Micro-local, non-linear, resolution analysis of generalised Radon transform inversions in anisotropic media

MAARTEN V. DE HOOP AND NORMAN BLEISTEIN
Center for Wave Phenomena,
Colorado School of Mines,
Golden CO 80401-1887, USA.

May 13, 1996

Abstract

The resolution analysis of generalised Radon transform (GRT) inversions of seismic data is carried out in general anisotropic media. The GRT inversion formula is derived from the ray-Born approximation of the wave field for volume scatterers. However, by considering in the resolution analysis scattering surfaces instead, the inversion provides reflection/transmission coefficients as functions of scattering angles and azimuths. Those coefficients can be subjected to any type of A(mplitude) V(ersus scattering) A(ngles) analysis, in a non-linear sense to obtain information about the medium discontinuities across the scattering surface.

Keywords: migration/inversion, Kirchhoff approximation, generalised Radon transform.

1 Introduction

In the process of enhancing resolution, both spatially as well as in elastic parameters, current acquisition techniques take reflection-seismic measurements at increasingly larger scattering angles. To achieve a better resolution, however, one must account for the interplay between heterogeneity and anisotropy, at different length scales. Also, to accommodate inversion of wide angle data, one has to consider a non-linear approach, at least near the relevant reflectors in the configuration. We use the *generalised Radon transform* or GRT formulation to achieve this goal.

First, the ray-Born approximation of the direct scattering problem is cast into an integral of surface integral representations. Then the inversion of the linearized scattering problem is carried out. The resolution analysis of the linearized GRT inversion serves as the basis to arrive at a micro-local non-linear formulation: its range of validity is extended by applying a stationary phase analysis with respect to migration dip, defined as the normalized gradient of total travel time along the scattering characteristic. In this process, carrying out the GRT inversion on Kirchhoff-type data, an explicit adjustment of the inversion formula is found.

By ordering the data, for each image point, into common scattering-angle/azimuth gathers, the GRT inversion formula can be modified to obtain reflection/transmission coefficients at specular ray geometries. Any amplitude versus scattering angles, AVA, analysis can then be applied to those coefficients to derive information about the medium perturbation. Several parametrizations of this perturbation have been developed to reveal to which quantities the coefficients are sensitive.

The idea of estimating angle-dependent reflectivity has been developed by various authors, see for example Geoltrain and Chovet [1] and Lumley [2]. A more rigorous discussion of such an approach can be found in the book of Bleistein *et al.* [3]. In our paper, we derive a GRT-based inversion procedure that accomplishes the goal of constructing full reflection, and possibly transmission, coefficients as functions of scattering angle and azimuth, in closed form.

GRT-type inversion formulas for the *linearized* inverse problem have been developed by Norton and Linzer [4], by Beylkin [5, 6, 7, 8], by Miller *et al.* [9, 10], by Beylkin *et al.* [11], and by Rakesh [12] for the acoustic case. The extension to the elastic case was discussed by Beylkin and Burridge [13], and anisotropy was considered by De Hoop *et al.* [14] and Spencer and De Hoop [15]. Inversion formulas aiming to estimate reflectivity rather than the medium perturbation were developed by Cohen and Bleistein [16], by Bleistein and Cohen [17], and by Bleistein [18]. This paper brings the two approaches together. How to implement the GRT inversion procedures numerically can be found in De Hoop and Spencer [19].

2 The basic equations

Notation

First, we introduce some basic notation. Choose coordinates in the configuration according to

$$\begin{aligned} \boldsymbol{x} = (x_1, x_2, x_3) &= \text{Cartesian position vector,} \\ \boldsymbol{s} = (s_1, s_2, s_3) &= \text{source point,} \\ \boldsymbol{r} = (r_1, r_2, r_3) &= \text{receiver point,} \\ t &= \text{time.} \end{aligned}$$

The medium is described by

$$\begin{aligned} \rho(\boldsymbol{x}) &= \text{density,} \\ c_{ijk\ell}(\boldsymbol{x}) &= \text{elastic stiffness tensor,} \end{aligned}$$

while the wave field is described by

$$\boldsymbol{u}(\boldsymbol{x}, t) = (u_1(\boldsymbol{x}, t), u_2(\boldsymbol{x}, t), u_3(\boldsymbol{x}, t)) = \text{displacement vector,}$$

and generated by a source distribution given by

$$\boldsymbol{f}(\boldsymbol{x}, t) = (f_1(\boldsymbol{x}, t), f_2(\boldsymbol{x}, t), f_3(\boldsymbol{x}, t)) = \text{body-force source density.}$$

In the remainder of the paper, we will employ the summation convention.

The displacement in a heterogeneous anisotropic medium satisfies the elastodynamic wave equation

$$\rho \partial_t^2 u_i - \partial_j (c_{ijk\ell} \partial_\ell u_k) = f_i \ , \tag{2.1}$$

with summation over repeated lower case indices, here and below. Let

$$\boldsymbol{G}(\boldsymbol{x}, \boldsymbol{x}', t) = (G_{ip}(\boldsymbol{x}, \boldsymbol{x}', t)) \tag{2.2}$$

be the causal Green's tensor, which satisfies (cf. Eq.(2.1))

$$\rho\, \partial_t^2 G_{ip} - \partial_j (c_{ijk\ell} \partial_\ell G_{kp}) = \delta_{ip} \delta(\boldsymbol{x} - \boldsymbol{x}') \delta(t) \ , \quad G_{ip} = 0 \ \text{ for } \ t < 0 \ . \tag{2.3}$$

Asymptotic ray theory

Here, we summarize the formulation of anisotropic ray theory for the evaluation of the Green's tensor. Let

$$G_{ip}(\boldsymbol{x},\boldsymbol{x}',t) = \sum_N A^{(N)}(\boldsymbol{x},\boldsymbol{x}')\,\xi_i^{(N)}(\boldsymbol{x})\xi_p^{(N)}(\boldsymbol{x}')\delta(t-\tau^{(N)}(\boldsymbol{x},\boldsymbol{x}')) + \text{ terms smoother in } t \; . \tag{2.4}$$

In this equation, the arrival time $\tau^{(N)}$ and the associated polarization vector $\boldsymbol{\xi}^{(N)}$ satisfy

$$(\rho\,\delta_{ik} - c_{ijk\ell}(\partial_\ell\tau^{(N)})(\partial_j\tau^{(N)}))\,\xi_k^{(N)} = 0 \quad (\text{at all } \boldsymbol{x}) \; , \tag{2.5}$$

which implies the eikonal equation

$$\det(\rho\,\delta_{ik} - c_{ijk\ell}(\partial_\ell\tau)\,(\partial_j\tau)) = 0 \quad (\text{at all } \boldsymbol{x}) \; . \tag{2.6}$$

The polarization vectors are assumed to be normalized so that $\xi_i^{(N)}\xi_i^{(N)} = 1$. Define the slowness vector $\boldsymbol{\gamma}^{(N)}$ by

$$\boldsymbol{\gamma}^{(N)}(\boldsymbol{x}) = \nabla_{\boldsymbol{x}}\tau^{(N)}(\boldsymbol{x},\boldsymbol{x}') \; . \tag{2.7}$$

Then, Eq.(2.6) constrains $\boldsymbol{\gamma}$ to lie on the sextic surface $\mathcal{A}(\boldsymbol{x})$ given by

$$\det(\rho\,\delta_{ik} - c_{ijk\ell}\gamma_\ell\gamma_j) = 0 \; . \tag{2.8}$$

$\mathcal{A}(\boldsymbol{x})$ consists of three sheets $\mathcal{A}^{(N)}(\boldsymbol{x})$, $N = 1, 2, 3$, each of which is a closed surface surrounding the origin. An individual sheet is described by (cf. Eq.(2.8))

$$2\,\mathcal{H} = \rho - \xi_i c_{ijk\ell}\gamma_\ell\gamma_j\xi_k = 0 \; . \tag{2.9}$$

The scalar amplitudes A must satisfy the transport equation

$$\partial_j(c_{ijk\ell}\xi_i^{(N)}\xi_k^{(N)}\,(A^{(N)})^2\,\partial_\ell\tau^{(N)}) = 0 \; , \tag{2.10}$$

where N, again, indicates the mode of propagation, that is, the sheet of the slowness surface on which the corresponding slowness vector lies.

The characteristic or group velocities $\boldsymbol{v}^{(N)}$ are normal to $\mathcal{A}^{(N)}(\boldsymbol{x})$ at $\boldsymbol{\gamma}^{(N)}$ and satisfy

$$\boldsymbol{v}^{(N)}\cdot\boldsymbol{\gamma}^{(N)} = 1 \; ; \quad \boldsymbol{v}^{(N)} = \left.\frac{\nabla_{\boldsymbol{\gamma}}\mathcal{H}}{\boldsymbol{\gamma}\cdot\nabla_{\boldsymbol{\gamma}}\mathcal{H}}\right|_{\mathcal{H}=0} \; . \tag{2.11}$$

The normal or phase speeds are given by

$$V^{(N)} = \frac{1}{|\boldsymbol{\gamma}^{(N)}|} \; . \tag{2.12}$$

The unit phase direction follows as

$$\boldsymbol{\alpha}^{(N)} = V^{(N)}\,\boldsymbol{\gamma}^{(N)} \,.$$

From Eq.(2.11) it follows that

$$V^{(N)} = |\boldsymbol{v}^{(N)}|\,\cos\chi\,, \tag{2.13}$$

where χ is the angle between $\boldsymbol{v}^{(N)}$ and $\boldsymbol{\gamma}^{(N)}$.

The amplitudes can be expressed in terms of certain Jacobians,

$$A = \frac{1}{4\pi[\rho(\boldsymbol{x})\rho(\boldsymbol{x}')\mathcal{M}]^{1/2}} \quad \text{with} \quad \mathcal{M} = \frac{|\boldsymbol{v}(\boldsymbol{x}')|V(\boldsymbol{x})\,\left|\dfrac{\partial\boldsymbol{x}}{\partial q_1}\wedge\dfrac{\partial\boldsymbol{x}}{\partial q_2}\right|_{\boldsymbol{x}}}{\left|\dfrac{\partial\boldsymbol{\gamma}}{\partial q_1}\wedge\dfrac{\partial\boldsymbol{\gamma}}{\partial q_2}\right|_{\boldsymbol{x}'}}\,, \tag{2.14}$$

in which A and $\mathcal{M}$ carry the superscript (N). Here, (q_1, q_2) parameterize the rays originating from the source. One can verify that the dimension of A is $[\text{time}]^2\times[\text{mass}]^{-1}$, which upon multiplication by force, with dimensions $[\text{mass}]\times[\text{length}]\times[\text{time}]^{-2}$, gives the dimension of displacement, i.e., [length].

Source and receiver Green's functions

In the integral representation for the scattered field, we need the Green's functions originating both at the source and the receiver points. Further, the gradient of total travel times from the source to a scattering point to the receiver are required in preparation of the GRT inversion. We introduce these functions here.

Set

$$\tilde{\boldsymbol{G}}(\boldsymbol{x},t) = \boldsymbol{G}(\boldsymbol{x},\boldsymbol{s},t)\,,\quad \hat{\boldsymbol{G}}(\boldsymbol{x},t) = \boldsymbol{G}(\boldsymbol{r},\boldsymbol{x},t)\,. \tag{2.15}$$

Employing asymptotic ray theory in both Green's functions, we introduce the notation

$$\tilde{A}^{(\tilde{N})}(\boldsymbol{x}) = A^{(N)}(\boldsymbol{x},\boldsymbol{s})\,,\quad \hat{A}^{(\hat{M})}(\boldsymbol{x}) = A^{(M)}(\boldsymbol{r},\boldsymbol{x}) \tag{2.16}$$

in the case of scattering from incident mode N to outgoing mode M.

According to Eq.(2.7), the slowness vectors at $\boldsymbol{x}$ are given by

$$\tilde{\boldsymbol{\gamma}}^{(\tilde{N})}(\boldsymbol{x}) = \nabla_{\boldsymbol{x}}\tau^{(N)}(\boldsymbol{x},\boldsymbol{s})\,,\quad \hat{\boldsymbol{\gamma}}^{(\hat{M})}(\boldsymbol{x}) = \nabla_{\boldsymbol{x}}\tau^{(M)}(\boldsymbol{r},\boldsymbol{x})\,; \tag{2.17}$$

the associated phase directions are given by

$$\tilde{\boldsymbol{\alpha}}^{(\tilde{N})} = \frac{\tilde{\boldsymbol{\gamma}}^{(\tilde{N})}}{|\tilde{\boldsymbol{\gamma}}^{(\tilde{N})}|}\,,\quad \hat{\boldsymbol{\alpha}}^{(\hat{M})} = \frac{\hat{\boldsymbol{\gamma}}^{(\hat{M})}}{|\hat{\boldsymbol{\gamma}}^{(\hat{M})}|} \tag{2.18}$$

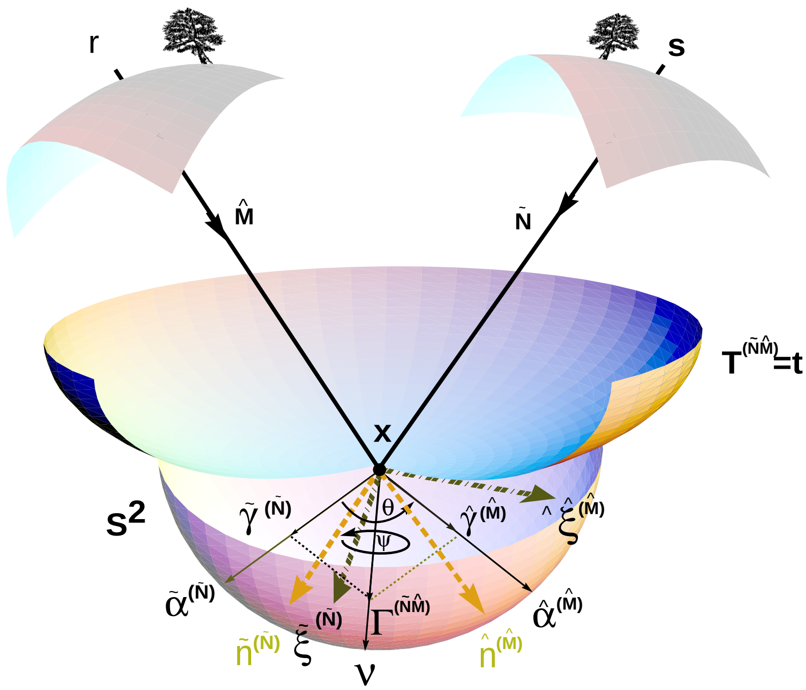

Figure 2.1: Source-receiver ray geometry.

and the phase speeds (cf. Eq.(2.12)) are given by

$$\tilde{V}^{(\tilde{N})} = \frac{1}{|\tilde{\boldsymbol{\gamma}}^{(\tilde{N})}|} \; , \quad \hat{V}^{(\hat{M})} = \frac{1}{|\hat{\boldsymbol{\gamma}}^{(\hat{M})}|} \; . \tag{2.19}$$

We also define the two-way travel time $T^{(\tilde{N}\hat{M})}$ and its gradient,

$$T^{(\tilde{N}\hat{M})}(\boldsymbol{r},\boldsymbol{y},\boldsymbol{s}) \equiv \tau^{(N)}(\boldsymbol{y},\boldsymbol{s}) + \tau^{(M)}(\boldsymbol{r},\boldsymbol{y}) \; , \quad \boldsymbol{\Gamma}^{(\tilde{N}\hat{M})}(\boldsymbol{r},\boldsymbol{x},\boldsymbol{s}) \equiv \nabla_{\boldsymbol{x}} T^{(\tilde{N}\hat{M})}(\boldsymbol{r},\boldsymbol{x},\boldsymbol{s}) \; . \tag{2.20}$$

From Eq.(2.17) we see that

$$\boldsymbol{\Gamma}^{(\tilde{N}\hat{M})}(\boldsymbol{r},\boldsymbol{x},\boldsymbol{s}) = \tilde{\boldsymbol{\gamma}}^{(\tilde{N})}(\boldsymbol{x}) + \hat{\boldsymbol{\gamma}}^{(\hat{M})}(\boldsymbol{x}) \; . \tag{2.21}$$

The direction of $\boldsymbol{\Gamma}^{(\tilde{N}\hat{M})}$,

$$\boldsymbol{\nu} \equiv \frac{\boldsymbol{\Gamma}^{(\tilde{N}\hat{M})}}{|\boldsymbol{\Gamma}^{(\tilde{N}\hat{M})}|} \; ,$$

will be the *migration dip*, which we referred to in the introduction. The different quantities are illustrated in Figure 2.1.

3 Medium description: micro-local perturbation

To analyze a typical geological setting, we consider the following, micro-local, representation,

$$\mathbf{c}^{(1)}(\boldsymbol{x}) \to \mathbf{c}^{(1)}(\boldsymbol{x}, \phi(\boldsymbol{x})) \ , \tag{3.1}$$

of the medium's perturbation. Here, ϕ is a smooth function of $\boldsymbol{x}$, some (curved) level surfaces of which describe a family of interfaces. The gradient of the perturbation is assumed to vary rapidly normal to the level surfaces of ϕ and smoothly along them, implying that

$$\nabla_{\boldsymbol{x}} \mathbf{c}^{(1)} = (\mathbf{c}^{(1)})' \, (\nabla_{\boldsymbol{x}} \phi) + \text{ smoother terms in } \boldsymbol{x} \ , \ \ (\mathbf{c}^{(1)})' = \partial_\phi \mathbf{c}^{(1)} \ . \tag{3.2}$$

This representation extends to a small ball around any point, in particular the image point $\boldsymbol{y}$, say, under investigation. The derivative $(\mathbf{c}^{(1)})'$ is understood in the distributional sense. Typically, it will have a Dirac-distribution type behavior across any geological interface. Loosely, the derivative can be interpreted as the difference in medium properties across a level surface.

We will omit the first argument of $\mathbf{c}^{(1)}$ in the remainder of this paper, and take only the most rapidly varying term of the perturbation's derivatives into account. Imaging reflectors or interfaces amounts to mapping this leading-order behavior, $(\mathbf{c}^{(1)})'$. We will confirm below that our inversion procedure does this, and also provides estimates of angularly dependent reflection coefficients.

4 The single scattering equation

In this section, we introduce the Born approximation representing the singly scattered wavefield. We then show how to recast this volume integral representation into a surface integral for the response to the most singular element of the scattering process, namely, reflecting the discontinuities of the medium parameters.

We begin the analysis with the volume-scattering representation of the ray-Born approximation for the scattered displacement field. Let $(\boldsymbol{r}, \boldsymbol{s}) \in \partial R \times \partial S$; ideally, the boundaries $\partial R, \partial S \sim S^2$ are closed surfaces surrounding the heterogeneous domain $\mathcal{D}$ in which the medium is unknown. Then (De Hoop *et al.* [14])

$$u^{(1)}_{pq}(\boldsymbol{r}, \boldsymbol{s}, t) = -\int_{\mathcal{D}} \hat{\xi}^{(\hat{M})}_p(\boldsymbol{r}) \tilde{\xi}^{(\tilde{N})}_q(\boldsymbol{s}) A^{(\tilde{N}\hat{M})}(\boldsymbol{x}) \ \times$$

$$(\mathbf{w}^{(\tilde{N}\hat{M})}(\boldsymbol{x}))^T \, \mathbf{c}^{(1)}(\boldsymbol{x}) \, \delta''(t - T^{(\tilde{N}\hat{M})}(\boldsymbol{r}, \boldsymbol{x}, \boldsymbol{s})) \ \mathrm{d}\boldsymbol{x} \ , \tag{4.1}$$

where $N, M \in \{1, 2, 3\}$,

$$A^{(\tilde{N}\hat{M})}(\boldsymbol{x}) = \rho(\boldsymbol{x}) \, \tilde{A}^{(\tilde{N})}(\boldsymbol{x}) \hat{A}^{(\hat{M})}(\boldsymbol{x}) \ , \tag{4.2}$$

contains the amplitudes,

$$\mathbf{w}^{(\tilde{N}\hat{M})} = \left\{ \tilde{\xi}_i^{(\tilde{N})} \hat{\xi}_i^{(\hat{M})}, \tfrac{1}{2} \left[\hat{a}_{ij}^{(\hat{M})} \tilde{a}_{k\ell}^{(\tilde{N})} + \hat{a}_{k\ell}^{(\hat{M})} \tilde{a}_{ij}^{(\tilde{N})} \right] \right\} ,$$

$$\hat{a}_{ij}^{(\hat{M})} = \tfrac{1}{2} V_\circ^{(\hat{M})} \left(\hat{\xi}_i^{(\hat{M})} \hat{\gamma}_j^{(\hat{M})} + \hat{\xi}_j^{(\hat{M})} \hat{\gamma}_i^{(\hat{M})} \right) , \quad \tilde{a}_{k\ell}^{(\tilde{N})} = \tfrac{1}{2} V_\circ^{(\tilde{N})} \left(\tilde{\xi}_k^{(\tilde{N})} \tilde{\gamma}_\ell^{(\tilde{N})} + \tilde{\xi}_\ell^{(\tilde{N})} \tilde{\gamma}_k^{(\tilde{N})} \right) ,$$

describes the contrast-source radiation patterns, and

$$\mathbf{c}^{(1)} = \left\{ \frac{\rho^{(1)}}{\rho}, \frac{c_{ijk\ell}^{(1)}}{\rho\, V_\circ^{(\hat{M})} V_\circ^{(\tilde{N})}} \right\} ,$$

represents the relative medium perturbation. Here, $V_\circ^{(L)}$ denotes the (local) normal speed of mode L in the background medium for $\boldsymbol{\gamma}$ averaged over all phase directions. The notation $_\circ$ is meant to emphasize that the quantity is angle independent, which is important for retaining the actual medium perturbation from $\mathbf{c}^{(1)}$ and the GRT inversion to be applicable.

In the micro-local setting, substituting Eq.(3.1) into Eq.(4.1), we have

$$u_{pq}^{(1)}(\boldsymbol{r}, \boldsymbol{s}, t) = -\int_{\mathcal{D}} \hat{\xi}_p^{(\hat{M})}(\boldsymbol{r}) \tilde{\xi}_q^{(\tilde{N})}(\boldsymbol{s}) A^{(\tilde{N}\hat{M})}(\boldsymbol{x}) \times$$
$$(\mathbf{w}^{(\tilde{N}\hat{M})}(\boldsymbol{x}, \tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(\boldsymbol{x}), \hat{\boldsymbol{\alpha}}^{(\hat{M})}(\boldsymbol{x})))^T \, \mathbf{c}^{(1)}(\phi(\boldsymbol{x}))\, \delta''(t - T(\boldsymbol{x}))\, \mathrm{d}\boldsymbol{x} , \tag{4.3}$$

where, for convenience, we employ the shorthand notation

$$T(\boldsymbol{x}) = T^{(\tilde{N}\hat{M})}(\boldsymbol{r}, \boldsymbol{x}, \boldsymbol{s}) , \quad \boldsymbol{\Gamma}(\boldsymbol{x}) = (\nabla_{\boldsymbol{x}} T^{(\tilde{N}\hat{M})})(\boldsymbol{r}, \boldsymbol{x}, \boldsymbol{s}) , \tag{4.4}$$

see Eq.(2.20). To make use of the properties of the gradient of the medium's perturbation (cf. Eq.(3.2)), we will partially integrate expression (4.3). Since

$$(\nabla_{\boldsymbol{x}} T)(\boldsymbol{x})\, \delta''(t - T(\boldsymbol{x})) = -\nabla_{\boldsymbol{x}} \delta'(t - T(\boldsymbol{x})) , \tag{4.5}$$

we have,

$$\delta''(t - T(\boldsymbol{x})) = -\frac{1}{(\bar{\boldsymbol{\nu}} \cdot \nabla_{\boldsymbol{x}} T)(\boldsymbol{x})}\, \bar{\boldsymbol{\nu}} \cdot \nabla_{\boldsymbol{x}} \delta'(t - T(\boldsymbol{x})) , \tag{4.6}$$

for arbitrary $\bar{\boldsymbol{\nu}} \in S^2$ as long as $\bar{\boldsymbol{\nu}} \cdot \nabla_{\boldsymbol{x}} T \neq 0$. Hence,

$$u_{pq}^{(1)}(\boldsymbol{r}, \boldsymbol{s}, t) \simeq -\int_{\mathcal{D}} \hat{\xi}_p^{(\hat{M})}(\boldsymbol{r}) \tilde{\xi}_q^{(\tilde{N})}(\boldsymbol{s}) A^{(\tilde{N}\hat{M})}(\boldsymbol{x}) \times \tag{4.7}$$
$$(\mathbf{w}^{(\tilde{N}\hat{M})}(\boldsymbol{x}, \tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(\boldsymbol{x}), \hat{\boldsymbol{\alpha}}^{(\hat{M})}(\boldsymbol{x})))^T \, (\mathbf{c}^{(1)})'(\phi(\boldsymbol{x})) \left. \frac{(\bar{\boldsymbol{\nu}} \cdot \nabla_{\boldsymbol{x}} \phi)}{(\bar{\boldsymbol{\nu}} \cdot \boldsymbol{\Gamma})} \right|_{\boldsymbol{x}} \delta'(t - T(\boldsymbol{x}))\, \mathrm{d}\boldsymbol{x} .$$

Here, the approximation arises from neglecting lower order terms as in Eq.(3.2). These will produce asymptotically lower order contributions to the wavefield.

It is assumed that $\bar{\boldsymbol{\nu}} = \bar{\boldsymbol{\nu}}(\boldsymbol{x})$ is slowly varying in space, and may be chosen equal to the local *geological dip*,

$$\boldsymbol{\nu}_\phi \equiv \frac{\nabla_{\boldsymbol{x}}\phi}{|\nabla_{\boldsymbol{x}}\phi|} \; . \tag{4.8}$$

On the other hand, we can choose $\bar{\boldsymbol{\nu}} = \boldsymbol{\nu}$, which we always know. In the inverse scattering problem, the geological dip is unknown and has to be determined. Below, we show that at stationarity the geological and migration dips must be parallel.

Surface integral representation

Now, we show how to recast the volume integral representation (4.7) into an integral over surface integrals over the level surfaces of ϕ. To this end, we choose curvi-linear coordinates $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{x})$, $\boldsymbol{\sigma} = (\sigma_\mu, \sigma_3)$, $\mu = 1, 2$, such that the σ_μ are coordinates in the level surfaces of ϕ and σ_3 is the local coordinate in the $\boldsymbol{\nu}_\phi$-direction. If σ_3 represents the actual value of the level of ϕ, we set $\sigma_3 = L$. The volume form is given by

$$\mathrm{d}\boldsymbol{x} = \frac{1}{|\nabla_{\boldsymbol{x}}\phi|}\,\mathrm{d}L\;\mathrm{d}\Sigma(\boldsymbol{x}) \; , \quad \mathrm{d}\Sigma(\boldsymbol{x}) = |\partial_{\sigma_1}\boldsymbol{x} \wedge \partial_{\sigma_2}\boldsymbol{x}|\;\mathrm{d}\sigma_1\mathrm{d}\sigma_2 \; ;$$

the transformation from Cartesian to curvi-linear coordinates yields the Jacobian

$$\frac{\partial(\boldsymbol{x})}{\partial(\boldsymbol{\sigma})} = |\partial_{\sigma_3}\boldsymbol{x} \cdot (\partial_{\sigma_1}\boldsymbol{x} \wedge \partial_{\sigma_2}\boldsymbol{x})| = \frac{|\partial_{\sigma_1}\boldsymbol{x} \wedge \partial_{\sigma_2}\boldsymbol{x}|}{|\nabla_{\boldsymbol{x}}\phi|} \tag{4.9}$$

Now write

$$(\mathbf{c}^{(1)})'(\phi(\boldsymbol{x})) = \int_{\mathrm{I\!R}} (\mathbf{c}^{(1)})'(L)\,\delta(\phi(\boldsymbol{x}) - L)\;\mathrm{d}L \; . \tag{4.10}$$

Substituting Eq.(4.10) into Eq.(4.7), interchanging the order of integration, and using the property (cf. Eq.(4.9))

$$\int_{\mathcal{D}} \cdots \delta(\phi(\boldsymbol{x}) - L)\,\mathrm{d}\boldsymbol{x} = \int_{\phi = L} \cdots \frac{1}{|\nabla_{\boldsymbol{x}}\phi|}\,\mathrm{d}\Sigma(\boldsymbol{x}) \; ,$$

we obtain

$$u^{(1)}_{pq}(\boldsymbol{r}, \boldsymbol{s}, t)$$

$$\simeq -\int_{\mathrm{I\!R}} \mathrm{d}L \left[\int_{\mathcal{D}} \hat{\xi}^{(\hat{M})}_p(\boldsymbol{r})\tilde{\xi}^{(\tilde{N})}_q(\boldsymbol{s}) A^{(\tilde{N}\hat{M})}(\boldsymbol{x})\,(\mathbf{w}^{(\tilde{N}\hat{M})}(\boldsymbol{x}, \tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(\boldsymbol{x}), \hat{\boldsymbol{\alpha}}^{(\hat{M})}(\boldsymbol{x})))^T\,(\mathbf{c}^{(1)})'(L) \; \times\right.$$

$$\left. \frac{(\bar{\boldsymbol{\nu}} \cdot \nabla_{\boldsymbol{x}} \phi)}{(\bar{\boldsymbol{\nu}} \cdot \boldsymbol{\Gamma})} \right|_{\boldsymbol{x}} \delta(\phi(\boldsymbol{x}) - L)\, \delta'(t - T(\boldsymbol{x}))\ \mathrm{d}\boldsymbol{x} \Big]$$

$$= -\int_{\mathrm{I\!R}} \mathrm{d}L \left[\int_{\phi=L} \hat{\xi}_p^{(\hat{M})}(\boldsymbol{r}) \tilde{\xi}_q^{(\tilde{N})}(\boldsymbol{s}) A^{(\tilde{N}\hat{M})}(\boldsymbol{x}) \ \times \right.$$

$$\left. (\mathbf{w}^{(\tilde{N}\hat{M})}(\boldsymbol{x}, \tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(\boldsymbol{x}), \hat{\boldsymbol{\alpha}}^{(\hat{M})}(\boldsymbol{x})))^T\, (\mathbf{c}^{(1)})'(L)\ \left. \frac{(\bar{\boldsymbol{\nu}} \cdot \nabla_{\boldsymbol{x}} \phi)}{(\bar{\boldsymbol{\nu}} \cdot \boldsymbol{\Gamma})\, |\nabla_{\boldsymbol{x}} \phi|} \right|_{\boldsymbol{x}} \delta'(t - T(\boldsymbol{x}))\, \mathrm{d}\Sigma(\boldsymbol{x}) \right]$$

$$= -\int_{\mathrm{I\!R}} \mathrm{d}L \left[\int_{\phi=L} \hat{\xi}_p^{(\hat{M})}(\boldsymbol{r}) \tilde{\xi}_q^{(\tilde{N})}(\boldsymbol{s}) A^{(\tilde{N}\hat{M})}(\boldsymbol{x}) \ \times \right.$$

$$\left. (\mathbf{w}^{(\tilde{N}\hat{M})}(\boldsymbol{x}, \tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(\boldsymbol{x}), \hat{\boldsymbol{\alpha}}^{(\hat{M})}(\boldsymbol{x})))^T\, (\mathbf{c}^{(1)})'(L)\ \left. \frac{(\bar{\boldsymbol{\nu}} \cdot \boldsymbol{\nu}_\phi)}{(\bar{\boldsymbol{\nu}} \cdot \boldsymbol{\Gamma})} \right|_{\boldsymbol{x}} \delta'(t - T(\boldsymbol{x}))\, \mathrm{d}\Sigma(\boldsymbol{x}) \right] . \tag{4.11}$$

As a consequence of Eq.(4.11), we also have

$$\partial_t u_{pq}^{(1)}(\boldsymbol{r}, \boldsymbol{s}, t) = -\int_{\mathrm{I\!R}} \mathrm{d}L \left[\int_{\phi=L} \hat{\xi}_p^{(\hat{M})}(\boldsymbol{r}) \tilde{\xi}_q^{(\tilde{N})}(\boldsymbol{s}) A^{(\tilde{N}\hat{M})}(\boldsymbol{x}) \ \times \right. \tag{4.12}$$

$$\left. (\mathbf{w}^{(\tilde{N}\hat{M})}(\boldsymbol{x}, \tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(\boldsymbol{x}), \hat{\boldsymbol{\alpha}}^{(\hat{M})}(\boldsymbol{x})))^T\, (\mathbf{c}^{(1)})'(L)\ \left. \frac{(\bar{\boldsymbol{\nu}} \cdot \boldsymbol{\nu}_\phi)}{(\bar{\boldsymbol{\nu}} \cdot \boldsymbol{\Gamma})} \right|_{\boldsymbol{x}} \delta''(t - T(\boldsymbol{x}))\, \mathrm{d}\Sigma(\boldsymbol{x}) \right] .$$

The singular support of the medium perturbation, $\phi(\boldsymbol{x}) = L$, and the isochrone surfaces, $T(\boldsymbol{x}) = t$, are illustrated in Figure 4.1.

5 Reflection and transmission coefficients

To identify reflection and transmission coefficients in Eq.(4.12), we first extract phase velocities at the scattering point from the amplitudes,

$$A_u(\boldsymbol{x}) = A^{(\tilde{N}\hat{M})}(\boldsymbol{x})\ \left[\tilde{V}^{(\tilde{N})}(\boldsymbol{x}) (\hat{V}^{(\hat{M})}(\boldsymbol{x}))^3 \right]^{1/2} \ ; \tag{5.1}$$

then, Eq.(4.12) can be written in the form

$$\partial_t u_{pq}^{(1)}(\boldsymbol{r}, \boldsymbol{s}, t) = -\int_{\mathrm{I\!R}} \left[\int_{\phi=L} \hat{\xi}_p^{(\hat{M})}(\boldsymbol{r}) \tilde{\xi}_q^{(\tilde{N})}(\boldsymbol{s}) A_u(\boldsymbol{x}) \ \times \right. \tag{5.2}$$

$$\left. R_L^{(\tilde{N}\hat{M})}(\boldsymbol{x}, \tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(\boldsymbol{x}), \hat{\boldsymbol{\alpha}}^{(\hat{M})}(\boldsymbol{x}))\ (\bar{\boldsymbol{\nu}} \cdot \boldsymbol{\nu}_\phi)(\bar{\boldsymbol{\nu}} \cdot \boldsymbol{\Gamma})\, |_{\boldsymbol{x}}\ \delta''(t - T(\boldsymbol{x}))\, \frac{\mathrm{d}\Sigma(\boldsymbol{x})}{|\nabla_{\boldsymbol{x}} \phi|} \right] \mathrm{d}L \ ,$$

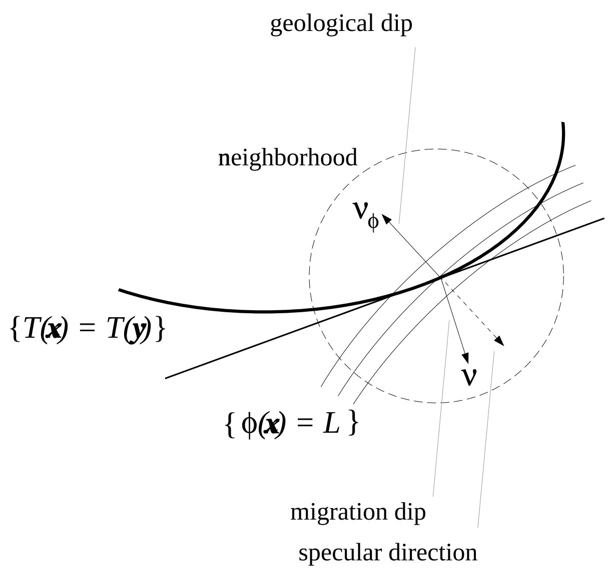

Figure 4.1: Micro-local medium perturbation.

in which,

$$R_L^{(\tilde{N}\hat{M})}(.,\tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(.),\hat{\boldsymbol{\alpha}}^{(\hat{M})}(.)) = \frac{(\mathbf{w}^{(\tilde{N}\hat{M})})^T\,(\mathbf{c}^{(1)})'|\nabla_{\boldsymbol{x}}\phi|}{\left[\tilde{V}^{(\tilde{N})}(\hat{V}^{(\hat{M})})^3\right]^{1/2}|\boldsymbol{\Gamma}|^2\,(\bar{\boldsymbol{\nu}}\cdot\boldsymbol{\nu})^2}\,, \tag{5.3}$$

represents the scattering coefficient for the N, M conversion at $\boldsymbol{x}$ with $\phi(\boldsymbol{x}) = L$, i.e., $R_L^{(\tilde{N}\hat{M})}$ really depends on $\sigma_\mu(\boldsymbol{x})$, $\mu = 1, 2$. Observe that the scattering coefficient contains a function that may be singular on the level surfaces of ϕ. In fact, the integration over L picks up the singular support of $(\mathbf{c}^{(1)})'$.

Specular reflection and transmission

Now, let $\mathbf{c}^{(1)}$ contain a step function in L (across a curved interface at $L = L_0$). Then $(\mathbf{c}^{(1)})'(L) = (\Delta\mathbf{c}^{(1)})\,\delta(L - L_0)$. At the specular point for given $\boldsymbol{\nu}_\phi$, the pair $\tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(.), \hat{\boldsymbol{\alpha}}^{(\hat{M})}(.)$ satisfies Snell's law,

$$\frac{\hat{\boldsymbol{\alpha}}^{(\hat{M})}}{\hat{V}^{(\tilde{M})}}\cdot(\mathbf{I} - \boldsymbol{\nu}_\phi\boldsymbol{\nu}_\phi) = -\frac{\tilde{\boldsymbol{\alpha}}^{(\tilde{N})}}{\tilde{V}^{(\tilde{N})}}\cdot(\mathbf{I} - \boldsymbol{\nu}_\phi\boldsymbol{\nu}_\phi)\ . \tag{5.4}$$

(In an isotropic medium with $M = N$ the solution is simply given by

$$\hat{\boldsymbol{\alpha}}_{\mathsf{s}}^{(\hat{N})} = -\tilde{\boldsymbol{\alpha}}^{(\tilde{N})}\cdot(\mathbf{I} - 2\boldsymbol{\nu}_\phi\boldsymbol{\nu}_\phi)\ ,$$

representing ordinary reflection). At the specular point, the migration dip coincides with the geological dip, $\boldsymbol{\nu} = \boldsymbol{\nu}_\phi$.

Substituting the solution, $\hat{\boldsymbol{\alpha}}_{\mathsf{s}}^{(\hat{M})}$, into the scattering matrix Eq.(5.3), yields the *linearized reflection/transmission coefficients* $\mathsf{R}^{(\tilde{N}\hat{M})}$,

$$\mathsf{R}_{L_0}^{(\tilde{N}\hat{M})}(., \tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(.))\ \delta(L - L_0) = R_L^{(\tilde{N}\hat{M})}(., \tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(.), \hat{\boldsymbol{\alpha}}_{\mathsf{s}}^{(\hat{M})}(.))\ . \tag{5.5}$$

In fact, with $\bar{\boldsymbol{\nu}} = \boldsymbol{\nu}_\phi$, Eq.(5.2) is equivalent to the Kirchhoff-Born approximation, which satisfies the principle of reciprocity. For notational convenience, we will absorb the Dirac distribution in $\mathsf{R}_L^{(\tilde{N}\hat{M})}$.

The Kirchhoff approximation

Here, we show how to go from the linearized Born representation Eq.(4.12) to the non-linear Kirchhoff approximation. For general reference, we introduce the relevant scattering and specular angles: in addition to $\boldsymbol{\nu}$, we set

$$\cos\theta = \tilde{\boldsymbol{\alpha}}^{(\tilde{N})} \cdot \hat{\boldsymbol{\alpha}}^{(\hat{M})},\ \ \psi = \text{third Euler angle}\ . \tag{5.6}$$

Then, at any point in $\mathcal{D}$, we have a mapping

$$(\tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(.), \hat{\boldsymbol{\alpha}}^{(\hat{M})}(.)) \rightarrow (\boldsymbol{\nu}_., \theta_., \psi_.)\ . \tag{5.7}$$

If $\boldsymbol{\nu} = \boldsymbol{\nu}_\phi$, the ray geometry is specular; let the associated specular scattering angles be defined as

$$\cos\tilde{\theta} = \tilde{\alpha}^{(\tilde{N})} \cdot \boldsymbol{\nu}_\phi,\ \ \cos\hat{\theta} = \hat{\alpha}_{\mathsf{s}}^{(\hat{M})} \cdot \boldsymbol{\nu}_\phi,\ \ \theta_{\mathsf{s}} = \tilde{\theta} - \hat{\theta}\ . \tag{5.8}$$

In the non-reciprocal, 'non-linear' Kirchhoff approximation, the scattering matrix is simply replaced by the *full* reflection/transmission coefficients at specular, cf. Eq.(5.2) with $\bar{\boldsymbol{\nu}} = \boldsymbol{\nu}_\phi$,

$$\partial_t u_{pq}^{(1)}(\boldsymbol{r}, \boldsymbol{s}, t) = -\int_{\mathbb{R}} \left[\int_{\phi=L} \hat{\xi}_p^{(\hat{M})}(\boldsymbol{r})\tilde{\xi}_q^{(\tilde{N})}(\boldsymbol{s}) A_u(\boldsymbol{x})\ \times \right. \tag{5.9}$$

$$\left. \mathsf{R}_L^{(\tilde{N}\hat{M})}(\boldsymbol{x}, \tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(\boldsymbol{x}))\ (\boldsymbol{\nu}_\phi \cdot \boldsymbol{\Gamma})\,|_{\boldsymbol{x}}\ \delta''(t - T(\boldsymbol{x}))\ \frac{\mathrm{d}\Sigma(\boldsymbol{x})}{|\nabla_{\boldsymbol{x}}\phi|}\right] \mathrm{d}L\ .$$

This representation is more adequate than Eq.(5.2) for wide-angle scattering. The time derivative is taken to pave the way for the Radon transform inversion. (In three dimensions, one needs the second derivative of the Dirac distribution.) In our further analysis, we actually employ the reciprocal representation Eq.(5.2) in which $R_L^{(\tilde{N}\hat{M})}$ is replaced by the full reflection/transmission coefficient at specular. Note that due to the singular function contained in $\mathsf{R}_L^{(\tilde{N}\hat{M})}$, cf. Eq.(5.5), the integration over L in Eq.(5.9) reduces to a sum over scattering surfaces or interfaces.

6 Stationary phase analysis of the direct scattering problem

By applying stationary phase arguments, the integral in Eq.(5.2) can be evaluated. The analysis confirms the consistency with asymptotic ray theory in configurations with a family of surface scatterers.

We choose rotated Cartesian coordinates (x_μ, z), $\mu = 1, 2$ in the neighborhood of a yet to be determined *specular point* $\boldsymbol{y}(L) \in \{\phi = L\}$, such that

$$z \parallel \boldsymbol{\nu}_\phi, \quad \{x_\mu\} \perp \boldsymbol{\nu}_\phi \tag{6.1}$$

and

$$\boldsymbol{\nu} = \boldsymbol{\nu}_\phi \; ;$$

see Figure 4.1. The function $T(\boldsymbol{y}(L))$ has a derivative given by

$$\frac{\mathrm{d}T}{\mathrm{d}L} = \left. \frac{|\nabla_{\boldsymbol{x}} T|}{|\nabla_{\boldsymbol{x}} \phi|} \right|_{\boldsymbol{y}(L)} , \tag{6.2}$$

while, by the implicit function theorem, the function $L_{\boldsymbol{y}}(t)$ satisfying

$$T\left(\boldsymbol{y}(L_{\boldsymbol{y}}(t))\right) = t$$

exists.

Taylor expansions of the level and isochrone surfaces including the curvature terms yield

$$\begin{aligned} 0 = \phi(\boldsymbol{x}) - \phi(\boldsymbol{y}) &= |\nabla_{\boldsymbol{x}} \phi(\boldsymbol{y})| \, z + \tfrac{1}{2} x_\mu \phi_{\mu\nu}(\boldsymbol{y}) x_\nu \quad , \quad \phi_{\mu\nu}(\boldsymbol{y}) = \left. \frac{\partial^2 \phi}{\partial_{x_\mu} \partial_{x_\nu}} \right|_{\boldsymbol{y}} \; ; \\ T(\boldsymbol{x}) &= T(\boldsymbol{y}) + |\nabla_{\boldsymbol{x}} T(\boldsymbol{y})| \, z + \tfrac{1}{2} x_\mu T_{\mu\nu}(\boldsymbol{y}) x_\nu \quad , \quad T_{\mu\nu}(\boldsymbol{y}) = \left. \frac{\partial^2 T}{\partial_{x_\mu} \partial_{x_\nu}} \right|_{\boldsymbol{y}} . \end{aligned} \tag{6.3}$$

(Summations are carried out over μ, ν.) The first equality in (6.3) amounts to the representation of the level surface $\{\phi = L\}$,

$$z = -\frac{x_\mu \phi_{\mu\nu} x_\nu}{2\, |\nabla_{\boldsymbol{x}} \phi|} \; ,$$

which upon substitution in the second equality yields

$$T(\boldsymbol{x}) = T(\boldsymbol{y}) + \tfrac{1}{2} |\nabla_{\boldsymbol{x}} T(\boldsymbol{y})| \, x_\mu \Upsilon_{\mu\nu}(\boldsymbol{y}) x_\nu \tag{6.4}$$

with

$$\Upsilon_{\mu\nu} \equiv \frac{T_{\mu\nu}}{|\nabla_{\boldsymbol{x}} T|} - \frac{\phi_{\mu\nu}}{|\nabla_{\boldsymbol{x}} \phi|}$$

and $\boldsymbol{x}, \boldsymbol{y}$ in the same level surface. Note that the matrix Υ may vary with the level L, and can be negative or positive definite, or indefinite. The case of vanishing Υ, leading to a caustic analysis, will be postponed to a future paper. Otherwise, for t near $T(\boldsymbol{y})$, we have the intermediate result

$$\partial_t^2 \int_{\phi=L} H(t - T(\boldsymbol{x}))\, \mathrm{d}\Sigma(\boldsymbol{x})$$

$$= \partial_t^2 \int_{\mathbb{R}^2} H\left(t - T(\boldsymbol{y}) - \tfrac{1}{2} |\nabla_{\boldsymbol{x}} T(\boldsymbol{y})|\, x_\mu \Upsilon_{\mu\nu}(\boldsymbol{y}) x_\nu\right)\, \mathrm{d}x_1\, \mathrm{d}x_2$$

$$= \frac{2\pi\, \delta^*(t - T(\boldsymbol{y}))}{|\boldsymbol{\Gamma}(\boldsymbol{y})| \sqrt{|\det(\Upsilon(\boldsymbol{y}))|}}\,, \tag{6.5}$$

where

$$\delta^* = \begin{cases} \delta & \text{if} \;\; \Upsilon \;\; \text{positive} \\ \mathcal{H}\delta & \text{if} \;\; \Upsilon \;\; \text{indefinite} \\ -\delta & \text{if} \;\; \Upsilon \;\; \text{negative}\,. \end{cases} \tag{6.6}$$

In the above $\mathcal{H}$ denotes the Hilbert transform; the notation * indicates an action on the time dependence.

We will use Eq.(6.5) to evaluate the integral in Eq.(4.11) eventually. First, note that Eq.(6.5) implies

$$\int_{\mathbb{R}} \mathrm{d}L \left[\partial_t^2 \int_{\phi=L} (\mathbf{c}^{(1)})'(L)\; \left.\frac{(\bar{\boldsymbol{\nu}} \cdot \boldsymbol{\nu}_\phi)}{(\bar{\boldsymbol{\nu}} \cdot \boldsymbol{\Gamma})}\right|_{\boldsymbol{x}}\; H(t - T(\boldsymbol{x}))\, \mathrm{d}\Sigma(\boldsymbol{x}) \right]$$

$$= \int_{\mathbb{R}} \mathrm{d}L\; (\mathbf{c}^{(1)})'(L)\, \frac{2\pi}{|\boldsymbol{\Gamma}| \sqrt{|\det(\Upsilon)|}}\; \left.\frac{(\bar{\boldsymbol{\nu}} \cdot \boldsymbol{\nu}_\phi)}{(\bar{\boldsymbol{\nu}} \cdot \boldsymbol{\Gamma})}\right|_{\boldsymbol{y}(L)}\; \delta^* \left[t - T(\boldsymbol{y}(L)) \right]$$

$$= \frac{2\pi}{|\boldsymbol{\Gamma}| \sqrt{|\det(\Upsilon)|}}\; \left.\frac{(\bar{\boldsymbol{\nu}} \cdot \boldsymbol{\nu}_\phi)}{(\bar{\boldsymbol{\nu}} \cdot \boldsymbol{\Gamma})}\right|_{\boldsymbol{y}(L_{\boldsymbol{y}}(t))}\; \left(\frac{\mathrm{d}T}{\mathrm{d}L}\right)^{-1} (\mathbf{c}^{(1)})' \Big|^{*}_{L_{\boldsymbol{y}}(t)}\,, \tag{6.7}$$

where, for any f,

$$\left(\frac{\mathrm{d}T}{\mathrm{d}L}\right)^{-1} \mathsf{f}\, \Big|^{*}_{L_{\boldsymbol{y}}(t)} = \int_{\mathbb{R}} \mathsf{f}(L)\, \delta^*[t - T(\boldsymbol{y}(L))]\; \mathrm{d}L\,.$$

Substituting Eq.(6.2) into Eq.(6.7), and using the result in Eq.(4.11) implies

$$u_{pq}^{(\tilde{N}\hat{M})}(\boldsymbol{r}, \boldsymbol{s}, t) = -\frac{2\pi\, |\nabla_{\boldsymbol{x}} \phi|}{\sqrt{|\det(\Upsilon)|}\; |\boldsymbol{\Gamma}|^3}\; \hat{\xi}_p^{(\hat{M})}(\boldsymbol{r}) \tilde{\xi}_q^{(\tilde{N})}(\boldsymbol{s}) A^{(\tilde{N}\hat{M})}(.)\; \times \tag{6.8}$$

$$(\mathbf{w}^{(\tilde{N}\hat{M})}(., \tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(.), \hat{\boldsymbol{\alpha}}_{\mathsf{s}}^{(\hat{M})}(.)))^T \Big|_{\boldsymbol{y}(L_{\boldsymbol{y}}(t))}\; (\mathbf{c}^{(1)})' \Big|^{*}_{L_{\boldsymbol{y}}(t)}\,,$$

since at the specular point we have $\boldsymbol{\nu} = \boldsymbol{\nu}_\phi$ and $\boldsymbol{\nu}_\phi \cdot \boldsymbol{\Gamma} = |\boldsymbol{\Gamma}|$. This formula is an extension of the *convolutional model* approximation in one-dimensional space to three dimensions.

In terms of the linearized reflection/transmission coefficients, we have

$$u_{pq}^{(\tilde{N}\hat{M})}(\boldsymbol{r},\boldsymbol{s},t) = -\frac{2\pi}{\sqrt{|\det(\Upsilon)|}\,|\boldsymbol{\Gamma}|}\,\hat{\xi}_p^{(\hat{M})}(\boldsymbol{r})\tilde{\xi}_q^{(\tilde{N})}(\boldsymbol{s})A_u(.)\,\mathsf{R}_L^{(\tilde{N}\hat{M})}\Big|^{*}_{\boldsymbol{y}=\boldsymbol{y}(L),\ L=L_{\boldsymbol{y}}(t)} . \tag{6.9}$$

Note that the * relates to the KMAH index.

7 Inversion based on the GRT

In preparation of the inverse transformation, we introduce the scalar quantity

$$\partial_t u^{(\tilde{N}\hat{M})}(\boldsymbol{r},\boldsymbol{s},\boldsymbol{y}) = \frac{\hat{\xi}_p^{(\hat{M})}(\boldsymbol{r})\,\partial_t u_{pq}^{(1)}(\boldsymbol{r},\boldsymbol{s},T^{(\tilde{N}\hat{M})}(\boldsymbol{r},\boldsymbol{y},\boldsymbol{s}))\,\tilde{\xi}_q^{(\tilde{N})}(\boldsymbol{s})}{A_u(\boldsymbol{y})} . \tag{7.1}$$

Then, using Eq.(5.2), we get

$$\partial_t u^{(\tilde{N}\hat{M})}(\boldsymbol{r},\boldsymbol{s},\boldsymbol{y}) \simeq -\int_{\mathbb{R}}\Big[\int_{\phi=L} R_L^{(\tilde{N}\hat{M})}(\boldsymbol{x},\tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(\boldsymbol{x}),\hat{\boldsymbol{\alpha}}^{(\hat{M})}(\boldsymbol{x}))\,(\bar{\boldsymbol{\nu}}\cdot\boldsymbol{\nu}_\phi)(\bar{\boldsymbol{\nu}}\cdot\boldsymbol{\Gamma})\,|_{\boldsymbol{x}} \times$$
$$\delta''(T(\boldsymbol{y})-T(\boldsymbol{x}))\,\frac{\mathrm{d}\Sigma(\boldsymbol{x})}{|\nabla_{\boldsymbol{x}}\phi|}\Big]\,\mathrm{d}L\ ; \tag{7.2}$$

we expand

$$T(\boldsymbol{y}) - T(\boldsymbol{x}) = \boldsymbol{\Gamma}(\boldsymbol{y})\cdot(\boldsymbol{y}-\boldsymbol{x}) + \ldots \simeq |\boldsymbol{\Gamma}(\boldsymbol{y})|\,(\boldsymbol{y}-\boldsymbol{x})\cdot\boldsymbol{\nu}\ , \tag{7.3}$$

which describes the isochrone tangent plane at $\boldsymbol{y}$. We have

$$\delta''(|\boldsymbol{\Gamma}(\boldsymbol{y})|\,(\boldsymbol{y}-\boldsymbol{x})\cdot\boldsymbol{\nu}) = |\boldsymbol{\Gamma}(\boldsymbol{y})|^{-3}\,\delta''((\boldsymbol{y}-\boldsymbol{x})\cdot\boldsymbol{\nu})\ .$$

Linearized inversion

The basis of the GRT inversion is Gel'fand's plane wave expansion, which we write in the form

$$\int_{S^2}\int_{\phi=L}\delta''((\boldsymbol{y}-\boldsymbol{x})\cdot\boldsymbol{\nu})\,\frac{\mathrm{d}\Sigma(\boldsymbol{x})}{|\nabla_{\boldsymbol{x}}\phi|}\,\mathrm{d}\boldsymbol{\nu} = \int_{S^2}\int_{\mathcal{D}}\delta(\phi(\boldsymbol{x})-L)\delta''((\boldsymbol{y}-\boldsymbol{x})\cdot\boldsymbol{\nu})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{\nu}$$
$$= -8\pi^2\delta(\phi(\boldsymbol{y})-L)\ . \tag{7.4}$$

We will use this expansion to invert Eq.(4.12). The inversion is accomplished by setting up a system of 22 equations, using the simplification according to Eq.(7.1), and integrating the result both over $\tilde{\boldsymbol{\alpha}}$ and $\hat{\boldsymbol{\alpha}}$, i.e.,

$$\mathbf{w}^{(\tilde{N}\hat{M})}(\boldsymbol{y},\tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(\boldsymbol{y}),\hat{\boldsymbol{\alpha}}^{(\hat{M})}(\boldsymbol{y}))\,\partial_t u^{(\tilde{N}\hat{M})}(\boldsymbol{r},\boldsymbol{s},\boldsymbol{y})\;\left.\frac{|\boldsymbol{\Gamma}|^4}{(\boldsymbol{\nu}\cdot\boldsymbol{\nu}_\phi)}\right|_{\boldsymbol{y}}\left.\frac{\partial(\tilde{\boldsymbol{\alpha}},\hat{\boldsymbol{\alpha}})}{\partial(\boldsymbol{s},\boldsymbol{r})}\right|_{\boldsymbol{y}}\mathrm{d}\boldsymbol{s}\mathrm{d}\boldsymbol{r}$$

$$= -\int_{\mathbb{R}}\mathrm{d}L\left[\int_{\phi=L}\mathbf{w}^{(\tilde{N}\hat{M})}(\boldsymbol{y},\tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(\boldsymbol{y}),\hat{\boldsymbol{\alpha}}^{(\hat{M})}(\boldsymbol{y}))\mathbf{w}^{(\tilde{N}\hat{M})}(\boldsymbol{x},\tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(\boldsymbol{x}),\hat{\boldsymbol{\alpha}}^{(\hat{M})}(\boldsymbol{x}))^T(\mathbf{c}^{(1)})'(L)\;\times\right.$$

$$\left.\left.\frac{(\boldsymbol{\nu}\cdot\boldsymbol{\Gamma})}{(\boldsymbol{\nu}\cdot\boldsymbol{\nu}_\phi)}\right|_{\boldsymbol{y}}\left.\frac{(\bar{\boldsymbol{\nu}}\cdot\boldsymbol{\nu}_\phi)}{(\bar{\boldsymbol{\nu}}\cdot\boldsymbol{\Gamma})}\right|_{\boldsymbol{x}}\frac{|\boldsymbol{\Gamma}(\boldsymbol{y})|^3}{|\boldsymbol{\Gamma}(\boldsymbol{x})|^3}\left.\frac{\partial(\tilde{\boldsymbol{\alpha}},\hat{\boldsymbol{\alpha}})}{\partial(\boldsymbol{\nu},\theta,\psi)}\right|_{\boldsymbol{y}}\delta''((\boldsymbol{y}-\boldsymbol{x})\cdot\boldsymbol{\nu})\,\mathrm{d}\Sigma(\boldsymbol{x})\right]\mathrm{d}\boldsymbol{\nu}\mathrm{d}\theta\mathrm{d}\psi\;. \qquad (7.5)$$

Define the matrix

$$\Lambda_{.}(\boldsymbol{\nu}_{.})\equiv\int_{S^2}\mathbf{w}^{(\tilde{N}\hat{M})}(.,\tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(.),\hat{\boldsymbol{\alpha}}^{(\hat{M})}(.))\mathbf{w}^{(\tilde{N}\hat{M})}(.,\tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(.),\hat{\boldsymbol{\alpha}}^{(\hat{M})}(.))^T\left.\frac{\partial(\tilde{\boldsymbol{\alpha}},\hat{\boldsymbol{\alpha}})}{\partial(\boldsymbol{\nu},\theta,\psi)}\right|_{.}\mathrm{d}\theta\mathrm{d}\psi \qquad (7.6)$$

at any image point $\boldsymbol{y}$. Then, employing Eq.(7.4) in Eq.(7.5) and setting $\bar{\boldsymbol{\nu}}=\boldsymbol{\nu}$ (recall that $\bar{\boldsymbol{\nu}}$ has been arbitrary until now), yields

$$(\mathbf{c}^{(1)})'(\phi(\boldsymbol{y}))\,|\nabla_{\boldsymbol{x}}\phi|_{\boldsymbol{y}}=\int_{\mathbb{R}}(\mathbf{c}^{(1)})'(L)\,\delta(\phi(\boldsymbol{y})-L)\,|\nabla_{\boldsymbol{x}}\phi|_{\boldsymbol{y}}\,\mathrm{d}L\simeq \qquad (7.7)$$

$$\frac{1}{8\pi^2}\int_{\partial S\times\partial R}\Lambda_{\boldsymbol{y}}^{-1}\mathbf{w}^{(\tilde{N}\hat{M})}(\boldsymbol{y},\tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(\boldsymbol{y}),\hat{\boldsymbol{\alpha}}^{(\hat{M})}(\boldsymbol{y}))\,\partial_t u^{(\tilde{N}\hat{M})}(\boldsymbol{r},\boldsymbol{s},\boldsymbol{y})\;\left.\frac{|\boldsymbol{\Gamma}|^4}{(\boldsymbol{\nu}\cdot\boldsymbol{\nu}_\phi)}\right|_{\boldsymbol{y}}\left.\frac{\partial(\tilde{\boldsymbol{\alpha}},\hat{\boldsymbol{\alpha}})}{\partial(\boldsymbol{s},\boldsymbol{r})}\right|_{\boldsymbol{y}}\mathrm{d}\boldsymbol{s}\mathrm{d}\boldsymbol{r}\;.$$

Note that in this inversion formula the actual shape of the level surfaces of ϕ is not used; just the local normal at the image point plays a rôle. However, in Section 9 it will be argued that the inner product, $(\boldsymbol{\nu}\cdot\boldsymbol{\nu}_\phi)$, can be removed from the formula without changing the resolution analysis in the high-frequency approximation. The inverse of Λ is understood in the *generalised* sense.

Through Eq.(7.5), we have the freedom to carry out the inversion Eq.(7.7) in two steps. For a given image point $\boldsymbol{y}$, we can sort the data ($(\boldsymbol{s},\boldsymbol{r})$ pairs) into common (θ,ψ) gathers. The variable in such gathers is the migration dip $\boldsymbol{\nu}$; formally, we denote the gathers by $(\partial S\times\partial R)'(\boldsymbol{y};\theta,\psi)$. The integration over dip is then carried out prior to the integration over scattering angle and azimuth.

To control the illumination of the image point at a given dip, we introduce the partition of unity, $\{\chi_J\}$, and

$$(\mathbf{c}^{(1)})'(\phi(\boldsymbol{y}))\,|\nabla_{\boldsymbol{x}}\phi|_{\boldsymbol{y}}\simeq\langle(\mathbf{c}^{(1)})'(\phi(\boldsymbol{y}))\,|\nabla_{\boldsymbol{x}}\phi|_{\boldsymbol{y}}\rangle=\sum_J\frac{1}{8\pi^2}\int_{\partial S\times\partial R} \qquad (7.8)$$

$$\Lambda_{\boldsymbol{y}}^{-1}\mathbf{w}^{(\tilde{N}\hat{M})}(\boldsymbol{y},\tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(\boldsymbol{y}),\hat{\boldsymbol{\alpha}}^{(\hat{M})}(\boldsymbol{y}))\,\partial_t u^{(\tilde{N}\hat{M})}(\boldsymbol{r},\boldsymbol{s},\boldsymbol{y})\;\left.\frac{|\boldsymbol{\Gamma}|^4}{(\boldsymbol{\nu}\cdot\boldsymbol{\nu}_\phi)}\right|_{\boldsymbol{y}}\chi_J(\boldsymbol{s},\boldsymbol{r})\left.\frac{\partial(\tilde{\boldsymbol{\alpha}},\hat{\boldsymbol{\alpha}})}{\partial(\boldsymbol{s},\boldsymbol{r})}\right|_{\boldsymbol{y}}\mathrm{d}\boldsymbol{s}\mathrm{d}\boldsymbol{r}\;.$$

Each term in the summation represents a partial reconstruction.

Resolution analysis

Backsubstituting the Kirchhoff-Born scattering formula Eq.(4.12) with $\bar{\boldsymbol{\nu}} = \boldsymbol{\nu}$, into the inversion formula Eq.(7.8) and substituting the one-sided Fourier representation of the Dirac distribution, yields the *resolution* operator,

$$\langle(\mathbf{c}^{(1)})'(\phi(\boldsymbol{y}))\, |\nabla_{\boldsymbol{x}}\phi|_{\boldsymbol{y}}\rangle = \int_{\rm I\!R}\int_{\phi=L} \mathcal{R}(\boldsymbol{y},\boldsymbol{x})\; (\mathbf{c}^{(1)})'(\phi(\boldsymbol{x}))\, |\nabla_{\boldsymbol{x}}\phi|_{\boldsymbol{x}}\, \frac{\mathrm{d}\Sigma(\boldsymbol{x})}{|\nabla_{\boldsymbol{x}}\phi|_{\boldsymbol{x}}}\, \mathrm{d}L \;, \tag{7.9}$$

with matrix kernel $\mathcal{R}$,

$$\mathcal{R}(\boldsymbol{y},\boldsymbol{x}) = \mathrm{Re}\sum_J \frac{1}{8\pi^3}\int \exp[i\Phi(\boldsymbol{y},\boldsymbol{x},\Theta)]\; a(\boldsymbol{y},\boldsymbol{x},\Theta)\; \chi_J(\Theta)\mathrm{d}\Theta \;. \tag{7.10}$$

Here, $\Theta = (\omega, \boldsymbol{s}, \boldsymbol{r})$ and

$$\int_{{\rm I\!R}^+\times\partial S\times\partial R} \cdots\, \mathrm{d}\Theta = \int_{\partial S\times\partial R}\int_{{\rm I\!R}^+} \cdots\, |\boldsymbol{\Gamma}^{(\tilde{N}\hat{M})}(\boldsymbol{r},\boldsymbol{y},\boldsymbol{s})|^3\omega^2\, \mathrm{d}\omega\; \mathrm{d}\boldsymbol{s}\, \mathrm{d}\boldsymbol{r} \;. \tag{7.11}$$

The resolution operator expresses how well the reconstruction can be accomplished within the framework of the linear theory.

The phase function Φ of the Fourier integral operator with kernel Eq.(7.10) is simply given by

$$\Phi(\boldsymbol{y},\boldsymbol{x},\Theta) \equiv \omega\, \left(T^{(\tilde{N}\hat{M})}(\boldsymbol{r},\boldsymbol{y},\boldsymbol{s}) - T^{(\tilde{N}\hat{M})}(\boldsymbol{r},\boldsymbol{x},\boldsymbol{s})\right) \;, \tag{7.12}$$

while the amplitude function arises as the matrix

$$\begin{aligned} a(\boldsymbol{y},\boldsymbol{x},\Theta) \equiv\; & \frac{A_u(\boldsymbol{x})}{A_u(\boldsymbol{y})}\; \hat{\xi}_p^{(\hat{M})}(\boldsymbol{r},\boldsymbol{y})\hat{\xi}_p^{(\hat{M})}(\boldsymbol{r},\boldsymbol{x})\; \tilde{\xi}_q^{(\tilde{N})}(\boldsymbol{s},\boldsymbol{y})\tilde{\xi}_q^{(\tilde{N})}(\boldsymbol{s},\boldsymbol{x}) \times \\ & \Big[(\Lambda_{\boldsymbol{y}}^{(\tilde{N}\hat{M})}(\boldsymbol{\nu}(\boldsymbol{r},\boldsymbol{y},\boldsymbol{s})))^{-1}\, \mathbf{w}^{(\tilde{N}\hat{M})}(\boldsymbol{y},\tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(\boldsymbol{y}),\hat{\boldsymbol{\alpha}}^{(\hat{M})}(\boldsymbol{y})) \\ & (\mathbf{w}^{(\tilde{N}\hat{M})}(\boldsymbol{x},\tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(\boldsymbol{x}),\hat{\boldsymbol{\alpha}}^{(\hat{M})}(\boldsymbol{x})))^T\Big] \times \\ & \left[\frac{\tilde{V}^{(\tilde{N})}(\boldsymbol{y})(\hat{V}^{(\hat{M})}(\boldsymbol{y}))^3}{\tilde{V}^{(\tilde{N})}(\boldsymbol{x})(\hat{V}^{(\hat{M})}(\boldsymbol{x}))^3}\right]^{1/2} \frac{|\boldsymbol{\Gamma}^{(\tilde{N}\hat{M})}(\boldsymbol{r},\boldsymbol{y},\boldsymbol{s})|}{|\boldsymbol{\Gamma}^{(\tilde{N}\hat{M})}(\boldsymbol{r},\boldsymbol{x},\boldsymbol{s})|}\, \frac{(\boldsymbol{\nu}\cdot\boldsymbol{\nu}_\phi)_{\boldsymbol{x}}}{(\boldsymbol{\nu}\cdot\boldsymbol{\nu}_\phi)_{\boldsymbol{y}}}\, \left.\frac{\partial(\tilde{\boldsymbol{\alpha}},\hat{\boldsymbol{\alpha}})}{\partial(\boldsymbol{s},\boldsymbol{r})}\right|_{\boldsymbol{y}} . \end{aligned} \tag{7.13}$$

In anticipation of introducing the scattering coefficients (cf. Eq.(5.3)), we introduce the one-dimensional array of functions a_R, satisfying

$$a_R(\boldsymbol{y},\boldsymbol{x},\Theta)\; R_L^{(\tilde{N}\hat{M})}(\boldsymbol{x},\tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(\boldsymbol{x}),\hat{\boldsymbol{\alpha}}^{(\hat{M})}(\boldsymbol{x})) = a(\boldsymbol{y},\boldsymbol{x},\Theta)\; (\mathbf{c}^{(1)})'(\phi(\boldsymbol{x}))\, |\nabla_{\boldsymbol{x}}\phi|_{\boldsymbol{x}} \;. \tag{7.14}$$

We will interpret Θ in the spatial Fourier domain. To this end, we carry out two coordinate transformations. First, we employ the ray induced mapping

$$\boldsymbol{s} = \boldsymbol{s}(\tilde{N}, \hat{M}, \boldsymbol{\nu}, \theta, \psi), \quad \boldsymbol{r} = \boldsymbol{r}(\tilde{N}, \hat{M}, \boldsymbol{\nu}, \theta, \psi) \quad \textit{for fixed } \boldsymbol{y} \; , \tag{7.15}$$

with

$$\int_{\partial S \times \partial R} \cdots \frac{\partial(\tilde{\boldsymbol{\alpha}}, \hat{\boldsymbol{\alpha}})}{\partial(\boldsymbol{s}, \boldsymbol{r})}\bigg|_{\boldsymbol{y}} \, \mathrm{d}\boldsymbol{s} \, \mathrm{d}\boldsymbol{r} = \int_{S^2 \times S^2} \cdots \frac{\partial(\tilde{\boldsymbol{\alpha}}, \hat{\boldsymbol{\alpha}})}{\partial(\boldsymbol{\nu}, \theta, \psi)}\bigg|_{\boldsymbol{y}} \, \mathrm{d}\boldsymbol{\nu} \, \mathrm{d}\theta \, \mathrm{d}\psi \; .$$

Thus, at $\boldsymbol{y}$, Θ is mapped on $(\omega, \boldsymbol{\nu}, \theta, \psi)$, and we set

$$a(\boldsymbol{y}, \boldsymbol{x}, \omega, \boldsymbol{\nu}, \theta, \psi) \, \frac{\partial(\tilde{\boldsymbol{\alpha}}, \hat{\boldsymbol{\alpha}})}{\partial(\boldsymbol{s}, \boldsymbol{r})}\bigg|_{\boldsymbol{y}} = a(\boldsymbol{y}, \boldsymbol{x}, \omega, \boldsymbol{s}, \boldsymbol{r}) \, \frac{\partial(\tilde{\boldsymbol{\alpha}}, \hat{\boldsymbol{\alpha}})}{\partial(\boldsymbol{\nu}, \theta, \psi)}\bigg|_{\boldsymbol{y}} \; .$$

We identify

$$T^{(\tilde{N}\hat{M})}(\boldsymbol{r}, \boldsymbol{x}, \boldsymbol{s}) \quad \text{with} \quad T^{(\tilde{N}\hat{M})}(\boldsymbol{x}, \boldsymbol{\nu}, \theta, \psi) \; ,$$

which function determines also the dual generalised Radon transform.

Second, the frequency ω is transformed to the wavenumber k_Γ according to

$$k_\Gamma \equiv \omega \, |\boldsymbol{\Gamma}^{(\tilde{N}\hat{M})}(\boldsymbol{r}, \boldsymbol{y}, \boldsymbol{s})| \; ; \tag{7.16}$$

then

$$\int_{\mathbb{R}^+} \cdots |\boldsymbol{\Gamma}^{(\tilde{N}\hat{M})}(\boldsymbol{r}, \boldsymbol{y}, \boldsymbol{s})|^3 \omega^2 \, \mathrm{d}\omega = \int_{\mathbb{R}^+} \cdots k_\Gamma^2 \, \mathrm{d}k_\Gamma \; . \tag{7.17}$$

We now identify the wave vector

$$\Theta' \equiv k_\Gamma \boldsymbol{\nu} \in \mathbb{R}^3 \quad \text{with} \quad \mathrm{d}\Theta' = k_\Gamma^2 \, \mathrm{d}k_\Gamma \, \mathrm{d}\boldsymbol{\nu} \; , \tag{7.18}$$

and we will consider (θ, ψ) as parameters, i.e., $\Theta \rightarrow (\omega, \boldsymbol{\nu}, \theta, \psi) \rightarrow (\Theta', \theta, \psi)$. The Jacobian of the latter transformation is written as (cf. Eq.(7.17))

$$\frac{\partial(\Theta')}{\partial(\omega, \boldsymbol{\nu})}\bigg|_{\boldsymbol{y}} = h(\boldsymbol{y}, \boldsymbol{\nu}) \, \omega^2 \; , \quad h(\boldsymbol{y}, \boldsymbol{\nu}) = |\boldsymbol{\Gamma}^{(\tilde{N}\hat{M})}(\boldsymbol{r}, \boldsymbol{y}, \boldsymbol{s})|^3 \; ;$$

formally,

$$h = \left| \det \begin{pmatrix} \boldsymbol{\Gamma}^{(\tilde{N}\hat{M})} & \partial_{\nu_1} \boldsymbol{\Gamma}^{(\tilde{N}\hat{M})} & \partial_{\nu_2} \boldsymbol{\Gamma}^{(\tilde{N}\hat{M})} \end{pmatrix} \right| \; . \tag{7.19}$$

The inverse transformation to frequency, the so-called Stolt mapping, is given by

$$\omega(\Theta') = \frac{\Theta' \cdot \boldsymbol{\Gamma}^{(\tilde{N}\hat{M})}(\boldsymbol{r}, \boldsymbol{y}, \boldsymbol{s})}{|\boldsymbol{\Gamma}^{(\tilde{N}\hat{M})}(\boldsymbol{r}, \boldsymbol{y}, \boldsymbol{s})|^2} \; , \tag{7.20}$$

since

$$\Theta' = \omega\, \mathbf{\Gamma}^{(\tilde{N}\hat{M})}(\boldsymbol{r}, \boldsymbol{y}, \boldsymbol{s}) \ .$$

We identify

$$\Phi(\boldsymbol{y}, \boldsymbol{x}, \Theta) \quad \text{with} \quad \Phi(\boldsymbol{y}, \boldsymbol{x}, \Theta', \theta, \psi) \ .$$

In the phase space with coordinates $(\boldsymbol{x}, \Theta')$, Eq.(7.10) gives rise to the resolution equation

$$\langle (\mathbf{c}^{(1)})'(\phi(\boldsymbol{y}))\, |\nabla_{\boldsymbol{x}}\phi|_{\boldsymbol{y}} \rangle = \mathrm{Re} \sum_J \int_{S^2} \frac{1}{8\pi^3} \int_{\mathrm{I\!R}} \int_{\phi=L}$$
$$\int_{\mathrm{I\!R}^+ \times S^2} \exp[i\Phi(\boldsymbol{y}, \boldsymbol{x}, \Theta', \theta, \psi)]\, a(\boldsymbol{y}, \boldsymbol{x}, \Theta', \theta, \psi)\, \chi_J(\Theta', \theta, \psi)\, \mathrm{d}\Theta' \ \times$$
$$(\mathbf{c}^{(1)})'(\phi(\boldsymbol{x}))\, |\nabla_{\boldsymbol{x}}\phi|_{\boldsymbol{x}}\ \frac{\mathrm{d}\Sigma(\boldsymbol{x})}{|\nabla_{\boldsymbol{x}}\phi|_{\boldsymbol{x}}}\, \mathrm{d}L\ \ \mathrm{d}\theta\, \mathrm{d}\psi \ . \tag{7.21}$$

In this expression, for $\boldsymbol{x}$ near $\boldsymbol{y}$, the dependencies of Φ on θ, ψ disappear to leading order: $\Phi = \Theta' \cdot (\boldsymbol{y} - \boldsymbol{x}) + \cdots$. The integration over Θ' represents the spatial resolution, whereas the integration over θ, ψ represents the parameter resolution per migration dip. Below, we will constrain χ_J to be a function of $k_\Gamma = |\Theta'|, \theta, \psi$ alone.

Stationary phase analysis of the resolution operator

Inside the integral over ω in Eqs.(7.10)-(7.11), we consider ω to be large. Then, we apply a four-dimensional stationary phase analysis with respect to the integrations over $(\sigma_1, \sigma_2, \boldsymbol{\nu}) \in \mathcal{S}(L) \times S^2$, cf. Eq.(7.21).

We choose polar coordinates on the $\boldsymbol{\nu}$-sphere,

$$\boldsymbol{\nu} = (\sin\theta^\nu \cos\psi^\nu, \sin\theta^\nu \sin\psi^\nu, \cos\theta^\nu) \ .$$

We extract $L = \sigma_3$ and $k_\Gamma = |\Theta'|$ from the set of phase space coordinates,

$$(\boldsymbol{x}, \Theta') \to (\sigma_1, \sigma_2, L, k_\Gamma, \theta^\nu, \psi^\nu) \ , \quad \eta \equiv (\sigma_1, \sigma_2, \theta^\nu, \psi^\nu) \ ,$$

resubstitute $k_\Gamma = \omega |\mathbf{\Gamma}(\boldsymbol{y})|$, and set

$$\Phi(\boldsymbol{y}, \boldsymbol{x}, \Theta', \theta, \psi) = \omega\, \Phi'(\boldsymbol{y}, L, \eta, k_\Gamma, \theta, \psi) \ .$$

Writing the coordinates explicitly, the resolution equation (7.21) takes the form

$$\langle (\mathbf{c}^{(1)})'(\phi(\boldsymbol{y}))\, |\nabla_{\boldsymbol{x}}\phi|_{\boldsymbol{y}} \rangle = \mathrm{Re} \sum_J \int_{S^2} \frac{1}{8\pi^3} \int_{\mathrm{I\!R}} \int_{\mathrm{I\!R}^+} \int_{\mathcal{S}(L)\times S^2}$$
$$\exp[i\omega\, \Phi'(\boldsymbol{y}, L, \eta, k_\Gamma, \theta, \psi)]\, a(\boldsymbol{y}, L, \eta, k_\Gamma, \theta, \psi)\, \chi_J(k_\Gamma, \theta, \psi)\, h(\boldsymbol{y}, \boldsymbol{\nu}) \ \times \tag{7.22}$$
$$(\mathbf{c}^{(1)})'(\phi(\boldsymbol{x}(\boldsymbol{\sigma})))\, |\nabla_{\boldsymbol{x}}\phi|_{\boldsymbol{x}(\boldsymbol{\sigma})}\ \frac{\mathrm{d}\eta}{|\nabla_{\boldsymbol{x}}\phi|_{\boldsymbol{x}(\boldsymbol{\sigma})}}\, \omega^2 \mathrm{d}\omega\, \mathrm{d}L\ \ \mathrm{d}\theta\, \mathrm{d}\psi \ ,$$

where

$$\mathrm{d}\eta = \mathrm{d}\Sigma(\boldsymbol{x}) \sin\theta^\nu \mathrm{d}\theta^\nu \mathrm{d}\psi^\nu \ . \tag{7.23}$$

For given (θ, ψ), the phase Φ' is stationary with respect to the variables of integration η if

$$\partial_{\sigma_\mu}\Phi' = 0 \quad \text{and} \quad \partial_{(\theta^\nu,\psi^\nu)}\Phi' = 0 \ ; \tag{7.24}$$

here,

$$\partial_{\sigma_\mu}\Phi' = -\boldsymbol{\Gamma}\cdot\partial_{\sigma_\mu}\boldsymbol{x} \ , \tag{7.25}$$

while

$$\partial_{\theta^\nu}\Phi' = [\hat{\boldsymbol{\gamma}}^{(\hat{M})}(\boldsymbol{y}) - \hat{\boldsymbol{\gamma}}^{(\hat{M})}(\boldsymbol{x})]\cdot\partial_{\theta^\nu}\boldsymbol{r} \ + [\tilde{\boldsymbol{\gamma}}^{(\tilde{N})}(\boldsymbol{y}) - \tilde{\boldsymbol{\gamma}}^{(\tilde{N})}(\boldsymbol{x})]\cdot\partial_{\theta^\nu}\boldsymbol{s} \ , \tag{7.26}$$

$$\partial_{\psi^\nu}\Phi' = [\hat{\boldsymbol{\gamma}}^{(\hat{M})}(\boldsymbol{y}) - \hat{\boldsymbol{\gamma}}^{(\hat{M})}(\boldsymbol{x})]\cdot\partial_{\psi^\nu}\boldsymbol{r} + [\tilde{\boldsymbol{\gamma}}^{(\tilde{N})}(\boldsymbol{y}) - \tilde{\boldsymbol{\gamma}}^{(\tilde{N})}(\boldsymbol{x})]\cdot\partial_{\psi^\nu}\boldsymbol{s} \ . \tag{7.27}$$

The stationary points are denoted as $\sigma_\mu^0 = \sigma_\mu^0(\boldsymbol{y})$ and induce the mapping $\boldsymbol{x}(\sigma_\mu^0, L)$ for any L. The solution of the first equation in (7.24) with (7.25) implies that the stationary migration dip, $\boldsymbol{\nu}^0$, must be parallel to the geological dip, $\boldsymbol{\nu}_\phi$, i.e.,

$$\boldsymbol{\nu}^0 = \pm\boldsymbol{\nu}_\phi \ ; \tag{7.28}$$

one solution, σ_μ^0, of the second equation is easy to identify:

$$\boldsymbol{x}(\sigma_\mu^0, L) = \boldsymbol{y} \ ;$$

at this value one expects the peak contribution. At $\boldsymbol{x}(\sigma_\mu^0, L)$, given $\boldsymbol{\nu}^0$, (θ, ψ) will determine the stationary values of $(\boldsymbol{s}, \boldsymbol{r})$. We denote the stationary points by $\eta^0 = (\sigma_\mu^0, \boldsymbol{\nu}^0(\boldsymbol{x}(\sigma_\mu^0, L)))$, and the stationary point set by H^0, which contains at least two elements (cf. Eq.(7.28)).

Applying the four-dimensional stationary phase approximation to Eq.(7.22) amounts to

$$\begin{aligned} \langle(\mathbf{c}^{(1)})'(\phi(\boldsymbol{y}))\,|\nabla_{\boldsymbol{x}}\phi|_{\boldsymbol{y}}\rangle = & \sum_{\eta^0\in\mathrm{H}^0} \mathrm{Re}\sum_J \int_{S^2} \frac{1}{8\pi^3}\int_{\mathrm{I\!R}}\int_{\mathrm{I\!R}^+} \left(\frac{2\pi}{\omega}\right)^2 \\ & \exp\left[i\omega\,\Phi'(\boldsymbol{y}, L, \eta^0, k_\Gamma, \theta, \psi) + i\frac{\pi}{4}\,\mathrm{sig}(\nabla_\eta\nabla_\eta\Phi')^0\right] \frac{a(\boldsymbol{y}, L, \eta^0, k_\Gamma, \theta, \psi)}{\sqrt{|\det(\nabla_\eta\nabla_\eta\Phi')^0|}} \\ & \chi_J(k_\Gamma, \theta, \psi)\, h(\boldsymbol{y}, \boldsymbol{\nu}^0) \ \times \\ & (\mathbf{c}^{(1)})'(\phi(\boldsymbol{x}(\sigma_\mu^0, L)))\,|\nabla_{\boldsymbol{x}}\phi|_{\boldsymbol{x}(\sigma_\mu^0,\, L)} \ \left.\frac{|\partial_{\sigma_1}\boldsymbol{x}\wedge\partial_{\sigma_2}\boldsymbol{x}|}{|\nabla_{\boldsymbol{x}}\phi|}\right|_{\boldsymbol{x}(\sigma_\mu^0,\, L)} \omega^2\mathrm{d}\omega\,\mathrm{d}L\ \ \mathrm{d}\theta\,\mathrm{d}\psi \ . \end{aligned} \tag{7.29}$$

Here sig denotes the number of positive eigenvalues minus the number of negative eigenvalues of a matrix. In Eq.(7.29), k_Γ is the stretch of frequency with $|\mathbf{\Gamma}(\boldsymbol{y})|^0$, the norm of the gradient of travel time in case the migration dip is stationary. In terms of scattering coefficients (cf. Eq.(7.14)), for $\boldsymbol{x}(\sigma^0_\mu, L)$ near $\boldsymbol{y} \in \{\phi = L\}$, expression (7.29) reduces to

$$\langle (\mathbf{c}^{(1)})'(\phi(\boldsymbol{y}))\,|\nabla_{\boldsymbol{x}}\phi|_{\boldsymbol{y}}\rangle \simeq \sum_{\boldsymbol{\nu}^0 = \pm\boldsymbol{\nu}_\phi} \tfrac{1}{2} \sum_J \int_{S^2} \int_{\mathrm{I\!R}}$$
$$\mathrm{Re}\,\frac{1}{\pi}\int_{\mathrm{I\!R}^+} \exp\left[i\omega\,\Phi'(\boldsymbol{y},L,\eta^0,k_\Gamma,\theta,\psi) + i\frac{\pi}{4}\,\mathrm{sig}(\nabla_\eta\nabla_\eta\Phi')^0\right] \chi_J(k_\Gamma,\theta,\psi)\mathrm{d}k_\Gamma \times$$
$$\frac{a_R(\boldsymbol{y},L,\eta^0,k_\Gamma,\theta,\psi)\,R_L^{(\tilde{N}\hat{M})}(\eta^0,k_\Gamma,\theta,\psi)\,h(\boldsymbol{y},\boldsymbol{\nu}^0)}{|\mathbf{\Gamma}(\boldsymbol{y})|^0\sqrt{|\det(\nabla_\eta\nabla_\eta\Phi')^0|}} \times$$
$$\left.\frac{|\partial_{\sigma_1}\boldsymbol{x}\wedge\partial_{\sigma_2}\boldsymbol{x}|}{|\nabla_{\boldsymbol{x}}\phi|}\right|_{\boldsymbol{x}(\sigma^0_\mu,\,L)} \mathrm{d}L\ \mathrm{d}\theta\,\mathrm{d}\psi\,. \tag{7.30}$$

Recognizing the bandlimited Dirac distribution,

$$I_{\Delta,J}(\boldsymbol{\nu}^0\cdot(\boldsymbol{y}-\boldsymbol{x})) \equiv \frac{1}{\pi}\times \tag{7.31}$$
$$\int_{\mathrm{I\!R}^+} \exp\left[i\omega\,\Phi'(\boldsymbol{y},L,\eta^0,k_\Gamma,\theta,\psi) + i\frac{\pi}{4}\,\mathrm{sig}(\nabla_\eta\nabla_\eta\Phi')^0\right] \chi_J(k_\Gamma,\theta,\psi)\,\mathrm{d}k_\Gamma\,,$$

we find that the reconstruction amounts to a weighted integration of scattering coefficients,

$$\langle (\mathbf{c}^{(1)})'(\phi(\boldsymbol{y}))\,|\nabla_{\boldsymbol{x}}\phi|_{\boldsymbol{y}}\rangle \simeq \sum_{\boldsymbol{\nu}^0 = \pm\boldsymbol{\nu}_\phi} \tfrac{1}{2} \sum_J \int_{E_\theta\times E_\psi} \int_{\mathrm{I\!R}}$$
$$\mathrm{Re}\,I_{\Delta,J}(\boldsymbol{\nu}^0\cdot(\boldsymbol{y}-\boldsymbol{x}))\left.\frac{1}{|\nabla_{\boldsymbol{x}}\phi|}\right|_{\boldsymbol{x}(\sigma^0_\mu,\,L)} \times$$
$$\frac{a_R(\boldsymbol{y},L,\eta^0,k_\Gamma,\theta,\psi)\,R_L^{(\tilde{N}\hat{M})}(\eta^0,k_\Gamma,\theta,\psi)\,h(\boldsymbol{y},\boldsymbol{\nu}^0)}{|\mathbf{\Gamma}(\boldsymbol{y})|^0\sqrt{|\det(\nabla_\eta\nabla_\eta\Phi')^0|}} \times$$
$$|\partial_{\sigma_1}\boldsymbol{x}\wedge\partial_{\sigma_2}\boldsymbol{x}|_{\boldsymbol{x}(\sigma^0_\mu,\,L)}\ \mathrm{d}L\ \mathrm{d}\theta\,\mathrm{d}\psi\,, \tag{7.32}$$

where we have accounted for the fact that the range of integration over θ, ψ will be limited by the acquisition geometry. The ranges are given by $E_\theta = E_\theta(\boldsymbol{\nu}^0)$, $E_\psi = E_\psi(\boldsymbol{\nu}^0, \theta)$.

In Section 9, we will evaluate the Hessian $\det(\nabla_\eta \nabla_\eta \Phi')^0$ and show that $\mathrm{sig}(\nabla_\eta \nabla_\eta \Phi')^0 = 0$ at $\boldsymbol{\nu}^0 = \pm \boldsymbol{\nu}_\phi$. Hence, in case the partition is complete, we have

$$\sum_J \mathrm{Re}\, I_{\Delta,J}(\boldsymbol{\nu}^0 \cdot (\boldsymbol{y} - \boldsymbol{x})) \frac{1}{|\nabla_{\boldsymbol{x}} \phi|} = \frac{1}{|\nabla_{\boldsymbol{x}} \phi|} \delta(\boldsymbol{\nu}^0 \cdot (\boldsymbol{y} - \boldsymbol{x})) \simeq \delta(\phi(\boldsymbol{y}) - \phi(\boldsymbol{x})) \; . \tag{7.33}$$

Due to the point symmetry of the slowness surface at the image point $\boldsymbol{y}$, we can replace the summation $\sum_{\boldsymbol{\nu}^0 = \pm \boldsymbol{\nu}_\phi} \frac{1}{2}$ by the substitution $\boldsymbol{\nu}^0 = \boldsymbol{\nu}_\phi$.

8 GRT inversion of Kirchhoff data

Now, we will analyze the resolution operator from a different perspective. To accommodate for the non-linear reflection/transmission coefficients, we substitute our Kirchhoff-like approximation, a mixture of Eqs.(5.2) and (5.9), into the inversion formula Eq.(7.7). The result is an equation very similar to Eq.(7.32). We obtain

$$\langle (\mathbf{c}^{(1)})'(\phi(\boldsymbol{y}))\, |\nabla_{\boldsymbol{x}} \phi|_{\boldsymbol{y}} \rangle \simeq \sum_{\boldsymbol{\nu}^0 = \pm \boldsymbol{\nu}_\phi} \tfrac{1}{2} \int_{E_\theta \times E_\psi} \int_{\mathbb{R}} r_L^{(\tilde{N}\hat{M})} \,\mathrm{d}L \;\, \mathrm{d}\theta \,\mathrm{d}\psi \; , \tag{8.1}$$

where

$$r_L^{(\tilde{N}\hat{M})} = \sum_J \;\mathrm{Re}\, I_{\Delta,J}(\boldsymbol{\nu}^0 \cdot (\boldsymbol{y} - \boldsymbol{x})) \left. \frac{1}{|\nabla_{\boldsymbol{x}} \phi|} \right|_{\boldsymbol{x}(\sigma_\mu^0, L)} \times$$
$$\frac{a_R(\boldsymbol{y}, L, \eta^0, k_\Gamma, \theta, \psi)\, \mathsf{R}_L^{(\tilde{N}\hat{M})}(\eta^0, k_\Gamma, \theta, \psi)\, h(\boldsymbol{y}, \boldsymbol{\nu}^0)}{|\boldsymbol{\Gamma}(\boldsymbol{y})|^0 \sqrt{|\det(\nabla_\eta \nabla_\eta \Phi')^0|}} \times$$
$$\left| \partial_{\sigma_1} \boldsymbol{x} \wedge \partial_{\sigma_2} \boldsymbol{x} \right|_{\boldsymbol{x}(\sigma_\mu^0, L)} \; . \tag{8.2}$$

Componentwise, the reflection/transmission coefficients can be identified, viz., by undoing the multiplications by a_R.

9 The Hessian

In this section we will evaluate the Hessian of Φ' at a stationary point. First, note that $\partial_{\nu_\mu} \partial_{\nu_\nu} \Phi' = 0$ at the stationary point $\boldsymbol{x} = \boldsymbol{y}$ (then $\Phi \equiv 0$). Hence,

$$\det\,(\nabla_\eta \nabla_\eta \Phi')^0 = \left[\det\,(\nabla_{\boldsymbol{\nu}} \nabla_{\boldsymbol{\sigma}} \Phi')^0 \right]^2 \; . \tag{9.1}$$

On the other hand, note that

$$\left(\partial_{\nu_\mu} \partial_{\sigma_\nu} \Phi' \right) = (\partial_{\nu_\mu} \boldsymbol{\Gamma}) \cdot (\partial_{\sigma_\nu} \boldsymbol{x}) \; . \tag{9.2}$$

This matrix can be written in the form

$$\begin{pmatrix} - & \partial_{\nu_1}\boldsymbol{\Gamma} & - \\ - & \partial_{\nu_2}\boldsymbol{\Gamma} & - \end{pmatrix} \begin{pmatrix} | & | \\ \partial_{\sigma_1}\boldsymbol{x} & \partial_{\sigma_2}\boldsymbol{x} \\ | & | \end{pmatrix} ,$$

hence, also,

$$|\boldsymbol{\Gamma}| \det\left[(\partial_{\boldsymbol{\nu}}\boldsymbol{\Gamma}) \cdot (\partial_{\boldsymbol{\sigma}}\boldsymbol{x})\right] = \det\left[\begin{pmatrix} - & \partial_{\nu_1}\boldsymbol{\Gamma} & - \\ - & \partial_{\nu_2}\boldsymbol{\Gamma} & - \\ - & \boldsymbol{\Gamma} & - \end{pmatrix} \begin{pmatrix} | & | & | \\ \partial_{\sigma_1}\boldsymbol{x} & \partial_{\sigma_2}\boldsymbol{x} & \boldsymbol{\nu}_\phi \\ | & | & | \end{pmatrix}\right] . \tag{9.3}$$

In this expression, the first matrix on the right-hand side can be identified as h (see Eq.(7.19)) and the second one with the Jacobian of the coordinate transformation (Eq.(4.9)). Hence, using Eq.(9.1), we arrive at the identity

$$|\boldsymbol{\Gamma}|^0 \sqrt{\det\left(\nabla_\eta\nabla_\eta\Phi'\right)^0} = h(.,\boldsymbol{\nu}^0) \; \left|\partial_{\sigma_1}\boldsymbol{x} \wedge \partial_{\sigma_2}\boldsymbol{x}\right|_{\boldsymbol{x}(\sigma^0_\mu,\,L)} \; . \tag{9.4}$$

In view of Eq.(9.1), the positive and negative eigenvalues must come in pairs, so that sig $(\nabla_\eta\nabla_\eta\Phi')^0$ must be equal to 0 or 4. On the other hand, we have intrinsically assumed that the gradient of two-way travel does not vanish, so that $h \not\equiv 0$. In accordance with Eq.(9.4), hence, the determinant cannot vanish. This implies that under continuous deformations of the interface and ray geometries, the signature of the Hessian cannot change from 0 to 4 or vice versa (this would require an eigenvalue to become zero). In the case of flat level surfaces, it can be shown that the signature equals zero, which now implies that

$$\text{sig}\left(\nabla_\eta\nabla_\eta\Phi'\right)^0 = 0$$

for any ϕ.

The modified GRT inversion

Substituting Eq.(9.4) into Eq.(8.2) yields

$$r_L^{(\tilde{N}\hat{M})} = \sum_J \text{Re}\; I_{\Delta,J}(\boldsymbol{\nu}^0 \cdot (\boldsymbol{y} - \boldsymbol{x})) \; \frac{1}{|\nabla_{\boldsymbol{x}}\phi|}\bigg|_{\boldsymbol{x}(\sigma^0_\mu,\,L)} \times$$
$$a_R(\boldsymbol{y}, L, \eta^0, k_\Gamma, \theta, \psi)\, \mathsf{R}_L^{(\tilde{N}\hat{M})}(\eta^0, k_\Gamma, \theta, \psi) \; . \tag{9.5}$$

To arrive at this expression, at the stationary dip $\boldsymbol{\nu}^0 = \boldsymbol{\nu}_\phi$, we could have set $(\boldsymbol{\nu} \cdot \boldsymbol{\nu}_\phi) = 1$ in Eq.(7.7) to begin with. Then, a priori knowledge about the geological dip is not required.

In fact, the dip is estimated from the image resulting from the partial reconstruction discussed in Section 7. The natural way to rewrite inversion formula (7.8) with Eq.(8.1) is

$$\int_{\mathbb{R}} r_L^{(\tilde{N}\hat{M})}\,\mathrm{d}L \simeq \frac{1}{8\pi^2}\int_{S^2} \tag{9.6}$$

$$\Lambda_{\boldsymbol{y}}^{-1}\mathbf{w}^{(\tilde{N}\hat{M})}(\boldsymbol{y},\tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(\boldsymbol{y}),\hat{\boldsymbol{\alpha}}^{(\hat{M})}(\boldsymbol{y}))\,\partial_t u^{(\tilde{N}\hat{M})}(\boldsymbol{r},\boldsymbol{s},\boldsymbol{y})\,|\boldsymbol{\Gamma}|^4\left.\frac{\partial(\tilde{\boldsymbol{\alpha}},\hat{\boldsymbol{\alpha}})}{\partial(\boldsymbol{\nu},\theta,\psi)}\right|_{\boldsymbol{y}}\mathrm{d}\boldsymbol{\nu}\ .$$

Here, we employ the mappings defined in Eq.(7.15). Note that this inversion formula comprises a system of equations; from each equation, in principle, the reflection/transmission coefficient at specular can be determined (we estimate the geological dip from the image and we have reordered the inversion by scattering angle and azimuth, which implies that we know a_R at stationary). The redundancy can be employed to verify or improve the estimate of the stationary dip. How the stationary dip appears in the spectrum of the medium's perturbation is discussed in Appendix A.

10 Discussion

We have shown, by carrying out a precise resolution analysis, that it is feasible to extract information about the angular dependent reflection/transmission coefficients from a GRT-based migration/inversion. We did not have to linearize nor expand the coefficients. In fact, the outcome of the resolution analysis is a multiple set of images for the reflection/transmission coefficients for the available range of specular scattering angles. Any type of AVA analysis can then be applied to interpret those images. In the derivation we have made use of the fact that the surface integral representations are linear in the scattering coefficients; these coefficients reduce, at specular, to the reflection/transmission coefficients.

The GRT approach employs a somewhat unusual ordering of data, viz., in common (θ,ψ) gathers. The inversion formula reduces to a two-dimensional integration over migration dip. The (θ,ψ) sorting, however, varies with the image point. It bears resemblance with the sorting in common offset, though. The use of such a sorting, however, necessitates the calculation of an additional Jacobian.

11 Acknowledgement

The authors would like to thank Herman Jaramillo for making Figure 2.1.

Appendix A. The spectrum of the medium perturbation

In principle, the geological dip can be directly estimated from an image. However, the geological dip is also hidden in the spectrum of the medium perturbation Eq.(4.10).

Setting (cf. Eq.(6.1))

$$z \parallel \boldsymbol{\nu}_\phi, \quad \{x_\mu\} \perp \boldsymbol{\nu}_\phi$$

at $\boldsymbol{y} \in \mathcal{D}$, the medium perturbation spectrum is of the form (cf. Eq.(4.10))

$$\widetilde{\partial_\phi \mathbf{c}^{(1)}}(\boldsymbol{k}) = \int_{\mathbb{R}} \mathrm{d}L\, (\mathbf{c}^{(1)})'(L) \int_{\mathcal{D}} \delta(\phi(\boldsymbol{x}) - L)\ \exp\left[-i\left(k_\mu x_\mu + k_z(z_{\boldsymbol{y}}(L) + z)\right)\right]\ \mathrm{d}\boldsymbol{x}\ , \quad \text{(A.1)}$$

where in fact $\boldsymbol{k} = \Theta' = k_\Gamma \boldsymbol{\nu}$. Near the level surface $\phi = L$, we employ the expansion

$$\phi(\boldsymbol{x}) = L + |\nabla_{\boldsymbol{x}}\phi| z + \tfrac{1}{2} x_\mu \phi_{\mu\nu} x_\nu\ ; \quad \text{(A.2)}$$

we implicitly assume that the integral over L is windowed around $\boldsymbol{y}$ such that in the window or neighborhood $\phi_{\mu\nu}$ may depend on L but $\boldsymbol{\nu}_\phi$ does not. At the level surface on which $\boldsymbol{y}$ lies, we have $z_{\boldsymbol{y}} = 0$. Then,

$$\widetilde{\partial_\phi \mathbf{c}^{(1)}}(\boldsymbol{k}) \simeq \int_{\mathbb{R}} \mathrm{d}L\, (\mathbf{c}^{(1)})'(L) \exp[-ik_z z_{\boldsymbol{y}}(L)]\ \times \quad \text{(A.3)}$$

$$\int_{\mathcal{D}} \delta(|\nabla_{\boldsymbol{x}}\phi| z + \tfrac{1}{2} x_\mu \phi_{\mu\nu} x_\nu)\ \exp\left[-i\left(k_\mu x_\mu + k_z z\right)\right]\ \mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}z$$

$$= \int_{\mathbb{R}} \mathrm{d}L\, (\mathbf{c}^{(1)})'(L) \exp[-ik_z z_{\boldsymbol{y}}(L)] \int_{\mathbb{R}^2} \exp\left[-i\left(k_\mu x_\mu - k_z \frac{x_\mu \phi_{\mu\nu} x_\nu}{2|\nabla_{\boldsymbol{x}}\phi|}\right)\right]\ \frac{\mathrm{d}x_1\, \mathrm{d}x_2}{|\nabla_{\boldsymbol{x}}\phi|}$$

$$= \int_{\mathbb{R}} \frac{2\pi\ \exp\left[\pi i\, \mathrm{sig}(k_z\phi_{\mu\nu})/4\right]}{|k_z| \sqrt{|\det(\phi_{\mu\nu}|}}\ \exp\left[ik_\mu \frac{|\nabla_{\boldsymbol{x}}\phi|}{k_z} \phi^{-1}_{\mu\nu} k_\nu\right]\ \times$$

$$(\mathbf{c}^{(1)})'(L) \exp[-ik_z z_{\boldsymbol{y}}(L)]\ \mathrm{d}L\ ,$$

where sig, as in the main text, represents the sum of signs (± 1) of eigenvalues of $k_z\phi_{\mu\nu}$. Note that $k_\mu = 0$ corresponds with the geological dip direction.

If the level surfaces of ϕ were *flat* and $\boldsymbol{\nu}_\phi = \boldsymbol{\nu}^1$ fixed, we would get

$$\widetilde{\partial_\phi \mathbf{c}^{(1)}}(\boldsymbol{k}) = \delta(\boldsymbol{k} - (\boldsymbol{k}\cdot\boldsymbol{\nu}^1)\boldsymbol{\nu}^1)\ (\widetilde{\mathbf{c}^{(1)}})'(\boldsymbol{k}\cdot\boldsymbol{\nu}^1) = \frac{1}{k_\Gamma^2}\, \delta(\boldsymbol{\nu} - (\boldsymbol{\nu}\cdot\boldsymbol{\nu}^1)\boldsymbol{\nu}^1)\ (\widetilde{\mathbf{c}^{(1)}})'(k_\Gamma(\boldsymbol{\nu}\cdot\boldsymbol{\nu}^1))\ ,$$

which, in the Radon domain, implies

$$\int_{\mathcal{D}} \partial_\phi \mathbf{c}^{(1)}(\phi(\boldsymbol{x}))\, \delta''(\boldsymbol{y}\cdot\boldsymbol{\nu} - \boldsymbol{x}\cdot\boldsymbol{\nu})\ \mathrm{d}\boldsymbol{x}$$

$$= -\frac{1}{\pi}\,\delta(\boldsymbol{\nu} - (\boldsymbol{\nu}\cdot\boldsymbol{\nu}^1)\boldsymbol{\nu}^1)\,\mathrm{Re}\int_{\mathbb{R}^+}(\widetilde{\mathbf{c}^{(1)}})'(k_\Gamma(\boldsymbol{\nu}\cdot\boldsymbol{\nu}^1))\,\exp(ik_\Gamma\varphi)\,\mathrm{d}k_\Gamma\Big|_{\varphi=\boldsymbol{y}\cdot\boldsymbol{\nu}}$$

$$= -\frac{1}{\pi}\,\delta(\boldsymbol{\nu} - (\boldsymbol{\nu}\cdot\boldsymbol{\nu}^1)\boldsymbol{\nu}^1)\,\mathrm{Re}\int_{\mathbb{R}^+}(\widetilde{\mathbf{c}^{(1)}})'(k_\Gamma)\,\exp(ik_\Gamma\varphi)\,\mathrm{d}k_\Gamma\Big|_{\varphi=\boldsymbol{y}\cdot\boldsymbol{\nu}^1}\ . \quad \text{(A.4)}$$

This formula shows that the GRT algorithm can reveal the geological dip explicitly. We assumed a proper coordinate system on S^2, such that

$$\int_{S^2}\delta(\boldsymbol{\nu} - (\boldsymbol{\nu}\cdot\boldsymbol{\nu}^1)\boldsymbol{\nu}^1)\ \mathrm{d}\boldsymbol{\nu} = 1\ .$$

References

[1] S. Geoltrain and E. Chovet, “Automatic association of kinematic information to prestack images”, 61st Ann. Int. Mtg. Soc. Explor. Geophys., *Expanded Abstracts*, 890-892 (1991).

[2] D.E. Lumley, “Angle-dependent reflectivity estimation”, 63rd Ann. Int. Mtg. Soc. Expl. Geophys., *Expanded Abstracts*, 746-749 (1993).

[3] N. Bleistein, J.K. Cohen and J. Stockwell, *Mathematics of multidimensional seismic inversion*, in preparation (1996).

[4] S.G. Norton and M. Linzer, “Ultrasonic scattering potential imaging in three dimensions: exact inverse scattering solutions for plane, cylindrical, and spherical apertures”, *IEEE Trans. on Biomedical Engineering, BME-28*, 202-220 (1981).

[5] G. Beylkin, *Generalized Radon Transform and its applications*, Ph.D. Thesis, New York University (1982).

[6] G. Beylkin, “The inversion problem and applications of the generalized Radon Transform”, *Comm. Pure Appl. Math., 37*, 579-599 (1984).

[7] G. Beylkin, “Imaging of discontinuities in the inverse scattering problem by inversion of a causal generalized Radon transform”, *J. of Math. Phys., 26*, 99-108 (1985).

[8] G. Beylkin, “Reconstructing discontinuities in multidimensional inverse scattering problems: smooth errors vs small errors”, *Applied Optics, 24*, 4086-4088 (1985).

[9] D.E. Miller, M. Oristaglio and G. Beylkin, “A new formalism and an old heuristic for seismic migration”, 54th Ann. Int. Mtg. Soc. Explor. Geophys., *Expanded Abstracts*, 704-707 (1984).

[10] D.E. Miller, M. Oristaglio and G. Beylkin, “A new slant on seismic imaging: migration and integral geometry”, *Geophysics, 52*, 943-964 (1987).

[11] G. Beylkin, M. Oristaglio and D. Miller, “Spatial resolution of migration algorithms”, in A.J. Berkhout, J. Ridder, and L.F. van der Waal, *Acoustical Imaging, 14*, Plenum Pub. Co., 155-167 (1985).

[12] Rakesh, “A linearised inverse problem for the wave equation”, *Comm. in Part. Diff. Eqs., 13*, 573-601 (1988).

[13] G. Beylkin and R. Burridge, “Linearized inverse scattering problems in acoustics and elasticity”, *Wave Motion, 12*, 15-52 (1990).

[14] M.V. de Hoop, R. Burridge, C. Spencer and D. Miller, "Generalized Radon Transform/amplitude versus angle (GRT/AVA) migration/inversion in anisotropic media", *SPIE proceedings, 2301*, 15-27 (1994).

[15] C. Spencer and M.V. de Hoop, "Linearity, resolution, and covariance in GRT inversions for anisotropic elastic moduli", *SPIE proceedings, 2571*, 110-122 (1995).

[16] J.K. Cohen and N. Bleistein, "An inverse method for determining small variations in propagation speed", *SIAM J. Appl. Math., 32*, 784-799 (1977).

[17] N. Bleistein and J.K. Cohen, "Velocity inversion for acoustic waves", *Geophysics, 44*, 1077-1087 (1979).

[18] N. Bleistein, "On imaging of reflectors in the earth", *Geophysics, 52*, 931-942 (1987).

[19] M.V. de Hoop and C. Spencer, "Quasi Monte-Carlo integration over $S^2 \times S^2$ for migration $\times$ inversion", accepted for publication in *Inverse Problems* (1996).