A simplified derivation to migration and demigration in isotropic inhomogeneous media

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ABSTRACT

We describe here some preliminary analysis that arose in our project to develop a true amplitude DMO operator in a depth-dependent medium in 2.5D and 3D. Our approach to this problem is to cascade integral operators to produce a transformation from an input data set at finite offset to an output data set at zero offset.

Much of the earlier work on this approach to DMO started from a Born approximate forward modeling formula. We propose to work with a Kirchhoff approximate forward modeling formula. Where the former representation is linear in the perturbation of the medium parameters, the latter is linear in the reflection coefficient that we seek, that coefficient, in turn, being a nonlinear function of the medium perturbations.

Thus, the purpose of our preliminary research was to develop useful forms of the modeling and inversion formalism based on the Kirchhoff approximation, rather than on the Born approximation. Recent research –notably Burridge et al, (1995), de Hoop and Bleistein (1996) and Tygel et al., (1996) – addresses the type of representations that we seek. In the first two papers, the Born representation is used to derive a form of Kirchhoff integral. There the Kirchhoff approximation is derived as an extension to the Born approximation formula. We present a simplified version of this derivation. In the third paper, Tygel et al., show a derivation of the isochron stack (demigration) and the diffraction stack (migration) operators using Kirchhoff–like integrals. While Burridge et al., and de Hoop and Bleistein did not derived a demigration operator, they use the isochron stack as a modeling operator. We combine Burridge et al. and de Hoop and Bleistein with Tygel et al.'s work to construct a simplified derivation of the isochron stack and the diffraction stack from the Born approximation. Our ultimate objective is to derive those operators directly from the Kirchhoff approximation. For that purpose we convert the Kirchhoff approximation from one integral over the reflecting surface, into a volume integral. We use the Tygel, et al. machinery to convert a volume modeling formula in an isochron stack and to invert volume integrals of the Generalized Radon Type and return to the final inversion formula as a surface integral.

Key words: migration/demigration, isochron, diffraction

1 INTRODUCTION

We describe here some preliminary analysis that arose in our project to develop a true amplitude DMO operator in a depth-dependent medium in 2.5D and 3D. Our approach to this problem is to cascade integral operators to produce a transformation from an input data set at finite offset to an output data set at zero–offset. The first operator is an inversion operator that creates an earth model from the input data. The second is a modeling operator that creates the zero–offset data from the physical model. By "true amplitude," we mean an operator that is consistent with a geometrical optics model of wave propagation. That is, (i) it should account for ray theoretical geometrical spreading changes in the transformation from finite offset to zero–offset and (ii) it should preserve a geometrical optics specular reflection coefficient at a determinable incidence angle of the data.

Much of the earlier work on this approach to DMO started from a Born–approximate forward modeling formula. We propose to work with a Kirchhoff–approximate forward modeling formula. Where the former representation is linear in the perturbation of the medium parameters, the latter is linear in the reflection coefficient that we seek, that coefficient, in turn, being a nonlinear function of the medium perturbations. Thus, for modeling the upward propagation data from a single reflector, we consider the Kirchhoff approximation to be the better choice. Below, we base our analysis on just this earth model.

Thus, the purpose of our preliminary research was to develop useful forms of the modeling and inversion formalism based on the Kirchhoff approximation, rather than on the Born approximation. Recent research –notably Burridge et al, (1995), de Hoop and Bleistein (1996) and Tygel et al., (1996) – addresses the type of representations that we seek. In the first two papers, the Born representation is used to derive a form of Kirchhoff integral. There the Kirchhoff approximation is derived as an extension to the Born approximation formula. Appendix A show a simplified version of this derivation. In the third paper, Tygel et al., show a derivation of the isochron stack (demigration) and the diffraction stack (migration) operators using Kirchhoff–like integrals. While Burridge et al., and de Hoop and Bleistein did not derived a demigration operator, they use the isochron stack as a modeling operator. We combined Burridge et al. and de Hoop and Bleistein with Tygel et al.'s work to construct a simplified derivation of the isochron stack and the diffraction stack from the Born approximation. Our ultimate objective is to derive those operators directly from the Kirchhoff approximation. For that purpose we convert the Kirchhoff integral from a scattering area into an integral over a scattering volume, we use the machinery developed to convert a volume modeling formula in an isochron stack and to invert volume integrals of the Generalized Radon Type and return to the final inversion formula as a surface integral.

Therefore, one purpose of this report is to provide simplified derivations of the isochron and diffraction stack operators, presented by de Hoop and Bleistein (1996). While de Hoop and Bleistein's operators are computed for a general anisotropic media, the operators derived here are constrained to isotropic media. Here, for simplicity, we describe a case in which the propagation mode of the wave is not converted (P–P or S–S), however the extension to elastic converted waves is readily made by changing the propagation speed along the rays from source or receiver accordingly to the P or S propagation mode.

The work presented here was originally motivated by the work of Tygel et al. (1996), who used cascaded forward and inverse operators to solve a wealth of imaging problems. Fabio Rocca (Claerbout, 1985), show that the Dip Moveout (DMO) impulse response could be constructed by tracing zero–offset rays (forward problem) to the prestack migration impulse response (inverse problem). Later, Jorden (1987) used Rocca's concept to introduce the true amplitude Transformation to Zero Offset (TZO) operator, as a cascade of two integral operators: zero–offset modeling, cascaded with finite offset inversion. Tygel et al. (1996) generalized the concept of cascading integral operators to solve a wealth of imaging problems such as: offset continuation, P–SV converted waves TZO, P–SV to P–P mode conversion, remigration etc. The forward and inverse operators used by Tygel et al. (1996) are the isochron stack (converted into a demigration operator) and the diffraction stack. An isochron is an equi-traveltime curve or surface. Thus, just as a diffraction stack is a summation over the traveltimes predicted for a point scatterer for a given source/receiver configuration and background velocity, the isochron stack is a summation over the equi-traveltime domain for the same source/receiver configuration and background velocity.

We should acknowledge not only the universality that the work of Tygel et al. provides to this imaging technique, but also note that they use Kirchhoff–type operators. As noted above, we consider them to be more convenient than Born–type operators, used by Jorden. While Born–type operators are volume integrals, Kirchhoff–type operators are surface integrals. This fact is important from the computational point of view. We will show here that on the one hand the regime of validity of the Kirchhoff approximation is larger than that of the Born approximation and, in addition, its use for deriving inverse and forward operators is actually simpler.

The concept of diffraction stack as a migration technique has been well established: Lindsey and Herman (1985); Rockwell (1971). The concept of the isochron stack as a demigration operator was introduced by Tygel et al. (1996).

Tygel et al. derived these fundamental operators by introducing unknown weights into the diffraction stack (migration) and isochron stack (demigration) operators. They derived the appropriate weights based on zero–order ray–theoretical results and the method of stationary phase. Our objective, here, is to present the diffraction stack and the isochron stack from a different perspective. Burridge et al. (1995), de Hoop et al. (1996) and de Hoop and Bleistein (1996) derived similar operators for general anisotropic media based on the Born–approximation forward model. Their generality is such that the specialization of their formulas to the isotropic case is not an easy matter. We present derivations of the diffraction stack and the isochron stack operators for the isotropic case, as seen from the Kirchhoff–approximate forward model. Not only is the range of validity of these derivations larger than for the Born—approximate forward model (in the sense that it is not constrained to small perturbations of the medium parameters), but it can be seen that the algebra will be simpler.

In Appendix A, we present the derivation of the Kirchhoff modeling formula from the Born modeling formula as a by-product of the derivation of the isochron stack. A general version of this derivation can be found in de Hoop and Bleistein (1996).

Finally, we use the isochron and diffraction stack operators, with the help of the superposition principle, to derive a new transform pair for doing both migration and demigration. The better use of one or another transform pair is dictated by the user interest in particular input or output data sets, as will be seen later.

Returning to our original objective, the cascading of these operators provides a formula for DMO processing. However, it involves integration over the upper surface (migration or inversion to an earth model) followed by an integration over the variables of the earth model (demigration or modeling) to obtain the zero–offset data. The latter integration does not involve the data. Thus, development of a DMO operator for DMO in a depth dependent medium amounts to finding an asymptotic approximation of that integral over the earth model, leaving only an integration over the coordinates of the input data with appropriate weights to assure a true amplitude DMO as output. That will be the subject of the follow-on research.

2 A NOTE ON DIFFRACTION THEORY

The basic modeling formulas that we use here are the Born modeling formula and the Kirchhoff modeling formula (Scales, 1995). Following is a brief introduction to both.

2.1 Born modeling

The Born modeling formula can be written as

$$u_S(\boldsymbol{x}_g, \boldsymbol{x}_s, \omega) \sim \omega^2 F(\omega) \int_D d^3x \frac{\alpha(\boldsymbol{x})}{c^2(\boldsymbol{x})} a(\boldsymbol{x}, \boldsymbol{\xi}) e^{i\omega\phi(\boldsymbol{x},\boldsymbol{\xi})}.$$

Here, $u_S(\boldsymbol{x}_g, \boldsymbol{x}_s, \omega)$ represents the monochromatic (ω) scattered field due to a source at $\boldsymbol{x}_s$ and recorded at the position $\boldsymbol{x}_g$; $c(\boldsymbol{x})$ is the "background" wave–speed; $\alpha(\boldsymbol{x}) = c^2(\boldsymbol{x})/v^2(\boldsymbol{x}) - 1$ is a perturbation in the slowness square field, where $v(\boldsymbol{x})$ is the medium propagation speed and D is the volume of scatterers. The spectrum function of the source signature is given by $F(\omega)$ and $\boldsymbol{\xi}$ is a vector defining the measurement configuration; i.e. in a common shot gather $\boldsymbol{\xi}$ would be the receiver location, while for a common offset configuration $\boldsymbol{\xi}$ represents the midpoint between source and receiver. Also,

$$\phi(\boldsymbol{x}, \boldsymbol{\xi}) = \tau_s + \tau_g, \quad a(\boldsymbol{x}, \boldsymbol{\xi}) = A(\boldsymbol{x}, \boldsymbol{x}_s(\boldsymbol{\xi}))A(\boldsymbol{x}_g(\boldsymbol{\xi}), \boldsymbol{x}), \tag{2.1}$$

are the WKBJ phase and amplitude factors. Here $\boldsymbol{\xi}$ is a two dimensional vector that parameterizes the measurement surface;

$$\tau_s = \tau(\boldsymbol{x}, \boldsymbol{x}_s(\boldsymbol{\xi})), \quad \tau_g = \tau(\boldsymbol{x_g}(\boldsymbol{\xi}), \boldsymbol{x}),$$

are the travel–times from source to scattering point $\boldsymbol{x}$ and from the scattering point $\boldsymbol{x}$ to the receiver, respectively. They are solutions of the eikonal equation:

$$\nabla\tau \cdot \nabla\tau = \frac{1}{c^2(\boldsymbol{x})}, \qquad \tau(\boldsymbol{x}_s, \boldsymbol{x}_s) = 0,$$

or similarly with $\boldsymbol{x}_s$ replaced by $\boldsymbol{x}_g$. The amplitudes,

$$A_s = A(\boldsymbol{x}, \boldsymbol{x}_s(\boldsymbol{\xi})), \quad A_g = A(\boldsymbol{x_g}(\boldsymbol{\xi}), \boldsymbol{x})$$

are the WKBJ amplitude solutions of the (first) transport equation,

$$2\nabla\tau \cdot \nabla A + A\nabla^2\tau = 0,$$

subject to the condition

$$4\pi A|\boldsymbol{x} - \boldsymbol{x}_s| \to 1, \qquad \text{as} \quad |\boldsymbol{x} - \boldsymbol{x}_s| \to 0,$$

or, similarly, with $\boldsymbol{x}_s$ replaced by $\boldsymbol{x}_g$.

2.2 Kirchhoff modeling

The Kirchhoff modeling formula is

$$u_S(\boldsymbol{x}_g, \boldsymbol{x}_s, \omega) \sim i\omega F(\omega) \int_\Sigma R(\boldsymbol{x}, \boldsymbol{x}_s) a(\boldsymbol{x}, \boldsymbol{\xi}) (\hat{\boldsymbol{n}} \cdot \nabla_x \phi(\boldsymbol{x}, \boldsymbol{\xi}))\ e^{i\omega\phi(\boldsymbol{x}, \boldsymbol{\xi})} d\Sigma. \tag{2.2}$$

Here, u_S is the monochromatic (ω) scattered field measured at $\boldsymbol{x}_g$ due to a source at $\boldsymbol{x}_s$; a and ϕ are defined in equation (2.1); $\hat{\boldsymbol{n}}$ is an upward unit vector normal to the reflecting surface Σ and R is the geometrical-optics reflection coefficient,

$$R(\boldsymbol{x}, \boldsymbol{x}_s) = \frac{\left|\frac{\partial}{\partial n}\tau(\boldsymbol{x}, \boldsymbol{x}_s)\right| - \left\{1/c_+^2(\boldsymbol{x}) - 1/c^2(\boldsymbol{x}) + \left[\frac{\partial}{\partial n}\tau(\boldsymbol{x}, \boldsymbol{x}_s)\right]^2\right\}^{1/2}}{\left|\frac{\partial}{\partial n}\tau(\boldsymbol{x}, \boldsymbol{x}_s)\right| + \left\{1/c_+^2(\boldsymbol{x}) - 1/c^2(\boldsymbol{x}) + \left[\frac{\partial}{\partial n}\tau(\boldsymbol{x}, \boldsymbol{x}_s)\right]^2\right\}^{1/2}}. \tag{2.3}$$

In this equation $c_+(\boldsymbol{x})$ is the propagation speed below the reflector, while $c(\boldsymbol{x})$ is the speed above. The reflection coefficient is computed by assuming continuity of the wave–field and its normal derivative across the reflecting interface (Bleistein, 1984). Figure 1 illustrates the different symbols defined above.

3 THE ISOCHRON STACK

The purpose of this section is to derive and study the true amplitude inverse migration operator from the inverse scattering point of view.

We start with the fact that data are bandlimited. Thus, below, when we use full–bandwidth distributions —Dirac delta functions, for example— we are describing only the "most singular part" or high–frequency part of the solution. By using the Dirac delta function, we are able to take advantage of the theory of distributions, carry out asymptotic approximations easily by recognizing the smooth and sharp operators (especially when taking derivatives, or introducing factors under the integral sign) and also by exploiting results from the Generalized Radon Transform (GRT) theory. Only after we obtain the final result, will we introduce a filter on the data. This is a valid argument, since convolution is a linear and commutative process. Bandlimiting amounts to multiplication by a frequency domain filter or convolution with a time domain source signature at the last

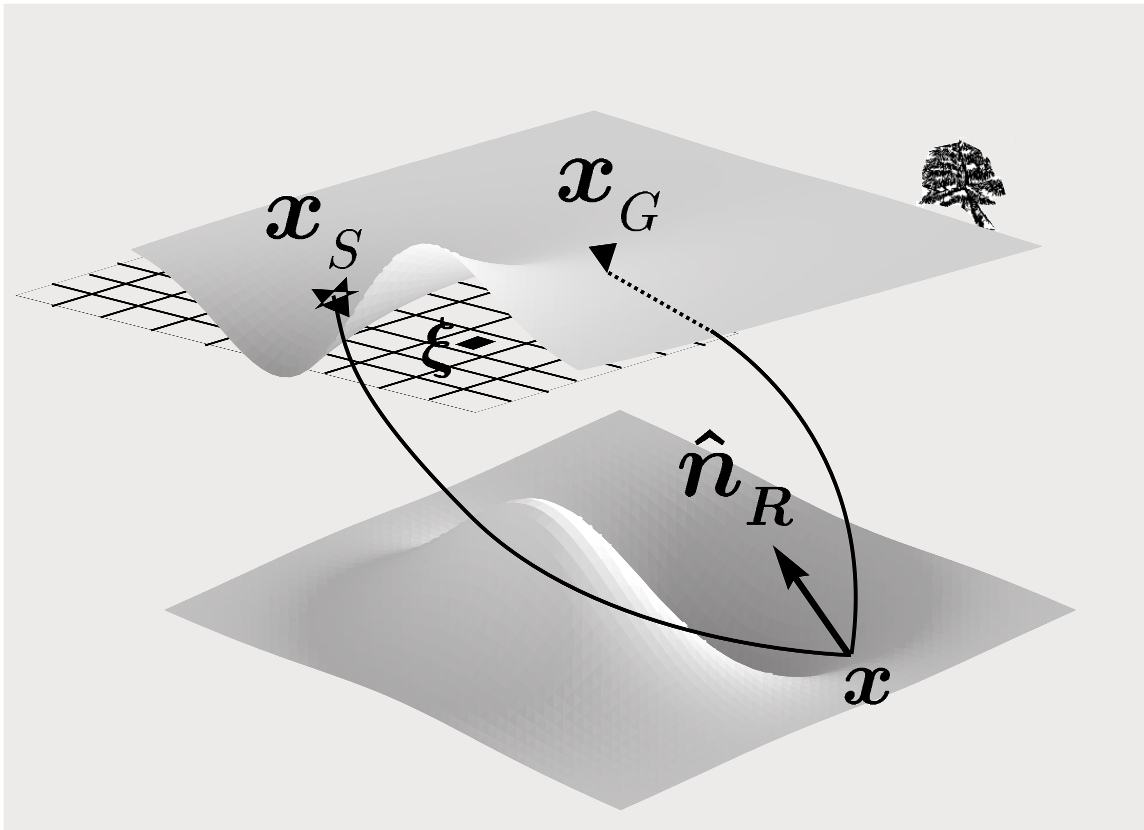

Figure 1. Description of symbols involved in the Kirchhoff modeling formula. Here x_S represents the source position, x_G the receiver position, x a reflector point and n a normal vector to the reflecting surface at x.

moment. This process commutes with all of the previous processes that we shall have employed.

3.1 Modeling versus inverse migration

Before studying the operators, we will point out some subtle differences between modeling and inverse migration processes. Seismic modeling maps acoustic, or elastic reflection data along a recording surface to a prescribed measurement configuration. Seismic imaging maps the recorded data from the surface to the discontinuity surfaces or singular functions of the modeling parameters. Seismic inversion maps the recorded data back into the earth–model parameters. Seismic true amplitude migration maps the recorded data into imaged data weighted by the oblique–incidence specular reflection coefficient. Seismic inverse migration (or demigration) maps true amplitude migrated data into data that would be recorded for a given set of earth model parameters and a defined measurement configuration along the recording surface. From the above, the output section of a modeling algorithm and an inverse migration algorithm should be the same; however the input section for the inverse migration algorithm consists of output data traces with format identical to that of the recorded data while the input of a modeling algorithm is an earth model. From the seismic data processing point of view, the use of cascade migration and demigration algorithms to solve practical problems such as DMO or offset continuation is preferred to the use of cascaded inversion and modeling algorithms. The reason for this is that the output of a migration program is ready to be used as input for a demigration program, while the output traces of an inversion program have to be pre–processed (travel times and amplitudes should be picked) before going into the modeling program. This pre–processing is not only tedious but a major source of human errors. Mathematically, there is no fundamental difference between cascading modeling and inversion operators or cascading demigration and migration operators for solution of imaging problems. The input and output data of the two cascading processes are the same. The intermediate data differ from one another but these data are used only while deriving the final composed operator. They will not be of any use after that.

3.2 The isochron stack as seen from the Born approximation

Here, we show how to recast the Born approximation in 3D – a volume integral – into an isochron stack – an integral over an equi–trave–time surface– plus an integral normal to the isochron, or in the direction of increasing time. Let us assume that a given reflecting surface is described by the equation $\Sigma_R(\boldsymbol{x}) = 0$. The perturbation α associated with that surface can be written as $\alpha(\boldsymbol{x})\ H(\Sigma_R(\boldsymbol{x}))$ where H is the step function; H is equal to zero above $\Sigma_R(\boldsymbol{x})$ and equal to unity below. For simplicity in our discussion, we assume here that the surface is of infinite extent, dividing all of our three dimensional space into two regions. The Born approximation (2.1) is rewritten as

$$D_B(\boldsymbol{\xi}, \omega) \sim \omega^2 F(\omega) \int_D d^3x\, \frac{\alpha(\boldsymbol{x})}{c^2(\boldsymbol{x})}\, H(\Sigma_R(\boldsymbol{x}))\, a(\boldsymbol{x}, \boldsymbol{\xi}) e^{i\omega\phi(\boldsymbol{x}, \boldsymbol{\xi})}. \tag{3.4}$$

Here D stands for Demigration, or Direct, or Data, and we will use the B for Born, and K for Kirchhoff, below. Taking the inverse Fourier transform to time domain, we obtain

$$D_B(\boldsymbol{\xi}, t) \sim -\int_D d^3x \, \frac{\alpha(\boldsymbol{x})}{c^2(\boldsymbol{x})} a(\boldsymbol{x}, \boldsymbol{\xi}) \, H(\Sigma_R(\boldsymbol{x})) \, \delta''(t - \phi(\boldsymbol{x}, \boldsymbol{\xi})). \tag{3.5}$$

Here, consistent with our discussion above of disregarding bandlimiting, we have taken $F(\omega) = 1$. Now, we use the identity,

$$\delta''(t - \phi) = \frac{-\boldsymbol{u} \cdot \nabla_x \delta'(t - \phi)}{\boldsymbol{u} \cdot \nabla_x \phi}.$$

Here ∇_x means gradient with respect to $\boldsymbol{x}$, and $\boldsymbol{u}$ is a unit vector chosen so that $\boldsymbol{u} \cdot \nabla_x \phi \neq 0$. We substitute this result in (3.5) to obtain

$$D_B(\boldsymbol{\xi}, t) \sim \int_D d^3x \, \frac{\alpha(\boldsymbol{x})}{c^2(\boldsymbol{x})} a(\boldsymbol{x}, \boldsymbol{\xi}) \, H(\Sigma_R(\boldsymbol{x})) \, \frac{\boldsymbol{u} \cdot \nabla_x \delta'(t - \phi)}{\boldsymbol{u} \cdot \nabla_x \phi}.$$

For the moment, we think of the integration variables as one coordinate along $\boldsymbol{u}$, and the other two orthogonal to $\boldsymbol{u}$. Then, we integrate by parts in the $\boldsymbol{u}$ direction and keep only the most singular term. The result is

$$D_B(\boldsymbol{\xi}, t) \sim \int_D d^3x \, \frac{\alpha(\boldsymbol{x})}{c^2(\boldsymbol{x})} a(\boldsymbol{x}, \boldsymbol{\xi}) \, \boldsymbol{u} \cdot \nabla_x \, H(\Sigma_R(\boldsymbol{x})) \, \frac{\delta'(t - \phi)}{\boldsymbol{u} \cdot \nabla_x \phi}. \tag{3.6}$$

Now, we rewrite

$$\begin{aligned} \nabla_x H(\Sigma_R(\boldsymbol{x})) &= H'(\Sigma_R(\boldsymbol{x})) \nabla_x \Sigma_R(\boldsymbol{x}) \\ &= \delta(\Sigma_R(\boldsymbol{x})) \nabla_x \Sigma_R(\boldsymbol{x}). \end{aligned} \tag{3.7}$$

Furthermore, we set

$$\delta'(t - \phi) = \frac{-\boldsymbol{v} \cdot \nabla_x \delta(t - \phi)}{\boldsymbol{v} \cdot \nabla_x \phi}, \tag{3.8}$$

where again we pick $\boldsymbol{v}$ such that $\boldsymbol{v}\cdot\nabla_x\phi \neq 0$. Using these results, we can rewrite equation (3.6) as

$$D_B(\boldsymbol{\xi},t) \sim -\int_D d^3x\, \frac{\alpha(\boldsymbol{x})}{c^2(\boldsymbol{x})} a(\boldsymbol{x},\boldsymbol{\xi})\delta(\Sigma_R(\boldsymbol{x}))\, \boldsymbol{u}\cdot\nabla_x\Sigma_R(\boldsymbol{x})\, \frac{\boldsymbol{v}\cdot\nabla_x\delta(t-\phi)}{(\boldsymbol{v}\cdot\nabla_x\phi)(\boldsymbol{u}\cdot\nabla_x\phi)}. \tag{3.9}$$

As we did before, let us think of the integration variables as having one coordinate directed along $\boldsymbol{v}$, and the others orthogonal to $\boldsymbol{v}$. Then, we integrate by parts and keep only the most singular term. The result is

$$D_B(\boldsymbol{\xi},t) \sim -\int_D d^3x\, \frac{\alpha(\boldsymbol{x})}{c^2(\boldsymbol{x})} a(\boldsymbol{x},\boldsymbol{\xi})\boldsymbol{v}\cdot\nabla_x\delta(\Sigma_R(\boldsymbol{x}))\, \boldsymbol{u}\cdot\nabla_x\Sigma_R(\boldsymbol{x})\, \frac{\delta(t-\phi)}{(\boldsymbol{v}\cdot\nabla_x\phi)(\boldsymbol{u}\cdot\nabla_x\phi)}. \tag{3.10}$$

Now, we use the identity

$$\int d^3x ......\delta(t-\phi) = \int ......\frac{d\Sigma_I}{|\nabla_x\phi\,|}, \tag{3.11}$$

where Σ_I is the locus of all $\boldsymbol{x}$ points such that $t=\phi(\boldsymbol{x},\boldsymbol{\xi})$, for each fixed t and fixed source receiver configuration dictated by $\boldsymbol{\xi}$. In other words, the set

$$\Sigma_I = \{\boldsymbol{x} : t = \phi(\boldsymbol{x},\boldsymbol{\xi})\} \tag{3.12}$$

represents the isochron (t–level surface) in the $\boldsymbol{x}$–domain for each fixed t and $\boldsymbol{\xi}$. Figure 2 shows a typical isochron for a constant velocity medium. Applying identity (3.11) to equation (3.10) we find

$$D_B(\boldsymbol{\xi},t) \sim -\int_{\Sigma_I} d\Sigma_I\, \frac{\alpha(\boldsymbol{x})}{c^2(\boldsymbol{x})}\frac{a(\boldsymbol{x},\boldsymbol{\xi})}{|\nabla_x\phi|}\boldsymbol{v}\cdot\nabla_x\delta(\Sigma_R(\boldsymbol{x}))\, \frac{\boldsymbol{u}\cdot\nabla_x\Sigma_R(\boldsymbol{x})}{(\boldsymbol{v}\cdot\nabla_x\phi)(\boldsymbol{u}\cdot\nabla_x\phi)}. \tag{3.13}$$

At this moment, we have freedom in picking $\boldsymbol{u}$ and $\boldsymbol{v}$ as long as they are not orthogonal to $\nabla_x\phi$. We pick $\boldsymbol{u}$ as the upward unit vector normal to the surface Σ_R and we pick $\boldsymbol{v}$ as the normal to the isochron Σ_I, that is, along $\nabla_x\phi$. The integral is dominated by the stationary value of $\Sigma_R(\boldsymbol{x})$ on the isochron Σ_I. One can show that, at stationarity $\boldsymbol{v} = -\boldsymbol{u} = \hat{\boldsymbol{n}}$. Thus, to leading order, we can modify the integrand as follows:

$$D_B(\boldsymbol{\xi},t) \;\sim\; \int_{\Sigma_I} d\Sigma_I\, \frac{\alpha(\boldsymbol{x})}{c^2(\boldsymbol{x})}\frac{a(\boldsymbol{x},\boldsymbol{\xi})}{|\nabla_x\phi)|^3}\frac{\partial\delta(\Sigma_R(\boldsymbol{x}))}{\partial n}\,|\nabla_x\Sigma_R(\boldsymbol{x})|$$

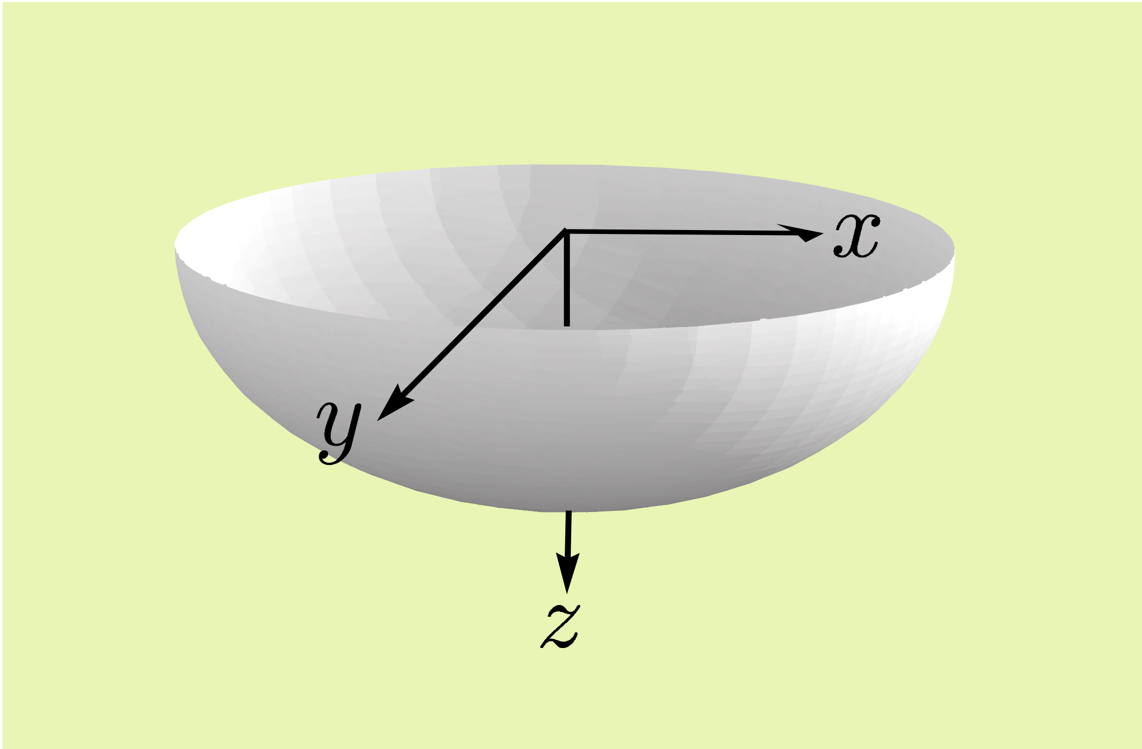

Figure 2. Isochron for a finite offset, constant velocity source receiver configuration.

$$\sim \int_{\Sigma_I} d\Sigma_I \, \frac{\alpha(\boldsymbol{x})}{c^2(\boldsymbol{x})|\nabla_x \phi|^2} \, \frac{a(\boldsymbol{x},\boldsymbol{\xi})}{|\nabla_x \phi|} \, \frac{\partial}{\partial n} \left\{ \delta(\Sigma_R(\boldsymbol{x})) \, |\nabla_x \Sigma_R(\boldsymbol{x})| \right\}, \tag{3.14}$$

where $\partial/\partial n$ means normal derivative along $\hat{\boldsymbol{n}}$.

The function,

$$\gamma(\boldsymbol{x}) \sim \delta(\Sigma_R(\boldsymbol{x})) \, |\nabla_x \Sigma_R(\boldsymbol{x})|, \tag{3.15}$$

is the singular function of the reflecting surface as defined in Bleistein (1987). Figure 3 shows an illustration of $\gamma(\boldsymbol{x})$. This function is a collection of bandlimited delta functions peaking at the reflector position and plotted along a normal line to the reflector. After using the singular function γ we find

$$D_B(\boldsymbol{\xi}, t) \sim \int_{\Sigma_I} d\Sigma_I \, \frac{\alpha(\boldsymbol{x})}{c^2(\boldsymbol{x})|\nabla_x \phi|^2} a(\boldsymbol{x},\boldsymbol{\xi}) \frac{\partial \gamma(\boldsymbol{x})}{\partial n}. \tag{3.16}$$

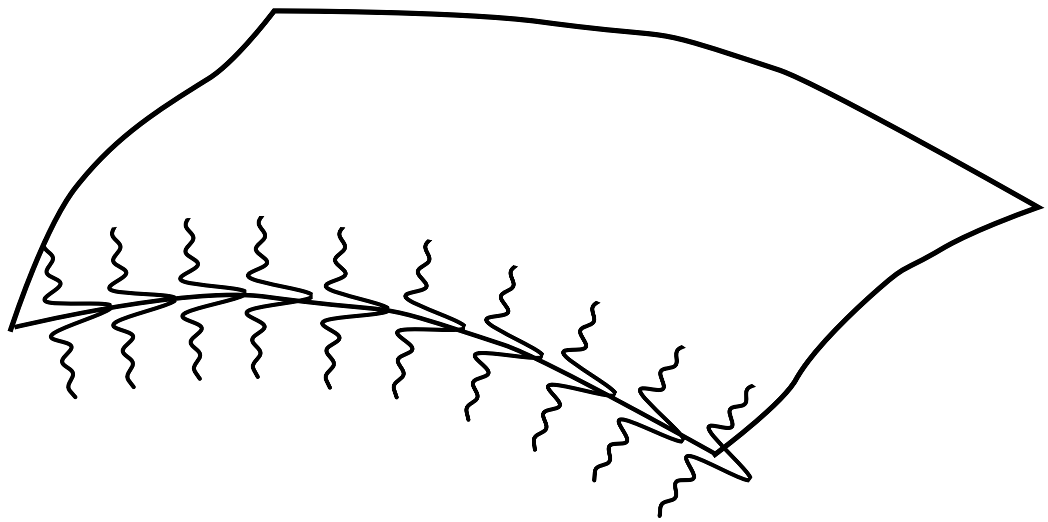

Figure 3. Sketch of the singular function of a given surface. As seen here the singular function is a collection of bandlimited delta functions plotted along a normal line to the surface

Now, we use the result,

$$
\begin{aligned}
|\nabla_x \phi|^2 = |\nabla_x \tau_s + \nabla_x \tau_g|^2 &= |\nabla_x \tau_s|^2 + |\nabla_x \tau_g|^2 + 2\nabla_x \tau_s \cdot \nabla_x \tau_g \\
&= \frac{2}{c^2(\boldsymbol{x})} + \frac{2\cos 2\theta}{c^2(\boldsymbol{x})} \\
&= \frac{4\cos^2\theta}{c^2(\boldsymbol{x})}.
\end{aligned}
$$

Here, θ is half the angle between the slowness vectors $\nabla_x \tau_s$ and $\nabla_x \tau_g$; [τ_s and τ_g are as in equation (2.2)]. Then, in place of equation (3.16), we obtain

$$
D_B(\boldsymbol{\xi}, t) \sim \int_{\Sigma_I} d\Sigma_I \, \frac{\alpha(\boldsymbol{x})}{4\cos^2\theta} \frac{a(\boldsymbol{x}, \boldsymbol{\xi})}{|\nabla_x \phi|} \, \frac{\partial \gamma(\boldsymbol{x})}{\partial n}. \tag{3.17}
$$

The coefficient,

$$
R_B = \frac{\alpha(\boldsymbol{x})}{4\cos^2\theta}, \tag{3.18}
$$

is the linearized Born reflection coefficient. The interpretation of R_B as a reflection coefficient is explained in Appendix A, where the Kirchhoff–approximate formula is derived from the Born–approximate formula by replacing R_B by R_K. With this representation of the reflection coefficient, equation (3.17) becomes

$$D_B(\boldsymbol{\xi}, t) \sim \int_{\Sigma_I} d\Sigma_I \; R_B \frac{a(\boldsymbol{x}, \boldsymbol{\xi})}{|\nabla_x \phi|} \frac{\partial \gamma(\boldsymbol{x})}{\partial n}. \tag{3.19}$$

If we define $\beta_B(\boldsymbol{x}) = R_B \; \gamma(\boldsymbol{x})$ and introduce the reflection coefficient into the derivative sign (which asymptotically is a valid step) then we find that

$$D_B(\boldsymbol{\xi}, t) \sim \int_{\Sigma_I} \frac{a(\boldsymbol{x}, \boldsymbol{\xi})}{|\nabla_x \phi|} \beta_B(\boldsymbol{x}). \tag{3.20}$$

This is an isochron stack and represents a mapping from the Born reflectivity data $\beta_B(\boldsymbol{x})$ to the Born modeled data $D_B(\boldsymbol{\xi}, t)$.

3.2.1 Comparison with Tygel, et al. demigration operator

Tygel, *et al.* (1996) developed a formula to do demigration. Their formula was derived by assuming that the demigration operator has the form of a Kirchhoff integral over the source–receiver isochron for each recording time t. The weight in this integral was an unknown. This weight was found by doing stationary phase analysis to their Kirchhoff–type integral and by knowing the expected zero order ray theory primary reflected wave–field. The matching of these two results at the stationary phase point provided a way to find the weight. A constrain imposed into their demigration operator is that the isochron is parameterized as $z = z_I(x, y)$. In this sense we want to obtain a formula that converts pres–tack depth migrated data into recorded data. Depth–migrated data are represented as vertical traces in depth. Therefore, we specialize equation (3.19) by projecting it into the depth (z) direction.

Let us introduce $\sqrt{g}$ as the metric correction factor for geological dip (reflector dip) * when we describe Σ in x, y with $z = z_\Sigma(x, y)$; that is, $\sqrt{g}\, dxdy = d\Sigma$. At stationarity, both the geological dip and the isochron dip coincide, so $\sqrt{g}$ is the same for both. We

* g is the discriminant of the first fundamental form of the differential geometry, (Kreyszig, 1991) have,

$$\sqrt{g} = \frac{1}{\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{n}}_z} = \frac{\partial z}{\partial n} = \frac{|\nabla_x \phi|}{\partial \phi / \partial z}, \tag{3.21}$$

where $\hat{\boldsymbol{n}}_z = (0, 0, 1)$, and $\hat{\boldsymbol{n}}$ is a unit vector in the direction of $\nabla_x \phi$. It then follows that

$$d\Sigma_I \;=\; \sqrt{g}\, dx dy = \frac{|\nabla_x \phi|}{\partial \phi / \partial z} dx dy, \tag{3.22}$$

$$\frac{\partial}{\partial n} \;=\; \frac{\partial z}{\partial n} \frac{\partial}{\partial z} = \sqrt{g}\, \frac{\partial}{\partial z} = \frac{|\nabla_x \phi|}{\partial \phi / \partial z} \frac{\partial}{\partial z}, \tag{3.23}$$

$$\gamma(\boldsymbol{x}) \;\sim\; \sqrt{g}\, \delta(z - z_{\Sigma_R}) = \frac{|\nabla_x \phi|}{\partial \phi / \partial z} \delta(z - z_{\Sigma_R}). \tag{3.24}$$

We use these results to replace $\partial \gamma / \partial n$ in equation (3.19) as follows.

$$\begin{aligned} R \frac{\partial \gamma}{\partial n} \;&\sim\; R\, |\nabla_x \phi|\, \frac{\partial}{\partial z} \frac{\delta(z - z_{\Sigma_R})}{\partial \phi / \partial z} \\ &\sim\; R\, |\nabla_x \phi|\, \frac{\partial}{\partial z} \delta \left\{ \frac{\partial \phi}{\partial z} (z - z_{\Sigma_R}) \right\}. \end{aligned} \tag{3.25}$$

Now, equation (3.19) can be rewritten as

$$D_B(\boldsymbol{\xi}, t) \sim \int dx dy\, \frac{a(\boldsymbol{x}, \boldsymbol{\xi})\, |\nabla_x \phi|^2}{(\partial \phi / \partial z)^2}\, \frac{\partial}{\partial z} R_B\, \delta(m_D (z - z_{\Sigma_R})) \Bigg|_{z = z_I}, \tag{3.26}$$

where $z_I = z_I(x, y)$ is the depth coordinate of the isochron, and

$$m_D = \partial \phi / \partial z \tag{3.27}$$

is called the migration stretching factor by Tygel, et al. (1994). Furthermore, we introduce the notation,

$$w_D(\boldsymbol{x}, \boldsymbol{\xi}) = \frac{a(\boldsymbol{x}, \boldsymbol{\xi})\, |\nabla_x \phi|^2}{(\partial \phi / \partial z)^2}, \tag{3.28}$$

and, following Tygel, et al. (1996), we define

$$M_B(\boldsymbol{x}_g) \sim R_B \delta(m_D(z - z_S)) \tag{3.29}$$

as the white (full–spectrum) migrated data. The isochron stack, equation (3.26), is then represented by the equation,

$$D_B(\boldsymbol{\xi}, t) \sim \int dxdy \; w_D(\boldsymbol{x}, \boldsymbol{\xi}) \; \frac{\partial}{\partial z} M_B(\boldsymbol{x}) \bigg|_{z=z_I} . \tag{3.30}$$

Now consider the structure of this isochron stack formula. It has two factors:

- The weighting factor (w_D) corrects for the geometrical spreading, directivity, and stretching and
- the migrated data, $M_B(\boldsymbol{x})$, are the true–amplitude migration data, correctly positioned at the reflector location, stretched by the stretching factor, m_D, and weighted by the oblique–incidence linearized Born reflection coefficient, R_B, but with no limits in spectra.

Equation (3.30) matches Tygel, *et al.* (1996) demigration operator. The differences on the weight are due to errors that are being documented in an errata that will be submitted to Geophysics in the near future.

The integration domain is an isochron Σ_I for each fixed time t and source/receiver parameter $\boldsymbol{\xi}$. This explains the name *isochron stack* . It is unfortunate, however, that we are not getting the true reflection coefficient but an approximation that is good only for small contrast α. It is evident that we cannot account for post–critical reflections, since R_B is a real number, while the post–critical reflection coefficient is complex. We are tempted to change the formula by replacing the linearized Born reflection coefficient R_B by the Kirchhoff reflection coefficient R_K defined in equation (2.3), and then extend the range of validity of this formula. However, instead of doing this we will derive the isochron stack formula directly from the Kirchhoff approximation. Thereby, we will confirm that the change from Born to Kirchhoff reflection coefficient is a natural action to take.

3.3 The isochron stack as seen from the Kirchhoff approximation

We now derive, the isochron stack equation based on the Kirchhoff approximation, equation (2.2),

$$D_K(\boldsymbol{\xi},\omega) \;\sim\; i\omega\int_{\Sigma_R} R(\boldsymbol{x},\boldsymbol{x}_s)a(\boldsymbol{x},\boldsymbol{\xi})(\hat{\boldsymbol{n}}\cdot\nabla_x\phi(\boldsymbol{x},\boldsymbol{\xi}))e^{i\omega\phi(\boldsymbol{x},\boldsymbol{\xi})}\,d\Sigma_R. \tag{3.31}$$

Similar to the Born approach, $D_K = u_S$, and we have taken the source signature $F(\omega)=1$. Our first objective is to recast this surface integral as a volume integral. To do this Let us introduce the singular function, $\gamma(\boldsymbol{x})$, as defined in equation(3.15). Then, we can rewrite the right side of equation(3.31), as

$$D_K(\boldsymbol{\xi},\omega) \;\sim\; i\omega\int d^3x\, R(\boldsymbol{x},\boldsymbol{x}_s)a(\boldsymbol{x},\boldsymbol{\xi})(\hat{\boldsymbol{n}}\cdot\nabla_x\phi(\boldsymbol{x},\boldsymbol{\xi}))\,\gamma(\boldsymbol{x})\,e^{i\omega\phi(\boldsymbol{x},\boldsymbol{\xi})}. \tag{3.32}$$

As in the discussion of Born–approximate data, we take the inverse Fourier transform from frequency (ω) to time (t), to obtain

$$D_K(\boldsymbol{\xi},t) \;\sim\; -\int d^3x\, R(\boldsymbol{x},\boldsymbol{x}_s)a(\boldsymbol{x},\boldsymbol{\xi})(\hat{\boldsymbol{n}}\cdot\nabla_x\phi(\boldsymbol{x},\boldsymbol{\xi}))\,\gamma(\boldsymbol{x})\,\delta'(t-\phi(\boldsymbol{x},\boldsymbol{\xi})). \tag{3.33}$$

Now, using identity (3.8) with $\boldsymbol{v}=\hat{\boldsymbol{n}}$, we obtain

$$D_K(\boldsymbol{\xi},t) \;\sim\; \int d^3x\, R(\boldsymbol{x},\boldsymbol{x}_s)a(\boldsymbol{x},\boldsymbol{\xi})\,\gamma(\boldsymbol{x})\,\hat{\boldsymbol{n}}\cdot\nabla_x\delta(t-\phi(\boldsymbol{x},\boldsymbol{\xi})). \tag{3.34}$$

Integrating by parts along $\hat{\boldsymbol{n}}$, and keeping the most singular term, yields

$$D_K(\boldsymbol{\xi},t) \;\sim\; \int d^3x\, R(\boldsymbol{x},\boldsymbol{x}_s)a(\boldsymbol{x},\boldsymbol{\xi})\,\frac{\partial\gamma(\boldsymbol{x})}{\partial n}\delta(t-\phi(\boldsymbol{x},\boldsymbol{\xi})). \tag{3.35}$$

After using the identity (3.11) in this equation, we find

$$D_K(\boldsymbol{\xi},t) \;\sim\; \int d\Sigma_I\, R(\boldsymbol{x},\boldsymbol{x}_s)\frac{a(\boldsymbol{x},\boldsymbol{\xi})}{|\nabla_x\phi|}\,\frac{\partial\gamma(\boldsymbol{x})}{\partial n}\bigg|_{t=\phi(\boldsymbol{x},\boldsymbol{\xi})}. \tag{3.36}$$

In Bleistein (1987), the stationary value of $R(\boldsymbol{x},\boldsymbol{x}_S)\,\gamma(\boldsymbol{x})$ was called the *reflectivity function* and denoted by $\beta(\boldsymbol{x})$. Since we are only using leading order asymptotics here, we use this replacement and further rewrite the previous result as

$$D_K(\boldsymbol{\xi},t) \sim \int d\Sigma_I \, \frac{a(\boldsymbol{x},\boldsymbol{\xi})}{|\nabla_x \phi|} \, \frac{\partial \beta(\boldsymbol{x},\boldsymbol{x}_S)}{\partial n}\bigg|_{t=\phi(\boldsymbol{x},\boldsymbol{\xi})} . \tag{3.37}$$

We interpret this demigration operator as a mapping function from the reflectivity data to modeled data.

We recognize that equation (3.37) is analogous to equation (3.19), with the only difference being the form of the reflection coefficient. The reflection coefficient $R(\boldsymbol{x},\boldsymbol{x}_s)$ is the Kirchhoff reflection coefficient defined in equation (2.3), while the reflection coefficient R_B in equation (3.19) is the linearized Born reflection coefficient. It is well known that the Kirchhoff approximation, using the geometrical optics reflection coefficient provides an accurate representation of the reflected field, up to and beyond the critical angle of reflection, while the linearized Born reflection coefficient is restricted to a narrow range of incidence angles near normal incidence. Thus, we extended the apparent range of validity of our problem, just by working with the Kirchhoff modeling (approximate) formula instead of the Born approximation formula. Another point that supports use the Kirchhoff formula is that this is a surface integral, while the Born approximation formula is itself a volume integral. The transformation of the Born formula into a surface integral is equivalent (after using the Kirchhoff reflection coefficient) to Kirchhoff modeling. Therefore, we abandon the Born approach and use only a Kirchhoff–type approach. We also drop the subscript K, below.

3.3.1 Comparison with Tygel, et al. demigration operator

As above, we will project the previous equation along the vertical z direction. We start with with the migrated data defined in equation (3.29) as

$$M(\boldsymbol{x}) \sim R(\boldsymbol{x},\boldsymbol{x_s})\delta(m_D(z - z_\Sigma)). \tag{3.38}$$

The isochron stack is then represented by the equation:

$$D(\boldsymbol{\xi},t) \sim \int dxdy \, w_D(\boldsymbol{x},\boldsymbol{\xi}) \, \frac{\partial}{\partial z} M(\boldsymbol{x})\bigg|_{z=z_I} . \tag{3.39}$$

As in Born modeling, the right side has only two factors. The factor $w_D(\boldsymbol{x}, \boldsymbol{\xi})$, reverses the geometrical spreading and obliquity effects of migrating the data. The other factor is the derivative of the white–reflectivity data, positioned at the right location, stretched with the correct stretching factor and weighted with the Kirchhoff–approximate reflection coefficient. This formula corresponds to the demigration formula derived by Tygel, *et al.* (1996). The differences on the weight are, as explained before, due to errors that are being documented in an errata that will be submitted to Geophysics in the near future.

In this and the previous section, then, we have provided simplified derivations of the more global results of de Hoop and Bleistein (1996). We have converted the Born and Kirchhoff modeling formulas into a demigration from data formula and also we showed that for the particular parameterization of the isochron as $z = z_I(x, y)$ that formula reduces to Tygel, *et al.* (1996) demigration formula.

4 THE DIFFRACTION STACK

4.1 Introduction

In this section we find an expression for the migration operator as a diffraction stack. We see that this diffraction stack operator coincides with the reflectivity function in Bleistein (1987). A different representation of the demigration and migration operators is found by using the superposition principle.

4.2 Previous work

It is a generally accepted principle that seismic migration can be carried out by stacking amplitudes along a diffraction curve and mapping its result into the corresponding diffractor location. See, for example, Hagedoorn (1954), Lindsey and Hermann (1985), Rockwell (1971), Schneider (1978), and Schleicher et al. (1993). This problem has been studied in a more formal way from the inverse–scattering theory point of view by using the Generalized Radon Transform (GRT), starting with Norton and Linzer (1981), who treated ultrasonic experiments in medical imaging and followed by Miller et al. (1984), Beylkin (1985) and others. Bleistein (1987) introduces the inverse scattering solution by using the Fourier transform instead of the GRT. An extension to general anisotropic media is obtained by Burridge et al. (1995).

4.3 Diffraction stack as inverse of the isochron stack

In this section we derive the diffraction stack operator as the asymptotic inverse of the isochron stack operator in the time domain by using the plane wave expansion of the δ function (Gel'fand & Shilov, 1964). This is at the heart of the GRT inverse.

To start, we rewrite equation (3.33) as

$$D(\boldsymbol{\xi},t) \sim -\int d^3x\, \beta(\boldsymbol{x})a(\boldsymbol{x},\boldsymbol{\xi})(\hat{\boldsymbol{n}}\cdot\nabla_x\phi(\boldsymbol{x},\boldsymbol{\xi}))\ \delta'(t-\phi(\boldsymbol{x},\boldsymbol{\xi})), \tag{4.40}$$

where $\beta(\boldsymbol{x}) = R(\boldsymbol{x}_s,\boldsymbol{x})\gamma(\boldsymbol{x})$. Now taking the time derivative:

$$\frac{\partial D(\boldsymbol{\xi},t)}{\partial t} \sim -\int d^3x\, \beta(\boldsymbol{x})a(\boldsymbol{x},\boldsymbol{\xi})(\hat{\boldsymbol{n}}\cdot\nabla_x\phi(\boldsymbol{x},\boldsymbol{\xi}))\ \delta''(t-\phi(\boldsymbol{x},\boldsymbol{\xi})). \tag{4.41}$$

The equation, $t=\phi(\boldsymbol{y},\boldsymbol{\xi})$, defines the t–isochron for a fixed source–receiver configuration dictated by $\boldsymbol{\xi}$. Let us pick a point $\boldsymbol{x}$ close to the isochron and denote by $\boldsymbol{y}$ the points on the isochron, that is, the points satisfying the equation $t=\phi(\boldsymbol{y},\boldsymbol{\xi})$. Now, for points $\boldsymbol{y}$ near $\boldsymbol{x}$ expand ϕ in a Taylor series,

$$\phi(\boldsymbol{x},\boldsymbol{\xi}) = \phi(\boldsymbol{y},\boldsymbol{\xi}) + \nabla_y\phi(\boldsymbol{y},\boldsymbol{\xi})\cdot(\boldsymbol{x}-\boldsymbol{y}) + \dots\ .$$

When considering full bandwidth data in the frequency domain or, equivalently, when considering the "most singular part" of the modeling and inversion output, the support of the delta function in the previous equation is the isochron itself. For high frequency data, we could say that $\boldsymbol{x}$ should be close to $\boldsymbol{y}$, in order to obtain some contribution from the integral. Based on this, we can assume the linear approximation,

$$t-\phi(\boldsymbol{x},\boldsymbol{\xi}) = \phi(\boldsymbol{y},\boldsymbol{\xi})-\phi(\boldsymbol{x},\boldsymbol{\xi}) \approx \nabla_y\phi(\boldsymbol{y},\boldsymbol{\xi})\cdot(\boldsymbol{y}-\boldsymbol{x}), \tag{4.42}$$

so that

$$\begin{aligned}
\delta''(t-\phi) &\approx \delta''(\nabla_y\phi\cdot(\boldsymbol{x}-\boldsymbol{y})) \\
&= \delta''(\,|\nabla_y\phi|\,\boldsymbol{\nu}\cdot(\boldsymbol{x}-\boldsymbol{y})) \\
&= |\nabla_y\phi|^{-3}\,\delta''(\boldsymbol{\nu}\cdot(\boldsymbol{x}-\boldsymbol{y})),
\end{aligned} \tag{4.43}$$

where $\boldsymbol{\nu} = \boldsymbol{\nu}(\boldsymbol{y})$ is a unit vector normal to the isochron. We remark that for fixed $\boldsymbol{x}$ and $\boldsymbol{y}$, we can vary $\boldsymbol{\nu}$ (over a hemisphere) by varying the source/receiver location through $\boldsymbol{\xi}$. Substituting equation (4.43) into equation (4.41) we obtain

$$\left.\frac{\partial D(\boldsymbol{\xi},t)}{\partial t}\right|_{t=\phi(\boldsymbol{y},\boldsymbol{\xi})} \sim -\int d^3x\, \beta(\boldsymbol{x}) a(\boldsymbol{x},\boldsymbol{\xi})(\hat{\boldsymbol{n}}\cdot\nabla_x\phi(\boldsymbol{x},\boldsymbol{\xi}))\ |\nabla_y\phi|^{-3}\,\delta''(\boldsymbol{\nu}\cdot(\boldsymbol{x}-\boldsymbol{y})). \tag{4.44}$$

For each output image point $\boldsymbol{y}$, We consider, for the moment, only those $\boldsymbol{\nu}$'s in the lower hemisphere (this would exclude migration dips larger than 90 degrees, imaged by turning rays. The reason for this will be explained below. We then multiply the previous equation by

$$|\nabla_y\phi|^3 / \left\{a(\boldsymbol{y},\boldsymbol{\xi})\, (\hat{\boldsymbol{n}}\cdot\nabla_y\phi(\boldsymbol{y},\boldsymbol{\xi}))\right\}, \tag{4.45}$$

and integrate over all $\boldsymbol{\nu}'s$ to obtain

$$\begin{aligned}\int d\boldsymbol{\nu}\quad & \frac{|\nabla_y\phi|^3}{a(\boldsymbol{y},\boldsymbol{\xi})\,(\hat{\boldsymbol{n}}\cdot\nabla_y\phi(\boldsymbol{y},\boldsymbol{\xi}))}\left.\frac{\partial D(\boldsymbol{\xi},t)}{\partial t}\right|_{t=\phi(\boldsymbol{y},\boldsymbol{\xi})} \sim \\ & -\int d\boldsymbol{\nu}\frac{|\nabla_y\phi|^3}{a(\boldsymbol{y},\boldsymbol{\xi})\,(\hat{\boldsymbol{n}}\cdot\nabla_y\phi(\boldsymbol{y},\boldsymbol{\xi}))} \\ & \cdot\int d^3x\, \beta(\boldsymbol{x})a(\boldsymbol{x},\boldsymbol{\xi})(\hat{\boldsymbol{n}}\cdot\nabla_x\phi(\boldsymbol{x},\boldsymbol{\xi}))\ |\nabla_y\phi|^{-3}\,\delta''(\boldsymbol{\nu}\cdot(\boldsymbol{x}-\boldsymbol{y})).\end{aligned} \tag{4.46}$$

At this point we can use the plane wave expansion representation of the delta function:

$$\delta(\boldsymbol{x}-\boldsymbol{y}) = \frac{(-1)^{\frac{n-1}{2}}}{2(2\pi)^{n-1}}\int \delta^{(n-1)}((\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{\nu})d\boldsymbol{\nu}, \tag{4.47}$$

where n is an odd number representing the dimension of the domain space (Gel'fand & Shilov, 1964). After substituting equation (4.47) for ($n = 3$) into equation (4.46), we have,

$$\int d\boldsymbol{\nu}\ \frac{|\nabla_x\phi|^3}{a(\boldsymbol{y},\boldsymbol{\xi})\,(\hat{\boldsymbol{n}}\cdot\nabla_y\phi(\boldsymbol{y},\boldsymbol{\xi}))}\left.\frac{\partial D(\boldsymbol{\xi},t)}{\partial t}\right|_{t=\phi(\boldsymbol{y},\boldsymbol{\xi})} \sim 4\pi^2\beta(\boldsymbol{y}). \tag{4.48}$$

The factor 4 in the right hand side comes from the fact that we are integrating only in half of the sphere; for the full sphere, this factor would be 8. We have then,

$$\beta(\boldsymbol{y}) \sim \frac{1}{4\pi^2}\int d\boldsymbol{\nu}\frac{|\nabla_x\phi|^3}{a(\boldsymbol{y},\boldsymbol{\xi})\,(\hat{\boldsymbol{n}}\cdot\nabla_y\phi(\boldsymbol{y},\boldsymbol{\xi}))}\frac{\partial D(\boldsymbol{\xi},t)}{\partial t}\bigg|_{t=\phi(\boldsymbol{y},\boldsymbol{\xi})}. \tag{4.49}$$

At this point we have an inversion formula in terms of an integral over all directions in the lower hemisphere. To derive the inversion formula in terms of an integration over the recording surface parameterized by $\boldsymbol{\xi}$, we should map the lower hemisphere into the recording surface. This can be done through the following Jacobian relations:

$$\begin{aligned}
d\boldsymbol{\nu} = \sqrt{g}_{\xi}d\boldsymbol{\xi} &= |\frac{\partial\boldsymbol{\nu}}{\partial\xi_1}\times\frac{\partial\boldsymbol{\nu}}{\partial\xi_2}|\,d\boldsymbol{\xi} \\
&= |\boldsymbol{\nu}\cdot\frac{\partial\boldsymbol{\nu}}{\partial\xi_1}\times\frac{\partial\boldsymbol{\nu}}{\partial\xi_2}|\,d\boldsymbol{\xi} \\
&= |\nabla_y\phi|^{-3}\,|\boldsymbol{p}\cdot\frac{\partial\boldsymbol{p}}{\partial\xi_1}\times\frac{\partial\boldsymbol{p}}{\partial\xi_2}|d\boldsymbol{\xi},
\end{aligned} \tag{4.50}$$

where $\boldsymbol{p} = \nabla_y\phi(\boldsymbol{y},\boldsymbol{\xi})$. By substituting equation (4.50) into equation (4.49) we find,

$$\beta(\boldsymbol{y}) \sim \frac{1}{4\pi^2}\int d\boldsymbol{\xi}\frac{h(\boldsymbol{y},\boldsymbol{\xi})}{a(\boldsymbol{y},\boldsymbol{\xi})\,(\hat{\boldsymbol{n}}\cdot\nabla_y\phi(\boldsymbol{y},\boldsymbol{\xi}))}\frac{\partial D(\boldsymbol{\xi},t)}{\partial t}\bigg|_{t=\phi(\boldsymbol{y},\boldsymbol{\xi})}. \tag{4.51}$$

where,

$$h(\boldsymbol{y},\boldsymbol{\xi}) = \det\begin{bmatrix}\nabla_y\phi(\boldsymbol{y},\boldsymbol{\xi}) \\ \frac{\partial}{\partial\xi_1}\nabla_y\phi(\boldsymbol{y},\boldsymbol{\xi}) \\ \frac{\partial}{\partial\xi_2}\nabla_y\phi(\boldsymbol{y},\boldsymbol{\xi})\end{bmatrix}, \tag{4.52}$$

is the Beylkin determinant (Beylkin, 1985; Bleistein *et al.*, 1996).

Given that $\hat{\boldsymbol{n}}\cdot\nabla_y\phi(\boldsymbol{y},\boldsymbol{\xi}) \sim -|\nabla_y\phi(\boldsymbol{y},\boldsymbol{\xi}|$ we can rewrite (4.51) as

$$\beta(\boldsymbol{y}) \sim -\frac{1}{4\pi^2}\int d\boldsymbol{\xi}\frac{h(\boldsymbol{y},\boldsymbol{\xi})}{a(\boldsymbol{y},\boldsymbol{\xi})\,|\nabla_y\phi(\boldsymbol{y},\boldsymbol{\xi}|}\frac{\partial D(\boldsymbol{\xi},t)}{\partial t}\bigg|_{t=\phi(\boldsymbol{y},\boldsymbol{\xi})}. \tag{4.53}$$

We have a two fold reason for choosing the integration of migration dips ($\boldsymbol{\nu}'s$) on the lower hemisphere. Since we allow both positive and negative frequencies (ω), we will have a maximum angular aperture of 360 degrees which is consistent with the physics of the problem. On the other hand, it can be shown that the Beylkin determinant h changes sign at horizontal migration dips. In other words, the mapping from the unit sphere to the recording surface parameter is singular. The meaning of this is that we will have to split the data set into two data sets, depending on whether the migration dips are smaller or larger than 90 degrees. Each data set can be processed with the same formula (4.53).

To find the frequency–domain counterpart of equation (4.53), we rewrite that equation as

$$\beta(\boldsymbol{y}) \sim -\frac{1}{4\pi^2}\int d\boldsymbol{\xi}\, dt \frac{h(\boldsymbol{y},\boldsymbol{\xi})}{a(\boldsymbol{y},\boldsymbol{\xi})\,|\nabla_y\phi(\boldsymbol{y},\boldsymbol{\xi}|}\frac{\partial D(\boldsymbol{\xi},t)}{\partial t}\delta(t-\phi(\boldsymbol{y},\boldsymbol{\xi})). \tag{4.54}$$

Now, we use the frequency representation of the delta function given by

$$\delta(t-\phi) = \frac{1}{2\pi}\int d\omega\ e^{i\omega(t-\phi)}.$$

Then

$$\begin{aligned}
\beta(\boldsymbol{y}) &\sim -\frac{1}{8\pi^3}\int d\boldsymbol{\xi} dt \frac{h(\boldsymbol{y},\boldsymbol{\xi})}{a(\boldsymbol{y},\boldsymbol{\xi})\,|\nabla_y\phi(\boldsymbol{y},\boldsymbol{\xi}|}\frac{\partial D(\boldsymbol{\xi},t)}{\partial t}\int d\omega\ e^{i\omega(t-\phi)} \\
&\sim -\frac{1}{8\pi^3}\int d\boldsymbol{\xi} \frac{h(\boldsymbol{y},\boldsymbol{\xi})}{a(\boldsymbol{y},\boldsymbol{\xi})\,|\nabla_y\phi(\boldsymbol{y},\boldsymbol{\xi}|}\int d\omega\ e^{-i\omega\phi}\int dt \frac{\partial D(\boldsymbol{\xi},t)}{\partial t}e^{i\omega t} \\
&\sim \frac{1}{8\pi^3}\int d\boldsymbol{\xi} \frac{h(\boldsymbol{y},\boldsymbol{\xi})}{a(\boldsymbol{y},\boldsymbol{\xi})\,|\nabla_y\phi(\boldsymbol{y},\boldsymbol{\xi}|}\int\ d\omega\ \ i\omega e^{-i\omega\phi}\hat{D}(\boldsymbol{\xi},\omega),
\end{aligned} \tag{4.55}$$

where $\hat{D}(\boldsymbol{\xi},\omega)$ is the Fourier transform function of $D(\boldsymbol{\xi},t)$. Equation (4.55) corresponds to equation (4) in Bleistein (1987), for $F(\omega)=1$.

4.3.1 Comparison with Tygel, et al. migration operator

In order to rewrite equation (4.53) in the notation of Tygel et al. (1996), we recast our distributional representation in terms of a distribution in z, as follows. Set

$$\begin{aligned} \beta(\boldsymbol{y}) \sim R(\boldsymbol{y},\boldsymbol{x}_s)\,\gamma(\boldsymbol{y}) &\sim R(\boldsymbol{y},\boldsymbol{x}_s)\,\sqrt{g}\,\delta(z-z_\Sigma) \\ &\sim R(\boldsymbol{y},\boldsymbol{x}_s)\frac{|\nabla_y\phi|}{\partial\phi/\partial z}\,\delta(z-z_\Sigma) \\ &\sim |\nabla_y\phi|\,R(\boldsymbol{y},\boldsymbol{x}_s)\,\delta(m_D(z-z_{\Sigma)})). \end{aligned} \tag{4.56}$$

Here g is as introduced above (3.21). Also we use the fact that the geological dip and the isochron dip coincide at stationarity, so that $\sqrt{g} = |\nabla_y\phi|/(\partial\phi/\partial z)$. The factor m_D, is defined by equation (3.27). Now from equations (3.38), (4.53), and (4.56) we find that

$$M(\boldsymbol{y}) \sim \frac{\beta(\boldsymbol{y})}{|\nabla_y\phi(\boldsymbol{y},\boldsymbol{\xi})|} \tag{4.57}$$

or

$$M(\boldsymbol{y}) \;\sim\; \frac{1}{4\pi^2}\int d\boldsymbol{\xi}\frac{h(\boldsymbol{y},\boldsymbol{\xi})}{a(\boldsymbol{y},\boldsymbol{\xi})\,|\nabla_y\phi|^2}\;\frac{\partial D(\boldsymbol{\xi},t)}{\partial t}\bigg|_{t=\phi(\boldsymbol{x},\boldsymbol{\xi})}. \tag{4.58}$$

Let us now define

$$w_M = \frac{1}{4\pi^2}\frac{h(\boldsymbol{y},\boldsymbol{\xi})}{a(\boldsymbol{y},\boldsymbol{\xi})\,|\nabla_x\phi|^2},$$

so that,

$$M(\boldsymbol{y}) \sim \int d\boldsymbol{\xi}\; w_M(\boldsymbol{y},\boldsymbol{\xi})\;\frac{\partial D(\boldsymbol{\xi},t)}{\partial t}\bigg|_{t=\phi(\boldsymbol{y},\boldsymbol{\xi})}. \tag{4.59}$$

This is the diffraction–stack operator as derived by Tygel, *et al.* (1996). The difference on the weight is due to errors that are being documented into an errata that will be submitted to Geophysics in the near future. This operator represents a weighted (with weight w_M) stack of data along diffraction curves $t = \phi(\boldsymbol{y},\boldsymbol{\xi})$ for each output point x. The output is a true amplitude migrated image corresponding to the reflecting model.

4.4 Discussion

We observe that the isochron stack operator, (3.39),

$$D(\boldsymbol{\xi}, t) \sim \int dxdy\, w_D(\boldsymbol{x}, \boldsymbol{\xi})\, \frac{\partial}{\partial z} M(\boldsymbol{x}) \Big|_{z=z_I}, \tag{4.60}$$

and the diffraction stack operator, (4.59),

$$M(\boldsymbol{x}) \sim \int d\boldsymbol{\xi}\, w_M(\boldsymbol{x}, \boldsymbol{\xi})\, \frac{\partial D(\boldsymbol{\xi}, t)}{\partial t} \Big|_{t=\phi}, \tag{4.61}$$

constitute an asymptotic transform pair between the space of seismic reflectors and the space of seismic reflections. The duality of these spaces is studied in Tygel et al. (1996). By back–substituting the isochron stack (3.39) into the diffraction stack (4.59) we can study the resolution operator for the inverse process. Due to limitations in aperture, this operator differs asymptotically from the identity operator. A normalization factor could be introduced to regularize the inversion procedure; however, we will not go that far in this investigation. If we do the back–substitution in the other direction (substituting the diffraction stack operator into the isochron operator) we can study the resolution in the data space. These types of substitutions could be introduced by modifying the configuration in one of the two operators. We could obtain "resolution" operators in the image space (reflectors) and in the data space (reflections). In this general context, processes such as remigration (Tygel *et al.*, 1996) can be seen as a "resolution" operator in the image space and TZO could be seen as a "resolution" operator in the data space. Being resolution operators, close to the identity, their spatial support should be small with respect to the support of its corresponding forward or inverse operators.

A different and more geometrical approach can be derived from the diffraction stack and isochron stack operators as follows. Both the diffraction stack and the isochron stack operators are linear. Thus, if we derive impulse responses for these operators, then, by applying the superposition principle, we can derive inversion operators from these impulse responses. Interestingly, they will turn out to be slightly different from those derived above, as we will see below.

Let us assume that $D(\boldsymbol{\xi}, t) = \delta(\boldsymbol{\xi} - \boldsymbol{\xi}_0, t - t_0)$, and find the migration impulse response by inserting these data into the diffraction stack operator (4.59). Then,

$$
\begin{aligned}
M_\delta(\boldsymbol{x};\boldsymbol{\xi}_0,t_0) &\sim \int d\boldsymbol{\xi} w_M(\boldsymbol{x},\boldsymbol{\xi}) \left.\frac{\partial\delta(\boldsymbol{\xi}-\boldsymbol{\xi}_0,t-t_0)}{\partial t}\right|_{t=\phi(\boldsymbol{x},\boldsymbol{\xi})} \\
&\sim \int d\boldsymbol{\xi}\frac{w_M(\boldsymbol{x},\boldsymbol{\xi})}{m_D(\boldsymbol{x},\boldsymbol{\xi})} \left.\frac{\partial\delta(\boldsymbol{\xi}-\boldsymbol{\xi}_0,t-t_0)}{\partial z}\right|_{t=\phi(\boldsymbol{x},\boldsymbol{\xi})} \\
&\sim -\int d\boldsymbol{\xi}\, \frac{\partial}{\partial z}\frac{w_M(\boldsymbol{x},\boldsymbol{\xi})}{m_D(\boldsymbol{x},\boldsymbol{\xi})}\delta(\boldsymbol{\xi}-\boldsymbol{\xi}_0,t-t_0)\Big|_{t=\phi(\boldsymbol{x},\boldsymbol{\xi})} \\
&\sim -\frac{\partial}{\partial z}\frac{w_M(\boldsymbol{x},\boldsymbol{\xi}_0)}{m_D(\boldsymbol{x},\boldsymbol{\xi}_0)}\Big|_{t_0=\phi(\boldsymbol{x},\boldsymbol{\xi}_0)} .
\end{aligned}
$$

For each fixed $\boldsymbol{\xi}_0$ and t_0, the equation $t_0 = \phi(\boldsymbol{x},\boldsymbol{\xi}_0)$ defines an isochron. That is, the support of a migration impulse response is an isochron; so, migration can be carried out as a superposition of isochrons according to the formula,

$$
M(\boldsymbol{x}) \sim \int d\boldsymbol{\xi_0} dt_0 M_\delta(\boldsymbol{x};\boldsymbol{\xi}_0,t_0) D(\boldsymbol{\xi_0},t_0).
$$

On the other hand, if $M(\boldsymbol{x}) \sim \delta(\boldsymbol{x}-\boldsymbol{x}_0)$, the inverse migration impulse response will be

$$
\begin{aligned}
D_\delta(\boldsymbol{\xi};\boldsymbol{t},\boldsymbol{x}_0) &\sim \int dxdy\;\, w_D(\boldsymbol{x},\boldsymbol{\xi})\left.\frac{\partial\delta(\boldsymbol{x}-\boldsymbol{x}_0)}{\partial z}\right|_{z=z_I(x,y,\boldsymbol{\xi},t)} \\
&\sim -\int dxdy\, \frac{\partial}{\partial z} w_D(\boldsymbol{x},\boldsymbol{\xi})\delta(\boldsymbol{x}-\boldsymbol{x}_0)\Big|_{z=z_I(x,y,\boldsymbol{\xi},t)} \\
&\sim -\frac{\partial}{\partial z} w_D(\boldsymbol{x}_0,\boldsymbol{\xi})\Big|_{z_0=z_I(x_0,y_0,\boldsymbol{\xi},t)} .
\end{aligned}
\tag{4.62}
$$

For each fixed $\boldsymbol{x}_0$ the equation $z_0 = z_I(x_0,y_0,\boldsymbol{\xi},t)$ defines a diffraction surface. That is, the support of the inverse migration impulse response is a diffraction surface; so inverse migration can be done as a superposition of diffraction surfaces according to the formula:

$$
D(\boldsymbol{\xi},t) \sim \int d\boldsymbol{x}_0 D_\delta(\boldsymbol{\xi};\boldsymbol{t},\boldsymbol{x}_0) M(\boldsymbol{x}_0).
$$

A simple geometrical interpretation of the equivalence between the diffraction stack and the isochron superposition is sketched in Figure 4. Here, for simplicity the velocity is assumed to be $v = 2$ km/s, and the offset is $h = 0$, so the time and depth plots coincide. Also, we display only the in–line direction of the experiment. The isochrons are concave semi–circles and the diffraction curves are hyperbolas. The illustration corresponds to the migration response of a point input that is an isochron (or semi–circle in this case) and to the migration response of a diffraction surface input (or hyperbola in this case), which is a point. A similar interpretation follows for the inverse migration process by interchanging the words superposition and stacking. When stacking is performed along diffraction surfaces we are doing migration, but when superposition is done with diffraction surfaces we are doing inverse migration. In this sense stacking along curves is the inverse process of superposition of those curves. Similarly, when stacking is performed along diffraction surfaces we are doing migration, but when stacking is performed along isochrons we are doing inverse migration. In this sense diffraction surfaces and isochrons are duals of each other. The following table shows a summary of these facts.

	stack	**superpose**
demig	isochron	diffraction
mig	diffraction	isochron

If the purpose is to migrate a particular subset of input data, then the isochron superposition is preferred. For example, if a file contains the picked reflections, then the isochron for each picked event is superposed to obtain the migrated data. On the other hand if the user wants to obtain a migrated section of a particular output subset, then the diffraction stack is more convenient. For example, if we want to obtain only a section in the in–line direction, we stack only on those diffraction surfaces corresponding to points in the in–line direction. These remarks apply in a similar way to the inverse migration process, by interchanging the words isochron and diffraction surfaces. In practice it is common to find migrated picked events for interpretation purposes. These data sets can be used to create superposition of diffractions to demigrate the data for different purposes. Migration by isochron superposition is defined by Schneider (1971) as input–trace viewpoint, and by Hubral et al., (1996) as migration with inplanats, while migration by diffraction stack is defined by Schneider as output–trace viewpoint and by Hubral et al. as migration with outplanats.

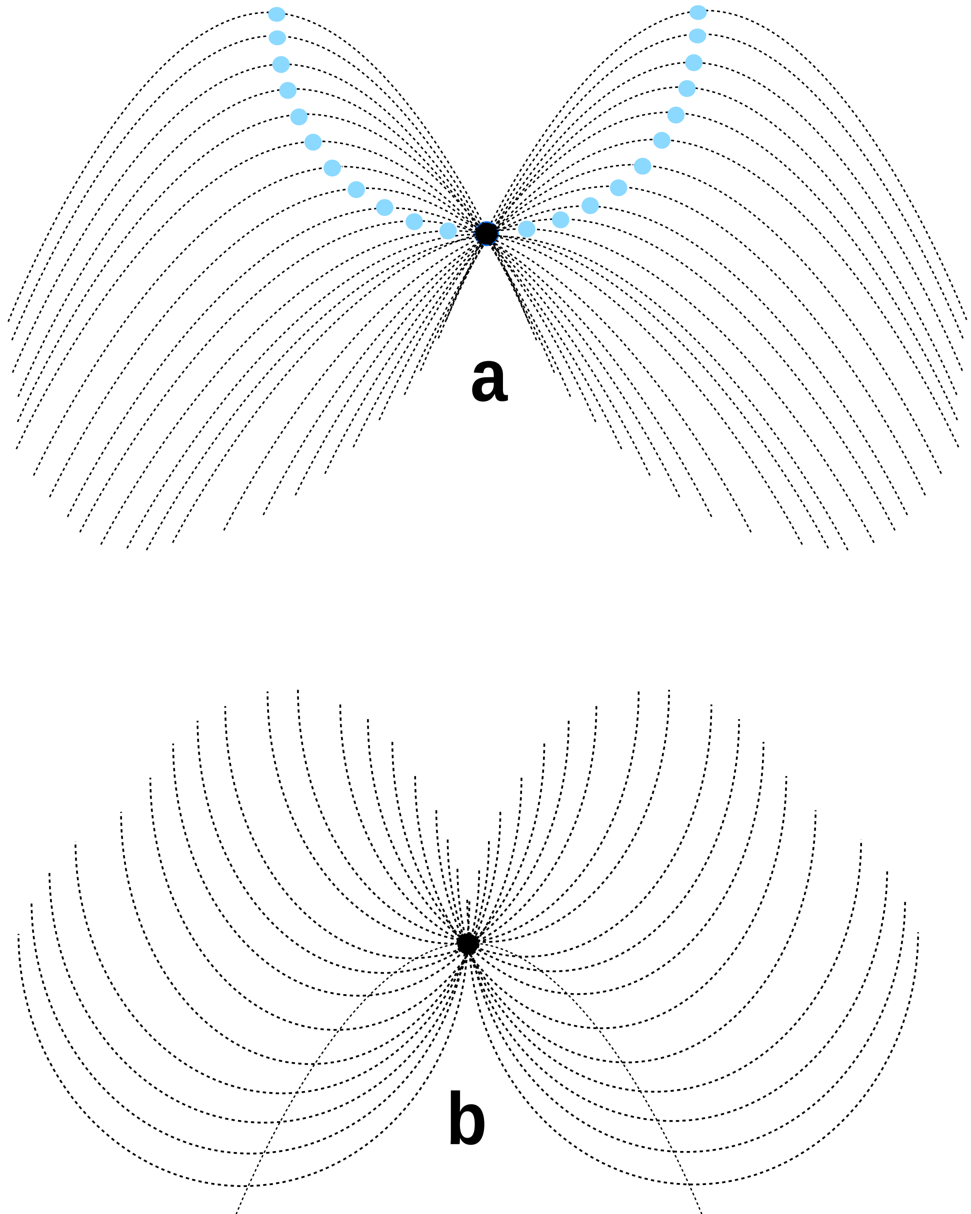

Figure 4. (a) The input data is a point (black) and the migration response is a concave semi–circle that can be constructed as i) a superposition of semi–circles (only one is needed, and here is represented by the gray dots) or ii) as a diffraction stack where each hyperbola smears the input point to its apex. (b) The output image is a point (black) and it can be computed i) as a superposition of semi–circles going through the output image point and with their minimum point sitting on a hyperbola(only one hyperbola is needed). Or ii) as a diffraction stack,where each point along the hyperbola is summed into its appex.

5 CONCLUSIONS

CONCLUSIONS

We carried out a new derivation of the diffraction stack and isochron stack (Tygel *et al.*, 1996) operators based on scattering theory. This derivation continues the work of Bleistein (1987) and builds a bridge through the more general derivation in de Hoop and Bleistein (1996) for anisotropic media. One particular difference between the derivations presented here, and those done previously using scattering theory, is that both the forward and the inverse operators are derived from the Kirchhoff approximate modeling formula, while the other derivations are based on the Born approximate modeling formula. A simplified derivation of the Kirchhoff approximate modeling formula is presented in Appendix A.

Finally, from the previous operators, we derived two different operators based on the superposition principle, to perform migration and demigration. These are the isochron superposition (to do migration) and the diffraction superposition (to do demigration). They are useful alternative when the user has previously selected a particular input data set to process.

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APPENDIX A:

A1 From Born to Kirchhoff

This appendix was "Born" as an effort to find justification for the name reflection coefficient to the factor R_B defined in equation (3.18). Here, we will show a derivation of the Kirchhoff approximate modeling formula (2.2) from the Born–approximate modeling formula (2.1). Start with equation (3.6) which we rewrite as

$$D_B(\boldsymbol{\xi},t) = \int d^3x \, \frac{\alpha(\boldsymbol{x})}{c^2(\boldsymbol{x})} a(\boldsymbol{x},\boldsymbol{\xi}) \nabla_x H(\Sigma_R(\boldsymbol{x})) \cdot \boldsymbol{u} \frac{\delta'(t-\phi)}{\boldsymbol{u}\cdot\nabla_x \phi}. \tag{A1}$$

Let us know pick $\boldsymbol{u} = \boldsymbol{u}_\Sigma$ where $\boldsymbol{u}_\Sigma$ is a normal vector to the surface pointing upwards. We have then that

$$\boldsymbol{u}\cdot\nabla_x H(\Sigma(\boldsymbol{x})) = -\gamma(\boldsymbol{x}), \tag{A2}$$

where $\gamma(\boldsymbol{x})$ is the singular function of the reflecting surface. The minus sign comes from the fact that $\nabla_x H(\Sigma(\boldsymbol{x}))$ is a unit vector normal to the surface pointing in the positive z direction; that is, downwards.

From the definition of the singular function we have then that:

$$D_B(\boldsymbol{\xi},t) = -\int d\Sigma_R \, \frac{\alpha(\boldsymbol{x})}{c^2(\boldsymbol{x})} a(\boldsymbol{x},\boldsymbol{\xi}) \gamma(\boldsymbol{x}) \frac{\delta'(t-\phi)}{\boldsymbol{u}_\Sigma\cdot\nabla_x \phi}. \tag{A3}$$

We use

$$\frac{\alpha(\boldsymbol{x})}{c^2(\boldsymbol{x})(\boldsymbol{u}_\Sigma\cdot\nabla_x\phi)^2} \sim \frac{\alpha(\boldsymbol{x})}{4\cos^2\theta} = R_B, \tag{A4}$$

where, as before, θ is half the angle between the source/receiver rays at the reflector. Then, we find

$$D_B(\boldsymbol{\xi},t) = -\int d\Sigma_R \, R_B \, \boldsymbol{u}_\Sigma\cdot\nabla_x\phi \, \delta'(t-\phi). \tag{A5}$$

Taking the direct Fourier transform of the previous equation we obtain:

$$D_B(\boldsymbol{\xi},\omega) = i\omega \int R_B \, a(\boldsymbol{x},\boldsymbol{\xi}) \, \boldsymbol{u}_\Sigma\cdot\nabla_x\phi \, e^{i\omega\phi(\boldsymbol{x},\boldsymbol{\xi})} \, d\Sigma.$$

If we replace R_B by R_K, then we will have that the previous equation is the Kirchhoff modeling (approximation) formula (2.2), where $\boldsymbol{u}_\Sigma = \hat{\boldsymbol{n}}$. This, then, explicitly shows that the Kirchhoff modeling formula is an extension to the Born approximation formula, with a less restrictive reflection coefficient, and also justifying the name “Linearized Born reflection coefficient”.

References

Beylkin, G. 1985. Imaging of discontinuities in the inverse scattering problem by inversion of a causal generalized Radon transform. *J. Math. Phys.*, **26**, 99–108.

Bleistein, N. 1984. *Mathematical methods for wave phenomena*. Academic Press, Inc.

Bleistein, N. 1987. On the imaging of reflectors in the Earth. *Geophysics*, **52**(7), 931–942.

Bleistein, N., Cohen, J.K., & Stockwell, J. 1996. *Mathematics of multidimensional seismic inversion*. In preparation.

Burridge, R., de Hoop M.V., & E., Miller D. 1995. Multi-parameter inversion in anisotropic media using the generalized Radon transform. *Geoph. J. Int., submitted.*

Claerbout, J. F. 1985. *Imaging the Earth's Interior*. Blackwell Scientific Publications. Pages 209–213.

de Hoop, M.V., & Bleistein, N. 1996. Generalized Radon Transform inversions for reflectivity in anisotropic elastic media. *Inverse Problems, in print.*

de Hoop, M.V., Spencer, C., & Burridge, R. 1996. The resolving power of seismic amplitude data: An anisotropic inversion/migration approach. *Geophysics, submitted.*

Gel'fand, I. M., & Shilov, G. E. 1964. *Generalized functions*. Academic Press.

Hagedoorn, J. G. 1954. A process of seismic reflection interpretation. *Geophys. Prosp.*, **2**(2), 85–127.

Hubral, P., Schleicher, J., & Tygel, M. 1996. A unified approach to 3-D seismic reflection imaging, Part I: Basic concepts. *Geophysics*, **61**(3), 742–758.

Jorden, T. E. 1987. *Transformation to zero offset*. M.Sc. thesis, Colorado School of Mines.

Kreyszig, E. 1991. *Differential Geometry*. Dover.

Lindsey, J. P., & Herman, A. 1985. Digital migration. *Pages 70–74 of:* Gardner, G. H. F. (ed), *Migration of seismic data*. Soc. Expl. Geophys.

Miller, D., Oristaglio, M., & Beylkin, G. 1984. A new formalism and an old heuristic for seismic migration. *Page Session:S19.6 of: 54th Annual Internat. Mtg., Soc. Expl. Geophys., Expanded Abstracts*, vol. 84.

Norton, S. G., & M., Linzer. 1981. Ultrasonic scattering potential imaging in three dimensions: Exact inverse scattering solutions for plane, cylindrical, and spherical apertures. *Intst. Elctr. Electron. Eng. Trans. on Biomedical Engineering*, **BME-28**, 202–220.

Rockwell, D. W. 1971. Migration stack aids interpretation. *Pages 75–81 of:* Gardner, G. H. F. (ed), *Migration stack aids interpretation*. Soc. Expl. Geophys.

Scales, J. A. 1995. *Theory of seismic imaging*. Springer-Verlag. Pages 113–118.

Schleicher, J., Tygel, M., & Hubral, P. 1993. 3-D true-amplitude finite-offset migration.

Geophysics, **58**(8), 1112–1126.

Schneider, William A. 1971. Developments in seismic data-processing and analysis (1968-1970). *Geophysics*, **36**(6), 1043–1073.

Schneider, William A. 1978. Integral formulation for migration in two-dimensions and three-dimensions. *Geophysics*, **43**(1), 49–76.

Tygel, M., Schleicher, J., & Hubral, P. 1994. Pulse distortion in depth migration. *Geophysics*, **59**(10), 1561–1569.

Tygel, M., Schleicher, J., & Hubral, P. 1996. A unified approach to 3-D seismic reflection imaging, Part II: Theory. *Geophysics*, **61**(3), 759–775.