, whose form in ω-$\mathbf{x}$ domain is:

$$(c_{ijkl}u_{k,l})_{,j} + \rho\omega^2 u_i = 0. \tag{2}$$

The WKBJ *ansätz*,

$$u_i(\mathbf{x},\omega) \approx e^{i\omega\tau(\mathbf{x})}U(\mathbf{x}), \tag{3}$$

yields Christoffel's matrix equation:

$$(c_{ijkl}\tau_{,l}\tau_{,j} - \rho\delta_{ik})U_k = 0. \tag{4}$$

Higher order terms would yield a "transport" equation that would complete the determination of the WKBJ amplitude, but here we need only the kinematic results implied by equation (4). Defining the *slowness vector*, $p_i = \tau_{,i}$, obtain

$$(\mathbb{M} - \rho \mathbb{I})\mathbf{U} = \mathbf{0}, \qquad \mathbb{M}_{ik} = c_{ijkl} p_j p_l, \ \mathbb{I}_{ik} = \delta_{ik}. \tag{5}$$

The symmetries of c_{ijkl} imply that $\mathbb{M}$ is a symmetric matrix, so its eigenvalues are real and its eigenvectors can be chosen to be an orthonormal basis in 3-space. Equation (5) implies that each eigenvalue, defines an eikonal equation,

$$\lambda(\mathbf{p}, \mathbf{x}) = \rho, \tag{6}$$

for the mode corresponding to that eigenvalue. The associated eigenvector $\mathbf{U}$ defines the polarization of the mode.

Since $\mathbb{M}$ is homogeneous of degree 2 in the p's, the λ's are also, and Euler's result for homogeneous functions yields the relation,

$$\mathbf{p} \cdot \nabla_{\mathbf{p}} \lambda = 2\lambda, \qquad \nabla_{\mathbf{p}} = (\partial_{p_1}, \partial_{p_2}, \partial_{p_3}). \tag{7}$$

PHASE AND GROUP VELOCITIES

The phase and group velocities can also be defined in terms of the eigenvalues defined by equation (5). The *group velocity* is defined by

$$\mathbf{g} = \nabla_{\mathbf{k}} \omega, \qquad \nabla_{\mathbf{k}} = (\partial_{k_1}, \partial_{k_2}, \partial_{k_3}). \tag{8}$$

Here, $\mathbf{k}$, the *wave vector*, is given by $\mathbf{k} = \omega \mathbf{p}$. The matrix with components,

$$\mathbb{M}_{ik}^{(k)} = c_{ijkl} k_j k_l, \tag{9}$$

has the same eigenvectors as $\mathbb{M}$, and its eigenvalues, $\lambda^{(k)}$, have the same functional form as the corresponding $\lambda(\mathbf{p}, \mathbf{x})$, and so are homogeneous of degree 2 in the $\mathbf{k}$ argument. Thus,

$$\lambda^{(k)} \equiv \lambda(\mathbf{k}, \mathbf{x}) = \omega^2 \lambda(\mathbf{p}, \mathbf{x}) = \rho \omega^2, \tag{10}$$

where the final result follows from the eikonal equation (6). Solving for ω, gives

$$\omega = \sqrt{\frac{\lambda^{(k)}}{\rho}}, \tag{11}$$

hence, from equation (8)

$$\mathbf{g} = \nabla_{\mathbf{k}} \sqrt{\frac{\lambda^{(k)}}{\rho}} = \frac{1}{\omega} \nabla_{\mathbf{p}} \sqrt{\frac{\omega^2 \lambda}{\rho}} = \nabla_{\mathbf{p}} \sqrt{\frac{\lambda}{\rho}} = \frac{\nabla_{\mathbf{p}} \lambda}{2\sqrt{\rho \lambda}}. \tag{12}$$

The eikonal equation (6) implies the following two alternative forms for the group velocity, $\mathbf{g}$:

$$\mathbf{g} = \frac{1}{2\lambda}\nabla_{\mathbf{p}}\lambda = \frac{1}{2\rho}\nabla_{\mathbf{p}}\lambda. \tag{13}$$

The *phase velocity* is defined by

$$\mathbf{v} = \frac{\omega}{k}\mathbf{n}, \qquad \mathbf{n}\cdot\mathbf{n} = 1, \tag{14}$$

with magnitude, or phase speed,

$$v = \frac{\omega}{k}. \tag{15}$$

Since $\mathbf{k} = \omega\mathbf{p}$, and $\mathbf{k} = k\mathbf{n}$, the unit normals $\mathbf{n}$ are given in terms of the slowness $\mathbf{p}$ as

$$\mathbf{n} = \frac{\omega}{k}\mathbf{p} = v\mathbf{p}. \tag{16}$$

The matrix with components,

$$\mathbb{M}^{(n)}_{ik} = c_{ijkl}n_j n_l, \tag{17}$$

again has the same eigenvectors as $\mathbb{M}$, and again its eigenvalues, $\lambda^{(n)}$, have the same functional form as the corresponding $\lambda(\mathbf{p},\mathbf{x})$, so

$$\lambda^{(n)} \equiv \lambda(\mathbf{n},\mathbf{x}) = v^2\lambda(\mathbf{p},\mathbf{x}) = \rho v^2. \tag{18}$$

Solving for v, gives

$$v = \sqrt{\frac{\lambda^{(n)}}{\rho}} = \sqrt{\frac{\lambda(\mathbf{n},\mathbf{x})}{\rho}}. \tag{19}$$

If we represent the unit vector $\mathbf{n}$ in spherical polar coordinates, equation (19) gives the usual phase angle representation of the phase velocity. On the other hand, since $\mathbf{n}\cdot\mathbf{n} = 1$, equation (16) yields an alternative expression for phase velocity as a function of the slowness vector:

$$v = \frac{1}{\sqrt{\mathbf{p}\cdot\mathbf{p}}}. \tag{20}$$

Remark: The definitions (8), (15) of group velocity and phase speed imply that $\mathbf{g} = \nabla_{\mathbf{k}}(kv)$. This is a key result for Tsvankin's derivation, but it will not be needed here.

NMO VELOCITY AS A FUNCTION OF RAY PARAMETER

First observe that

$$\mathbf{p}\cdot\mathbf{g} = \frac{\mathbf{p}\cdot\nabla_{\mathbf{p}}\lambda}{2\lambda} = 1, \tag{21}$$

where the final equality follows from equation (7). This well-known result will be critical to the derivation.

The eikonal equation, $\lambda = \rho$, is a first order partial differential equation for the traveltime τ and hence can be reduced to solving a system of ordinary differential equations by the "method of characteristics" (Courant & Hilbert, 1962). Denoting differentiation along the characteristics (or "rays") with a dot, this system can be written:

$$\begin{aligned} &\dot{\mathbf{x}} = \nabla_{\mathbf{p}}\lambda = 2\rho\mathbf{g}, \qquad \mathbf{x}(0) = \mathbf{0}, \\ &\dot{\mathbf{p}} = -\nabla_{\mathbf{x}}\lambda, \qquad \mathbf{p}(0) = \mathbf{p}_0, \\ &\dot{\tau} = \nabla_{\mathbf{x}}\tau \cdot \dot{\mathbf{x}} = 2\rho\mathbf{p}\cdot\mathbf{g} = 2\rho, \qquad \tau(0) = 0. \end{aligned} \tag{22}$$

Here, equation (21) has been used to simplify the $\dot{\tau}$ equation.

Remark: The ray equations imply that

$$\frac{d\mathbf{x}}{d\tau} = \frac{2\rho\mathbf{g}}{2\rho} = \mathbf{g}. \tag{23}$$

This could serve as an alternative definition of group velocity if one wished to avoid introducing the wave vector $\mathbf{k}$.

Now confine attention to a symmetry plane where $p_2 = 0$ and $x_2 = 0$, and since the ray problem is now two dimensional, introduce the special notations,

$$\mathbf{x} = (x, 0, z), \qquad \mathbf{p} = (p, 0, q), \tag{24}$$

here, $p = p_1$, is called the ray parameter and appears already in equation (1). The group velocity likewise simplifies and takes the form,

$$\mathbf{g} = \frac{1}{2\rho}\nabla_{\mathbf{p}}\lambda = \frac{1}{2\rho}(\lambda_p, 0, \lambda_q). \tag{25}$$

Using z as the parameter along rays, equations (22) simplify to

$$\begin{aligned} &\frac{dx}{dz} = \frac{g_1}{g_3}, \qquad x(0) = 0, \\ &\frac{dp}{dz} = -\frac{\lambda_x}{2\rho g_3}, \qquad p(0) = p_0, \\ &\frac{d\tau}{dz} = \frac{1}{g_3}, \qquad \tau(0) = 0. \end{aligned} \tag{26}$$

Here, in place of a differential equation for q, we conceive of solving the eikonal equation, $\lambda = \rho$, for $q = q(p, x, z)$. The implicit function theorem implies that

$$\frac{\lambda_p}{\lambda_q} = -q', \tag{27}$$

where we use a prime to denote differentiation with respect to the ray parameter, p. Thus,

$$\frac{g_1}{g_3} = \frac{\lambda_p}{\lambda_q} = -q'. \tag{28}$$

This result and equation (21) imply that

$$\frac{1}{g_3} = q - pq'. \tag{29}$$

Hence the ray equations become

$$\begin{aligned} \frac{dx}{dz} &= -q', \qquad x(0) = 0 \\ \frac{dp}{dz} &= -\frac{\lambda_x}{\lambda_q}, \qquad p(0) = p_0, \\ \frac{d\tau}{dz} &= q - pq', \qquad \tau(0) = 0, \end{aligned} \tag{30}$$

with q determined by solving $\lambda = \rho$. Notice that an analogous form of the ray equations (30) can be derived in a three-dimensional setting using the solution $p_3 = q(p_1, p_2, x, y, z)$ of $\lambda = \rho$.

The Hale formula (1) requires the derivative of x with respect to p. The first of the above ray equations yields the needed equation as

$$\frac{d}{dz}\left(\frac{dx}{dp}\right) = -q'', \qquad x'(0) = 0, \tag{31}$$

For a laterally homogenous medium, p is a constant along the rays, and the solutions of the ray equations can be expressed as ordinary integrations over z. Here, we specialize further to the case of a completely homogeneous medium, in which case, $q = q(p) =$ constant on rays and explicit solutions are immediately available:

$$x = -q'z, x' = -q''z, \tau = (q - pq')z. \tag{32}$$

Thus for a homogeneous medium, the Hale formula (1) gives

$$V_{\rm nmo}^2(p) = \frac{q''}{pq' - q}, \tag{33}$$

where again, $q = q(p)$ is determined by $\lambda = \rho$ for each mode. The derivation of (33) is the key result of this paper and applications are shown in the next section.

As contrasted to the methodologies in Tsvankin (1995) and Cohen (1996), the moveout function is here obtained in terms of $q(p)$, instead of in terms of the phase speed, $v(\theta)$ or $v(p)$. The expression for the moveout function in terms of $v(p)$ follows from solving equation (20) for q to obtain

$$q(p) = \sqrt{1/v^2(p) - p^2}. \tag{34}$$

Inserting this expression into equation (33), gives the result

$$V_{\rm nmo}^2(p) = \frac{(1 - p^2v^2)vv'' + (3p^2v^2 - 2)v'^2 + 2pv^3v' + v^4}{(1 - p^2v^2)v(pv)'} \tag{35}$$

in agreement with the moveout function derived in Cohen (1996).

APPLICATION

As a simple example of the results obtained above, consider the pure shear mode in a VTI medium (i.e., a medium that is transversely isotropic with a vertical axis of symmetry). For corresponding *quasi-P* wave results based on the equivalent, but more complicated, equation (35), see Cohen (1996).

The moveout function

For a VTI medium, the $\mathbb{M}$-matrix is

$$\mathbb{M} = \begin{pmatrix} c_{11}p_1^2 + c_{66}p_2^2 + c_{55}p_3^2 & (c_{11} - c_{66})p_1p_2 & (c_{13} + c_{55})p_1p_3 \\ (c_{11} - c_{66})p_1p_2 & c_{66}p_1^2 + c_{11}p_2^2 + c_{55}p_3^2 & (c_{13} + c_{55})p_2p_3 \\ (c_{13} + c_{55})p_1p_3 & (c_{13} + c_{55})p_2p_3 & c_{55}p_1^2 + c_{55}p_2^2 + c_{33}p_3^2 \end{pmatrix}. \tag{36}$$

For the pure shear mode, the eigenvalue, expressed in terms of the isotropic shear speed c_S and Thomsen's (1986) parameter γ is

$$\lambda_S = \rho c_S^2[(1 + 2\gamma)p^2 + q^2]. \tag{37}$$

Solving the eikonal equation, $\lambda_S = \rho$, for q yields

$$q(p) = \frac{1}{c_S}\sqrt{1 - (1 + 2\gamma)c_S^2p^2}. \tag{38}$$

This solution only involves solving the linear equation (37) in the quantity q^2. For the other modes, the *quasi-P* and *quasi-S*, the corresponding equation is a quadratic in q^2 and so, can again be solved explicitly. Indeed, the solution of this quadratic for q^2 was a step in the phase velocity-based methodologies (Alkhalifah and Tsvankin, 1995; Cohen, 1966).

With $q(p)$ given by equation (38), equation (33) gives

$$V_{\text{nmo}}^2(p) = \frac{c_S^2(1 + 2\gamma)}{1 - (1 + 2\gamma)c_S^2p^2}. \tag{39}$$

Introducing the zero-dip moveout,

$$V_{\text{nmo}}(0) = c_S\sqrt{1 + 2\gamma}, \tag{40}$$

and the dimensionless variable,

$$y = V_{\text{nmo}}^2(0)p^2, \tag{41}$$

one obtains the result in Alkhalifah-Tsvankin format as

$$V_{\text{nmo}}(p) = \frac{V_{\text{nmo}}(0)}{\sqrt{1 - y}}. \tag{42}$$

Notice that $V_{\text{nmo}}(p)$ has been obtained without first determining the phase speed.

Phase and group velocity

Although the phase speed is not required for the determination of the moveout function, the q-function does also determine the phase speed as a function of ray parameter. Indeed, equation (34) gives $v(p)$ as

$$v(p) = \frac{1}{\sqrt{p^2 + q^2(p)}}. \tag{43}$$

For example, using the $q(p)$ derived above for the shear mode in a transversely isotropic medium gives

$$v_S(p) = \frac{c_S}{\sqrt{1 - 2\gamma c_S^2 p^2}}. \tag{44}$$

Alternatively, to obtain the phase speed as a function of angle, equation (19) allows direct use of the eigenvalue. Indeed, writing the S-wave eigenvalue shown above for the formulation in terms of the unit normal gives

$$\lambda_S(\mathbf{n}) = \rho c_S^2[(1 + 2\gamma)n_1^2 + n_3^2], \tag{45}$$

so

$$v_S(\theta) = c_S\sqrt{1 + 2\gamma \sin^2\theta} \tag{46}$$

Finally, equation (25) allows writing the group velocity in terms of partial derivatives of the eigenvalue in equation (37) as

$$\mathbf{g} = c_S^2((1 + 2\gamma)p, 0, q), \tag{47}$$

or using the solution, $q(p)$, as

$$\mathbf{g}(p) = ((1 + 2\gamma)c_S^2 p, 0, c_S\sqrt{1 - (1 + 2\gamma)c_S^2 p^2}). \tag{48}$$

The substitution, $p = \sin\theta / v_S(\theta)$, produces

$$\mathbf{g}(\theta) = \frac{c_s}{\sqrt{1 + 2\gamma \sin^2\theta}}((1 + 2\gamma)\sin\theta, 0, \cos\theta). \tag{49}$$

CONCLUSIONS

A new and simpler expression for the moveout velocity in a symmetry plane as a function of ray parameter has been derived from first principles. Its use has been illustrated with an example from transversely isotropic media.

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