

Uniform asymptotic expansion of the generalized Bremmer series

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ABSTRACT

The Bremmer coupling series solution of the wave equation, in generally inhomogeneous media, requires the introduction of pseudo-differential operators. In this paper, in two dimensions, uniform asymptotic expansions of the Schwartz kernels of these operators are derived. Also, we derive a uniform asymptotic expansion of the one-way propagator appearing in the series, which overlaps with the phase shift / phase screen / split-step Fourier approximations in homogeneous media but is entirely superior in laterally varying media. We focus on designing closed-form representations, valid in the high-frequency limit, taking into account critical scattering-angle phenomena. Our expansion is not limited by propagation angle. In principle, the uniform asymptotic expansion of a kernel follows from matching its asymptotic behaviors away and near its diagonal. The Bremmer series solver consists of three steps: directional decomposition into up- and downgoing waves, one-way propagation, and interaction of the counter-propagating constituents. Each of these steps is here represented by a uniform asymptotic expansion. The resulting algorithm provides an improvement of the parabolic equation method, in particular for transient wave phenomena, and extends the latter method, systematically, for backscattered and critical-angle wave constituents.

Key words: wavefield decomposition, Bremmer series, uniform asymptotics

1 INTRODUCTION

Directional wave field decomposition is a tool for analyzing and computing the propagation of waves in configurations with a certain directionality, such as the waveguiding structure. The method consists of three main steps: (i) decomposing the field into two constituents, propagating upward or downward along a preferred direction, (ii) computing the interaction of the counterpropagating constituents and (iii) recomposing the constituents into observables at the positions of interest. The method is beneficial because it can be compu-

tationally efficient and it can be used to separate constituent propagation phenomena, which is of importance in the interpretation and imaging/inversion of remote measurements.

The generalized Bremmer series (De Hoop, 1996) synthesizes the constituents obtained with the directional decomposition procedure to the full solution of the wave equation. In the Bremmer series approach to modelling, in the time-*Laplace* domain, we encounter pseudo-differential operators in the directional (de)composition, in the downward and upward propagation or continuation, and in the reflections and transmissions due to variations in medium properties in the preferred direction (De Hoop, 1996). Various approaches have been developed over the years to approximate these operators in the Bremmer series to make *numerical* computations feasible. An overview of these approaches can be found in Van Stralen *et al.* (1997).

In this paper, our goal is to gain *analytic* insight in the propagation and scattering of waves on the basis of the generalized Bremmer series – and at the same time developing a time-*Fourier* analysis of the constituent operators. Thus, rather than using the time-Laplace domain we will here divert to the time-Fourier domain and employ spectral theory rather than the theory of pseudo-differential operators. The advantage is that we can then derive uniform asymptotic expansions of the various operators generating the scattering process as well as the one-way propagators. A key reference in this respect is the paper by Fishman *et al.* (1997).

The uniform asymptotic expansions do not only allow us to gain analytic insight, they can also be used in a numerical scheme. Though our current theory is based on WKB solutions and as such is essentially two-dimensional, a paper on the three-dimensional case based on Maslov solutions is in preparation.

The outline of the paper is as follows. In the next section, a summary of the method of directional decomposition, leading to a coupled system of one-way wave equations is given. In Section 3, the medium is decomposed into thin slabs. In each thin slab we introduce a ‘characteristic’ Green’s function. In Sections 4 and 5 we discuss characteristic Green’s function (integral) representations for the operators arising in the directional decomposition. In Section 6 the uniform asymptotic expansion is introduced, and applied to the (de)composition and interaction operators and the one-way propagator in Sections 7, 8 and 9. In Section 10, finally, the Bremmer series solution procedure is summarized. The proof of the uniform asymptotic expansion discussed in Section 6 is given in the Appendix.

2 DIRECTIONAL WAVE FIELD DECOMPOSITION

For the details on the derivation of the Bremmer coupling series solution of the acoustics wave equation, we refer the reader to De Hoop (1996). Here, we restrict ourselves to a summary of this *wave field* decomposition method.

2.1 Notation, transformations

We consider acoustic waves in a two-dimensional configuration. In this configuration, let p denote the pressure and (v_1, v_3) the particle velocity. We introduce the Fourier transformation with respect to time t as

$$(\mathcal{F}\{p, v_1, v_3\})(x_1, x_3, \omega) = \int_{t \in \mathbb{R}_{\geq 0}} \{p, v_1, v_3\}(x_1, x_3, t) \exp(i\omega t) dt \quad (2.1)$$

for $\text{Im}\{\omega\} > 0$. Under this transformation, assuming zero initial conditions, we have $\partial_t \rightarrow -i\omega$.

In each subdomain of the configuration where the acoustic properties vary continuously with position, the acoustic wave field $\{p, v_1, v_3\}$ satisfies the system of partial differential equations

$$\partial_k p - i\omega \rho v_k = f_k, \quad (2.2)$$

$$-i\omega \kappa p + \partial_1 v_1 + \partial_3 v_3 = q. \quad (2.3)$$

Here, ρ denotes the volume density of mass, κ the compressibility, q the volume source density of injection rate, and f_k the volume source density of force.

The spatial variation of the wave field along a direction of preference can now be expressed in terms of the variation of the wave field in the direction perpendicular to it. The direction of preference is taken along the x_3 -axis (or ‘vertical’ axis) and the remaining (‘transverse’ or ‘horizontal’) coordinate is denoted by x_1 .

2.2 The reduced system of equations

Directional decomposition requires a separate handling of the horizontal or transverse component of the particle velocity. From Eqs.(2.2) and (2.3) we obtain

$$v_1 = -i\rho^{-1}\omega^{-1}(\partial_1 p - f_1), \quad (2.4)$$

leaving, upon substitution, the matrix differential equation

$$(\partial_3 \delta_{I,J} - i\omega A_{I,J}) F_J = N_I, \quad A_{I,J} = A_{I,J}(x_1, D_1; x_3), \quad D_1 \equiv \frac{-i}{\omega} \partial_1, \quad (2.5)$$

in which the elements of the acoustic field matrix are given by

$$F_1 = p, \quad F_2 = v_3, \quad (2.6)$$

the elements of the acoustic system’s matrix operator by

$$A_{1,1} = A_{2,2} = 0, \quad (2.7)$$

$$A_{1,2} = \rho, \quad (2.8)$$

$$A_{2,1} = -D_1(\rho^{-1} D_1) + \kappa, \quad (2.9)$$

and the elements of the notional source matrix by

$$N_1 = f_3, \quad N_2 = D_1(\rho^{-1} f_1) + q. \quad (2.10)$$

It is observed that the right-hand side of Eq.(2.4) and $A_{I,J}$ contain the spatial derivative D_1 with respect to the horizontal coordinate only. In the sequel of the paper it will become clear that D_1 has the interpretation of *horizontal slowness* operator. Further, it is noted that $A_{1,2}$ is simply a multiplicative operator.

2.3 The coupled system of one-way wave equations

To distinguish up- and downgoing constituents in the wave field, we shall construct an appropriate linear operator $L_{I,J}$ with

$$F_I = L_{I,J} W_J, \quad (2.11)$$

that, with the aid of the commutation relation ($[\cdot, \cdot]$ denotes the commutator)

$$(\partial_3 L_{I,J}) = [L_{I,J}, \partial_3] \quad (2.12)$$

transforms Eq.(2.5) into

$$L_{I,J} (\partial_3 \delta_{J,M} + i\omega \Lambda_{J,M}) W_M = -(\partial_3 L_{I,J}) W_J + N_I, \quad (2.13)$$

as to make $\Lambda_{J,M}$, satisfying

$$A_{I,J}L_{J,M} = L_{I,J}\Lambda_{J,M} , \quad (2.14)$$

a diagonal matrix of operators. We denote $L_{I,J}$ as the composition operator and W_M as the wave matrix. The expression in parentheses on the left-hand side of Eq.(2.13) represents the two so-called *one-way* wave operators. The first term on the right-hand side of Eq.(2.13) is representative for the scattering due to variations of the medium properties in the vertical direction. The diffraction due to variations of the medium properties in the horizontal directions is contained in $\Lambda_{J,M}$ and, implicitly, in $L_{I,J}$. This diffraction comprises the multi-pathing of characteristics that commonly occurs in geophysical configurations.

To investigate whether solutions of Eq.(2.14) exist, we introduce the column matrix operators $L_I^{(\pm)}$ according to

$$L_I^{(+)} = L_{I,1} , \quad L_I^{(-)} = L_{I,2} . \quad (2.15)$$

Upon writing the diagonal elements of $\Lambda_{J,M}$ as

$$\Lambda_{1,1} = \Gamma^{(+)} , \quad \Lambda_{2,2} = \Gamma^{(-)} , \quad (2.16)$$

Eq.(2.14) decomposes into the two systems of equations

$$A_{I,J}L_J^{(\pm)} = L_I^{(\pm)}\Gamma^{(\pm)} . \quad (2.17)$$

By analogy with the case where the medium is translationally invariant in the horizontal directions, we shall denote $\Gamma^{(\pm)}$ as the *vertical slowness* operators. Notice that the operators $L_1^{(\pm)}$ compose the acoustic pressure and that the operators $L_2^{(\pm)}$ compose the vertical particle velocity. Through mutual elimination, the equations for $L_1^{(\pm)}$ and $L_2^{(\pm)}$ can be decoupled as follows:

$$A_{1,2}A_{2,1}L_1^{(\pm)} = L_1^{(\pm)}\Gamma^{(\pm)}\Gamma^{(\pm)} , \quad (2.18)$$

$$A_{2,1}A_{1,2}L_2^{(\pm)} = L_2^{(\pm)}\Gamma^{(\pm)}\Gamma^{(\pm)} . \quad (2.19)$$

The partial differential operators on the left-hand sides differ from one another in the case where the volume density of mass does vary in the horizontal directions.

To ensure that non-trivial solutions of Eqs.(2.18)-(2.19) exist, one equation must imply the other. To construct a formal solution, an Ansatz is introduced in the form of a commutation relation for one of the components $L_J^{(\pm)}$ that restricts the freedom in the choice for the other component. In the *acoustic-pressure normalization* analog one assumes that $L_2^{(\pm)}$ can be chosen such that

$$[A_{1,2}L_2^{(\pm)} , A_{1,2}A_{2,1}] = 0 . \quad (2.20)$$

In view of Eq.(2.19) the $\Gamma^{(\pm)}$ must then satisfy

$$A_{1,2}A_{2,1} - \Gamma^{(\pm)}\Gamma^{(\pm)} = 0 . \quad (2.21)$$

The commutation relation for $L_1^{(\pm)}$ follows as $[L_1^{(\pm)} , A_{1,2}A_{2,1}] = 0$ and a possible solution of Eq.(2.17) is

$$L_2^{(\pm)} = A_{1,2}^{-1}\Gamma^{(\pm)} , \quad L_1^{(\pm)} = I . \quad (2.22)$$

Since $L_2^{(\pm)}$ as given by Eq.(2.22) satisfies Eq.(2.20), the Ansatz is justified. The solutions of Eq.(2.21) are written as

$$\Gamma^{(+)} = -\Gamma^{(-)} = \Gamma = A^{1/2} \quad \text{with } A = A_{1,2}A_{2,1}. \quad (2.23)$$

Thus, the *composition* operator becomes

$$L = \begin{pmatrix} I & I \\ A_{1,2}^{-1}\Gamma & -A_{1,2}^{-1}\Gamma \end{pmatrix}. \quad (2.24)$$

Note that we have decomposed the pressure field according to

$$F_1 = F_1^{(+)} + F_1^{(-)} \quad \text{with } F_1^{(+)} = W_1, \quad F_1^{(-)} = W_2.$$

In terms of the inverse vertical slowness operator, $\Gamma^{-1} = A^{-1/2}$, the *decomposition* operator then follows as

$$L^{-1} = \frac{1}{2} \begin{pmatrix} I & \Gamma^{-1}A_{1,2} \\ I & -\Gamma^{-1}A_{1,2} \end{pmatrix}. \quad (2.25)$$

Using the decomposition operator, Eq.(2.13) transforms into

$$(\partial_3 \delta_{I,M} - i\omega \Lambda_{I,M}) W_M = -(L^{-1})_{I,M} (\partial_3 L_{M,J}) W_J + (L^{-1})_{I,M} N_M, \quad (2.26)$$

which can be interpreted as a coupled system of one-way wave equations. The coupling between the counter-propagating components, W_1 and W_2 , is apparent in the first source-like term on the right-hand side. We have

$$-L^{-1}(\partial_3 L) = \begin{pmatrix} T & R \\ R & T \end{pmatrix}, \quad (2.27)$$

in which T and R represent the *transmission* and *reflection* operators, respectively: let $Y = A_{1,2}^{-1}\Gamma$ denote the *admittance* operator, then

$$R = -T = \frac{1}{2} Y^{-1} (\partial_3 Y). \quad (2.28)$$

2.4 The two-way Helmholtz equation

Suppose that the medium does not vary with x_3 . Eliminating v_3 from Eqs.(2.5) then leads to the second-order equation,

$$[\partial_3^2 + \omega^2 A(x_1, D_1)] p = i\omega \rho [q + D_1(\rho^{-1} f_1)] + \partial_3 f_3, \quad (2.29)$$

the *two-way* Helmholtz equation.

3 DECOMPOSITION INTO THIN SLABS

In preparation of the development of the Bremmer series representation, we will now decompose the *medium* into (thin) slabs. Each slab in our 2-dimensional configuration is assumed to be *invariant* in the direction of

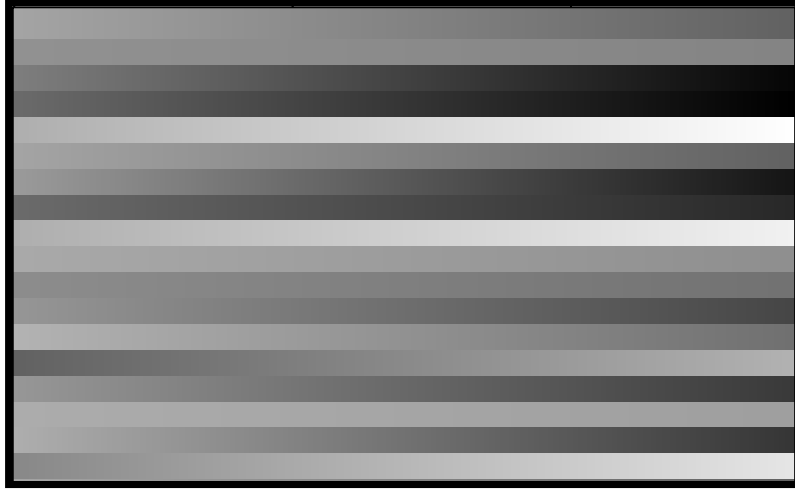


Figure 1. Variation of medium properties.

preference, x_3 : the compressibility, κ , may vary in the transverse direction, whereas the density is assumed to be constant all together, see Figure 1. However, the medium may vary from slab to slab, and hence the vertical coordinate x_3 becomes a parameter that identifies the slab in our further analysis.

The Bremmer series evaluation consists of three main steps (i) decomposing the field into two constituents and propagating these constituents upward or downward along the preferred direction, (ii) letting the counterpropagating constituents interact, and (iii) recomposing the constituents into observables at the positions of interest. Step (i) will be carried out in each slab separately, and we will then accumulate the contributions from the slabs composing a stack; step (ii) will be evaluated at any boundary separating neighboring slabs; step (iii) again will be carried out in each slab individually.

3.1 The characteristic operator

As mentioned, in our thin-slab analysis, we will consider the following medium profile,

$$\rho = \text{const.}, \quad (3.1)$$

$$\kappa(x_1) = \kappa_0 n^2(x_1) \quad (3.2)$$

thus, setting $\kappa_0 = \rho^{-1}c_0^{-2}$, the wave speed follows from

$$c^{-2}(x_1) = c_0^{-2}n^2(x_1),$$

where n denotes the index of refraction. The operator in Eq.(2.23) is then given by

$$A(x_1, D_1) = -D_1^2 + c_0^{-2}n^2(x_1). \quad (3.3)$$

We will denote A as the transverse Helmholtz or *characteristic* operator.

3.2 Factorization, Green's functions

Introduce the well-known Helmholtz-equation Green's function as (cf. Eq.(2.29))

$$[\partial_3^2 + \omega^2 A(x_1, D_1)] G(x_1, x_3 - x'_3; x'_1) = -\delta(x_1 - x'_1)\delta(x_3 - x'_3). \quad (3.4)$$

The vertical slowness operators $\Gamma^{(\pm)}$ factorize the Helmholtz operator (cf. Eq.(2.23))

$$\partial_3^2 + \omega^2 A(x_1, D_1) = [\partial_3 - i\omega \Gamma^{(+)}(x_1, D_1)] [\partial_3 - i\omega \Gamma^{(-)}(x_1, D_1)]. \quad (3.5)$$

The one-way Green's functions $\mathcal{G}^{(\pm)}$ associated with the two factors satisfy

$$[\partial_3 - i\omega \Gamma^{(\pm)}(x_1, D_1)] \mathcal{G}^{(\pm)}(x_1, x_3 - x'_3; x'_1) = \delta(x_1 - x'_1) \delta(x_3 - x'_3). \quad (3.6)$$

To arrive at a uniform asymptotic description of the generalized Bremmer series, we have to find uniform asymptotic representations for L^{-1} (i.e. Γ^{-1}), $\mathcal{G}^{(\pm)}$, R and L (i.e. Γ). This is the subject of the remaining sections.

4 THE SQUARE-ROOT OR VERTICAL SLOWNESS OPERATOR

The vertical slowness or square-root operator Γ (Eq.(2.23)) acts on the wave field as

$$(\Gamma\{W_1, W_2\})(x_1) = \int_{x'_1 \in \mathbb{R}} \mathcal{C}(x_1, x'_1) \{W_1, W_2\}(x'_1) dx'_1, \quad (4.1)$$

where \mathcal{C} denotes a Schwartz kernel. From this operator representation, we extract the left vertical slowness symbol through the Fourier transformation

$$\gamma(x_1, p_1) = \int_{x'_1 \in \mathbb{R}} \mathcal{C}(x_1, x'_1) \exp[-i\omega(x_1 - x'_1)p_1] dx'_1. \quad (4.2)$$

The left symbol of the horizontal slowness operator D_1 appears to be simply p_1 . The relation between the left vertical slowness symbol and the horizontal slowness symbol constitutes the generalized slowness surface.

In the remainder of this section we will focus on finding integral representations for the Schwartz kernel.

4.1 Green's function representation

To begin with, the Schwartz kernel can be expressed in terms of the one-way Green's function,

$$\mathcal{C}^{(+)}(x_1, x'_1; x'_3) = - \lim_{x_3 \downarrow x'_3} \frac{i}{\omega} \partial_3 \mathcal{G}^{(+)}(x_1, x_3 - x'_3; x'_1), \quad (4.3)$$

$$\mathcal{C}^{(-)}(x_1, x'_1; x'_3) = - \lim_{x_3 \uparrow x'_3} \frac{i}{\omega} \partial_3 \mathcal{G}^{(-)}(x_1, x_3 - x'_3; x'_1). \quad (4.4)$$

Using the image principle, we can express the one-way Green's functions in terms of the Green's function of the second-order Helmholtz equation,

$$\mathcal{G}^{(+)}(x_1, x_3 - x'_3; x'_1) + \mathcal{G}^{(-)}(x_1, x_3 - x'_3; x'_1) = -2 \partial_3 G(x_1, x_3 - x'_3; x'_1). \quad (4.5)$$

Hence, for $x_3 > x'_3$,

$$\mathcal{G}^{(+)}(x_1, x_3 - x'_3; x'_1) = -2 \partial_3 G(x_1, x_3 - x'_3; x'_1), \quad (4.6)$$

so that with Eq.(4.3)

$$\mathcal{C}(x_1, x'_1; x'_3) = - \lim_{x_3 \downarrow x'_3} \frac{2}{i\omega} \partial_3^2 G(x_1, x_3 - x'_3; x'_1). \quad (4.7)$$

In our notation, we will suppress the dependency on x'_3 . In fact, $\mathcal{G} \equiv \mathcal{G}^{(+)}$ is the kernel of the (downward) one-way wave propagator. In view of Eq.(4.6) this kernel satisfies the property

$$\partial_3^{2j} \mathcal{G} = [-\omega^2 A(x_1, D_1)]^j \mathcal{G}, \quad j = 1, 2, \dots, \quad (4.8)$$

for $x_3 > x'_3$.

4.2 Fourier-integral representation

We will cast Eq.(4.7) into a spatial Fourier representation. To this end, note that the Fourier representation of the causal Green's function G yields

$$G(x_1, x_3 - x'_3; x'_1) = \frac{\omega}{2\pi c_0} \int_{\zeta \in \mathcal{Z}} \tilde{G}(x_1, x'_1; \zeta) \exp[i(\omega/c_0) |x_3 - x'_3| \zeta] d\zeta. \quad (4.9)$$

Since

$$\omega^2 A(x_1, D_1) = \partial_1^2 + (\omega/c_0)^2 n^2(x_1), \quad (4.10)$$

\tilde{G} satisfies (Eq.(3.4))

$$[\partial_1^2 + (\omega/c_0)^2 (n^2(x_1) - \zeta^2)] \tilde{G}(x_1, x'_1; \zeta) = -\delta(x_1 - x'_1), \quad (4.11)$$

or, more formally,

$$-\omega^2 [A(x_1, D_1) - c_0^{-2} \zeta^2] \tilde{G}(x_1, x'_1; \zeta) = \delta(x_1 - x'_1). \quad (4.12)$$

Observe the symmetry $\tilde{G}(x_1, x'_1; -\zeta) = \tilde{G}(x_1, x'_1; \zeta)$. Also note that Eq.(4.11) is an ordinary differential equation, i.e., a differential equation in a space of dimension one less than the one of the original configuration. The contour \mathcal{Z} follows the real axis in the complex ζ -plane, viz.,

$$\mathcal{Z} = \{\zeta - i0 \mid \zeta \in \mathbb{R}_{<0}\} \cup \{0\} \cup \{\zeta + i0 \mid \zeta \in \mathbb{R}_{>0}\}.$$

The solution to Eq.(4.11) can in generality be represented by

$$\tilde{G}(x_1, x'_1; \zeta) = U_1(x_{1>}; \zeta) U_2(x_{1<}; \zeta), \quad (4.13)$$

where $x_{1<} = \min(x_1, x'_1)$, $x_{1>} = \max(x_1, x'_1)$, and U_1 and U_2 are homogeneous solutions of Eq.(4.11).

Our aim will be to replace the exact solutions U_1 and U_2 by uniform asymptotic approximations for $\zeta \in \mathcal{Z}$. In this respect note the occurrence of *turning points* at $\zeta = \pm n(x_1)$, $\pm n(x'_1)$. Fishman *et al.* (1997) have shown that in the first and the third quadrants of the complex ζ -plane $\tilde{G}(x_1, x'_1; \zeta)$ is analytic and approaches zero as $|\zeta| \rightarrow \infty$. Therefore the contour in Eq.(4.9) can be deformed to the contour \mathcal{Z}' on which the distance from a turning point is always greater than some finite number, as shown in Figure 2. Then Eq.(4.9) becomes

$$G(x_1, x_3 - x'_3; x'_1) = \frac{k_0}{2\pi} \int_{\zeta \in \mathcal{Z}'} \exp[ik_0 |x_3 - x'_3| \zeta] \tilde{G}(x_1, x'_1; \zeta) d\zeta \quad (4.14)$$

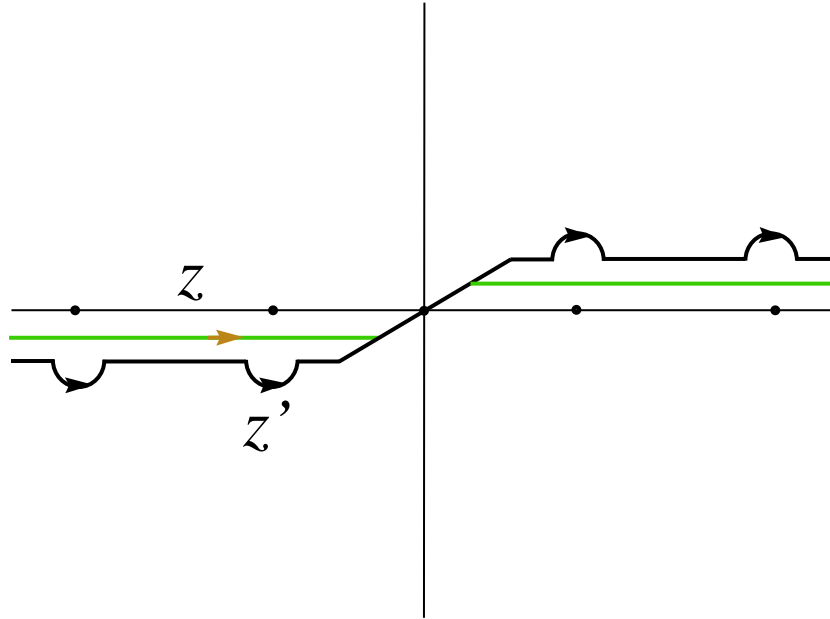


Figure 2. Contours of integration in the complex ζ -plane for the Helmholtz-equation Green's function representation.

when the background wave number is

$$k_0 \equiv \omega/c_0 . \quad (4.15)$$

Substituting Eq.(4.14) into Eq.(4.7) yields

$$\mathcal{C}(x_1, x'_1) = -\frac{1}{\pi i c_0} \lim_{x_3 \downarrow 0} \partial_3^2 \int_{\zeta \in \mathcal{Z}'} \exp[ik_0 x_3 \zeta] \tilde{G}(x_1, x'_1; \zeta) d\zeta . \quad (4.16)$$

In this integral, in distributional sense, we can use

$$\lim_{x_3 \downarrow 0} \partial_3^2 \exp[ik_0 x_3 \zeta] = -k_0^2 \zeta^2 \quad (4.17)$$

so that

$$\mathcal{C}(x_1, x'_1) = \frac{k_0^2}{\pi i c_0} \int_{\zeta \in \mathcal{Z}'} \zeta^2 \exp[ik_0 x_3 \zeta] \tilde{G}(x_1, x'_1; \zeta) d\zeta . \quad (4.18)$$

4.3 Fourier-integral representation of the one-way propagator

Substituting Eq.(4.14) into Eq.(4.6) leads to a Fourier-integral representation of the one-way propagator,

$$\mathcal{G}(x_1, x_3 - x'_3; x'_1) = -\partial_3 \left[\frac{k_0}{\pi} \int_{\zeta \in \mathcal{Z}'} \tilde{G}(x_1, x'_1; \zeta) \exp[ik_0 |x_3 - x'_3| \zeta] d\zeta \right] , \quad (4.19)$$

where we can use

$$\partial_3 \exp[ik_0 |x_3 - x'_3| \zeta] \exp[ik_0 |x_3 - x'_3| \zeta] = ik_0 \zeta \quad \text{for} \quad x_3 > x'_3 . \quad (4.20)$$

To arrive at our uniform asymptotic representation of G , we will have to make the assumption that the propagation distance

$$k_0 |x_3 - x'_3| = \mathcal{O}(1) \tag{4.21}$$

to guarantee that the stationary point of the integral representation remains at $\zeta = 0$, and that

$$|\exp[ik_0 |x_3 - x'_3| \zeta]| = \mathcal{O}(1) \quad \text{for} \quad \zeta \in \mathcal{Z}' . \tag{4.22}$$

5 GENERAL FRACTIONAL POWERS OF A

Equation (4.16) is a special form of the Dunford integral representation for powers of A . To arrive at this general representation, let R_λ denote the *resolvent* of A , i.e.,

$$(A - \lambda I) R_\lambda = I .$$

Then the resolvent kernel \mathcal{R}_λ must satisfy

$$[A(x_1, D_1) - \lambda I] \mathcal{R}_\lambda(x_1, x'_1) = \delta(x_1 - x'_1) . \tag{5.1}$$

Upon comparing this equation with Eq.(4.12), in an x_3 -invariant profile, it follows that the resolvent kernel \mathcal{R}_λ is proportional to the Green's function and in fact equals $-\omega^2 \tilde{G}$: map

$$\lambda \leftrightarrow c_0^{-2} \zeta^2 : \mathcal{L} \leftrightarrow \mathcal{Z} .$$

Instead of rewriting Eq.(4.16), we will consider spectral representations for *general* fractional powers of A .

5.1 Negative fractional powers

Let the power λ^z of a complex variable λ with $z \in \mathbb{R}$ be defined as

$$\lambda^z = |\lambda|^z \exp[iz \arg(\lambda)] , \tag{5.2}$$

with $\arg(\lambda) \in (0, 2\pi)$. With this definition, the branch cut of λ^z is along the positive real axis. \mathcal{L} is a contour of integration in the λ -plane around the spectrum, counter-clockwise oriented, staying away a small but finite distance from the origin (the branch point). Then, for $z \in \mathbb{R}_{<0}$, the Dunford integral

$$A_z = -\frac{1}{2\pi i} \int_{\lambda \in \mathcal{L}} \lambda^z R_\lambda \, d\lambda \tag{5.3}$$

converges in the operator norm $\|\cdot\|_{r,r-2z}$ on the Sobolev space H^r . The Schwartz kernel of A_z is given by

$$\begin{aligned} \mathcal{A}_z(x_1, x'_1) &= -\frac{1}{2\pi i} \int_{\lambda \in \mathcal{L}} \lambda^z \mathcal{R}_\lambda(x_1, x'_1) \, d\lambda \\ &= \frac{k_0^2}{\pi i \, c_0^{2z}} \int_{\zeta \in \mathcal{Z}'} \zeta^{2z+1} \tilde{G}(x_1, x'_1; \zeta) \, d\zeta , \end{aligned} \tag{5.4}$$

which expression is consistent with the Green's function representation Eq.(4.18). The Dunford integral representation satisfies the composition equation

$$A_z A_w = A_{z+w} \quad (5.5)$$

for $z, w \in \mathbb{R}_{<0}$.

5.2 Non-negative fractional powers

With the aid of Eq.(5.3) a non-negative fractional power of A can be readily introduced through

$$A^z = A^j A_{z-j}, \quad (5.6)$$

where j is an integer such that $j > z$. A similar representation for the associated Schwartz kernels is found:

$$\mathcal{A}^z = A^j \mathcal{A}_{z-j}. \quad (5.7)$$

The resulting operators behave, again, like ordinary powers, i.e.,

$$A^z A^w = A^{z+w} \quad (5.8)$$

(note that A and its resolvent commute). Thus

$$\Gamma = A^{1/2} = A A_{-1/2}. \quad (5.9)$$

Note that the operator A takes over the role of ∂_3^2 if compared with Eqs.(4.16) and (4.18):

$$A(x_1, D_1)(-\omega^2 \tilde{G}(x_1, x'_1; \zeta)) = -k_0^{-2} \zeta^2 \tilde{G}(x_1, x'_1; \zeta) + \delta(x_1 - x'_1) \quad (5.10)$$

cf. Eq.(4.12). The regularization of the Schwartz kernels follows the canonical regularization of distributions.

For any medium profile for which the Green's function $G(x_1, x_3 - x'_3; x'_1)$ is known in closed form, closed-form expressions for all the kernels relevant to the Bremmer series can be found.

6 UNIFORM ASYMPTOTIC EXPANSION OF THE 'CHARACTERISTIC' GREEN'S FUNCTION

Here, we will develop a uniform asymptotic expansion of the Green's function $G(x_1, x_3 - x'_3; x'_1)$ in *general* medium profiles – in the high-frequency approximation, i.e., k_0 large – that will lead us to uniform asymptotic expansions for the Bremmer series kernels. For a general background we refer the reader to Bleistein and Handelsman (1967) and Handelsman and Bleistein (1969).

6.1 Finite vertical offset

Substituting the leading order WKB approximation to the causal Green's function of Eq.(4.12) at finite vertical offset into Eq.(4.14) yields

$$G(x_1, x_3 - x'_3; x'_1) \sim \frac{i}{2\pi} \int_{\zeta \in \mathcal{Z}'} \frac{\exp\left[ik_0 \int_{\xi_1=x_{1<}}^{x_{1>}} (n^2(\xi_1) - \zeta^2)^{1/2} d\xi_1\right]}{2 \left[(n^2(x_1) - \zeta^2)(n^2(x'_1) - \zeta^2)\right]^{1/4}} \exp[ik_0 |x_3 - x'_3| \zeta] d\zeta, \quad (6.1)$$

where as before $x_{1<} = \min(x_1, x'_1)$ and $x_{1>} = \max(x_1, x'_1)$. To get a feeling for the uniform asymptotic approximation to G , we will consider three limiting cases.

Case 1: $k_0^{1/2}|x_1 - x'_1| \gg 1$ (away from the diagonal). Then the principal contribution to the Green's function comes from the stationary point at $\zeta = 0$ (according to Eq.(4.21) we have $|x_3 - x'_3| = \mathcal{O}(k_0^{-1})$ and hence the term $|x_3 - x'_3|\zeta$ does not play a role in the phase). Denote the phase in the integral representation (6.1) by Φ ,

$$\Phi(x_1, x'_1; \zeta) = \int_{\xi_1=x_{1<}}^{x_{1>}} (n^2(\xi_1) - \zeta^2)^{1/2} d\xi_1. \quad (6.2)$$

Then

$$\partial_\zeta \Phi(x_1, x'_1; \zeta) = - \int_{\xi_1=x_{1<}}^{x_{1>}} \frac{\zeta}{(n^2(\xi_1) - \zeta^2)^{1/2}} d\xi_1, \quad (6.3)$$

and

$$\partial_\zeta^2 \Phi(x_1, x'_1; \zeta) = - \int_{\xi_1=x_{1<}}^{x_{1>}} \frac{n^2(\xi_1)}{(n^2(\xi_1) - \zeta^2)^{3/2}} d\xi_1. \quad (6.4)$$

Indeed, $(\partial_\zeta \Phi)(x_1, x'_1; 0) = 0$. For the stationary phase analysis we introduce the notation

$$I_0(x_1, x'_1) = \Phi(x_1, x'_1; 0), \quad I_1(x_1, x'_1) = -(\partial_\zeta^2 \Phi)(x_1, x'_1; 0), \quad (6.5)$$

or in general,

$$I_j(x_1, x'_1) = \int_{\xi_1=x_{1<}}^{x_{1>}} [n(\xi_1)]^{1-2j} d\xi_1. \quad (6.6)$$

Carrying out the stationary phase analysis yields,

$$G(x_1, x_3 - x'_3; x'_1) \sim \left[\frac{i}{2\pi k_0 I_1(x_1, x'_1)} \right]^{1/2} \frac{\exp[ik_0 I_0(x_1, x'_1)]}{2 [n(x_1)n(x'_1)]^{1/2}}. \quad (6.7)$$

Case 2: $k_0^{1/2}|x_1 - x'_1| \ll 1$ (near the diagonal). Let $\xi_1 = \frac{1}{2}(x_1 + x'_1) + \frac{1}{2}(x_1 - x'_1) \sigma_1$. Then the phase (Eq.(6.2))

of Eq.(6.1) can be approximated by

$$\begin{aligned}\Phi(x_1, x'_1; \zeta) &= \frac{1}{2}|x_1 - x'_1| \int_{\sigma_1=-1}^1 [n^2(\frac{1}{2}(x_1 + x'_1) + \frac{1}{2}(x_1 - x'_1)\sigma_1) - \zeta^2]^{1/2} d\sigma_1 \\ &= |x_1 - x'_1| \{ [n^2(\frac{1}{2}(x_1 + x'_1)) - \zeta^2]^{1/2} + \mathcal{O}((x_1 - x'_1)^2) \},\end{aligned}\quad (6.8)$$

while the denominator of the integrand in Eq.(6.1) can be approximated by

$$[(n^2(x_1) - \zeta^2)(n^2(x'_1) - \zeta^2)]^{1/4} = [n^2(\frac{1}{2}(x_1 + x'_1)) - \zeta^2]^{1/2} \{1 + \mathcal{O}((x_1 - x'_1)^2)\} . \quad (6.9)$$

Using Eqs.(6.8) and (6.9) in Eq.(6.1), we find that

$$\begin{aligned}G(x_1, x_3 - x'_3; x'_1) &\sim \frac{i}{4\pi} \int_{\zeta \in \mathcal{Z}'} \frac{\exp[ik_0 ([n^2(\frac{1}{2}(x_1 + x'_1)) - \zeta^2]^{1/2}|x_1 - x'_1| + \zeta|x_3 - x'_3|)]}{[n^2(\frac{1}{2}(x_1 + x'_1)) - \zeta^2]^{1/2}} d\zeta \\ &= \frac{i}{4} H_0^{(1)} \left(k_0 n(\frac{1}{2}(x_1 + x'_1)) [(x_1 - x'_1)^2 + (x_3 - x'_3)^2]^{1/2} \right)\end{aligned}\quad (6.10)$$

(see Morse and Feshbach, 1953, p.823).

Case 3: $|x_1 - x'_1| = \mathcal{O}(k_0^{-1/2})$. On this *overlap* region the asymptotic expansions (6.7) and (6.10) must match.

On the one hand, in Eq.(6.7) note that on this region

$$I_j(x_1, x'_1) = [n(\frac{1}{2}(x_1 + x'_1))]^{-j} |x_1 - x'_1| \{1 + \mathcal{O}((x_1 - x'_1)^2)\}, \quad j = 0, 1. \quad (6.11)$$

Also,

$$[n(x_1)n(x'_1)]^{1/2} = n(\frac{1}{2}(x_1 + x'_1)) \{1 + \mathcal{O}((x_1 - x'_1)^2)\}, \quad (6.12)$$

cf. Eq.(6.9).

On the other hand, in Eq.(6.10) we have

$$H_0^{(1)}(k_0 \Phi) \sim \left(\frac{2}{\pi}\right)^{1/2} (-i)^{1/2} \frac{\exp(ik_0 \Phi)}{(k_0 \Phi)^{1/2}} \{1 + \mathcal{O}(k_0^{-1})\} .$$

In Appendix A we show how this leads to the uniform asymptotic expansion

$$G(x_1, x_3 - x'_3; x'_1) \sim \frac{i}{4} \nu(x_1, x'_1) \frac{H_0^{(1)}(k_0 \nu(x_1, x'_1) r(x_1, x'_1))}{[n(x_1)n(x'_1)]^{1/2}}, \quad (6.13)$$

where, for notational convenience, we have introduced the *effective* index of refraction and *effective* horizontal distance as

$$\nu(x_1, x'_1) \equiv \left[\frac{I_0(x_1, x'_1)}{I_1(x_1, x'_1)} \right]^{1/2}, \quad (6.14)$$

$$\chi_1(x_1, x'_1) \equiv \left[I_0(x_1, x'_1) I_1(x_1, x'_1) \right]^{1/2}, \quad (6.15)$$

with limiting behaviors

$$\nu(x_1, x_1) = n(x_1), \quad (6.16)$$

$$\lim_{x_1 \rightarrow x'_1} \frac{\chi_1(x_1, x'_1)}{|x_1 - x'_1|} = 1 \quad (6.17)$$

and the distance

$$r(x_1, x'_1) = [(\chi_1(x_1, x'_1))^2 + (x_3 - x'_3)^2]^{1/2} \quad (6.18)$$

with limiting behaviors

$$\begin{aligned} r(x_1, x'_1) &\sim \chi_1(x_1, x'_1) \left\{ 1 + \frac{(x_3 - x'_3)^2}{2(\chi_1(x_1, x'_1))^2} + \mathcal{O}((\chi_1(x_1, x'_1))^{-4}) \right\}, & k_0^{1/2}|x_1 - x'_1| \gg 1, \\ r(x_1, x'_1) &\sim [(x_1 - x'_1)^2 + (x_3 - x'_3)^2]^{1/2} \{1 + \mathcal{O}((x_1 - x'_1)^2)\}, & k_0^{1/2}|x_1 - x'_1| \ll 1. \end{aligned}$$

Indeed, with these expansions, from Eq.(6.13) both limiting equations (6.7) and (6.10) can be recovered.

In Appendix A, the next order term of the uniform asymptotic expansion of the characteristic-equation Green's function has been derived as well. To provide insight in the results, we introduce the function of indices of refraction as

$$b_{j-1}(x_1, x'_1) = (2j-1)[\beta_1(x_1, x'_1) + (2j-3)\beta_2(x_1, x'_1) + (2j-5)(2j-3)\beta_3(x_1, x'_1)], \quad (6.19)$$

where

$$\beta_3(x_1, x'_1) = -\frac{1}{8} \left[1 - \frac{\nu(x_1, x'_1)}{\nu'(x_1, x'_1)} \right], \quad (6.20)$$

$$\beta_2(x_1, x'_1) = -\frac{1}{4} \left[\left(\frac{\nu(x_1, x'_1)}{n(x_1)} \right)^2 + \left(\frac{\nu(x_1, x'_1)}{n(x'_1)} \right)^2 - 2 \right] + 12\beta_3(x_1, x'_1), \quad (6.21)$$

$$\beta_1(x_1, x'_1) = -\frac{1}{8} \frac{\nu(x_1, x'_1)}{\nu''(x_1, x'_1)} + 4\beta_2(x_1, x'_1) - 24\beta_3(x_1, x'_1), \quad (6.22)$$

with

$$[\nu'(x_1, x'_1)]^{-1} = \frac{[\nu(x_1, x'_1)]^2}{\chi_1(x_1, x'_1)} \int_{\xi_1=x_1 <}^{x_1 >} [n(\xi_1)]^{-3} d\xi_1, \quad (6.23)$$

$$[\nu''(x_1, x'_1)]^{-1} = \chi_1(x_1, x'_1) \int_{\xi_1=x_1 <}^{x_1 >} [n(\xi_1)]^{-1} \left[2 \frac{\partial_1^2 n(\xi_1)}{n(\xi_1)} - 3 \left(\frac{\partial_1 n(\xi_1)}{n(\xi_1)} \right)^2 \right] d\xi_1. \quad (6.24)$$

The limiting behaviors β_j^0 of the β_j ,

$$\beta_j^0(x'_1) = \lim_{x_1 \rightarrow x'_1} \frac{\beta_j(x_1, x'_1)}{[\chi_1(x_1, x'_1)]^2} < \infty, \quad j = 1, 2, 3, \quad (6.25)$$

are found to be

$$\beta_3^0(x'_1) = \frac{1}{24} \left(\frac{\partial_1 n(x'_1)}{n(x'_1)} \right)^2, \quad (6.26)$$

$$\beta_2^0(x'_1) = \frac{1}{12} \left[\frac{\partial_1^2 n(x'_1)}{n(x'_1)} + 2 \left(\frac{\partial_1 n(x'_1)}{n(x'_1)} \right)^2 \right], \quad (6.27)$$

$$\beta_1^0(x'_1) = \frac{1}{24} \left[\frac{2 \partial_1^2 n(x'_1)}{n(x'_1)} + \left(\frac{\partial_1 n(x'_1)}{n(x'_1)} \right)^2 \right]. \quad (6.28)$$

Then Eq.(6.13) extends to

$$\begin{aligned} G(x_1, x_3 - x'_3; x'_1) \sim \frac{i}{4} \frac{\nu(x_1, x'_1)}{[n(x_1)n(x'_1)]^{1/2}} \left\{ \right. \\ \left. \left[1 + (6\beta_3(x_1, x'_1) - \beta_2(x_1, x'_1)) \left(\frac{x_3 - x'_3}{\chi_1(x_1, x'_1)} \right)^2 \right] B_0(k_0\nu(x_1, x'_1) r(x_1, x'_1)) + \right. \\ \left. \left[b_{-1}(x_1, x'_1) + \beta_3(x_1, x'_1) \left(\frac{k_0\nu(x_1, x'_1) (x_3 - x'_3)^2}{r(x_1, x'_1)} \right)^2 \right] \frac{B_{-1}(k_0\nu(x_1, x'_1) r(x_1, x'_1))}{(k_0\nu(x_1, x'_1) \chi_1(x_1, x'_1))^2} + \dots \right\}, \end{aligned} \quad (6.29)$$

where

$$B_j(y) \equiv \frac{(2j-1)!! H_j^{(1)}(y)}{y^j}, \quad (2j-1)!! = \frac{(j - \frac{1}{2})! 2^j}{\sqrt{\pi}}. \quad (6.30)$$

(Note that $(-3)!! = -1$ while $H_{-j}(y) = (-1)^j H_j(y)$.)

6.2 Zero vertical offset

To find a uniform asymptotic expansion of the Green's function at zero vertical offset, we can simply take the limit

$$x_3 - x'_3 \rightarrow 0$$

in the results of the previous subsection. In particular, the distance r reduces to χ_1 . From Eq.(6.29), we obtain

$$\begin{aligned} G(x_1, 0; x'_1) \sim \frac{i}{4} \frac{\nu(x_1, x'_1)}{[n(x_1)n(x'_1)]^{1/2}} \left\{ \right. \\ \left. B_0(k_0\nu(x_1, x'_1) \chi_1(x_1, x'_1)) + b_{-1}(x_1, x'_1) \frac{B_{-1}(k_0\nu(x_1, x'_1) \chi_1(x_1, x'_1))}{(k_0\nu(x_1, x'_1) \chi_1(x_1, x'_1))^2} + \dots \right\}. \end{aligned} \quad (6.31)$$

7 UNIFORM ASYMPTOTIC EXPANSION OF THE (DE)COMPOSITION OPERATOR KERNELS

7.1 Uniform asymptotic expansion of $\mathcal{A}_{-1/2}$

On the basis of Eq.(5.4) we find that the Schwartz kernel of the inverse vertical slowness operator directly follows from Eq.(6.31):

$$\begin{aligned} \mathcal{A}_{-1/2}(x_1, x'_1) = -2i\omega G(x_1, 0; x'_1) \sim \omega \frac{\nu(x_1, x'_1)}{2[n(x_1)n(x'_1)]^{1/2}} \left\{ \right. \\ \left. B_0(k_0\nu(x_1, x'_1) \chi_1(x_1, x'_1)) + b_{-1}(x_1, x'_1) \frac{B_{-1}(k_0\nu(x_1, x'_1) \chi_1(x_1, x'_1))}{(k_0\nu(x_1, x'_1) \chi_1(x_1, x'_1))^2} + \dots \right\}. \end{aligned} \quad (7.1)$$

7.2 Uniform asymptotic expansion of $\mathcal{A}^{j-1/2}$, $j \in \mathbb{Z}$

According to Eq.(5.7) we have

$$\mathcal{A}^{j-1/2} = A^j \mathcal{A}_{-1/2}. \tag{7.2}$$

For the Schwartz kernels associated with the odd powers of the vertical slowness operator, with Eq.(7.1), we thus find the following rule

$$\begin{aligned} \mathcal{A}^{j-1/2}(x_1, x'_1) \sim \omega \frac{\nu(x_1, x'_1)}{2 [n(x_1)n(x'_1)]^{1/2}} [c_0^{-1}\nu(x_1, x'_1)]^{2j} \left\{ \right. \\ \left. B_j(k_0\nu(x_1, x'_1) \chi_1(x_1, x'_1)) + b_{j-1}(x_1, x'_1) \frac{B_{j-1}(k_0\nu(x_1, x'_1) \chi_1(x_1, x'_1))}{(k_0\nu(x_1, x'_1) \chi_1(x_1, x'_1))^2} + \dots \right\}. \end{aligned} \tag{7.3}$$

So far, the kernels have been implicitly parameterized by x'_3 .

8 UNIFORM ASYMPTOTIC EXPANSION OF THE ONE-WAY PROPAGATOR KERNEL

Substituting Eq.(6.29) into Eq.(4.6), we obtain

$$\begin{aligned} \mathcal{G}(x_1, x_3 - x'_3; x'_1) \sim \frac{i}{2} \frac{k_0^2 \nu^3(x_1, x'_1) (x_3 - x'_3)}{[n(x_1)n(x'_1)]^{1/2}} \left\{ \right. \\ \left. \left[1 + \left(\frac{x_3 - x'_3}{\chi_1(x_1, x'_1)} \right)^2 \left(2\beta_3(x_1, x'_1) - \beta_2(x_1, x'_1) + 2\beta_3(x_1, x'_1) \left(\frac{x_3 - x'_3}{r(x_1, x'_1)} \right)^2 \right) \right] \right. \\ \left. \times B_1(k_0\nu(x_1, x'_1) r(x_1, x'_1)) + \right. \\ \left. \frac{1}{(k_0\nu(x_1, x'_1) \chi_1(x_1, x'_1))^2} \left[b_0(x_1, x'_1) - \beta_3(x_1, x'_1) \left(\frac{k_0\nu(x_1, x'_1) (x_3 - x'_3)^2}{r(x_1, x'_1)} \right)^2 \right] \right. \\ \left. \times B_0(k_0\nu(x_1, x'_1) r(x_1, x'_1)) + \dots \right\}. \end{aligned} \tag{8.1}$$

Note that $\mathcal{A}^{1/2}(x_1, x'_1) = -i\omega^{-1}\partial_3\mathcal{G}(x_1, 0; x'_1)$ in agreement with Eq.(7.3). We also observe that

$$\lim_{x_3 \downarrow x'_3} \mathcal{G}(x_1, x_3 - x'_3; x'_1) = \delta(x_1 - x'_1) \tag{8.2}$$

and we can show that

$$\mathcal{G}(x_1, x_3 - x'_3; x'_1) \sim i \sum_{j=1}^{\infty} \frac{(-)^{j-1}}{(2j-1)!} (\omega(x_3 - x'_3))^{2j-1} \mathcal{A}^{j-1/2}(x_1, x'_1), \tag{8.3}$$

which expansion is in agreement with the product integral representation of De Hoop (1996).

9 UNIFORM ASYMPTOTIC EXPANSION OF THE REFLECTION/TRANSMISSION OPERATOR KERNEL

To incorporate the up/down interaction, we allow the transverse Helmholtz or characteristic operator to vary from one thin slab to another. Though strictly speaking, we could now take the finite difference of two neighboring operators only, we will in fact replace this difference by a derivative.

9.1 Uniform asymptotic expansion of $\partial_3 \mathcal{A}^{1/2}$

For the interaction operator we need the vertical derivative of the vertical slowness operator, assuming that the index of refraction is now x_3 dependent. By differentiation, the asymptotic expansion of its Schwartz kernel follows from Eq.(7.1) as

$$(\partial_3 \mathcal{A}^{1/2})(x_1, x'_1) \sim \frac{k_0}{2c_0} \frac{I_0(x_1, x'_1)}{I_1(x_1, x'_1)} d(x_1, x'_1) \left\{ \frac{\partial_3 I_0(x_1, x'_1)}{I_0(x_1, x'_1)} B_0(k_0 I_0(x_1, x'_1)) + b_{0,3}(x_1, x'_1) B_1(k_0 I_0(x_1, x'_1)) + \dots \right\}, \quad (9.1)$$

where

$$d(x_1, x'_1) = \left[\frac{I_0(x_1, x'_1)}{n(x_1)n(x'_1)I_1(x_1, x'_1)} \right]^{1/2} \quad (9.2)$$

and

$$b_{0,3}(x_1, x'_1) = -\frac{1}{2} \times \left[\frac{\partial_3 n(x_1)}{n(x_1)} + \frac{\partial_3 n(x'_1)}{n(x'_1)} + (1 + 2b_0(x_1, x'_1)) \frac{\partial_3 I_0(x_1, x'_1)}{I_0(x_1, x'_1)} + 3 \frac{\partial_3 I_1(x_1, x'_1)}{I_1(x_1, x'_1)} \right]. \quad (9.3)$$

9.2 Uniform asymptotic expansion of \mathcal{R}

For constant density, we can rewrite Eq.(2.28) as

$$R = \frac{1}{2} \Gamma^{-1}(\partial_3 \Gamma). \quad (9.4)$$

To arrive at the kernel for this reflection (transmission) operator, we thus have to compose the Schwartz kernels $\mathcal{A}^{-1/2}$ and $\partial_3 \mathcal{A}^{1/2}$ numerically. That is

$$\begin{aligned} \mathcal{R}(x_1, x'_1) &= \frac{1}{2} \int_{x''_1 \in \mathbb{R}} \mathcal{A}_{-1/2}(x_1, x''_1) (\partial_3 \mathcal{A}^{1/2})(x''_1, x'_1) dx''_1 \\ &\sim \frac{k_0^2}{8} \int_{x''_1 \in \mathbb{R}} d(x_1, x''_1) d(x''_1, x'_1) \frac{I_0(x''_1, x'_1)}{I_1(x''_1, x'_1)} \\ &\quad \times \left[\frac{\partial_3 I_0(x''_1, x'_1)}{I_0(x''_1, x'_1)} B_0(k_0 I_0(x_1, x''_1)) B_0(k_0 I_0(x''_1, x'_1)) + \right. \\ &\quad \left. b_{0,3}(x''_1, x'_1) B_0(k_0 I_0(x_1, x''_1)) B_1(k_0 I_0(x''_1, x'_1)) + \right. \\ &\quad \left. b_{-1}(x_1, x''_1) \frac{\partial_3 I_0(x''_1, x'_1)}{I_0(x''_1, x'_1)} B_1(k_0 I_0(x_1, x''_1)) B_0(k_0 I_0(x''_1, x'_1)) + \dots \right] dx''_1. \end{aligned} \quad (9.5)$$

Since

$$3\pi (k_0 n(x_1))^2 \lim_{x'_1 \rightarrow x_1} b_{-1}(x_1, x'_1) B_1(k_0 I_0(x_1, x'_1)) = -i \left[\frac{\partial_1^2 n(x_1)}{n(x_1)} - \left(\frac{\partial_1 n(x_1)}{n(x_1)} \right)^2 \right], \quad (9.6)$$

$$3\pi (k_0 n(x_1))^2 \lim_{x'_1 \rightarrow x_1} b_{0,3}(x_1, x'_1) B_1(k_0 I_0(x_1, x'_1)) = -i \partial_3 \left(\frac{\partial_1^2 n(x_1)}{n(x_1)} \right), \quad (9.7)$$

the only singularities of the kernel in Eq.(9.5) are the ones contained in the factors $B_0(k_0 I_0(x''_1, x'_1))$, which are logarithmic at $x_1 = x'_1$.

10 THE BREMMER COUPLING SERIES

10.1 The coupled system of integral equations

Applying the operators with kernels Eq.(4.6) to Eq.(2.26) we obtain a coupled system of integral equations. In operator form, they are given by

$$(\delta_{I,J} - K_{I,J})W_J = W_I^{(0)}, \tag{10.1}$$

in which $W^{(0)}$ denotes the incident field. In our configuration the domain of heterogeneity will be restricted to the slab $(0, x_3^{\text{exit}}]$, and the excitation of the waves will be specified through an initial condition at the level $x_3 = 0$, viz.

$$W_1^{(0)}(x_1, x_3) = \int_{x'_1 \in \mathbb{R}} \mathcal{G}^{(+)}(x_1, x_3; x'_1, 0) W_1(x'_1, 0) dx'_1, \tag{10.2}$$

$$W_2^{(0)}(x_1, x_3) = 0, \tag{10.3}$$

in the range of interest, $x_3 \in [0, x_3^{\text{exit}}]$; the second equation reflects the assumption that there is no excitation below the heterogeneous slab. The integral operators in Eq.(10.1) are given by

$$(K_{1,1}W_1)(x_1, x_3) = \int_{\zeta=0}^{x_3} \int_{x'_1 \in \mathbb{R}} \mathcal{G}^{(+)}(x_1, x_3; x'_1, \zeta) (TW_1)(x'_1, \zeta) dx'_1 d\zeta, \tag{10.4}$$

$$(K_{1,2}W_2)(x_1, x_3) = \int_{\zeta=0}^{x_3} \int_{x'_1 \in \mathbb{R}} \mathcal{G}^{(+)}(x_1, x_3; x'_1, \zeta) (RW_2)(x'_1, \zeta) dx'_1 d\zeta, \tag{10.5}$$

$$(K_{2,1}W_1)(x_1, x_3) = \int_{\zeta=x_3}^{x_3^{\text{exit}}} \int_{x'_1 \in \mathbb{R}} \mathcal{G}^{(-)}(x_1, x_3; x'_1, \zeta) (RW_1)(x'_1, \zeta) dx'_1 d\zeta, \tag{10.6}$$

$$(K_{2,2}W_2)(x_1, x_3) = \int_{\zeta=x_3}^{x_3^{\text{exit}}} \int_{x'_1 \in \mathbb{R}} \mathcal{G}^{(-)}(x_1, x_3; x'_1, \zeta) (TW_2)(x'_1, \zeta) dx'_1 d\zeta. \tag{10.7}$$

They describe the interaction between the counter-propagating constituent waves.

10.2 Bremmer series

If $\omega = -is$ (and $p_1 = i\alpha_1 \in i\mathbb{R}$) with s real and sufficiently large, the Neumann expansion can be employed to invert $(\delta_{I,J} - K_{I,J})$ in Eq.(10.1), see De Hoop (1996). Such a procedure leads to the Bremmer coupling series,

$$W_I = \sum_{j=0}^{\infty} (K^j)_{I,J} W_J^{(0)} = W_I^{(0)} + K_{I,J} W_J^{(0)} + (K^2)_{I,J} W_J^{(0)} + \dots \tag{10.8}$$

To emphasize the physical nature of the expansion, we write

$$W_I = \sum_{j=0}^{\infty} W_I^{(j)}, \tag{10.9}$$

in which

$$W_I^{(j)} = K_{I,J} W_J^{(j-1)} \text{ for } j \geq 1, \tag{10.10}$$

can be interpreted as the j -times reflected or scattered wave. This equation indicates that the solution of Eq.(10.1) can be found with the aid of an iterative scheme.

11 DISCUSSION

The Bremmer series expansion of the solution to the multi-dimensional wave equation provides insight in and control over multiple scattering. As such the expansion is a useful tool for analyzing and interpreting (migrating) wave fields in multi-dimensional configurations. In this paper we have established a closed-form uniform asymptotic expansion of constituent kernels of the Bremmer series.

On the one hand, our analysis gave us insight in the multi-dimensional scattering process; on the other hand, we have extended propagation schemes, like the phase-shift-plus-interpolation (Gazdag and Sguazzero, 1984) (or the McClellan transform approach in three dimensions [Hale, 1990]), the split-step Fourier (Stoffa and Fokkema, 1990) and the phase screen (Wu, 1994) methods.

Our Bremmer series solution procedure can be applied to the fields of integrated optics, ocean acoustics and seismics. In particular, in imaging where multi-pathing and post-critical angle phenomena play a role, our uniform asymptotic approach provides a useful basis for the underlying migration procedure.

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APPENDIX A: Derivation of the uniform asymptotic expansion of the characteristic-equation Green's function

We employ the WKB method to obtain the high-frequency asymptotic solution to Eq.(4.11):

$$\tilde{G}(x_1, x'_1; \zeta) = \frac{i}{k_0} \frac{\exp[ik_0 \Phi(x_1, x'_1; \zeta)]}{2 [\tilde{n}(x_1; \zeta) \tilde{n}(x'_1; \zeta)]^{1/2}} \left\{ 1 + \frac{1}{8ik_0} \int_{\xi_1=x_1 <}^{x_1 >} F(\xi_1; \zeta) d\xi_1 + \frac{1}{16k_0^2} \left(\frac{F(x_1; \zeta)}{\tilde{n}(x_1; \zeta)} + \frac{F(x'_1; \zeta)}{\tilde{n}(x'_1; \zeta)} + H(x_1, x'_1; \zeta) \right) + \mathcal{O}(k_0^{-3}) \right\}, \quad (\text{A1})$$

where Φ is given by Eq.(6.2),

$$\tilde{n}(\xi_1; \zeta) \equiv (n^2(\xi_1) - \zeta^2)^{1/2}, \quad (\text{A2})$$

and

$$F(\xi_1; \zeta) = [\tilde{n}(\xi_1; \zeta)]^{-5} [2[\tilde{n}(\xi_1; \zeta)]^2 (n(\xi_1) \partial_1^2 n(\xi_1) + [\partial_1 n(\xi_1)]^2) - 5 [n(\xi_1) \partial_1 n(\xi_1)]^2]. \quad (\text{A3})$$

The term $H(x_1, x'_1; \zeta)$ has the property that $H(x_1, x_1; \zeta) = H(x_1, x'_1; 0) = 0$. Thus an integration by parts in Eq.(4.14) reveals that this term does not contribute to the order in k_0^{-1} considered.

In the following asymptotic analysis we will introduce the average horizontal coordinate,

$$\bar{x}_1 = \frac{1}{2}(x_1 + x'_1), \quad (\text{A4})$$

the background radial distance,

$$r_0(x_1, x'_1) = [(x_1 - x'_1)^2 + (x_3 - x'_3)^2]^{1/2}, \quad (\text{A5})$$

the derivative functions of the index of refraction,

$$\delta_2(x_1) = \left(\frac{\partial_1 n(x_1)}{n(x_1)} \right)^2, \quad (\text{A6})$$

$$\delta_1(x_1) = \frac{\partial_1^2 n(x_1)}{n(x_1)} + \delta_2(x_1), \quad (\text{A7})$$

and the scaled, dimensionless, vertical wavenumber

$$\zeta_n \equiv \frac{\zeta}{n(\bar{x}_1)}. \quad (\text{A8})$$

Further, we set

$$\tilde{n}_0(\zeta_n) = (1 - \zeta_n^2)^{1/2} \quad \text{with} \quad n(\bar{x}_1) \tilde{n}_0(\zeta_n) = \tilde{n}(\bar{x}_1; \zeta). \quad (\text{A9})$$

In the asymptotic analysis we will repeatedly consider the behaviors of the phase $\Phi(x_1, x'_1; \zeta)$, of the denominator $[\tilde{n}(x_1; \zeta)\tilde{n}(x'_1; \zeta)]^{1/2}$, of the function $F(\xi_1; \zeta)$, and of the integral of F .

Case 1: $k_0^{1/2}|x_1 - x'_1| \gg 1$ (away from the diagonal). We expand about the stationary point at $\zeta = 0$,

$$\Phi(x_1, x'_1; \zeta) \sim I_0(x_1, x'_1) - \frac{1}{2}\zeta^2 I_1(x_1, x'_1) - \frac{1}{8}\zeta^4 I_2(x_1, x'_1) + \dots, \quad (\text{A10})$$

where the I_j are given by Eq.(6.6), while

$$[\tilde{n}(x_1; \zeta)]^{-1/2} \sim [n(x_1)]^{-1/2} \left\{ 1 + \left(\frac{\zeta}{2n(x_1)} \right)^2 + \dots \right\} \quad (\text{A11})$$

and

$$F(\xi_1; \zeta) \sim [n(\xi_1)]^{-1} (2\delta_1(\xi_1) - 5\delta_2(\xi_1)). \quad (\text{A12})$$

Substituting these expansions in the expression (A1) for \tilde{G} yields,

$$\begin{aligned} \tilde{G}(x_1, x'_1; \zeta) \sim & \frac{i}{k_0} \frac{\exp[ik_0 (I_0(x_1, x'_1) - \frac{1}{2}\zeta^2 I_1(x_1, x'_1))]}{2 [n(x_1)n(x'_1)]^{1/2}} \left\{ \right. \\ & 1 + \frac{1}{8ik_0} \int_{\xi_1=x_1<}^{x_1>} [n(\xi_1)]^{-1} (2\delta_1(\xi_1) - 5\delta_2(\xi_1)) d\xi_1 + \\ & \left. + \frac{\zeta^2}{4} \left(\frac{1}{[n(x_1)]^2} + \frac{1}{[n(x'_1)]^2} \right) - \frac{ik_0 \zeta^4}{8} I_2(x_1, x'_1) + \mathcal{O}(k_0^{-2}) \right\}. \end{aligned} \quad (\text{A13})$$

Expanding the exponential further in a Taylor series and substituting the result into Eq.(4.14) then leads to

$$\begin{aligned} G(x_1, x_3 - x'_3; x'_1) \sim & \left[\frac{i}{2\pi k_0 I_1(x_1, x'_1)} \right]^{1/2} \frac{\exp[ik_0 I_0(x_1, x'_1)]}{2 [n(x_1)n(x'_1)]^{1/2}} \left\{ \right. \\ & 1 + \frac{1}{ik_0 I_0(x_1, x'_1)} \left[b_{-1}(x_1, x'_1) + \frac{1}{8} - \frac{[k_0 (x_3 - x'_3)]^2 I_0(x_1, x'_1)}{2I_1(x_1, x'_1)} \right] + \mathcal{O}(k_0^{-2}) \left. \right\}, \end{aligned} \quad (\text{A14})$$

where b_{-1} is given by Eq.(6.19).

Case 2: $k_0^{1/2}|x_1 - x'_1| \ll 1$ (near the diagonal). We now expand in $|x_1 - x'_1|$,

$$\Phi(x_1, x'_1; \zeta) \sim n(\bar{x}_1) |x_1 - x'_1| \left\{ \tilde{n}_0(\zeta_n) + \frac{(x_1 - x'_1)^2}{24\tilde{n}_0(\zeta_n)} \left(\delta_1(\bar{x}_1) - \frac{\delta_2(\bar{x}_1)}{[\tilde{n}_0(\zeta_n)]^2} \right) + \dots \right\}, \quad (\text{A15})$$

while

$$\begin{aligned} [\tilde{n}(x_1; \zeta)\tilde{n}(x'_1; \zeta)]^{-1/2} \sim & [n(\bar{x}_1)\tilde{n}_0(\zeta_n)]^{-1} \\ & \times \left\{ 1 - \frac{1}{8}(x_1 - x'_1)^2 \left(\frac{\delta_1(\bar{x}_1)}{[\tilde{n}_0(\zeta_n)]^2} - \frac{2\delta_2(\bar{x}_1)}{[\tilde{n}_0(\zeta_n)]^4} \right) + \dots \right\} \end{aligned} \quad (\text{A16})$$

and

$$\frac{F(\xi_1; \zeta)}{\tilde{n}(\xi_1; \zeta)} \sim [n(\bar{x}_1)]^{-2} \left(\frac{2\delta_1(\bar{x}_1)}{[\tilde{n}_0(\zeta_n)]^4} - \frac{5\delta_2(\bar{x}_1)}{[\tilde{n}_0(\zeta_n)]^6} \right), \quad (\text{A17})$$

so that

$$\int_{\xi_1=x_1<}^{x_1>} F(\xi_1; \zeta) d\xi_1 \sim [n(\bar{x}_1)]^{-1} |x_1 - x'_1| \left(\frac{2\delta_1(\bar{x}_1)}{[\tilde{n}_0(\zeta_n)]^3} - \frac{5\delta_2(\bar{x}_1)}{[\tilde{n}_0(\zeta_n)]^5} \right). \tag{A18}$$

In anticipation of the asymptotic expansion of the characteristic Green's function represented by Eq.(4.14) in the limiting case under consideration, we introduce integrals Q_j^{\ll} over ζ as,

$$Q_j^{\ll}(x_1, x_3 - x'_3; x'_1) = \frac{i}{4\pi} \int_{\zeta \in \mathcal{Z}'} [\tilde{n}_0(\zeta_n)]^{-j} \exp[ik_0 n(\bar{x}_1) (|x_1 - x'_1| \tilde{n}_0(\zeta_n) + |x_3 - x'_3| \zeta_n)] d\zeta_n. \tag{A19}$$

Note that

$$Q_1^{\ll}(x_1, x_3 - x'_3; x'_1) = \frac{i}{4} H_0^{(1)} \left(k_0 n(\bar{x}_1) [(x_1 - x'_1)^2 + (x_3 - x'_3)^2]^{1/2} \right), \tag{A20}$$

while the Q_j^{\ll} satisfy the recursion relation

$$jQ_{j+2}^{\ll} - ik_0 n(\bar{x}_1) |x_1 - x'_1| Q_{j+1}^{\ll} = (j - 1)Q_j^{\ll} - [(x_1 - x'_1)\partial_1 + |x_3 - x'_3|\partial_3]Q_j^{\ll}. \tag{A21}$$

Equation (4.14) now leads to the expansion

$$\begin{aligned} G(x_1, x_3 - x'_3; x'_1) &\sim Q_1^{\ll} - \frac{\delta_1(\bar{x}_1)}{24 [k_0 n(\bar{x}_1)]^2} \left\{ -6 Q_5^{\ll} + 6ik_0 n(\bar{x}_1) |x_1 - x'_1| Q_4^{\ll} \right. \\ &\quad \left. - 3[ik_0 n(\bar{x}_1) |x_1 - x'_1|]^2 Q_3^{\ll} + [ik_0 n(\bar{x}_1) |x_1 - x'_1|]^3 Q_2^{\ll} \right\} - \\ &\frac{\delta_2(\bar{x}_1)}{24 [k_0 n(\bar{x}_1)]^2} \left\{ -15 Q_7^{\ll} + 15ik_0 n(\bar{x}_1) |x_1 - x'_1| Q_6^{\ll} \right. \\ &\quad \left. + 6[ik_0 n(\bar{x}_1) |x_1 - x'_1|]^2 Q_5^{\ll} - [ik_0 n(\bar{x}_1) |x_1 - x'_1|]^3 Q_4^{\ll} \right\} + \\ &\mathcal{O}(k_0^{-3}). \end{aligned} \tag{A22}$$

Substituting the recursion (A20)-(A21) into this equation, simplifies the expansion to

$$\begin{aligned} G(x_1, x_3 - x'_3; x'_1) &\sim \frac{i}{4} \left\{ \left[1 + \frac{-\delta_1(\bar{x}_1) + 2\delta_2(\bar{x}_1)}{12} (r_0(x_1, x'_1))^2 \right] B_0(k_0 n(\bar{x}_1) r_0(x_1, x'_1)) + \right. \\ &\left[4(\delta_1(\bar{x}_1) - 2\delta_2(\bar{x}_1)) + (-\delta_1(\bar{x}_1)(x_1 - x'_1)^2 + \delta_2(\bar{x}_1)(r_0(x_1, x'_1))^2)(k_0 n(\bar{x}_1))^2 \right] \times \\ &\quad \left. \frac{B_{-1}(k_0 n(\bar{x}_1) r_0(x_1, x'_1))}{24 (k_0 n(\bar{x}))^2} + \dots \right\}, \end{aligned} \tag{A23}$$

where the B_j are given by Eq.(6.30).

Case 3: $|x_1 - x'_1| = \mathcal{O}(k_0^{-1/2})$. We expand (cf. Eq.(A9))

$$\begin{aligned} \Phi(x_1, x'_1; \zeta) &= I_0(x_1, x'_1) \tilde{n}_0(\zeta_n) + \int_{\xi_1=x_1<}^{x_1>} \tilde{n}(\xi_1; \zeta) d\xi_1 - I_0(x_1, x'_1) [n(\bar{x}_1)]^{-1} \tilde{n}(\bar{x}_1; \zeta) \\ &\sim I_0(x_1, x'_1) \tilde{n}_0(\zeta_n) + \frac{1}{48} n(\bar{x}_1) |x_1 - x'_1|^3 \zeta_n^2 ([2 + \zeta_n^2] \delta_1(\bar{x}_1) - 4[1 + \zeta_n^2] \delta_2(\bar{x}_1)) + \dots, \end{aligned} \quad (\text{A24})$$

while

$$\begin{aligned} [\tilde{n}(x_1; \zeta) \tilde{n}(x'_1; \zeta)]^{-1/2} &\sim [n(\bar{x}_1) \tilde{n}_0(\zeta_n)]^{-1} \\ &\times \left\{ 1 - \frac{1}{8} (x_1 - x'_1)^2 ([1 + \zeta_n^2] \delta_1(\bar{x}_1) - 2[1 + 2\zeta_n^2] \delta_2(\bar{x}_1)) + \dots \right\} \end{aligned} \quad (\text{A25})$$

and

$$\int_{\xi_1=x_1<}^{x_1>} F(\xi_1; \zeta) d\xi_1 \sim F(\bar{x}_1; 0) |x_1 - x'_1| \sim [n(\bar{x}_1)]^{-1} |x_1 - x'_1| (2\delta_1(\bar{x}_1) - 5\delta_2(\bar{x}_1)). \quad (\text{A26})$$

Also, note that

$$\begin{aligned} \zeta |x_3 - x'_3| &= \zeta_n \nu(x_1, x'_1) |x_3 - x'_3| + \zeta_n n(\bar{x}_1) [1 - n^{-1}(\bar{x}_1) \nu(x_1, x'_1)] |x_3 - x'_3| \\ &\sim \zeta_n \nu(x_1, x'_1) |x_3 - x'_3| - \frac{1}{24} \zeta_n n(\bar{x}_1) (x_1 - x'_1)^2 (\delta_1(\bar{x}_1) - 2\delta_2(\bar{x}_2)) |x_3 - x'_3|. \end{aligned} \quad (\text{A27})$$

In anticipation of the asymptotic expansion of the characteristic Green's function in the limiting case under consideration, we introduce integrals Q_j^\sim as,

$$Q_j^\sim(x_1, x_3 - x'_3; x'_1) = \frac{i}{4\pi} \int_{\zeta \in \mathcal{Z}'} \zeta_n^{j-1} \exp[ik_0 \nu(x_1, x'_1) (\chi_1(x_1, x'_1) \tilde{n}_0(\zeta_n) + |x_3 - x'_3| \zeta_n)] d\zeta_n. \quad (\text{A28})$$

As before Eq.(4.14) then leads to the expansion

$$\begin{aligned} G(x_1, x_3 - x'_3; x'_1) &\sim \left\{ 1 - \frac{1}{8} (x_1 - x'_1)^2 - \frac{i|x_1 - x'_1|}{8 k_0 n(\bar{x}_1)} (2\delta_1(\bar{x}_1) - 5\delta_2(\bar{x}_1)) \right\} Q_1^\sim - \\ &\frac{1}{24} i k_0 n(\bar{x}_1) (x_1 - x'_1)^2 (\delta_1(\bar{x}_1) - 2\delta_2(\bar{x}_1)) Q_2^\sim + \\ \frac{1}{24} (x_1 - x'_1)^2 &\left\{ i k_0 n(\bar{x}_1) |x_1 - x'_1| (\delta_1(\bar{x}_1) - 2\delta_2(\bar{x}_1)) - 3\delta_1(\bar{x}_1) + 6\delta_2(\bar{x}_1) \right\} Q_3^\sim + \\ &\frac{1}{48} i k_0 n(\bar{x}_1) |x_1 - x'_1|^3 (\delta_1(\bar{x}_1) - 4\delta_2(\bar{x}_1)) Q_5^\sim + \\ &\mathcal{O}(k_0^{-2}). \end{aligned} \quad (\text{A29})$$

The Q_j^\sim in this expansion are known in closed form. However, to the order of approximation considered here, we can approximate these integrals by the following expressions,

$$Q_1^\sim(x_1, x_3 - x'_3; x'_1) = \frac{i}{4} H_0^{(1)}(k_0 \nu(x_1, x'_1) r(x_1, x'_1)),$$

$$\begin{aligned}
 Q_2^\sim(x_1, x_3 - x'_3; x'_1) &= \left| \frac{x_3 - x'_3}{x_1 - x'_1} \right| \left\{ 1 + \mathcal{O}(k_0^{-1/2}) \right\} Q_1^\sim(x_1, x_3 - x'_3; x'_1), \\
 Q_3^\sim(x_1, x_3 - x'_3; x'_1) &= \left\{ \frac{-i}{k_0 n(\bar{x}_1) |x_1 - x'_1|} + \frac{1}{2 [k_0 n(\bar{x}_1) |x_1 - x'_1|]^2} \right. \\
 &\quad \left. + \left(\frac{x_3 - x'_3}{x_1 - x'_1} \right)^2 + \mathcal{O}(k_0^{-3/2}) \right\} Q_1^\sim(x_1, x_3 - x'_3; x'_1), \\
 Q_5^\sim(x_1, x_3 - x'_3; x'_1) &= \frac{-3}{[k_0 n(\bar{x}_1) |x_1 - x'_1|]^2} \left\{ 1 + \mathcal{O}(k_0^{-1/2}) \right\} Q_1^\sim(x_1, x_3 - x'_3; x'_1),
 \end{aligned}$$

where r has been replaced by

$$r(x_1, x'_1) = |x_3 - x'_3| \left\{ 1 + \mathcal{O}(k_0^{-1}) \right\}, \tag{A30}$$

everywhere except in the arguments of the Hankel functions, and

$$H_1^{(1)}(y) = -iH_0^{(1)}(y) \left\{ 1 + \frac{i}{2y} + \mathcal{O}(y^{-2}) \right\}. \tag{A31}$$

Uniform expansion. The uniform expansion is then given by Eq.(6.29). To verify the inner and outer expansions of this expression, we need to consider the following approximations.

Outer expansion: then we have

$$\begin{aligned}
 B_0(k_0\nu(x_1, x'_1) r(x_1, x'_1)) &\sim \left(\frac{2}{\pi} \right)^{1/2} (-i)^{1/2} \frac{\exp[k_0\nu(x_1, x'_1) \chi_1(x_1, x'_1)]}{[k_0\nu(x_1, x'_1) \chi_1(x_1, x'_1)]^{1/2}} \left\{ \right. \\
 &\quad \left. 1 + \frac{k_0\nu(x_1, x'_1) (x_3 - x'_3)^2}{2 \chi_1(x_1, x'_1)} - \frac{i}{8\nu(x_1, x'_1) \chi_1(x_1, x'_1)} \right\}, \tag{A32}
 \end{aligned}$$

$$\begin{aligned}
 B_{-1}(k_0\nu(x_1, x'_1) r(x_1, x'_1)) &\sim \\
 &\quad \left(\frac{2}{\pi} \right)^{1/2} (-i)^{3/2} [k_0\nu(x_1, x'_1) \chi_1(x_1, x'_1)]^{1/2} \exp[k_0\nu(x_1, x'_1) \chi_1(x_1, x'_1)], \tag{A33}
 \end{aligned}$$

with $|x_3 - x'_3| = \mathcal{O}(k_0^{-1})$.

Inner expansion: then we have

$$\begin{aligned}
 B_0(k_0\nu(x_1, x'_1) r(x_1, x'_1)) &\sim B_0(k_0\nu(x_1, x'_1) r_0(x_1, x'_1)) - \\
 &\quad \left(\delta_1(\bar{x}_1) - 2\delta_2(\bar{x}_1) + \frac{\delta_2(\bar{x}_1)(x_1 - x'_1)^2}{(r_0(x_1, x'_1))^2} \right) B_{-1}(k_0\nu(x_1, x'_1) r_0(x_1, x'_1)), \tag{A34}
 \end{aligned}$$

$$B_{-1}(k_0\nu(x_1, x'_1) r(x_1, x'_1)) \sim B_{-1}(k_0\nu(x_1, x'_1) r_0(x_1, x'_1)), \tag{A35}$$

while

$$\frac{\nu(x_1, x'_1)}{[n(x_1)n(x'_1)]^{1/2}} \sim 1 - \frac{1}{12}(\delta_1(\bar{x}_1) - 2\delta_2(\bar{x}_1))(x_1 - x'_1)^2, \tag{A36}$$

$$\frac{\beta_j(x_1, x'_1)}{(\chi_1(x_1, x'_1))^2} \sim \beta_j^0(\bar{x}_1), \tag{A37}$$

$$\frac{b_{-1}(x_1, x'_1)}{(\nu(x_1, x'_1) \chi_1(x_1, x'_1))^2} \sim \frac{1}{6n^2(\bar{x}_1)} (\delta_1(\bar{x}_1) - 2\delta_2(\bar{x}_1)), \tag{A38}$$

where the β_j^0 are given in Section 6 of the main text.

Overlapping region: use Eqs.(A36)-(A38) together with Eqs.(A30)-(A31).

The error in our expansion is $\mathcal{O}(k_0^{-2})$ uniformly in x_1 and x'_1 . More precisely, the error is $\mathcal{O}(k_0^{-5/2})$ on the outer region, $\mathcal{O}(k_0^{-3})$ on the inner region, and $\mathcal{O}(k_0^{-9/4})$ on the overlapping region.

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