Multi-parameter DMO
in anisotropic media
using the generalized Radon transform

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Abstract

Dip moveout (DMO), as part of the transformation to zero-offset (TZO), is a process that transforms data collected at finite source-receiver offsets to data that would be collected at zero-offset. The kinematic or ray geometric principles underlying the DMO process have been well established. The aim of this paper is to extend the TZO and DMO processes to elastic media and develop dynamically correct transformations of wave field data. We accomplish this goal with the aid of the generalized Radon transform (GRT). We allow the media to be generically anisotropic.

Keywords: dip moveout, migration/demigration, generalized Radon transform.

1 Introduction

Dip moveout (DMO), as an intermediate step in the transformation to zero-offset (TZO), is a process that transforms data collected at finite source-receiver offsets to data that would be collected at zero-offset. In practice, true zero-offset data cannot be measured, but even if this were possible, the use of TZO applied to data from mulitple offsets would enhance the signal-to-noise ratio in the data as in the conventional stack. In the past decade, the kinematic or ray-geometric principles underlying the DMO process have been well established. It is only recently, however, that the associated amplitude aspects have received attention. The aim of this paper is to extend the Kirchhoff-style TZO and DMO processes to elastic media, and to develop dynamically correct transformations of wavefield data. We accomplish this goal with the aid of the generalized Radon transform (GRT). We consider general, anisotropic media, and provide closed-form expressions for the constant-background-medium case.

For an overview of papers on DMO, we refer the reader to Hale [1]. The key idea of a wave-theoretic representation of TZO is to apply a migration/inversion to finite-offset data, followed by a modelling of or demigrating to, in the Born approximation, the zero-offset configuration; see Sullivan and Cohen [2] and Bleistein and Jorden [3]. In the acoustic case, Miller and Burridge [4] applied the GRT for this purpose. A different approach, in the context of a more general offset continuation, using differential equations rather than integral transforms, was followed by Fomel [5]. The extension to depth-varying media, as far as the kinematics is concerned, was accomplished by Artley and Hale [6], and an extension to transversely isotropic media with vertical symmetry axis can be found in Uren *et al.* [7] and Alkhalifah [8]. Certain true-amplitude aspects were discussed by Black *et al.* [9] and Bleistein and Cohen [10]. Here, we will account for the full radiation characteristics of the wavefields in anisotropic media, and derive an elastic DMO procedure. Our theory is based on the ray-Born approximation. Related work on DMO and Born inversion can be found in Liner [11].

In this paper, we consider single-mode scattering, in particular the single scattering associated with the fastest characteristics (qP). We also briefly indicate how to extend the theory to the multi-mode, or mode-converted, case.

2 The basic equations

Notation

First, we introduce some basic notation. Choose coordinates in the configuration according to

$$\begin{aligned} \boldsymbol{x} = (x_1, x_2, x_3) &= \text{Cartesian position vector,} \\ \boldsymbol{s} = (s_1, s_2, s_3) &= \text{source point,} \\ \boldsymbol{r} = (r_1, r_2, r_3) &= \text{receiver point,} \\ t &= \text{recording time.} \end{aligned}$$

The medium is described by

$$\begin{aligned} \rho(\boldsymbol{x}) &= \text{density,} \\ c_{ijk\ell}(\boldsymbol{x}) &= \text{elastic stiffness tensor,} \end{aligned}$$

while the wavefield is described by

$$\boldsymbol{u}(\boldsymbol{x}, t) = (u_1(\boldsymbol{x}, t), u_2(\boldsymbol{x}, t), u_3(\boldsymbol{x}, t)) = \text{displacement vector,}$$

and generated by a source distribution given by

$$\boldsymbol{f}(\boldsymbol{x}, t) = (f_1(\boldsymbol{x}, t), f_2(\boldsymbol{x}, t), f_3(\boldsymbol{x}, t)) = \text{body-force source density.}$$

In the remainder of the paper, we will employ the summation convention.

The displacement in an inhomogeneous, anisotropic medium satisfies the elastodynamic wave equation

$$\rho \partial_t^2 u_i - \partial_j (c_{ijk\ell} \partial_\ell u_k) = f_i \ , \tag{2.1}$$

with summation over repeated lower case indices, here and below. Let

$$\boldsymbol{G}(\boldsymbol{x}, \boldsymbol{x}', t) = (G_{ip}(\boldsymbol{x}, \boldsymbol{x}', t)) \tag{2.2}$$

be the causal Green's tensor, which satisfies (cf. Eq.(2.1))

$$\rho\, \partial_t^2 G_{ip} - \partial_j (c_{ijk\ell} \partial_\ell G_{kp}) = \delta_{ip} \delta(\boldsymbol{x} - \boldsymbol{x}') \delta(t) \ , \ \ G_{ip} = 0 \ \text{ for } \ t < 0 \ . \tag{2.3}$$

Asymptotic ray theory

Here, we summarize the formulation of anisotropic ray theory for the evaluation of the Green's tensor (see Kendall *et al.* [12] for details). Let

$$\begin{aligned} G_{ip}(\boldsymbol{x}, \boldsymbol{x}', t) &= \sum_N A^{(N)}(\boldsymbol{x}, \boldsymbol{x}')\, \xi_i^{(N)}(\boldsymbol{x})\xi_p^{(N)}(\boldsymbol{x}')\delta(t - \tau^{(N)}(\boldsymbol{x}, \boldsymbol{x}')) \\ &\quad + \text{ terms smoother in } t\ . \end{aligned} \tag{2.4}$$

In this equation, the arrival time $\tau^{(N)}$ and the associated polarization vector $\boldsymbol{\xi}^{(N)}$ satisfy

$$(\rho\, \delta_{ik} - c_{ijk\ell}(\partial_\ell \tau^{(N)})(\partial_j \tau^{(N)}))\, \xi_k^{(N)} = 0 \quad \text{(at all } \boldsymbol{x}) \ , \tag{2.5}$$

which implies the eikonal equation

$$\det(\rho\, \delta_{ik} - c_{ijk\ell}(\partial_\ell \tau)\, (\partial_j \tau)) = 0 \quad \text{(at all } \boldsymbol{x}) \ . \tag{2.6}$$

The polarization vectors are assumed to be normalized so that $\xi_i^{(N)}\xi_i^{(N)} = 1$. Define the slowness vector $\boldsymbol{\gamma}^{(N)}$ by

$$\boldsymbol{\gamma}^{(N)}(\boldsymbol{x}) = \nabla_{\boldsymbol{x}} \tau^{(N)}(\boldsymbol{x}, \boldsymbol{x}') \ . \tag{2.7}$$

Then, Eq.(2.6) constrains $\boldsymbol{\gamma}$ to lie on the sextic surface $\mathcal{A}(\boldsymbol{x})$ given by

$$\det(\rho\, \delta_{ik} - c_{ijk\ell}\gamma_\ell\gamma_j) = 0 \ . \tag{2.8}$$

$\mathcal{A}(\boldsymbol{x})$ consists of three sheets $\mathcal{A}^{(N)}(\boldsymbol{x})$, $N = 1, 2, 3$, each of which is a closed surface surrounding the origin. An individual sheet is described by (cf. Eq.(2.8))

$$2\, \mathcal{H} = \rho - \xi_i c_{ijk\ell}\gamma_\ell\gamma_j\xi_k = 0 \ . \tag{2.9}$$

The scalar amplitudes A must satisfy the transport equation

$$\partial_j(c_{ijk\ell}\xi_i^{(N)}\xi_k^{(N)}\, (A^{(N)})^2\, \partial_\ell \tau^{(N)}) = 0 \ , \tag{2.10}$$

where N, again, indicates the mode of propagation, that is, the sheet of the slowness surface on which the corresponding slowness vector lies.

The characteristic or group velocities $\boldsymbol{v}^{(N)}$ are normal to $\mathcal{A}^{(N)}(\boldsymbol{x})$ at $\boldsymbol{\gamma}^{(N)}$ and satisfy

$$\boldsymbol{v}^{(N)} \cdot \boldsymbol{\gamma}^{(N)} = 1 \ ; \quad \boldsymbol{v}^{(N)} = \left. \frac{\nabla_{\boldsymbol{\gamma}} \mathcal{H}}{\boldsymbol{\gamma} \cdot \nabla_{\boldsymbol{\gamma}} \mathcal{H}} \right|_{\mathcal{H}=0} \ . \tag{2.11}$$

The normal or phase speeds are given by

$$V^{(N)} = \frac{1}{|\boldsymbol{\gamma}^{(N)}|} \ . \tag{2.12}$$

The unit phase direction follows as

$$\boldsymbol{\alpha}^{(N)} = V^{(N)}\,\boldsymbol{\gamma}^{(N)} \; .$$

From Eq.(2.11) it follows that

$$V^{(N)} = |\boldsymbol{v}^{(N)}| \, \cos\chi \; , \tag{2.13}$$

where χ is the angle between $\boldsymbol{v}^{(N)}$ and $\boldsymbol{\gamma}^{(N)}$.

The amplitudes can be expressed in terms of certain Jacobians,

$$A = \frac{1}{4\pi[\rho(\boldsymbol{x})\rho(\boldsymbol{x}')\mathcal{M}]^{1/2}} \quad \text{with} \quad \mathcal{M} = \frac{|\boldsymbol{v}(\boldsymbol{x}')| V(\boldsymbol{x}) \left| \dfrac{\partial \boldsymbol{x}}{\partial q_1} \wedge \dfrac{\partial \boldsymbol{x}}{\partial q_2} \right|_{\boldsymbol{x}}}{\left| \dfrac{\partial \boldsymbol{\gamma}}{\partial q_1} \wedge \dfrac{\partial \boldsymbol{\gamma}}{\partial q_2} \right|_{\boldsymbol{x}'}} \; , \tag{2.14}$$

in which A and $\mathcal{M}$ carry the superscript (N). Here, (q_1, q_2) parametrize the rays originating from the source. We will substitute the phase angles, $\boldsymbol{\alpha}$, at the source for those parameters. The wedge denotes the vector cross product.

Source and receiver Green's functions

In the integral representation for the scattered field, we need the Green's functions originating both at the source and the receiver points. Further, the gradient of total travel times from the source to a scattering point to the receiver are required in preparation of the inverse GRT. We introduce these functions here.

Set

$$\tilde{\boldsymbol{G}}(\boldsymbol{x},t) = \boldsymbol{G}(\boldsymbol{x},\boldsymbol{s},t) \; , \quad \hat{\boldsymbol{G}}(\boldsymbol{x},t) = \boldsymbol{G}(\boldsymbol{r},\boldsymbol{x},t) \; . \tag{2.15}$$

Employing asymptotic ray theory in both Green's functions, we introduce the notation

$$\tilde{\tau}^{(\tilde{N})}(\boldsymbol{x}) = \tau^{(N)}(\boldsymbol{x},\boldsymbol{s}) \; , \quad \hat{\tau}^{(\hat{M})}(\boldsymbol{x}) = \tau^{(M)}(\boldsymbol{r},\boldsymbol{x}) \tag{2.16}$$

and

$$\tilde{A}^{(\tilde{N})}(\boldsymbol{x}) = A^{(N)}(\boldsymbol{x},\boldsymbol{s}) \; , \quad \hat{A}^{(\hat{M})}(\boldsymbol{x}) = A^{(M)}(\boldsymbol{r},\boldsymbol{x}) \tag{2.17}$$

in the case of scattering from incident mode N to outgoing mode M.

According to Eq.(2.7), the slowness vectors at $\boldsymbol{x}$ are given by

$$\tilde{\boldsymbol{\gamma}}^{(\tilde{N})}(\boldsymbol{x}) = \nabla_{\boldsymbol{x}}\tau^{(N)}(\boldsymbol{x},\boldsymbol{s}) \; , \quad \hat{\boldsymbol{\gamma}}^{(\hat{M})}(\boldsymbol{x}) = \nabla_{\boldsymbol{x}}\tau^{(M)}(\boldsymbol{r},\boldsymbol{x}) \; ; \tag{2.18}$$

the associated phase directions are given by

$$\tilde{\boldsymbol{\alpha}}^{(\tilde{N})} = \frac{\tilde{\boldsymbol{\gamma}}^{(\tilde{N})}}{|\tilde{\boldsymbol{\gamma}}^{(\tilde{N})}|} \ , \quad \hat{\boldsymbol{\alpha}}^{(\hat{M})} = \frac{\hat{\boldsymbol{\gamma}}^{(\hat{M})}}{|\hat{\boldsymbol{\gamma}}^{(\hat{M})}|} \tag{2.19}$$

and the normal (phase) speeds (cf. Eq.(2.12)) are given by

$$\tilde{V}^{(\tilde{N})} = \frac{1}{|\tilde{\boldsymbol{\gamma}}^{(\tilde{N})}|} \ , \quad \hat{V}^{(\hat{M})} = \frac{1}{|\hat{\boldsymbol{\gamma}}^{(\hat{M})}|} \ . \tag{2.20}$$

The unit group directions are given by

$$\tilde{\boldsymbol{n}}^{(\tilde{N})} = \frac{\tilde{\boldsymbol{v}}^{(\tilde{N})}}{|\tilde{\boldsymbol{v}}^{(\tilde{N})}|} \ , \quad \hat{\boldsymbol{n}}^{(\hat{M})} = \frac{\hat{\boldsymbol{v}}^{(\hat{M})}}{|\hat{\boldsymbol{v}}^{(\hat{M})}|} \ . \tag{2.21}$$

We also define the two-way travel time $T^{(\tilde{N}\hat{M})}$ and its gradient,

$$T^{(\tilde{N}\hat{M})}(\boldsymbol{r},\boldsymbol{y},\boldsymbol{s}) \equiv \tau^{(N)}(\boldsymbol{y},\boldsymbol{s}) + \tau^{(M)}(\boldsymbol{r},\boldsymbol{y}) \ , \quad \boldsymbol{\Gamma}^{(\tilde{N}\hat{M})}(\boldsymbol{r},\boldsymbol{x},\boldsymbol{s}) \equiv \nabla_{\boldsymbol{x}} T^{(\tilde{N}\hat{M})}(\boldsymbol{r},\boldsymbol{x},\boldsymbol{s}) \ . \tag{2.22}$$

From Eq.(2.18) we see that

$$\boldsymbol{\Gamma}^{(\tilde{N}\hat{M})}(\boldsymbol{r},\boldsymbol{x},\boldsymbol{s}) = \tilde{\boldsymbol{\gamma}}^{(\tilde{N})}(\boldsymbol{x}) + \hat{\boldsymbol{\gamma}}^{(\hat{M})}(\boldsymbol{x}) \ . \tag{2.23}$$

The direction of $\boldsymbol{\Gamma}^{(\tilde{N}\hat{M})}$,

$$\boldsymbol{\nu} \equiv \frac{\boldsymbol{\Gamma}^{(\tilde{N}\hat{M})}}{|\boldsymbol{\Gamma}^{(\tilde{N}\hat{M})}|} \ ,$$

will be the *migration dip*; it is the normal to the hypersurface of constant two-way travel time.

3 The single scattering equations

In this section, we introduce the ray-Born approximation representing the singly scattered wavefield (see De Hoop *et al.* [13] and Burridge *et al.* [14]). For non-coincident, finite-offset, source-receiver pairs, we let $(\boldsymbol{r},\boldsymbol{s}) \in \partial R \times \partial S$; ideally, the boundaries $\partial R, \partial S \sim S^2$ are closed surfaces surrounding a heterogeneous domain $\mathcal{D}$. For coinciding, zero-offset, source-receiver pairs, we employ the notation $\boldsymbol{s} = \boldsymbol{r} = \boldsymbol{S} \in \partial S$. We restrict ourselves to waves propagating in mode N.

The wavefield at finite offset

In the volume-scattering, ray-Born approximation, the multi-component (q source and p receiver) scattered displacement field is given by

$$u^{(1)}_{pq}(\boldsymbol{r},\boldsymbol{s},t) \simeq -\int_{\mathcal{D}} \hat{\xi}^{(\hat{N})}_p(\boldsymbol{r}) \tilde{\xi}^{(\tilde{N})}_q(\boldsymbol{s}) \, \mathcal{I}^{(\tilde{N}\hat{N})}(\boldsymbol{r},\boldsymbol{x},\boldsymbol{s}) \ \times$$
$$(\mathbf{w}^{(\tilde{N}\hat{N})}(\boldsymbol{x}, \tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(\boldsymbol{x}), \hat{\boldsymbol{\alpha}}^{(\hat{N})}(\boldsymbol{x})))^T \, \mathbf{c}^{(1)}(\boldsymbol{x}) \, \delta''(t - T^{(\tilde{N}\hat{N})}(\boldsymbol{r},\boldsymbol{x},\boldsymbol{s})) \ \mathrm{d}\boldsymbol{x} \ , \tag{3.1}$$

where $N \in \{1, 2, 3\}$,

$$\mathcal{I}^{(\tilde{N}\hat{N})}(\boldsymbol{r}, \boldsymbol{x}, \boldsymbol{s}) = \rho(\boldsymbol{x})\, \tilde{A}^{(\tilde{N})}(\boldsymbol{x})\hat{A}^{(\hat{N})}(\boldsymbol{x}) \tag{3.2}$$

contains the geometrical spreading, and

$$\begin{aligned} \mathbf{w}^{(\tilde{N}\hat{N})} &= \left\{ \tilde{\xi}_i^{(\tilde{N})}\hat{\xi}_i^{(\hat{N})}, \tfrac{1}{2}\left[\hat{a}_{ij}^{(\hat{N})}\tilde{a}_{k\ell}^{(\tilde{N})} + \hat{a}_{k\ell}^{(\hat{N})}\tilde{a}_{ij}^{(\tilde{N})} \right] \right\} , \\ \hat{a}_{ij}^{(\hat{N})} &= \tfrac{1}{2} V_\circ^{(\hat{N})} \left(\hat{\xi}_i^{(\hat{N})}\hat{\gamma}_j^{(\hat{N})} + \hat{\xi}_j^{(\hat{N})}\hat{\gamma}_i^{(\hat{N})} \right) , \ \ \tilde{a}_{k\ell}^{(\tilde{N})} = \tfrac{1}{2} V_\circ^{(\tilde{N})} \left(\tilde{\xi}_k^{(\tilde{N})}\tilde{\gamma}_\ell^{(\tilde{N})} + \tilde{\xi}_\ell^{(\tilde{N})}\tilde{\gamma}_k^{(\tilde{N})} \right) , \end{aligned}$$

describe the contrast-source radiation patterns, and

$$\mathbf{c}^{(1)} = \left\{ \frac{\rho^{(1)}}{\rho}, \frac{c_{ijk\ell}^{(1)}}{\rho\, V_\circ^{(\hat{N})} V_\circ^{(\tilde{N})}} \right\} ,$$

represents the relative medium perturbation ($\rho^{(1)}$ is the absolute perturbation in density and $c_{ijk\ell}^{(1)}$ is the absolute perturbation in stiffness with respect to the background medium). Here, $V_\circ^{(N)}$ denotes the local phase velocity of mode N in the background medium for $\boldsymbol{\gamma}$ averaged over all phase directions. The hypersurface,

$$\{\boldsymbol{x} \in \mathcal{D} \mid T^{(\tilde{N}\hat{N})}(\boldsymbol{r}, \boldsymbol{x}, \boldsymbol{s}) = t\} ,$$

over which the integral in Eq.(3.1) is taken, is denoted as the finite-offset isochron.

The wavefield at zero offset

To obtain the wavefield at zero offset, we substitute $\boldsymbol{r} = \boldsymbol{s} = \boldsymbol{S}$ in Eq.(3.1). To distinguish the zero-offset configuration from the finite-offset one, we use capital letters to indicate the components here. The rays emanating from the source or the receiver connect the same points; excluding the possibility of multi-pathing, those rays must coincide. Thus, we can omit $\tilde{}$ and $\hat{}$. In fact, we have

- $\hat{N} = \tilde{N} \equiv N$;
- $\tilde{\boldsymbol{\gamma}} = \hat{\boldsymbol{\gamma}} \equiv \boldsymbol{\gamma}$; $\tilde{\boldsymbol{\alpha}} = \hat{\boldsymbol{\alpha}} \equiv \boldsymbol{\alpha}$;
- $\tilde{\boldsymbol{\xi}} = \hat{\boldsymbol{\xi}} \equiv \boldsymbol{\xi}$.

Set the two-way travel time and its gradient to (cf. Eq.(2.22))

$$T^{(NN)}(\boldsymbol{s}, \boldsymbol{x}, \boldsymbol{s}) \equiv T^{(N)}(\boldsymbol{x}, \boldsymbol{S}) , \quad \boldsymbol{\Gamma}^{(N)}(\boldsymbol{x}, \boldsymbol{S}) = \nabla_{\boldsymbol{x}} T^{(N)}(\boldsymbol{x}, \boldsymbol{S}) = 2\,\boldsymbol{\gamma}^{(N)}(\boldsymbol{x}) . \tag{3.3}$$

It hence follows that

$$\boldsymbol{\alpha}^{(N)} \parallel \nabla_{\boldsymbol{x}} T^{(NN)} ,$$

expressing that the incident and scattered rays coincide. For the geometrical spreading, we have

$$\mathcal{I}^{(NN)}(\boldsymbol{S},\boldsymbol{x},\boldsymbol{S}) \equiv \mathcal{I}^{(N)}(\boldsymbol{x},\boldsymbol{S}) \ . \tag{3.4}$$

Also $\tilde{a} = \hat{a} \equiv a$. Thus, we can write

$$\mathbf{w}^{(NN)}(\boldsymbol{x},\boldsymbol{\alpha}^{(N)}(\boldsymbol{x}),\boldsymbol{\alpha}^{(N)}(\boldsymbol{x})) \equiv \mathbf{w}^{(N)}(\boldsymbol{x},\boldsymbol{\alpha}^{(N)}(\boldsymbol{x})) \tag{3.5}$$

for the radiation patterns. (Note that the first element of $\mathbf{w}^{(N)}$, the inner product of the polarization vector with itself, becomes unity.)

With $u^{(1)}_{PQ}(\boldsymbol{S},\tau) \equiv u^{(1)}_{PQ}(\boldsymbol{S},\boldsymbol{S},\tau)$, we then obtain

$$u^{(1)}_{PQ}(\boldsymbol{S},\tau) \simeq -\partial_\tau^2 \int_{\mathcal{D}} \xi^{(N)}_P(\boldsymbol{S})\xi^{(N)}_Q(\boldsymbol{S})\, \mathcal{I}^{(N)}(\boldsymbol{x},\boldsymbol{S}) \ \times$$
$$(\mathbf{w}^{(N)}(\boldsymbol{x},\boldsymbol{\alpha}^{(N)}(\boldsymbol{x}))^T\, \mathbf{c}^{(1)}(\boldsymbol{x})\, \delta(\tau - T^{(N)}(\boldsymbol{x},\boldsymbol{S}))\ \mathrm{d}\boldsymbol{x} \ . \tag{3.6}$$

The hypersurface,

$$\{\boldsymbol{x} \in \mathcal{D} \mid T^{(N)}(\boldsymbol{x},\boldsymbol{S}) = \tau\} \ ,$$

over which the integral in Eq.(3.6) is taken, is denoted as the zero-offset isochron. Here, τ is the zero-offset time.

4 Transformation to zero offset

The idea of transformation to zero offset is to eliminate the medium perturbation from the equations representing the finite-offset (Eq.(3.1)) and zero-offset (Eq.(3.6)) scattered fields. This is achieved by cascading finite-offset GRT inversion with zero-offset GRT modelling.

Linearized inversion

The general expression for GRT inversion of the scattered field Eq.(3.1), an integration over the phase directions (associated with two source coordinates and two receiver coordinates), is given in De Hoop *et al.* [13] and Burridge *et al.* [14]. First, set

$$\mathcal{J}^{(\tilde{N}\hat{N})}(\boldsymbol{r},\boldsymbol{x},\boldsymbol{s}) = \frac{\left|\boldsymbol{\Gamma}^{(\tilde{N}\hat{N})}(\boldsymbol{r},\boldsymbol{x},\boldsymbol{s})\right|^3}{\mathcal{I}^{(\tilde{N}\hat{N})}(\boldsymbol{r},\boldsymbol{x},\boldsymbol{s})} \ . \tag{4.1}$$

Then

$$\mathbf{c}^{(1)}(\boldsymbol{x}) \simeq \frac{1}{8\pi^2} \int_{\partial S \times \partial R} [\Lambda^{(\tilde{N}\hat{N})}_{\boldsymbol{x}}(\boldsymbol{\nu}(\boldsymbol{r},\boldsymbol{x},\boldsymbol{s}))]^{-1} \mathbf{w}^{(\tilde{N}\hat{N})}(\boldsymbol{x},\tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(\boldsymbol{x}),\hat{\boldsymbol{\alpha}}^{(\hat{N})}(\boldsymbol{x})) \ \times \tag{4.2}$$
$$\mathcal{J}^{(\tilde{N}\hat{N})}(\boldsymbol{r},\boldsymbol{x},\boldsymbol{s})\ \hat{\xi}^{(\hat{N})}_p(\boldsymbol{r}) u^{(1)}_{pq}(\boldsymbol{r},\boldsymbol{s},T^{(\tilde{N}\hat{N})}(\boldsymbol{r},\boldsymbol{x},\boldsymbol{s}))\tilde{\xi}^{(\tilde{N})}_q(\boldsymbol{s}) \ \left.\frac{\partial(\tilde{\boldsymbol{\alpha}},\hat{\boldsymbol{\alpha}})}{\partial(\boldsymbol{s},\boldsymbol{r})}\right|_{\boldsymbol{x}} \ \mathrm{d}\boldsymbol{s}\mathrm{d}\boldsymbol{r} \ ,$$

where $\boldsymbol{\nu}$ is the migration dip. The scattering angle θ and azimuth ψ associated with the dip, are defined through

$$\cos\theta = \hat{\boldsymbol{\alpha}} \cdot \tilde{\boldsymbol{\alpha}} \ , \ \ \text{while} \ \ \psi = \text{third Euler angle around } \boldsymbol{\nu} \ ; \tag{4.3}$$

in Eq.(4.2),

$$\Lambda^{(\tilde{N}\hat{N})}(\boldsymbol{\nu}) = \int_{E_\theta(\boldsymbol{\nu})} \int_{E_\psi(\boldsymbol{\nu},\theta)} \mathbf{w}^{(\tilde{N}\hat{N})} (\mathbf{w}^{(\tilde{N}\hat{N})})^T \frac{\partial(\tilde{\boldsymbol{\alpha}}, \hat{\boldsymbol{\alpha}})}{\partial(\boldsymbol{\nu}, \theta, \psi)} \, \mathrm{d}\psi \mathrm{d}\theta \tag{4.4}$$

is used to carry out the inversion of amplitude versus scattering-angle/azimuth. Here, we have explicitly incorporated the limitations of the acquisition geometry by restricting the integrations over scattering angle and azimuth to finite domains, viz. E_θ and E_ψ respectively. Note that the inversion integral is effectively carried out over diffraction hypersurfaces,

$$\{(\boldsymbol{r}, t) \in \partial R \times \mathbb{R}_{\geq 0} \mid t = T^{(\tilde{N}\hat{N})}(\boldsymbol{r}, \boldsymbol{x}, \boldsymbol{s})\} \ .$$

Jacobians

For the inversion we have to evaluate the Jacobians introduced in Eqs.(4.2) and (4.4). The Jacobian occurring in the integrated normal matrix Eq.(4.4) has been derived by Burridge *et al.* [14] and is given by

$$\frac{\partial(\tilde{\boldsymbol{\alpha}}, \hat{\boldsymbol{\alpha}})}{\partial(\boldsymbol{\nu}, \theta, \psi)} = \frac{\sin\theta}{1 + (|\tilde{\boldsymbol{\gamma}}||\hat{\boldsymbol{\gamma}}|/|\boldsymbol{\Gamma}|^2)\,(\tan\hat{\chi} - \tan\tilde{\chi})\,\sin\theta} \ , \tag{4.5}$$

where

$$\boldsymbol{\Gamma} = \boldsymbol{\Gamma}^{(\tilde{N}\hat{N})} \ , \ \ \cos\tilde{\chi} = \tilde{\boldsymbol{n}}_{\|} \cdot \tilde{\boldsymbol{\alpha}} \ \ \text{and} \ \ \cos\hat{\chi} = \hat{\boldsymbol{n}}_{\|} \cdot \hat{\boldsymbol{\alpha}} \ . \tag{4.6}$$

Here, $\tilde{\boldsymbol{n}}_{\|}$ and $\hat{\boldsymbol{n}}_{\|}$ denote the normals to the slowness surface in the azimuth plane ψ at the scattering point.

The Jacobian in Eq.(4.2) is directly related to dynamic ray theory. We have the factorization

$$\left.\frac{\partial(\tilde{\boldsymbol{\alpha}}, \hat{\boldsymbol{\alpha}})}{\partial(\boldsymbol{s}, \boldsymbol{r})}\right|_{\boldsymbol{x}} = \left.\frac{\partial(\tilde{\boldsymbol{\alpha}})}{\partial(\boldsymbol{s})}\right|_{\boldsymbol{x}} \left.\frac{\partial(\hat{\boldsymbol{\alpha}})}{\partial(\boldsymbol{r})}\right|_{\boldsymbol{x}} \ . \tag{4.7}$$

In general, the factors can be expressed in terms of the dynamic ray amplitudes since, like the amplitudes, they follow from a variation of the anisotropic ray tracing equations (see Eq.(2.14)). For the source side:

$$\left.\frac{\partial(\boldsymbol{s}^\Sigma)}{\partial(\tilde{\boldsymbol{\alpha}})}\right|_{\boldsymbol{x}} = \frac{1}{16\pi^2 \rho(\boldsymbol{s})\rho(\boldsymbol{x})\,\tilde{V}^{(\tilde{N})}(\boldsymbol{s})(\tilde{V}^{(\tilde{N})}(\boldsymbol{x}))^3\,(\tilde{A}^{(\tilde{N})}(\boldsymbol{x}))^2} \tag{4.8}$$

as long as ∂S in the neighborhood of $\boldsymbol{s}$ coincides with the wave front $\Sigma(\boldsymbol{x}, \tau^{(\tilde{N})}(\boldsymbol{s}, \boldsymbol{x}))$ originating at $\boldsymbol{x}$; $\boldsymbol{s}^\Sigma$ denotes the coordinates on $\Sigma(\boldsymbol{x}, \tau^{(\tilde{N})}(\boldsymbol{s}, \boldsymbol{x}))$. If this is not the case, we have to correct for the ratio of the area on ∂S to the area on the wave front $\Sigma(\boldsymbol{x}, \tau^{(\tilde{N})}(\boldsymbol{s}, \boldsymbol{x}))$ at $\boldsymbol{s}$ onto which it is mapped by projection along the rays. This arises from the fact that ∂S is not necessarily tangent to $\Sigma(\boldsymbol{x}, \tau^{(\tilde{N})}(\boldsymbol{s}, \boldsymbol{x}))$ at $\boldsymbol{s}$. It amounts to dividing the previous Jacobian by the Jacobian

$$\frac{\partial(\boldsymbol{s}^{\Sigma})}{\partial(\boldsymbol{s})} = (\tilde{\boldsymbol{\alpha}}(\boldsymbol{s}) \cdot \tilde{\boldsymbol{\beta}}(\boldsymbol{s})) \ ,$$

where

$$\tilde{\boldsymbol{\beta}}(\boldsymbol{s}) = \text{normal to } \partial S \sim \partial \mathcal{D} \text{ at the source.}$$

Note that $\tilde{\boldsymbol{\alpha}}(\boldsymbol{s})$ is the normal to the wave front at $\boldsymbol{s}$. Similar expressions hold for the receiver side.

In the case of a homogeneous background medium (in which the rays are straight), we have the following simplications

$$\frac{\partial(\tilde{\boldsymbol{\alpha}})}{\partial(\tilde{\boldsymbol{n}})} = \frac{(\tilde{\boldsymbol{n}} \cdot \tilde{\boldsymbol{\alpha}})\, \tilde{V}^2}{\tilde{\kappa}} \ , \quad \left.\frac{\partial(\tilde{\boldsymbol{n}})}{\partial(\boldsymbol{s})}\right|_{\boldsymbol{y}} = \frac{(\tilde{\boldsymbol{n}} \cdot \tilde{\boldsymbol{\beta}})}{|\boldsymbol{s} - \boldsymbol{y}|^2} \ , \tag{4.9}$$

where

$$\tilde{\kappa} = \text{Gaussian curvature of the slowness surface at } \tilde{\boldsymbol{\gamma}}^{(\tilde{N})} \ .$$

If ∂S coincides with $\Sigma(\boldsymbol{x}, \tau^{(\tilde{N})}(\boldsymbol{s}, \boldsymbol{x}))$, i.e., $\boldsymbol{s} = \boldsymbol{s}^{\Sigma}$, then $\tilde{\boldsymbol{\beta}} = \tilde{\boldsymbol{\alpha}}$ and

$$\left.\frac{\partial(\boldsymbol{s})}{\partial(\tilde{\boldsymbol{\alpha}})}\right|_{\boldsymbol{x}} = \frac{\tilde{\kappa} |\boldsymbol{s} - \boldsymbol{x}|^2 |\tilde{\boldsymbol{v}}|^2}{\tilde{V}^4} \ . \tag{4.10}$$

In applications, the constant background is an effective medium rather than the actual one. For the process of transformation to zero offset, we will employ a *constant background*.

Zero-offset modelling of the inverted data

The Dirac distribution in the integral representation Eq.(3.6) can be used to reduce the dimensionality of the integration explicitly. Reconsider the zero-offset isochron defined by

$$\left\{ \boldsymbol{x} \in \mathbb{R}^3 \mid T^{(N)}(\boldsymbol{x}, \boldsymbol{S}) = \tau \right\} \ ,$$

where $\boldsymbol{S}$ denotes the zero-offset position and τ the zero-offset time. In a homogeneous medium, this surface is just the wave front or the group-velocity surface scaled by τ, i.e.,

$$\boldsymbol{x} - \boldsymbol{S} = \tfrac{1}{2} \boldsymbol{v}^{(N)}\, \tau \ . \tag{4.11}$$

We choose local, curvi-linear coordinates $\boldsymbol{\sigma}$ such that (σ_1, σ_2) span the group surface and σ_3 is the coordinate in the direction normal to it, parallel to the phase direction. The origin of the coordinate system is chosen to be $\boldsymbol{S}$. We write $\sigma_\perp = (\sigma_1, \sigma_2)$. Then, for fixed σ_3, $\boldsymbol{x}(\boldsymbol{\sigma}) = \frac{1}{2}\boldsymbol{s}^\Sigma$ of the previous subsection. Let $\mathrm{d}\Sigma$ denote a surface element in the wave-front surface, then

$$\mathrm{d}\boldsymbol{x} = \frac{1}{|\nabla_{\boldsymbol{x}} T^{(N)}|}\,\mathrm{d}\tau\;\mathrm{d}\Sigma(\boldsymbol{x})\ , \quad \mathrm{d}\Sigma = g\;\mathrm{d}\sigma_1\mathrm{d}\sigma_2\ , \tag{4.12}$$

with

$$\frac{1}{|\nabla_{\boldsymbol{x}} T^{(N)}|} = \tfrac{1}{2} V^{(N)}\ , \quad g = |\partial_{\sigma_1}\boldsymbol{x} \wedge \partial_{\sigma_2}\boldsymbol{x}|\ .$$

Choosing $\sigma_\perp$ to be equal to the zero-offset phase angles, $\boldsymbol{\alpha}^{(N)}$, at the source, the Jacobian g is directly related to the zero-offset ray amplitude, $g \sim [A^{(N)}]^{-2}$, and is given by Eq.(4.10) upon replacing $|\boldsymbol{s} - \boldsymbol{x}|$ with $\frac{1}{2}\boldsymbol{v}\,\tau$, using the principle of reciprocity.

In Eq.(3.6) we will now use the property

$$\int_{\mathcal{D}} \cdots \delta(\tau - T^{(N)}(\boldsymbol{x},\boldsymbol{S}))\,\mathrm{d}\boldsymbol{x} = \int_{T^{(N)}(.,\boldsymbol{S})=\tau} \cdots \frac{1}{|\nabla_{\boldsymbol{x}} T^{(N)}|}\,\mathrm{d}\Sigma(\boldsymbol{x})\ .$$

The desired zero-offset experiment then becomes

$$u^{(1)}_{PQ}(\boldsymbol{S},\tau) \simeq -\partial_\tau^2 \int_{T^{(N)}(.,\boldsymbol{S})=\tau} \xi_P^{(N)}(\boldsymbol{S})\xi_Q^{(N)}(\boldsymbol{S})\,\mathcal{I}^{(N)}(.,\boldsymbol{S})\ \times$$
$$(\mathbf{w}^{(N)}(.,\boldsymbol{\alpha}(.)))^T\,\mathbf{c}^{(1)}(.)\ \tfrac{1}{2}V^{(N)}(.)\Big|_{\boldsymbol{x}(\boldsymbol{\sigma})}\ \mathrm{d}\Sigma\ . \tag{4.13}$$

Substituting Eq.(4.2) into Eq.(4.13) and interchanging the order of integrations, yields

$$u^{(1)}_{PQ}(\boldsymbol{S},\tau) \simeq -\frac{1}{16\pi^2}\int_{\partial S\times\partial R}\int_{t\in\mathbb{R}_{\geq 0}} \partial_\tau^2 \int_{T^{(N)}(.,\boldsymbol{S})=\tau} \mathcal{I}^{(N)}(.,\boldsymbol{S})\,\mathcal{J}^{(\tilde{N}\hat{N})}(\boldsymbol{r},.,\boldsymbol{s})\ \times$$
$$\xi_P^{(N)}(\boldsymbol{S})\xi_Q^{(N)}(\boldsymbol{S})\,\hat{\xi}_p^{(\hat{N})}(\boldsymbol{r})u^{(1)}_{pq}(\boldsymbol{r},\boldsymbol{s},t)\tilde{\xi}_q^{(\tilde{N})}(\boldsymbol{s})\ \times$$
$$(\mathbf{w}^{(N)}(.,\boldsymbol{\alpha}^{(N)}(.)))^T\,[\Lambda_.^{(\tilde{N}\hat{N})}(\boldsymbol{\nu}(\boldsymbol{r},.,\boldsymbol{s}))]^{-1}\mathbf{w}^{(\tilde{N}\hat{N})}(.,\tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(.),\hat{\boldsymbol{\alpha}}^{(\hat{N})}(.))\ \times$$
$$\delta(t - T^{(\tilde{N}\hat{N})}(\boldsymbol{r},.,\boldsymbol{s}))\,V^{(N)}(.)\,\frac{\partial(\tilde{\boldsymbol{\alpha}},\hat{\boldsymbol{\alpha}})}{\partial(\boldsymbol{s},\boldsymbol{r})}\bigg|_{\boldsymbol{x}(\boldsymbol{\sigma})}\ \mathrm{d}\Sigma\,\mathrm{d}t\ \mathrm{d}\boldsymbol{s}\mathrm{d}\boldsymbol{r}\ . \tag{4.14}$$

This expression shows that for each point in the zero-offset isochron surface, the original field at all nonzero-offset isochron surfaces through that point contribute to the zero-offset field. We next carry out a stationary-phase analysis for the integrations over (σ_1, σ_2).

5 Stationary phase analysis

We employ the one-sided Fourier representation

$$\delta(t - T^{(\tilde{N}\hat{N})}(\boldsymbol{r},.,\boldsymbol{s})) = \frac{1}{\pi}\mathrm{Re}\int_{\mathbb{R}_{\geq 0}} \exp\left[i\omega(t - T^{(\tilde{N}\hat{N})}(\boldsymbol{r},.,\boldsymbol{s}))\right]\,\mathrm{d}\omega\ , \tag{5.1}$$

for the Dirac distribution in Eq.(4.14). Then we apply a two-dimensional stationary-phase analysis to the integration over (σ_1, σ_2). The stationary points follow from the equations

$$\partial_{\sigma_1} T^{(\tilde{N}\hat{N})}(\boldsymbol{r}, \boldsymbol{x}(\sigma_1, \sigma_2, \tau), \boldsymbol{s}) \;=\; 0 \;, \tag{5.2}$$

$$\partial_{\sigma_2} T^{(\tilde{N}\hat{N})}(\boldsymbol{r}, \boldsymbol{x}(\sigma_1, \sigma_2, \tau), \boldsymbol{s}) \;=\; 0 \;, \tag{5.3}$$

where $\boldsymbol{x}(\sigma_1, \sigma_2, \tau)$ follows the zero-offset isochron of Eq.(4.11). The stationary points are denoted by $(\sigma_\perp^0, \tau)$. Then (cf. Eq.(4.14))

$$\begin{aligned} u_{PQ}^{(1)}(\boldsymbol{S}, \tau) \simeq -\frac{1}{8\pi} \int_{\partial S \times \partial R'} \int_{t \in \mathbb{R}_{\geq 0}} \partial_\tau^2 \, \mathcal{I}^{(N)}(., \boldsymbol{S}) \, \mathcal{J}^{(\tilde{N}\hat{N})}(\boldsymbol{r}, ., \boldsymbol{s}) \; \times \\ \xi_P^{(N)}(\boldsymbol{S}) \xi_Q^{(N)}(\boldsymbol{S}) \, \hat{\xi}_p^{(\hat{N})}(\boldsymbol{r}) U_{pq}^{(1)}(\boldsymbol{r}, \boldsymbol{s}, t) \tilde{\xi}_q^{(\tilde{N})}(\boldsymbol{s}) \; \times \\ (\mathbf{w}^{(N)}(., \boldsymbol{\alpha}^{(N)}(.)))^T \, [\Lambda_.^{(\tilde{N}\hat{N})}(\boldsymbol{\nu}(\boldsymbol{r}, ., \boldsymbol{s}))]^{-1} \mathbf{w}^{(\tilde{N}\hat{N})}(., \tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(.), \hat{\boldsymbol{\alpha}}^{(\hat{N})}(.)) \; \times \\ \frac{\delta^*(t - T^{(\tilde{N}\hat{N})}(\boldsymbol{r}, ., \boldsymbol{s}))}{\sqrt{\left| \det(\nabla_{\sigma_\perp} \nabla_{\sigma_\perp} T^{(\tilde{N}\hat{N})})^0 \right|}} V^{(N)}(.) \, \left. \frac{\partial(\tilde{\boldsymbol{\alpha}}, \hat{\boldsymbol{\alpha}})}{\partial(\boldsymbol{s}, \boldsymbol{r})} \right|_{\boldsymbol{x}(\sigma_\perp^0, \tau)} g(\sigma_\perp^0, \tau) \, \mathrm{d}t \, \mathrm{d}\boldsymbol{s} \mathrm{d}\boldsymbol{r} \;, \end{aligned} \tag{5.4}$$

where

$$U_{pq}^{(1)}(\boldsymbol{r}, \boldsymbol{s}, t) = \int_0^t u_{pq}^{(1)}(\boldsymbol{r}, \boldsymbol{s}, t') \, \mathrm{d}t' \;, \tag{5.5}$$

and

$$\delta^*(t) = \frac{1}{\pi} \, \mathrm{Re} \int_{\mathbb{R}_{\geq 0}} i \, \exp\left[\frac{i\pi}{4} \, \mathrm{sig}(\nabla_{\sigma_\perp} \nabla_{\sigma_\perp} T^{(\tilde{N}\hat{N})})^0 \right] \, \exp(i\omega t) \, \mathrm{d}\omega \;. \tag{5.6}$$

Here, sig denotes the signature of a matrix. Observe that the stationary point $\sigma_\perp^0$ varies with $(\boldsymbol{s}, \boldsymbol{r})$. Equation (5.4) shows that for a fixed zero-offset isochron, we have to integrate over all finite-offset isochrons that 'kiss', or come tangent to, the zero-offset one, see the subsection below.

The TZO in Eq.(5.4) contains in fact four mappings: one for the geometrical spreading, one for the polarizations, one for the radiation patterns (or reflection coefficients), and one for the coordinates. The bare TZO accounts for the final mapping only, see Deregowski and Rocca [15].

The geometry of the stationary point set

In Eqs.(5.2)-(5.3), we have

$$\partial_{\sigma_{1,2}} T^{(\tilde{N}\hat{N})} = (\nabla_{\boldsymbol{x}} T^{(\tilde{N}\hat{N})}) \cdot (\partial_{\sigma_{1,2}} \boldsymbol{x}) = (\nabla_{\boldsymbol{x}} T^{(\tilde{N}\hat{N})}) \cdot (\partial_{\sigma_{1,2}} (\boldsymbol{x} - \boldsymbol{S})) \;, \tag{5.7}$$

where $\boldsymbol{x}$ is given by Eq.(4.11). Note that $\nabla_{\boldsymbol{x}} T^{(\tilde{N}\hat{N})}$ is normal to the finite-offset isochron surface for fixed $(\boldsymbol{s}, \boldsymbol{r})$, while $\partial_{\sigma_1}\boldsymbol{x}$ and $\partial_{\sigma_2}\boldsymbol{x}$ are tangent to the zero-offset isochron surface for fixed $(\boldsymbol{S}, \tau)$. Hence, the condition of stationarity Eqs.(5.2)-(5.3) implies that the zero-offset isochron surface must be tangent to the finite-offset isochron surface.

The stationary geometry can be captured in a system of non-linear equations (see Artley *et al.* [16]). The first subsystem arises from the condition that the gradient of finite-offset travel time should be parallel to the slowness vector of the zero-offset ray. The second subsystem arises from the condition that the rays of the finite-offset configuration should connect at the stationary scattering point. Let us choose Cartesian coordinates such that the 1-direction coincides with the line connecting $\boldsymbol{s}$ with $\boldsymbol{r}$, and let $\boldsymbol{h} = \frac{1}{2}(\boldsymbol{r} - \boldsymbol{s})$. Let the $(1,2)$-plane be defined by $(\boldsymbol{s}, \boldsymbol{S}, \boldsymbol{r})$; at stationarity, the points $\boldsymbol{s}, \boldsymbol{r}, \boldsymbol{S}, \boldsymbol{x}$ define a tetrahedron. Then, the first condition yields

$$\frac{\Gamma_1^{(\tilde{N}\hat{N})}}{\Gamma_3^{(\tilde{N}\hat{N})}} = \frac{\gamma_1^{(N)}}{\gamma_3^{(N)}} \,, \quad \frac{\Gamma_2^{(\tilde{N}\hat{N})}}{\Gamma_3^{(\tilde{N}\hat{N})}} = \frac{\gamma_2^{(N)}}{\gamma_3^{(N)}} \,. \tag{5.8}$$

The second condition yields (see also Eq.(2.16))

$$\begin{aligned} \tilde{v}_1^{(\tilde{N})}\tilde{\tau}^{(\tilde{N})} - \hat{v}_1^{(\hat{N})}\hat{\tau}^{(\hat{N})} &= 2\,h \quad , \\ \tilde{v}_{2,3}^{(\tilde{N})}\tilde{\tau}^{(\tilde{N})} - \hat{v}_{2,3}^{(\hat{N})}\hat{\tau}^{(\hat{N})} &= 0 \quad , \end{aligned} \tag{5.9}$$

while the times are constrained by

$$\tilde{\tau}^{(\tilde{N})} + \hat{\tau}^{(\hat{N})} = T^{(\tilde{N}\hat{N})} \,. \tag{5.10}$$

The zero-offset position and time follow from the third subsystem of equations, arising from the condition that the ray of the zero-offset configuration connects with the rays of the finite-offset configuration at the stationary scattering point:

$$\begin{aligned} \tilde{v}_1^{(\tilde{N})}\tilde{\tau}^{(\tilde{N})} - \tfrac{1}{2}v_1^{(N)}\tau &= S_1 + h \quad , \\ \tilde{v}_2^{(\tilde{N})}\tilde{\tau}^{(\tilde{N})} - \tfrac{1}{2}v_2^{(N)}\tau &= S_2 \quad , \\ \tilde{v}_3^{(\tilde{N})}\tilde{\tau}^{(\tilde{N})} - \tfrac{1}{2}v_3^{(N)}\tau &= 0 \quad . \end{aligned} \tag{5.11}$$

As before, all the group velocities are taken in the background medium; the ray directions, appearing in Eqs.(5.9)-(5.11), coincide with the group velocities.

Eliminating the finite-offset times with the aid of Eq.(5.10)) from equations (5.9), leads to

$$\begin{aligned} \frac{\hat{v}_3^{(\hat{N})}}{\tilde{v}_3^{(\tilde{N})}+\hat{v}_3^{(\hat{N})}}\,\tilde{v}_1^{(\tilde{N})} - \frac{\tilde{v}_3^{(\tilde{N})}}{\tilde{v}_3^{(\tilde{N})}+\hat{v}_3^{(\hat{N})}}\,\hat{v}_1^{(\hat{N})} &= \frac{2\,h}{T^{(\tilde{N}\hat{N})}} \quad , \\ \frac{\tilde{v}_2^{(\tilde{N})}}{\tilde{v}_3^{(\tilde{N})}} - \frac{\hat{v}_2^{(\hat{N})}}{\hat{v}_3^{(\hat{N})}} &= 0 \quad . \end{aligned} \tag{5.12}$$

Eliminating the finite-offset times from equations (5.11), leads to

$$\begin{aligned} S_1 &= \tfrac{1}{2}\left(\frac{\tilde{v}_1^{(\tilde{N})}}{\tilde{v}_3^{(\tilde{N})}}\,v_3^{(N)} - v_1^{(N)}\right)\tau - h \quad , \\ S_2 &= \tfrac{1}{2}\left(\frac{\tilde{v}_2^{(\tilde{N})}}{\tilde{v}_3^{(\tilde{N})}}\,v_3^{(N)} - v_2^{(N)}\right)\tau \quad , \\ \tau &= \frac{2\,\tilde{v}_3^{(\tilde{N})}\hat{v}_3^{(\hat{N})}}{v_3^{(N)}\,(\tilde{v}_3^{(\tilde{N})}+\hat{v}_3^{(\hat{N})})}\,T^{(\tilde{N}\hat{N})} \quad . \end{aligned} \tag{5.13}$$

In Eqs.(5.8)-(5.13) we have an implicit relationship between group velocity and slowness vector in the background medium. Convenient variables are the two horizontal components of the three slowness vectors: $\tilde{\gamma}_{1,2}^{(\tilde{N})}$, $\hat{\gamma}_{1,2}^{(\hat{N})}$, $\gamma_{1,2}^{(N)}$. The components $\gamma_{1,2}^{(N)}$ parameterize the stationary point set; h and $T^{(\tilde{N}\hat{N})}$ are given. Then, the four equations (5.8), (5.12) constitute a system with four unknowns, that has to be solved numerically. Equations (5.13) then give the zero-offset-point/zero-offset-time pair, $(\boldsymbol{S},\tau)$, at stationarity.

The Hessian can be written as

$$\begin{aligned} \partial_{\sigma_1}\partial_{\sigma_1}T^{(\tilde{N}\hat{N})} &= \nabla_{\boldsymbol{x}}\nabla_{\boldsymbol{x}}T^{(\tilde{N}\hat{N})} : (\partial_{\sigma_1}\boldsymbol{x})\,(\partial_{\sigma_1}\boldsymbol{x}) + \nabla_{\boldsymbol{x}}T^{(\tilde{N}\hat{N})}\cdot\partial_{\sigma_1}\partial_{\sigma_1}\boldsymbol{x} \\ &= \left(\tfrac{1}{2}\tau\right)^2 |\nabla_{\boldsymbol{x}}T^{(\tilde{N}\hat{N})}| \left(\frac{2}{\tau}\boldsymbol{\nu}\cdot\partial_{\sigma_1}\partial_{\sigma_1}\boldsymbol{v}^{(N)} + (\partial_{\sigma_1}\boldsymbol{v}^{(N)})\,(\partial_{\sigma_1}\boldsymbol{v}^{(N)}) : \frac{\nabla_{\boldsymbol{x}}\nabla_{\boldsymbol{x}}T^{(\tilde{N}\hat{N})}}{|\nabla_{\boldsymbol{x}}T^{(\tilde{N}\hat{N})}|}\right) , \end{aligned} \tag{5.14}$$

etc. Here, we have separated the Hessian into the obliquity factor, the Hessian attached to the zero-offset isochron surface, and the Hessian attached to the finite-offset isochron surface. Note that the obliquity factor, $|\nabla_{\boldsymbol{x}}T^{(\tilde{N}\hat{N})}|$, can be taken together with $\mathcal{J}^{(\tilde{N}\hat{N})}$ and that the scaling $(\frac{1}{2}\tau)^2$ can be taken together with g (Eq.(5.4)).

It is noted, that given the zero-offset ray, only a restricted range of $(\boldsymbol{s},\boldsymbol{r})$ will allow for a stationary geometry to exist. We have indicated this restriction by replacing $\partial S\times\partial R$ with $\partial S\times\partial R'$ in Eq.(5.4). In Appendix A, we give the expressions for an isotropic medium in two dimensions, which establishes the correspondence to the conventional DMO procedure.

Kirchhoff-style Dip MoveOut

In this subsection, we review the concept of Dip MoveOut. For each point on a fixed finite-offset isochron surface (fixed $(\boldsymbol{s}, \boldsymbol{r})$ and travel time $T^{(\tilde{N}\hat{N})}(\boldsymbol{r}, ., \boldsymbol{s}) = t$) we construct the combination $\boldsymbol{S}, \tau$ such that the zero-offset isochron surface is tangent to or touches the finite-offset one (cf. Eq.(5.4))

$$\Big((\boldsymbol{r}, \boldsymbol{s}), T^{(\tilde{N}\hat{N})}(\boldsymbol{r}, \boldsymbol{x}^0, \boldsymbol{s}) = t\Big) \rightarrow (\boldsymbol{S}, \tau) \quad \text{for variable} \quad \boldsymbol{x}^0 \; . \tag{5.15}$$

As $\boldsymbol{x}^0$ varies along the finite-offset isochron surface, the stationary dip $\boldsymbol{\nu}(\boldsymbol{r}, \boldsymbol{x}^0, \boldsymbol{s})$ varies according to the conditions of tangency. (In the previous subsection, we employed the horizontal components of the zero-offset slowness vector at the stationary point for this rôle.) Hence, the latter equation can be viewed as a mapping from dip to a zero-offset-point/zero-offset-time pair. Often, in this mapping, the midpoint $\boldsymbol{y} = \frac{1}{2}(\boldsymbol{r} + \boldsymbol{s})$ is chosen to be the origin in $\boldsymbol{S}$-space. It is also clear that each finite-offset time t maps on a set of zero-offset times τ. Mapping (5.15) illustrates that a particular sample of the original field contributes to the zero-offset field at various space-time combinations. In fact, in the implementation of transformation (5.4) one would loop over the finite-offset isochrons labelled by $(\boldsymbol{r}, \boldsymbol{s}, t)$, and accumulate the contributions from any finite-offset isochron to the zero-offset field at the associated values of $(\boldsymbol{S}, \tau)$.

Observe now, that inversion formula (4.2) produces an image for each (θ, ψ) pair by integrating over the dip. This fact is commonly exploited in DMO, viz., by integrating the integrand of Eq.(5.4) over the midpoints $\boldsymbol{y}$ only, keeping the half-offset $\boldsymbol{h} = \frac{1}{2}(\boldsymbol{r} - \boldsymbol{s})$ fixed. The result is a family of intermediate zero-offset fields $u^{(1)}_{PQ}(\boldsymbol{S}, \tau; \boldsymbol{h})$, the stack of which is the true zero-offset field,

$$u^{(1)}_{PQ}(\boldsymbol{S}, \tau) = \int_{\partial H} u^{(1)}_{PQ}(\boldsymbol{S}, \tau; \boldsymbol{h}) \, \mathrm{d}\boldsymbol{h} \; . \tag{5.16}$$

Here,

$$u^{(1)}_{PQ}(\boldsymbol{S}, \tau; \boldsymbol{h}) = -\frac{1}{2\pi} \int_{\partial Y(\boldsymbol{h})'} \int_{t \in \mathbb{R}_{\geq 0}} \partial^2_\tau \, \mathcal{S}_{PQpq}(\boldsymbol{S}, \tau; \boldsymbol{y}, t; \boldsymbol{h}) \, U^{(1)}_{pq}(\boldsymbol{y} + \boldsymbol{h}, \boldsymbol{y} - \boldsymbol{h}, t) \, \mathrm{d}t \mathrm{d}\boldsymbol{y} \; , \tag{5.17}$$

where the smear kernel $\mathcal{S}_{PQpq}$ (Deregowski [15]) is given by (cf. Eq.(5.4))

$$\begin{aligned} \mathcal{S}_{PQpq}(\boldsymbol{S}, \tau; \boldsymbol{y}(\boldsymbol{r}, \boldsymbol{s}), t; \boldsymbol{h}(\boldsymbol{r}, \boldsymbol{s})) = {} & \xi^{(N)}_P(\boldsymbol{S}) \xi^{(N)}_Q(\boldsymbol{S}) \, \hat{\xi}^{(\hat{N})}_p(\boldsymbol{r}) \tilde{\xi}^{(\tilde{N})}_q(\boldsymbol{s}) \; \times \\ & (\mathbf{w}^{(N)}(., \boldsymbol{\alpha}^{(N)}(.)))^T \, [\Lambda^{(\tilde{N}\hat{N})}_{.}(\boldsymbol{\nu}(\boldsymbol{r}, ., \boldsymbol{s}))]^{-1} \mathbf{w}^{(\tilde{N}\hat{N})}(., \tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(.), \hat{\boldsymbol{\alpha}}^{(\hat{N})}(.)) \; \times \\ & \delta^*(t - T^{(\tilde{N}\hat{N})}(\boldsymbol{r}, ., \boldsymbol{s})) \, \frac{V^{(N)}(.)}{\sqrt{\left| \det(\nabla_{\sigma_\perp} \nabla_{\sigma_\perp} T^{(\tilde{N}\hat{N})})^0 \right|}} \; \times \end{aligned}$$

$$g\,\mathcal{I}^{(N)}(.,\boldsymbol{S})\,\mathcal{J}^{(\tilde{N}\hat{N})}(\boldsymbol{r},.,\boldsymbol{s})\;\left.\frac{\partial(\tilde{\boldsymbol{\alpha}},\hat{\boldsymbol{\alpha}})}{\partial(\boldsymbol{s},\boldsymbol{r})}\right|_{\boldsymbol{x}(\sigma_\perp^0,\tau)}\,, \tag{5.18}$$

using that

$$\mathrm{d}\boldsymbol{s}\mathrm{d}\boldsymbol{r} = 4\,\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{h}\ .$$

Having solved for the slowness vectors at the stationary point, the remaining quantities are readily obtained.

The DMO impulse response is the zero-offset field obtained by substituting in Eq.(5.17) the non-physical field (unit data sample)

$$U^{(1)}_{pq}(\boldsymbol{r},\boldsymbol{s},t) = \delta\left(\boldsymbol{r}-[\boldsymbol{y}^{\mathrm{i}}+\boldsymbol{h}]\right)\,\delta\left(\boldsymbol{s}-[\boldsymbol{y}^{\mathrm{i}}-\boldsymbol{h}]\right)\,\delta\left(t-t^{\mathrm{i}}\right)\ ,$$

and is given by

$$-\frac{1}{2\pi}\,\partial_\tau^2\mathcal{S}_{PQpq}(\boldsymbol{S},\tau;\boldsymbol{y}^{\mathrm{i}},t^{\mathrm{i}};\boldsymbol{h})\ . \tag{5.19}$$

The singular support of this function can be multi-valued. Equation (5.17) represents an integration of DMO impulse responses over midpoints and offset times.

In practical acquisition geometries, we have a preferred orientation in the configuration: $\boldsymbol{s}$ and $\boldsymbol{r}$ are constrained to a horizontal plane. This implies that there is a preferred migration dip, viz. the one parallel to vertical. The ‘Normal MoveOut’ now follows from the mapping $t\to\tau$ in (5.15) at $\boldsymbol{x}^0 = \boldsymbol{x}_n$ where the stationary dip $\boldsymbol{\nu}(\boldsymbol{r},\boldsymbol{x}_n,\boldsymbol{s}) = \boldsymbol{\nu}_n$ coincides with the vertical direction (the isochron tangent plane is horizontal); we write

$$((\boldsymbol{r},\boldsymbol{s}),t)\to(\boldsymbol{S}_n,\tau_n)\quad\text{for any }\boldsymbol{y}\text{ (fixing }\boldsymbol{x}^0=\boldsymbol{x}_n)$$

and find $t = t(\tau_n,\boldsymbol{h})$. In Eq.(5.17) we can now replace the integration variable t by τ_n, and set

$$U^{(1)}_{n,pq}(\boldsymbol{y},\boldsymbol{h},\tau_n) \equiv U^{(1)}_{pq}(\boldsymbol{y}+\boldsymbol{h},\boldsymbol{y}-\boldsymbol{h},t(\tau_n,\boldsymbol{h}))\,\frac{\partial t}{\partial\tau_n}\ .$$

The field $U^{(1)}_{n,pq}$ can be viewed as the NMO corrected finite-offset data $U^{(1)}_{pq}$. The mapping Eq.(5.15) can be decomposed into a mapping based at $\boldsymbol{x}_n$ or $\boldsymbol{\nu}_n$ followed by a mapping from $\boldsymbol{x}_n$ or $\boldsymbol{\nu}_n$ to $\boldsymbol{x}^0$ or $\boldsymbol{\nu}(\boldsymbol{r},\boldsymbol{x}^0,\boldsymbol{s})$. The second, correcting map,

$$(\boldsymbol{S}_n,\tau_n)\to(\boldsymbol{S},\tau)$$

represents the geometry underlying the DMO procedure.

6 Multi-mode scattering

To account for mode-converted waves, we replace $\hat{N}$ by $\hat{M}$. In a homogeneous background medium, for the zero-offset configuration, the incident and scattered ray directions would still coincide. However, the slowness vectors at the scattering point are no longer the same and may point in different directions. The zero-offset isochron surface,

$$\left\{ \boldsymbol{x} \in \mathbb{R}^3 \mid T^{(NM)}(\boldsymbol{x}, \boldsymbol{S}) = \tau \right\} ,$$

is now given by

$$\boldsymbol{x} - \boldsymbol{S} = \left[1 + \frac{|\boldsymbol{v}^{(M)}|}{|\boldsymbol{v}^{(N)}|} \right]^{-1} \boldsymbol{v}^{(N)} \, \tau \ . \tag{6.1}$$

It is realistic that this surface may become multi-valued. We will not deal with that case here. If the zero-offset isochron surface is single-valued, the analysis of the previous sections applies, with the modification that

$$\nabla_{\boldsymbol{x}} T^{(N)} = \boldsymbol{\gamma}^{(N)} + \boldsymbol{\gamma}^{(M)} \not\parallel \boldsymbol{\gamma}^{(N)} ,$$

in general.

7 Discussion

We have derived a DMO/TZO procedure that corrects for radiation characteristics and is valid in anisotropic, elastic media. We have used generalized Radon transforms, focussing on the Kirchhoff-style representation of the procedure.

Fundamentally, the TZO contains four mappings, one for the geometrical spreading, one for the polarizations, one for the radiation patterns, and one for the coordinates. The mapping of radiation patterns establishes a mapping between reflection coefficients from the finite-offset to the zero-offset configuration. The formalism is complete within the ray-Born scattering theory. We argue that the TZO should be correct at least within this approximation, but this does not assure that the procedure is valid beyond. Perhaps the various characteristics of the procedure (i.e., spreading, polarizations, radiation patterns and coordinates) have different sensitivities to the background model. It is still to be analyzed how large the contributions from the constituent mappings are going to be in a practical setting.

There are various ways to address numerical implementation of the TZO process. The Kirchhoff-style representation lends itself for a quasi-Monte Carlo discretisation [17]. Within certain limits, analytic solutions for the stationary-point set can be found, which significantly speeds up the computations.

Appendix A. The stationary phase analysis for an isotropic medium in the two dimensions

In an isotropic medium the group and phase directions coincide. Let the medium's velocity be $c^{(N)}$. Then

$$\alpha = \text{polar angle} \; , \quad v_1^{(N)} = c^{(N)} \cos\alpha \; , \; v_3^{(N)} = c^{(N)} \sin\alpha \; ,$$

while

$$g = \tfrac{1}{2} c^{(N)} \tau^{(N)} \; .$$

Note that in a two-dimensional configuration $U^{(1)}$ is the Hilbert transformed scattered displacement field.

The time convolution

The time function with which the data has to be convolved (cf. Eq.(5.4)), in two dimensions, is given by

$$\frac{1}{\pi} \operatorname{Re} \int_{\mathbb{R}_{\geq 0}} \sqrt{\frac{\pi}{\omega}} \, \exp\left[-\frac{i\pi}{4} \right] \, \exp(i\omega t) \, \mathrm{d}\omega = \frac{H(t)}{\sqrt{t}} \; . \tag{A.1}$$

The Hessian

In the Hessian, Eq.(5.14), we have the following substitutions. In an isotropic medium, we have

$$|\nabla_{\boldsymbol{x}} T^{(\tilde{N}\hat{N})}| = \frac{2}{c^{(N)}} \, \cos\left(\frac{\theta}{2} \right) \; . \tag{A.2}$$

Further, note that

$$\partial_{\sigma_1}^2 \boldsymbol{v}^{(N)} = \partial_\alpha^2 \boldsymbol{v}^{(N)} = -\boldsymbol{v}^{(N)} \; .$$

Since the phase and group directions coincide, at stationary, $\boldsymbol{\nu} \parallel \boldsymbol{v}^{(N)}$. Hence,

$$\boldsymbol{\nu} \cdot \partial_{\sigma_1} \partial_{\sigma_1} \boldsymbol{v}^{(N)} = -c^{(N)} \; . \tag{A.3}$$

The unit tangent vector to the isochrons at the stationary point is given by

$$[c^{(N)}]^{-1} \partial_{\sigma_1} \boldsymbol{v}^{(N)} = (-\sin\alpha, \cos\alpha) \; .$$

Hence,

$$[c^{(N)}]^{-1} (\partial_{\sigma_1} \boldsymbol{v}^{(N)}) \; (\partial_{\sigma_1} \boldsymbol{v}^{(N)}) \; : \; \frac{\frac{1}{2} \nabla_{\boldsymbol{x}} \nabla_{\boldsymbol{x}} T^{(\tilde{N}\hat{N})}}{|\nabla_{\boldsymbol{x}} T^{(\tilde{N}\hat{N})}|} = \frac{1}{r^{(\tilde{N}\hat{N})}} \; , \tag{A.4}$$

where $r^{(\tilde{N}\hat{N})}$ is twice the radius of curvature of the finite-offset isochron at stationary. We obtain

$$\partial_{\sigma_1} \partial_{\sigma_1} T^{(\tilde{N}\hat{N})} = [\tau^{(N)}]^2 \cos\left(\frac{\theta}{2} \right) \left[\frac{1}{\tau^{(N)}} - \frac{1}{r^{(\tilde{N}\hat{N})}} \right] \; . \tag{A.5}$$

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