UNIFORM ASYMPTOTIC EXPANSION OF THE GENERALIZED BREMMER SERIES

MAARTEN V. DE HOOP* AND A.K. GAUTESEN†

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Abstract

The Bremmer coupling series solution of the wave equation, in generally inhomogeneous media, requires the introduction of pseudo-differential operators. In this paper, in two dimensions, uniform asymptotic expansions of the Schwartz kernels of these operators are derived. Also, we derive a uniform asymptotic expansion of the one-way propagator appearing in the series. We focus on designing closed-form representations, valid in the high-frequency limit, taking into account critical scattering-angle phenomena. Our expansion is not limited by propagation angle. In principle, the uniform asymptotic expansion of a kernel follows by matching its asymptotic behaviors away and near its diagonal. The Bremmer series solver consists of three steps: directional decomposition into up- and downgoing waves, one-way propagation, and interaction of the counter-propagating constituents. Each of these steps is here represented by a kernel for which a uniform asymptotic expansion is found. The associated algorithm provides a fundamental improvement of the parabolic-equation and phase-shift/phase-screen style methods applied in ocean acoustics, integrated optics and exploration seismology.

*Center for Wave Phenomena and Department of Mathematical and Computer Sciences, Colorado School of Mines, Golden, CO 80401-1887, USA (mdehoop@dix.Mines.EDU). M.V. d.H. would like to thank Mobil for financial support of this research.

†Applied Mathematical Sciences, Ames Laboratory and Department of Mathematics, Iowa State University, Ames, IA 50011-3020, USA (gautesen@ameslab.gov). The work of A.K. G. was supported by the Applied Mathematical Sciences subprogram of the Office of Energy Research of the United States Department of Energy under contract No. W-7405-Eng-82.

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1 Introduction

Directional wave field decomposition is a tool for analyzing and computing wave propagation in configurations with a special directionality, such as the waveguiding structure. Such method consists of three main steps: (i) decomposing the field into two constituents, propagating upward or downward along a preferred direction, (ii) computing the interaction of the counterpropagating constituents and (iii) recomposing the constituents into observables at the positions of interest. The Bremmer series [1] then synthesizes the constituents into a full wave solution. The series representation is illustrated in Figure 1.1. Each term in the series represents a wave constituent that has traveled up and down a number of times equal to its order.

Applications of the generalized Bremmer series solution to the wave equation include (i) the identification of multiple scattered wave constituents, and (ii) the formulation of various imaging and inverse scattering procedures in remote sensing. In general, the inverse scattering problem can be decomposed into a coupled inverse (contrast) source – inverse constituency problem. With the aid of time-reversed mirrors, successive terms in the Bremmer series can be exploited to construct the (contrast) source (De Hoop [2]). On the other hand, the generalized Bremmer series can be viewed as a full-wave extension of the (high-frequency) geometrical ray series representation of the wave field. As such, the Bremmer series can be exploited to formulate the full-wave analogue of the high-frequency Generalized Radon Transform inversion procedure decribed by De Hoop and Bleistein [3]. In the latter framework, the attraction of the generalized Bremmer series based inversion procedure is the natural inclusion of caustics. Extensive lists of references to applications of the Bremmer series in exploration seismics, ocean acoustics and integrated optics can be found in De Hoop [1], Van Stralen, De Hoop and Blok [4] and Fishman, Gautesen and Sun [5].

De Hoop [1] originally formulated the generalized Bremmer series modeling method in the time-*Laplace* domain. Owing to the fact that the medium can vary in the direction tranverse to the preferred direction, pseudodifferential calculus became a neccesary tool to introduce the up- and downgoing Green's functions: pseudodifferential operators appear in the directional (de)composition, in the downward and upward propagation or continuation, and in the interaction (reflection and transmission) between the counterpropagating constituents due to variations in medium properties in the preferred direction.

Various approaches have been developed over the years to approximate the operators appearing in the Bremmer series to make *numerical* computations feasible. An overview of the approaches based on rational (paraxial) expansions of the operator symbols can be found in Van Stralen, De Hoop and Blok [4]. An overview of approaches based on phase-screen-like approximations of the operator symbols can be found in De Hoop, Wu and Le Rousseau [6]. With these numerical approaches, however, critical scattering-angle phenomena cannot be modeled.

In this paper, our goal is to gain *analytic* insight into the propagation and scattering of waves as described by the generalized Bremmer series – and at the same time developing a time-*Fourier* analysis of the constituent operators. Thus, instead of using pseudodifferential operators in the time-Laplace domain, we will employ spectral theory in the time-Fourier domain. The advantage is that we can then derive uniform asymptotic expansions of both the scattering and propagation operator kernels. A key reference in this respect is the paper by Fishman, Gautesen and Sun [5]. The authors of this paper constructed uniform asymptotic approximations of operator 'symbols', whereas here we consider uniform asymptotic approximations of operator kernels. Our approach and the approach of Fishman, Gautesen and Sun are complementary also in as much as they propose the Dirichlet-to-Neumann map as the tool to model solutions whereas we propose a series solution allowing greater insight into the multiple-scattering process.

The uniform asymptotic expansions also provide the basis for a numerical scheme. Such a scheme would involve the computations of (i) a spatially varying effective index of refraction and (ii) a spatially varying effective 'distance' in the transverse direction, and (iii) applying the kernel. The accuracy of such a numerical scheme does surpass the one following from the rational-expansion or screen-approximation approaches, not only because critical scattering-angle phenomena are included in our uniform asymptotic expansions.

Our current theory is based on one-dimensional WKB solutions and as such applies to *two-dimensional* configurations. We consider acoustic waves in fluids. A paper on the three-dimensional case based on two-dimensional uniform asymptotic expansions is in preparation.

The outline of this paper is as follows. In the next section a summary of the method of directional decomposition, leading to a coupled system of one-way wave equations is given. In Section 3, the medium is decomposed into thin slabs. In each thin slab we introduce a 'characteristic' Green's function. In Sections 4 and 5 we discuss characteristic Green's function (integral) representations for the operators arising in the directional decomposition. In Section 6 the uniform asymptotic expansion is introduced, and applied to the (de)composition and interaction operators and the one-way propagator in Sections 7, 8 and 9, respectively. In Section 10, finally, the Bremmer series solution procedure is summarized. The proof of the uniform asymptotic expansion discussed in Section 6 is given in the Appendix.

2 Directional wave field decomposition

For the details on the derivation of the Bremmer coupling series solution of the acoustic wave equation, we refer the reader to De Hoop [1]. Here, we restrict ourselves to a summary of this *wave field* decomposition method.

Notation, transformations

We consider acoustic waves in a two-dimensional configuration. In this configuration, let p denote the pressure and (v_1, v_3) the particle velocity. We introduce the Fourier transformation with respect to time t as

$$(2.1)\qquad (\mathcal{F}\{p, v_1, v_3\})(x_1, x_3, \omega) = \int_{t \in \mathbb{R}_{\geq 0}} \{p, v_1, v_3\}(x_1, x_3, t)\ \exp(\mathrm{i}\omega t)\,\mathrm{d}t$$

for $\mathrm{Im}\{\omega\} > 0$. Under this transformation, assuming zero initial conditions, we have $\partial_t \to -\mathrm{i}\omega$.

In each subdomain of the configuration where the acoustic properties vary continuously with position, the acoustic wave field $\{p, v_1, v_3\}$ satisfies the system of partial differential equations

$$(2.2)\qquad \partial_k p - \mathrm{i}\omega\rho\, v_k = f_k\ ,$$

$$(2.3)\qquad -\mathrm{i}\omega\kappa\, p + \partial_1 v_1 + \partial_3 v_3 = q\ .$$

Here, ρ denotes the volume density of mass, κ the compressibility, q the volume source density of injection rate, and f_k the volume source density of force.

The spatial variation of the wave field along a direction of preference can now be expressed in terms of the variation of the wave field in the direction perpendicular to it. The direction of preference is taken along the x_3-axis (or 'vertical' axis) and the remaining ('transverse' or 'horizontal') coordinate is denoted by x_1.

The reduced system of equations

Directional decomposition requires a separate handling of the horizontal or transverse component of the particle velocity. From Eqs.(2.2) and (2.3) we obtain

$$(2.4)\qquad v_1 = -\mathrm{i}\rho^{-1}\omega^{-1}(\partial_1 p - f_1)\ ,$$

leaving, upon substitution, the matrix differential equation

$$(2.5)\qquad (\partial_3\delta_{IJ} - \mathrm{i}\omega A_{IJ})\, F_J = N_I\ , \quad A_{IJ} = A_{IJ}(x_1, D_1; x_3)\ , \quad D_1 \equiv -\frac{\mathrm{i}}{\omega}\partial_1\ ,$$

in which the elements of the acoustic field matrix[1] are given by

$$(2.6)\qquad F_1 = p\ , \quad F_2 = v_3\ ,$$

[1]Present ocean-bottom seismic acquisition technology allows both p and v_3 to be measured.

the elements of the acoustic system's matrix operator by

$$A_{11} = A_{22} = 0 \,, \tag{2.7}$$

$$A_{12} = \rho \,, \tag{2.8}$$

$$A_{21} = -D_1(\rho^{-1} D_1\,) + \kappa \,, \tag{2.9}$$

and the elements of the notional source matrix by

$$N_1 = f_3, \quad N_2 = D_1(\rho^{-1} f_1) + q \,. \tag{2.10}$$

It is observed that the right-hand side of Eq.(2.4) and A_{IJ} contain the spatial derivative D_1 with respect to the horizontal coordinate only. In the sequel of the paper it will become clear that D_1 has the interpretation of *horizontal slowness* operator. Further, it is noted that A_{12} is simply a multiplicative operator.

The coupled system of one-way wave equations

To distinguish up- and downgoing constituents in the wave field, we shall construct an appropriate linear operator L_{IJ} with

$$F_I = L_{IJ} W_J \,, \tag{2.11}$$

that, with the aid of the commutation relation ($[.,.]$ denotes the commutator)

$$(\partial_3 L_{IJ}) = [\partial_3, L_{IJ}] \tag{2.12}$$

transforms Eq.(2.5) into

$$L_{IJ}\,(\partial_3 \delta_{JM} - \mathrm{i}\omega \Lambda_{JM}) W_M = -(\partial_3 L_{IJ})\, W_J + N_I \,, \tag{2.13}$$

as to make Λ_{JM}, satisfying

$$A_{IJ} L_{JM} = L_{IJ} \Lambda_{JM} \,, \tag{2.14}$$

a diagonal matrix of operators. We denote L_{IJ} as the composition operator and W_M as the wave matrix. The expression in parentheses on the left-hand side of Eq.(2.13) represents the two so-called *one-way* wave operators. The first term on the right-hand side of Eq.(2.13) is representative for the scattering due to variations of the medium properties in the vertical direction. The diffraction due to variations of the medium properties in the horizontal directions is contained in Λ_{JM} and, implicitly, in L_{IJ}. This diffraction comprises the multi-pathing of characteristics that commonly occurs in geophysical configurations.

To investigate whether solutions of Eq.(2.14) exist, we introduce the column matrix operators $L_I^{(\pm)}$ according to

(2.15) $$L_I^{(+)} = L_{I1} \ , \quad L_I^{(-)} = L_{I2} \ .$$

Upon writing the diagonal elements of Λ_{JM} as

(2.16) $$\Lambda_{11} = \Gamma^{(+)} \ , \quad \Lambda_{22} = \Gamma^{(-)} \ ,$$

Eq.(2.14) decomposes into the two systems of equations

(2.17) $$A_{IJ} L_J^{(\pm)} = L_I^{(\pm)} \Gamma^{(\pm)} \ .$$

By analogy with the case where the medium is translationally invariant in the horizontal directions, we shall denote $\Gamma^{(\pm)}$ as the *vertical slowness* operators. Notice that the operators $L_1^{(\pm)}$ compose the acoustic pressure and that the operators $L_2^{(\pm)}$ compose the vertical particle velocity. Through mutual elimination, the equations for $L_1^{(\pm)}$ and $L_2^{(\pm)}$ can be decoupled as follows:

(2.18) $$A_{12} A_{21} L_1^{(\pm)} = L_1^{(\pm)} \Gamma^{(\pm)} \, \Gamma^{(\pm)} \ ,$$

(2.19) $$A_{21} A_{12} L_2^{(\pm)} = L_2^{(\pm)} \Gamma^{(\pm)} \, \Gamma^{(\pm)} \ .$$

The partial differential operators on the left-hand sides differ from one another in the case where the volume density of mass does vary in the horizontal directions.

To ensure that non-trivial solutions of Eqs.(2.18)-(2.19) exist, one equation must imply the other. To construct a formal solution, an Ansatz is introduced in the form of a commutation relation for one of the components $L_J^{(\pm)}$ that restricts the freedom in the choice for the other component. In the *acoustic-pressure normalization* analog one assumes that $L_2^{(\pm)}$ can be chosen such that

(2.20) $$[A_{12} L_2^{(\pm)}, A_{12} A_{21}] = 0 \ .$$

In view of Eq.(2.19) the $\Gamma^{(\pm)}$ must then satisfy

(2.21) $$A_{12} A_{21} - \Gamma^{(\pm)} \, \Gamma^{(\pm)} = 0 \ .$$

The commutation relation for $L_1^{(\pm)}$ follows as $[L_1^{(\pm)}, A_{12} A_{21}] = 0$ and a possible solution of Eq.(2.17) is

(2.22) $$L_2^{(\pm)} = A_{12}^{-1} \Gamma^{(\pm)} \ , \ L_1^{(\pm)} = I \ .$$

Since $L_2^{(\pm)}$ as given by Eq.(2.22) satisfies Eq.(2.20), the Ansatz is justified. The solutions of Eq.(2.21) are written as

(2.23) $$\Gamma^{(+)} = -\Gamma^{(-)} = \Gamma = \mathrm{A}^{1/2} \ \text{ with } \ \mathrm{A} = A_{12} A_{21} \ .$$

Thus, the *composition* operator becomes

$$L = \begin{pmatrix} I & I \\ A_{12}^{-1}\Gamma & -A_{12}^{-1}\Gamma \end{pmatrix} . \tag{2.24}$$

Note that we have decomposed the pressure field according to

$$F_1 = F_1^{(+)} + F_1^{(-)} \text{ with } F_1^{(+)} = W_1 \ , \ \ F_1^{(-)} = W_2 \ .$$

In terms of the inverse vertical slowness operator, $\Gamma^{-1} = \mathrm{A}^{-1/2}$, the *decomposition* operator then follows as

$$L^{-1} = \tfrac{1}{2} \begin{pmatrix} I & \Gamma^{-1}A_{12} \\ I & -\Gamma^{-1}A_{12} \end{pmatrix} . \tag{2.25}$$

Using the decomposition operator, Eq.(2.13) transforms into

$$(\partial_3 \delta_{IM} - \mathrm{i}\omega \Lambda_{IM})\, W_M = -(L^{-1})_{IM}(\partial_3 L_{MJ})W_J + (L^{-1})_{IM} N_M \ , \tag{2.26}$$

which can be interpreted as a coupled system of one-way wave equations. The propagation is captured by the left-hand side. The coupling between the counter-propagating components, W_1 and W_2, is apparent in the first source-like term on the right-hand side, see Figure 1.1. The waves are excited by the second term on the right-hand side. We have

$$-L^{-1}(\partial_3 L) = \begin{pmatrix} T & R \\ R & T \end{pmatrix} , \tag{2.27}$$

in which T and R represent the *transmission* and *reflection* operators, respectively: let $Y = A_{12}^{-1}\Gamma$ denote the *admittance* operator, then

$$R = -T = \tfrac{1}{2} Y^{-1} (\partial_3 Y) \ . \tag{2.28}$$

The two-way Helmholtz equation

Suppose that the medium does *not* vary with x_3. Eliminating F_2 or v_3 from Eqs.(2.5) then leads to the second-order equation for the pressure,

$$[\partial_3^2 + \omega^2 \mathrm{A}(x_1, D_1)]\, F_1 = \mathrm{i}\omega\, \rho N_2 + \partial_3 N_1 \ , \tag{2.29}$$

the *two-way* Helmholtz equation, where A is given by Eq.(2.23).

3 Decomposition of the configuration into thin slabs

In preparation of the development of the Bremmer series representation, we will now decompose the *medium* into (thin) slabs. Each slab in our 2-dimensional configuration is assumed to be *invariant* in the direction of preference, x_3: the compressibility, κ, may vary in the transverse direction, whereas the density is assumed to be constant all together, see Figure 3.1. However, the medium may vary from slab to slab, and hence the vertical coordinate x_3 becomes a parameter that identifies the slab in our further analysis.

The Bremmer series evaluation consists of three main steps (i) decomposing the field into two constituents and propagating these constituents upward or downward along the preferred direction, (ii) letting the counterpropagating constituents interact, and (iii) recomposing the constituents into observables at the positions of interest. Step (i) will be carried out in each slab separately, and we will then accumulate the contributions from the slabs composing a stack; step (ii) will be evaluated at any boundary separating neighboring slabs; step (iii) again will be carried out in each slab individually.

The characteristic operator

As mentioned, in our thin-slab analysis, we will consider the following medium profile,

$$\rho = \text{const.} , \tag{3.1}$$

$$\kappa(x_1) = \kappa_0 \, n^2(x_1) \tag{3.2}$$

thus, setting $\kappa_0 = \rho^{-1} c_0^{-2}$, the wave speed follows from

$$c^{-2}(x_1) = c_0^{-2} n^2(x_1) ,$$

where n denotes the index of refraction. The operator in Eq.(2.23) is then given by

$$\mathrm{A}(x_1, D_1) = -D_1^2 + c_0^{-2} n^2(x_1) . \tag{3.3}$$

We will denote A as the transverse Helmholtz or *characteristic* operator.

Factorization, Green's functions

Introduce the well-known Helmholtz-equation Green's function as (cf. Eq.(2.29))

$$[\partial_3^2 + \omega^2 \mathrm{A}(x_1, D_1)] \, G(x_1, x_3 - x_3'; x_1') = -\delta(x_1 - x_1')\delta(x_3 - x_3') . \tag{3.4}$$

The vertical slowness operators $\Gamma^{(\pm)}$ factorize the Helmholtz operator (cf. Eq.(2.23))

$$\partial_3^2 + \omega^2 \mathrm{A}(x_1, D_1) = [\partial_3 - \mathrm{i}\omega \, \Gamma^{(+)}(x_1, D_1)] \, [\partial_3 - \mathrm{i}\omega \, \Gamma^{(-)}(x_1, D_1)] . \tag{3.5}$$

The one-way Green's functions $\mathcal{G}^{(\pm)}$ associated with the two factors satisfy

$$[\partial_3 - \mathrm{i}\omega\,\Gamma^{(\pm)}(x_1, D_1)]\,\mathcal{G}^{(\pm)}(x_1, x_3 - x_3'; x_1') = \delta(x_1 - x_1')\delta(x_3 - x_3') \ . \tag{3.6}$$

To arrive at a uniform asymptotic description of the generalized Bremmer series, we have to find uniform asymptotic representations for L^{-1} (i.e. Γ^{-1}), $\mathcal{G}^{(\pm)}$, R and L (i.e. Γ). This is the subject of the remaining sections. Also, Γ determines the thin-slab *Dirichlet-to-Neumann map*, defined by the admittance operator Y [5].

4 The vertical slowness or square-root operator

The vertical slowness or square-root operator Γ (Eq.(2.23)) acts on the wave field as

$$(\Gamma\{W_1, W_2\})(x_1) = \int_{x_1' \in \mathbb{R}} \mathcal{C}(x_1, x_1')\,\{W_1, W_2\}(x_1')\,\mathrm{d}x_1' \ , \tag{4.1}$$

where $\mathcal{C}$ denotes a Schwartz kernel. From this operator representation, we extract the left vertical slowness symbol through the Fourier transformation

$$\gamma(x_1, p_1) = \int_{x_1' \in \mathbb{R}} \mathcal{C}(x_1, x_1') \exp[-\mathrm{i}\omega(x_1 - x_1')p_1]\,\mathrm{d}x_1' \ . \tag{4.2}$$

The left symbol of the horizontal slowness operator D_1 appears to be simply p_1. The relation between the left vertical slowness symbol and the horizontal slowness symbol constitutes the generalized slowness surface.

In the remainder of this section we will focus on finding integral representations for the Schwartz kernel.

Green's function representation

To begin with, the Schwartz kernel can be expressed in terms of the one-way Green's function,

$$\mathcal{C}^{(+)}(x_1, x_1'; x_3') = -\lim_{x_3 \downarrow x_3'} \frac{\mathrm{i}}{\omega}\partial_3 \mathcal{G}^{(+)}(x_1, x_3 - x_3'; x_1') \ , \tag{4.3}$$

$$\mathcal{C}^{(-)}(x_1, x_1'; x_3') = -\lim_{x_3 \uparrow x_3'} \frac{\mathrm{i}}{\omega}\partial_3 \mathcal{G}^{(-)}(x_1, x_3 - x_3'; x_1') \ . \tag{4.4}$$

Using the image principle, we can express the one-way Green's functions in terms of the Green's function of the second-order Helmholtz equation,

$$\mathcal{G}^{(+)}(x_1, x_3 - x_3'; x_1') + \mathcal{G}^{(-)}(x_1, x_3 - x_3'; x_1') = -2\,\partial_3 G(x_1, x_3 - x_3'; x_1') \ . \tag{4.5}$$

Hence, for $x_3 > x_3'$,

$$\mathcal{G}^{(+)}(x_1, x_3 - x_3'; x_1') = -2\,\partial_3 G(x_1, x_3 - x_3'; x_1') \ , \tag{4.6}$$

so that with Eq.(4.3)

$$\mathcal{C}(x_1, x_1'; x_3') = -\lim_{x_3 \downarrow x_3'} \frac{2}{\mathrm{i}\omega}\,\partial_3^2 G(x_1, x_3 - x_3'; x_1') \ . \tag{4.7}$$

Note that $\mathcal{C}$ is dependent on x_3' through the index of refraction. We will suppress this dependence in our notation.

In fact, $\mathcal{G} \equiv \mathcal{G}^{(+)}$ is the kernel of the (upward) one-way wave propagator. In view of Eq.(4.6) this kernel satisfies the property

$$\partial_3^{2j}\mathcal{G} = [-\omega^2 \mathrm{A}(x_1, D_1)]^j \mathcal{G} \ , \qquad j = 1, 2, \cdots \ , \tag{4.8}$$

for $x_3 > x_3'$.

Fourier-integral representation of the vertical slowness kernel

We will cast Eq.(4.7) into a spatial Fourier representation. To this end, note that the Fourier representation of the causal Green's function G yields

$$G(x_1, x_3 - x_3'; x_1') = \frac{\omega}{2\pi c_0} \int_{\zeta \in \mathcal{Z}} \tilde{G}(x_1, x_1'; \zeta)\ \exp[\mathrm{i}(\omega/c_0)\,|x_3 - x_3'|\zeta]\,\mathrm{d}\zeta \ . \tag{4.9}$$

Since

$$\omega^2 \mathrm{A}(x_1, D_1) = \partial_1^2 + (\omega/c_0)^2\, n^2(x_1) \ , \tag{4.10}$$

$\tilde{G}$ satisfies (Eq.(3.4))

$$[\partial_1^2 + (\omega/c_0)^2\,(n^2(x_1) - \zeta^2)]\,\tilde{G}(x_1, x_1'; \zeta) = -\delta(x_1 - x_1') \ , \tag{4.11}$$

or, more formally,

$$-\omega^2[\mathrm{A}(x_1, D_1) - c_0^{-2}\zeta^2]\,\tilde{G}(x_1, x_1'; \zeta) = \delta(x_1 - x_1') \ . \tag{4.12}$$

Observe the symmetry $\tilde{G}(x_1, x_1'; -\zeta) = \tilde{G}(x_1, x_1'; \zeta)$. Also note that Eq.(4.11) is an ordinary differential equation, i.e., a differential equation in a space of dimension one less than the one of the original configuration. The contour $\mathcal{Z}$ follows the real axis in the complex ζ-plane, viz.,

$$\mathcal{Z} = \{\zeta + \mathrm{i}0 \mid \zeta \in \mathbb{R}_{<0}\} \cup \{0\} \cup \{\zeta - \mathrm{i}0 \mid \zeta \in \mathbb{R}_{>0}\} \ ,$$

see Figure 4.1. The solution to Eq.(4.11) can in generality be represented by

$$\tilde{G}(x_1, x_1'; \zeta) = U_1(x_{1>}; \zeta)\, U_2(x_{1<}; \zeta) \ , \tag{4.13}$$

where $x_{1<} = \min(x_1, x_1')$, $x_{1>} = \max(x_1, x_1')$, and U_1 and U_2 are homogeneous solutions of Eq.(4.11).

Our aim will be to replace the exact solutions U_1 and U_2 by uniform asymptotic approximations for $\zeta \in \mathcal{Z}$. In this respect note the occurrence of *turning points* at $\zeta = \pm n(x_1)\,,\, \pm n(x_1')$. Fishman, Gautesen and Sun [5] have shown that in the second and the fourth quadrants of the complex ζ-plane $\tilde{G}(x_1, x_1'; \zeta)$ is analytic and approaches zero as $|\zeta| \to \infty$. Therefore the contour in Eq.(4.9) can be deformed to the contour $\mathcal{Z}'$ on which the distance from a turning point is always greater than some finite number, as shown in Figure 4.1. Then Eq.(4.9) becomes

(4.14) $$G(x_1, x_3 - x_3'; x_1') = \frac{k_0}{2\pi} \int_{\zeta \in \mathcal{Z}'} \exp[\mathrm{i}k_0\,|x_3 - x_3'|\zeta]\,\tilde{G}(x_1, x_1'; \zeta)\,\mathrm{d}\zeta$$

where the background wave number is

(4.15) $$k_0 \equiv \omega / c_0 \ .$$

Substituting Eq.(4.14) into Eq.(4.7) yields

(4.16) $$\mathcal{C}(x_1, x_1') = -\frac{1}{\pi\mathrm{i}\,c_0} \lim_{x_3 \downarrow 0} \partial_3^2 \int_{\zeta \in \mathcal{Z}'} \exp[\mathrm{i}k_0\,x_3\zeta]\,\tilde{G}(x_1, x_1'; \zeta)\,\mathrm{d}\zeta \ .$$

In this integral, in distributional sense, we can use

(4.17) $$\lim_{x_3 \downarrow 0} \partial_3^2 \exp[\mathrm{i}k_0\,x_3\zeta] = -k_0^2\zeta^2$$

so that

(4.18) $$\mathcal{C}(x_1, x_1') = \frac{k_0^2}{\pi\mathrm{i}\,c_0} \int_{\zeta \in \mathcal{Z}'} \zeta^2 \tilde{G}(x_1, x_1'; \zeta)\,\mathrm{d}\zeta \ .$$

Fourier-integral representation of the one-way propagator

Substituting Eq.(4.14) into Eq.(4.6) leads to a Fourier-integral representation of the one-way propagator,

(4.19)

$$\mathcal{G}(x_1, x_3 - x_3'; x_1') = -H(x_3 - x_3')\partial_3\left[\frac{k_0}{\pi} \int_{\zeta \in \mathcal{Z}'} \tilde{G}(x_1, x_1'; \zeta)\ \exp[\mathrm{i}k_0\,|x_3 - x_3'|\zeta]\,\mathrm{d}\zeta\right] ,$$

where H denotes the Heaviside function. In this expression, in distributional sense, we can use

(4.20) $$\partial_3 \exp[\mathrm{i}k_0\,|x_3 - x_3'|\zeta] = \mathrm{i}k_0\,\zeta \exp[\mathrm{i}k_0\,|x_3 - x_3'|\zeta] \qquad \text{for} \qquad x_3 > x_3' \ .$$

To arrive at our uniform asymptotic representation of G, we will have to make the assumption that the propagation distance satisfies

(4.21) $$k_0\,|x_3 - x_3'| = \mathcal{O}(1)$$

to guarantee that the stationary point of the integral representation remains at $\zeta = 0$, and that

(4.22) $$|\exp[\mathrm{i}k_0\,|x_3 - x_3'|\zeta]| = \mathcal{O}(1) \qquad \text{for} \qquad \zeta \in \mathcal{Z}' \ .$$

5 General fractional powers of A

Equation (4.16) is a special form of the Dunford integral representation for powers of A. To arrive at this general representation, let R_λ denote the *resolvent* of A, i.e.,

$$(\mathrm{A} - \lambda\,\mathrm{I})\,R_\lambda = \mathrm{I}\ .$$

Then the resolvent kernel $\mathcal{R}_\lambda$ must satisfy the partial differential equation

$$[\mathrm{A}(x_1, D_1) - \lambda\,\mathrm{I}]\,\mathcal{R}_\lambda(x_1, x_1') = \delta(x_1 - x_1')\ . \tag{5.1}$$

Upon comparing this equation with Eq.(4.12), in an x_3-invariant profile, it follows that the resolvent kernel $\mathcal{R}_\lambda$ is proportional to the Green's function and in fact equals $-\omega^2\tilde{G}$: map

$$\lambda \leftrightarrow c_0^{-2}\zeta^2\ :\ -\mathcal{L} \leftrightarrow \mathcal{Z}'\ .$$

Instead of rewriting Eq.(4.18), we will consider integral representations for *general* fractional powers of A.

Negative fractional powers

Let the power λ^z of a complex variable λ with $z \in \mathbb{R}$ be defined as

$$\lambda^z = |\lambda|^z \exp[\mathrm{i}z\arg(\lambda)]\ , \tag{5.2}$$

with $\arg(\lambda) \in (-\pi, \pi)$. With this definition, the branch cut of λ^z is along the negative real axis. $\mathcal{L}$ is a contour of integration in the λ-plane around the spectrum of A, clockwise oriented, staying away a small but finite distance from the branch point. Then, for $z \in \mathbb{R}_{<0}$, the Dunford integral

$$\mathrm{A}_z = \frac{1}{2\pi\mathrm{i}}\int_{\lambda\in\mathcal{L}} \lambda^z R_\lambda\,\mathrm{d}\lambda \tag{5.3}$$

converges on a Sobolev space. The kernel of A_z is given by

$$\begin{aligned}\mathcal{A}_z(x_1, x_1') &= \frac{1}{2\pi\mathrm{i}}\int_{\lambda\in\mathcal{L}} \lambda^z \mathcal{R}_\lambda(x_1, x_1')\,\mathrm{d}\lambda \\ &= \frac{k_0^2}{\pi\mathrm{i}\ c_0^{2z}}\int_{\zeta\in\mathcal{Z}'} \zeta^{2z+1}\tilde{G}(x_1, x_1'; \zeta)\,\mathrm{d}\zeta\ ,\end{aligned} \tag{5.4}$$

which expression is consistent with the Green's function representation Eq.(4.18). The Dunford integral representation satisfies the composition equation

$$\mathrm{A}_z\,\mathrm{A}_w = \mathrm{A}_{z+w} \tag{5.5}$$

for $z, w \in \mathbb{R}_{<0}$.

Non-negative fractional powers

With the aid of Eq.(5.3) a non-negative fractional power of A can be readily introduced through

(5.6) $$\mathrm{A}^z = \mathrm{A}^j \, \mathrm{A}_{z-j} \ ,$$

where j is an integer such that $j > z$. A similar representation for the associated kernels is found (note that the superscripts of $\mathcal{A}$ do not correspond to simple powers):

(5.7) $$\mathcal{A}^z(x_1, x_1') = \mathrm{A}^j(x_1, D_1) \, \mathcal{A}_{z-j}(x_1, x_1') \ .$$

The resulting operators behave, again, like ordinary powers, i.e.,

(5.8) $$\mathrm{A}^z \, \mathrm{A}^w = \mathrm{A}^{z+w}$$

(note that A and its resolvent commute). In particular,

(5.9) $$\Gamma = \mathrm{A}^{1/2} = \mathrm{A} \, \mathrm{A}_{-1/2} \ .$$

Note that the operator A takes over the role of ∂_3^2 in comparison with Eqs.(4.16) and (4.18):

(5.10) $$\mathrm{A}(x_1, D_1)(-\omega^2 \tilde{G}(x_1, x_1'; \zeta)) = -k_0^{-2} \zeta^2 \tilde{G}(x_1, x_1'; \zeta) + \delta(x_1 - x_1')$$

cf. Eq.(4.12). The regularization of the Schwartz kernels follows the canonical regularization of distributions.

For any medium profile for which the Green's function $G(x_1, x_3 - x_3'; x_1')$ is known in closed form, closed-form expressions for all the kernels relevant to the Bremmer series can be found.

6 Uniform asymptotic expansion of the 'characteristic' Green's function

Here, we will develop a uniform asymptotic expansion of the Green's function $G(x_1, x_3 - x_3'; x_1')$ in *general* medium profiles – in the high-frequency approximation, i.e., k_0 large – that will lead us to uniform asymptotic expansions for the Bremmer series kernels. For a general background we refer the reader to Bleistein and Handelsman [7] and Handelsman and Bleistein [8].

Scales

We assume that our wave field is a transient phenomenon with dominant wave number k_0. Our medium is supposed to vary on a characteristic length ℓ – of $\mathcal{O}(1)$ in k_0 – which can be associated with a reciprocal wave number dominant in the spectrum of the index of refraction. Note that $\delta_2^{1/2} \sim \ell^{-1}$, cf. Eq.(A.6). In the uniform asymptotic analysis, we will distinguish three regions exhibiting an interplay between the two scales, k_0 and ℓ^{-1}.

Finite vertical offset

Substituting the leading order WKB approximation to the causal Green's function of Eq.(4.12) at finite vertical offset into Eq.(4.14) yields

$$G(x_1, x_3 - x_3'; x_1') \sim \frac{\mathrm{i}}{2\pi} \int_{\zeta \in \mathcal{Z}'} \frac{\exp\left[\mathrm{i}k_0 \int_{\xi_1 = x_{1<}}^{x_{1>}} (n^2(\xi_1) - \zeta^2)^{1/2}\, \mathrm{d}\xi_1\right]}{2\left[(n^2(x_1) - \zeta^2)(n^2(x_1') - \zeta^2)\right]^{1/4}} \exp[\mathrm{i}k_0\, |x_3 - x_3'|\zeta]\, \mathrm{d}\zeta \ ,$$

(6.1)

where as before $x_{1<} = \min(x_1, x_1')$ and $x_{1>} = \max(x_1, x_1')$. To develop the uniform asymptotic approximation to G, we will introduce three regions. The outer region is defined by the condition $\ell^{-1}|x_1 - x_1'| = \mathcal{O}(1)$; then the phase will vary rapidly. Thus an inner region is defined by the condition $k_0|x_1 - x_1'| = \mathcal{O}(1)$. The asymptotic expansions on the outer and inner regions will appear to be valid for $k_0|x_1 - x_1'| \gg 1$ and $k_0\ell^{-2}|x_1 - x_1'|^3 \ll 1$, respectively. The overlapping region is defined by the condition $(k_0\ell^{-1})^{1/2}|x_1 - x_1'| = \mathcal{O}(1)$, where the outer and inner expansion should match to the order in k_0 considered.

<u>*Case 1: $\ell^{-1}|x_1 - x_1'| = \mathcal{O}(1)$ (away from the diagonal)*</u>. Then the principal contribution to the Green's function comes from the stationary point at $\zeta = 0$ (according to Eq.(4.21) we have $|x_3 - x_3'| = \mathcal{O}(k_0^{-1})$ and hence the term $|x_3 - x_3'|\zeta$ does not play a role in the phase). Denote the phase in the integral representation (6.1) by Φ,

(6.2) $$\Phi(x_1, x_1'; \zeta) = \int_{\xi_1 = x_{1<}}^{x_{1>}} (n^2(\xi_1) - \zeta^2)^{1/2}\, \mathrm{d}\xi_1 \ .$$

Then

(6.3) $$\partial_\zeta \Phi(x_1, x_1'; \zeta) = -\int_{\xi_1 = x_{1<}}^{x_{1>}} \frac{\zeta}{(n^2(\xi_1) - \zeta^2)^{1/2}}\, \mathrm{d}\xi_1 \ ,$$

and

(6.4) $$\partial_\zeta^2 \Phi(x_1, x_1'; \zeta) = -\int_{\xi_1 = x_{1<}}^{x_{1>}} \frac{n^2(\xi_1)}{(n^2(\xi_1) - \zeta^2)^{3/2}}\, \mathrm{d}\xi_1 \ .$$

Indeed, $(\partial_\zeta \Phi)(x_1, x_1'; 0) = 0$. For the stationary phase analysis we introduce the notation

(6.5) $$I_0(x_1, x_1') = \Phi(x_1, x_1'; 0) \ , \qquad I_1(x_1, x_1') = -(\partial_\zeta^2 \Phi)(x_1, x_1'; 0) \ ,$$

or in general,

(6.6) $$I_j(x_1, x_1') = \int_{\xi_1 = x_{1<}}^{x_{1>}} [n(\xi_1)]^{1-2j}\, \mathrm{d}\xi_1 \ .$$

Carrying out the stationary phase analysis then yields,

(6.7) $$G(x_1, x_3 - x_3'; x_1') \sim \left[\frac{\mathrm{i}}{2\pi k_0\, I_1(x_1, x_1')}\right]^{1/2} \frac{\exp[\mathrm{i}k_0\, I_0(x_1, x_1')]}{2\,[n(x_1)n(x_1')]^{1/2}} \ .$$

<u>*Case 2:* $k_0|x_1 - x_1'| = \mathcal{O}(1)$ *(near the diagonal)*</u>. Let $\xi_1 = \frac{1}{2}(x_1 + x_1') + \frac{1}{2}(x_1 - x_1')\,\sigma_1$. Then the phase (Eq.(6.2)) of Eq.(6.1) can be approximated by

$$\Phi(x_1, x_1'; \zeta) = \tfrac{1}{2}|x_1 - x_1'| \int_{\sigma_1=-1}^{1} [n^2(\tfrac{1}{2}(x_1 + x_1') + \tfrac{1}{2}(x_1 - x_1')\,\sigma_1) - \zeta^2]^{1/2}\,\mathrm{d}\sigma_1$$

(6.8) $$= |x_1 - x_1'|\,\{[n^2(\tfrac{1}{2}(x_1 + x_1')) - \zeta^2]^{1/2} + \mathcal{O}((x_1 - x_1')^2)\} \ ,$$

while the denominator of the integrand in Eq.(6.1) can be approximated by

(6.9)

$$[(n^2(x_1) - \zeta^2)(n^2(x_1') - \zeta^2)]^{1/4} = [n^2(\tfrac{1}{2}(x_1 + x_1')) - \zeta^2]^{1/2}\{1 + \mathcal{O}((x_1 - x_1')^2)\} \ .$$

Using Eqs.(6.8) and (6.9) in Eq.(6.1), we find that

$$G(x_1, x_3 - x_3'; x_1') \sim \frac{\mathrm{i}}{4\pi} \int_{\zeta \in \mathcal{Z}'} \frac{\exp[\mathrm{i}k_0\,([n^2(\frac{1}{2}(x_1 + x_1')) - \zeta^2]^{1/2}|x_1 - x_1'| + \zeta|x_3 - x_3'|)]}{[n^2(\frac{1}{2}(x_1 + x_1')) - \zeta^2]^{1/2}}\,\mathrm{d}\zeta$$

(6.10) $$= \frac{\mathrm{i}}{4}\, H_0^{(1)}\left(k_0\, n(\tfrac{1}{2}(x_1 + x_1'))\,[(x_1 - x_1')^2 + (x_3 - x_3')^2]^{1/2}\right)$$

(see Morse and Feshbach [9], p.823).

<u>*Case 3:* $(k_0\ell^{-1})^{1/2}|x_1 - x_1'| = \mathcal{O}(1)$</u>. On this *overlapping* region the asymptotic expansions (6.7) and (6.10) must match. On the one hand, in Eq.(6.7) note that on this region

(6.11) $$I_j(x_1, x_1') = [n(\tfrac{1}{2}(x_1 + x_1'))]^{-j}\,|x_1 - x_1'|\,\{1 + \mathcal{O}((x_1 - x_1')^2)\} \ , \quad j = 0, 1 \ .$$

Also,

(6.12) $$[n(x_1)n(x_1')]^{1/2} = n(\tfrac{1}{2}(x_1 + x_1'))\{1 + \mathcal{O}((x_1 - x_1')^2)\} \ ,$$

cf. Eq.(6.9). On the other hand, in Eq.(6.10) we have

$$H_0^{(1)}(k_0\Phi) \sim \left(\frac{2}{\pi}\right)^{1/2} (-\mathrm{i})^{1/2} \frac{\exp(\mathrm{i}k_0\,\Phi)}{(k_0\,\Phi)^{1/2}}\,\{1 + \mathcal{O}(k_0^{-1})\} \ .$$

In the Appendix we show how this leads to the uniform asymptotic expansion

$$G(x_1, x_3 - x_3'; x_1') \sim \frac{\mathrm{i}}{4} \frac{\nu(x_1, x_1')}{[n(x_1)n(x_1')]^{1/2}} H_0^{(1)}(k_0 \nu(x_1, x_1') \, r(x_1, x_1')) \; , \tag{6.13}$$

where, for notational convenience, we have introduced the *effective* index of refraction and *effective* horizontal distance as

$$\nu(x_1, x_1') \equiv \left[\frac{I_0(x_1, x_1')}{I_1(x_1, x_1')}\right]^{1/2} , \tag{6.14}$$

$$\chi_1(x_1, x_1') \equiv [I_0(x_1, x_1') I_1(x_1, x_1')]^{1/2} \; , \tag{6.15}$$

with limiting behaviors

$$\nu(x_1, x_1) = n(x_1) \; , \tag{6.16}$$

$$\lim_{x_1 \to x_1'} \frac{\chi_1(x_1, x_1')}{|x_1 - x_1'|} = 1 \tag{6.17}$$

and the *effective* distance

$$r(x_1, x_1') = [(\chi_1(x_1, x_1'))^2 + (x_3 - x_3')^2]^{1/2} \tag{6.18}$$

with limiting behaviors

$$r(x_1, x_1') \sim \chi_1(x_1, x_1') \left\{1 + \frac{(x_3 - x_3')^2}{2\,(\chi_1(x_1, x_1'))^2} + \mathcal{O}((\chi_1(x_1, x_1'))^{-4})\right\}, (k_0/\ell)^{1/2}|x_1 - x_1'| \gg 1,$$

$$r(x_1, x_1') \sim [(x_1 - x_1')^2 + (x_3 - x_3')^2]^{1/2} \{1 + \mathcal{O}((x_1 - x_1')^2)\}, \qquad (k_0/\ell)^{1/2}|x_1 - x_1'| \ll 1.$$

Indeed, with these expansions, from Eq.(6.13) both limiting equations (6.7) and (6.10) can be recovered. Note that the distance function r (and the function r_0 in the Appendix) not only depends on x_1, x_1' but also on $x_3 - x_3'$. To elucidate the 'matrix' structure of our kernels, however, we suppress the dependence on the vertical coordinate in our notation.

In Appendix A, the next order term of the uniform asymptotic expansion of the characteristic-equation Green's function has been derived as well. To provide insight in the results, we introduce functionals b_j of index of refraction as

(6.19)

$$b_{j-1}(x_1, x_1') = (2j - 1)\,[\beta_1(x_1, x_1') + (2j - 3)\,\beta_2(x_1, x_1') + (2j - 5)(2j - 3)\,\beta_3(x_1, x_1')] \; ,$$

where

$$\beta_3(x_1,x_1') = -\frac{1}{8}\left[1-\frac{\nu(x_1,x_1')}{\nu'(x_1,x_1')}\right] , \tag{6.20}$$

$$\beta_2(x_1,x_1') = -\frac{1}{4}\left[\left(\frac{\nu(x_1,x_1')}{n(x_1)}\right)^2+\left(\frac{\nu(x_1,x_1')}{n(x_1')}\right)^2-2\right]+12\beta_3(x_1,x_1') , \tag{6.21}$$

$$\beta_1(x_1,x_1') = -\frac{1}{8}\,\frac{\nu(x_1,x_1')}{\nu''(x_1,x_1')}+4\beta_2(x_1,x_1')-24\beta_3(x_1,x_1') , \tag{6.22}$$

with

$$[\nu'(x_1,x_1')]^{-1} = \frac{[\nu(x_1,x_1')]^2}{\chi_1(x_1,x_1')}\int_{\xi_1=x_{1<}}^{x_{1>}}[n(\xi_1)]^{-3}\,\mathrm{d}\xi_1 , \tag{6.23}$$

$$[\nu''(x_1,x_1')]^{-1} = \chi_1(x_1,x_1')\int_{\xi_1=x_{1<}}^{x_{1>}}[n(\xi_1)]^{-1}\left[2\,\frac{\partial_1^2 n(\xi_1)}{n(\xi_1)}-3\left(\frac{\partial_1 n(\xi_1)}{n(\xi_1)}\right)^2\right]\mathrm{d}\xi_1 . \tag{6.24}$$

The limiting behaviors β_i^0 of the β_i,

$$\beta_i^0(x_1') = \lim_{x_1\to x_1'}\frac{\beta_i(x_1,x_1')}{[\chi_1(x_1,x_1')]^2}<\infty , \qquad i=1,2,3 , \tag{6.25}$$

are found to be

$$\beta_3^0(x_1') = \frac{1}{24}\left(\frac{\partial_1 n(x_1')}{n(x_1')}\right)^2 , \tag{6.26}$$

$$\beta_2^0(x_1') = \frac{1}{12}\left[\frac{\partial_1^2 n(x_1')}{n(x_1')}+2\left(\frac{\partial_1 n(x_1')}{n(x_1')}\right)^2\right] , \tag{6.27}$$

$$\beta_1^0(x_1') = \frac{1}{24}\left[\frac{2\,\partial_1^2 n(x_1')}{n(x_1')}+\left(\frac{\partial_1 n(x_1')}{n(x_1')}\right)^2\right] . \tag{6.28}$$

Then Eq.(6.13) extends to

$$G(x_1,x_3-x_3';x_1') \sim \frac{\mathrm{i}}{4}\,\frac{\nu(x_1,x_1')}{[n(x_1)n(x_1')]^{1/2}}\Bigg\{$$
$$\left[1+(6\beta_3(x_1,x_1')-\beta_2(x_1,x_1'))\left(\frac{x_3-x_3'}{\chi_1(x_1,x_1')}\right)^2\right]B_0(k_0\nu(x_1,x_1')\,r(x_1,x_1'))+$$
$$\left[b_{-1}(x_1,x_1')+\beta_3(x_1,x_1')\left(\frac{k_0\nu(x_1,x_1')\,(x_3-x_3')^2}{r(x_1,x_1')}\right)^2\right]\frac{B_{-1}(k_0\nu(x_1,x_1')\,r(x_1,x_1'))}{(k_0\nu(x_1,x_1')\,\chi_1(x_1,x_1'))^2}+\cdots\Bigg\} , \tag{6.29}$$

where

(6.30) $$B_j(y) \equiv \frac{(2j-1)!!\, H_j^{(1)}(y)}{y^j} \,, \qquad (2j-1)!! = \frac{(j-\frac{1}{2})!\, 2^j}{\sqrt{\pi}} \,.$$

(Note that $(-3)!! = -1$ while $H_{-j}^{(1)}(y) = (-1)^j H_j^{(1)}(y)$.)

Zero vertical offset

To find a uniform asymptotic expansion of the Green's function at zero vertical offset, we can simply take the limit

$$x_3 - x_3' \to 0$$

in the results obtained in the previous subsection. In particular, the distance r reduces to χ_1. From Eq.(6.29), we obtain

(6.31) $$G(x_1, 0; x_1') \sim \frac{\mathrm{i}}{4} \frac{\nu(x_1, x_1')}{[n(x_1)n(x_1')]^{1/2}} \Big\{ B_0(k_0\nu(x_1, x_1')\,\chi_1(x_1, x_1')) + b_{-1}(x_1, x_1')\, \frac{B_{-1}(k_0\nu(x_1, x_1')\,\chi_1(x_1, x_1'))}{(k_0\nu(x_1, x_1')\,\chi_1(x_1, x_1'))^2} + \cdots \Big\} \,.$$

7 Uniform asymptotic expansion of the (de)composition operator kernels

Uniform asymptotic expansion of $\mathcal{A}_{-1/2}$

On the basis of Eq.(5.4) we find that the Schwartz kernel of the inverse vertical slowness operator directly follows from Eq.(6.31):

(7.1) $$\mathcal{A}_{-1/2}(x_1, x_1') = -2\mathrm{i}\omega\, G(x_1, 0; x_1') \sim \omega\, \frac{\nu(x_1, x_1')}{2\,[n(x_1)n(x_1')]^{1/2}} \Big\{ B_0(k_0\nu(x_1, x_1')\,\chi_1(x_1, x_1')) + b_{-1}(x_1, x_1')\, \frac{B_{-1}(k_0\nu(x_1, x_1')\,\chi_1(x_1, x_1'))}{(k_0\nu(x_1, x_1')\,\chi_1(x_1, x_1'))^2} + \cdots \Big\} \,.$$

Uniform asymptotic expansion of $\mathcal{A}^{j-1/2}$, $j \in \mathbb{Z}$

According to Eq.(5.7) we have

(7.2) $$\mathcal{A}^{j-1/2} = \mathrm{A}^j \mathcal{A}_{-1/2} \,.$$

For the Schwartz kernels associated with the odd powers of the vertical slowness operator, with Eq.(7.1), we thus find the following rule

(7.3) $$\mathcal{A}^{j-1/2}(x_1, x_1') \sim \omega\, \frac{\nu(x_1, x_1')}{2\,[n(x_1)n(x_1')]^{1/2}}\, [c_0^{-1}\nu(x_1, x_1')]^{2j} \Big\{$$

$$B_j(k_0\nu(x_1,x_1')\,\chi_1(x_1,x_1')) + b_{j-1}(x_1,x_1')\,\frac{B_{j-1}(k_0\nu(x_1,x_1')\,\chi_1(x_1,x_1'))}{(k_0\nu(x_1,x_1')\,\chi_1(x_1,x_1'))^2} + \cdots \Bigg\} \,.$$

Again, all these kernels are implicitly dependent on x_3 through the index of refraction.

8 Uniform asymptotic expansion of the one-way propagator kernel

Substituting Eq.(6.29) into Eq.(4.6), we obtain

$$\begin{aligned}
\mathcal{G}(x_1, x_3 - x_3'; x_1') \sim \frac{\mathrm{i}}{2}\,\frac{k_0^2\nu^3(x_1,x_1')\,(x_3-x_3')}{[n(x_1)n(x_1')]^{1/2}} \Bigg\{ \\
\left[1 + \left(\frac{x_3-x_3'}{\chi_1(x_1,x_1')}\right)^2 \left(2\beta_3(x_1,x_1') - \beta_2(x_1,x_1') + 2\beta_3(x_1,x_1')\left(\frac{x_3-x_3'}{r(x_1,x_1')}\right)^2\right)\right] \\
\times B_1(k_0\nu(x_1,x_1')\,r(x_1,x_1')) + \\
\frac{1}{(k_0\nu(x_1,x_1')\,\chi_1(x_1,x_1'))^2}\left[b_0(x_1,x_1') - \beta_3(x_1,x_1')\,\left(\frac{k_0\nu(x_1,x_1')\,(x_3-x_3')^2}{r(x_1,x_1')}\right)^2\right] \\
\times B_0(k_0\nu(x_1,x_1')\,r(x_1,x_1')) + \cdots \Bigg\} \,. \qquad (8.1)
\end{aligned}$$

Note that $\mathcal{A}^{1/2}(x_1,x_1') = -\mathrm{i}\omega^{-1}\partial_3\mathcal{G}(x_1,0;x_1')$ in agreement with Eq.(7.3). We also observe that

$$\lim_{x_3 \downarrow x_3'} \mathcal{G}(x_1, x_3 - x_3'; x_1') = \delta(x_1 - x_1') \,, \qquad (8.2)$$

while we can show that

$$\mathcal{G}(x_1, x_3 - x_3'; x_1') \sim \mathrm{i}\sum_{j=1}^{\infty} \frac{(-)^{j-1}}{(2j-1)!}\,(\omega(x_3-x_3'))^{2j-1}\mathcal{A}^{j-1/2}(x_1,x_1') \,, \qquad (8.3)$$

which expansion is in agreement with the product integral representation of De Hoop [1].

9 Uniform asymptotic expansion of the reflection/transmission operator kernel

To incorporate the up/down interaction, we allow the transverse Helmholtz or characteristic operator to vary from one thin slab to another. Though strictly speaking, we could now take the finite difference of two neighboring operators only, we will in fact replace this difference by a derivative.

Uniform asymptotic expansion of $\partial_3 \mathcal{A}^{1/2}$

For the interaction operator we need the vertical derivative of the vertical slowness operator, assuming that the index of refraction is now x_3 dependent. By differentiation, the asymptotic expansion of its Schwartz kernel follows from Eq.(7.1) as

$$(\partial_3 \mathcal{A}^{1/2})(x_1, x_1') \sim \frac{k_0}{2c_0} \frac{I_0(x_1, x_1')}{I_1(x_1, x_1')}\, d(x_1, x_1') \Bigg\{$$

$$\frac{\partial_3 I_0(x_1, x_1')}{I_0(x_1, x_1')}\, B_0(k_0 I_0(x_1, x_1')) + b_{0,3}(x_1, x_1')\, B_1(k_0 I_0(x_1, x_1')) + \cdots \Bigg\}\,, \tag{9.1}$$

where

$$d(x_1, x_1') = \left[\frac{I_0(x_1, x_1')}{n(x_1) n(x_1')\, I_1(x_1, x_1')}\right]^{1/2} \tag{9.2}$$

and

$$b_{0,3}(x_1, x_1') = -\frac{1}{2}\, \times \tag{9.3}$$

$$\left[\frac{\partial_3 n(x_1)}{n(x_1)} + \frac{\partial_3 n(x_1')}{n(x_1')} + (1 + 2 b_0(x_1, x_1'))\, \frac{\partial_3 I_0(x_1, x_1')}{I_0(x_1, x_1')} + 3\, \frac{\partial_3 I_1(x_1, x_1')}{I_1(x_1, x_1')}\right]\,.$$

Uniform asymptotic expansion of $\mathcal{R}$

Since we assumed the density to be constant, we can rewrite Eq.(2.28) as

$$R = \tfrac{1}{2} \Gamma^{-1} (\partial_3 \Gamma)\,. \tag{9.4}$$

To arrive at the kernel for this reflection (and transmission) operator, we thus have to compose the Schwartz kernels $\mathcal{A}^{-1/2}$ and $\partial_3 \mathcal{A}^{1/2}$ numerically. That is

$$\mathcal{R}(x_1, x_1') = \tfrac{1}{2} \int_{x_1'' \in \mathbb{R}} \mathcal{A}_{-1/2}(x_1, x_1'')\, (\partial_3 \mathcal{A}^{1/2})(x_1'', x_1')\, \mathrm{d}x_1''$$

$$\sim \frac{k_0^2}{8} \int_{x_1'' \in \mathbb{R}} d(x_1, x_1'') d(x_1'', x_1')\, \frac{I_0(x_1'', x_1')}{I_1(x_1'', x_1')}$$

$$\times \Bigg[\frac{\partial_3 I_0(x_1'', x_1')}{I_0(x_1'', x_1')}\, B_0(k_0 I_0(x_1, x_1'')) B_0(k_0 I_0(x_1'', x_1')) +$$

$$b_{0,3}(x_1'', x_1')\, B_0(k_0 I_0(x_1, x_1'')) B_1(k_0 I_0(x_1'', x_1')) +$$

$$b_{-1}(x_1, x_1'') \frac{\partial_3 I_0(x_1'', x_1')}{I_0(x_1'', x_1')}\, B_1(k_0 I_0(x_1, x_1'')) B_0(k_0 I_0(x_1'', x_1')) + \cdots \Bigg]\, \mathrm{d}x_1''\,. \tag{9.5}$$

Since

$$3\pi\,(k_0 n(x_1))^2 \lim_{x_1'\to x_1} b_{-1}(x_1,x_1')B_1(k_0 I_0(x_1,x_1')) = -\mathrm{i}\left[\frac{\partial_1^2 n(x_1)}{n(x_1)} - \left(\frac{\partial_1 n(x_1)}{n(x_1)}\right)^2\right], \tag{9.6}$$

$$3\pi\,(k_0 n(x_1))^2 \lim_{x_1'\to x_1} b_{0,3}(x_1,x_1')B_1(k_0 I_0(x_1,x_1')) = -\mathrm{i}\,\partial_3\left(\frac{\partial_1^2 n(x_1)}{n(x_1)}\right), \tag{9.7}$$

the only singularities of the kernel in Eq.(9.5) are the ones contained in the factors $B_0(k_0 I_0(x_1,x_1'))$, which are logarithmic at $x_1 = x_1'$.

Constraining our analysis to the behaviors of the kernels of our operators *near their diagonals*, we make the following observation. Up to leading-order asymptotics, ignoring the commutators of any couple of operators, in Eq.(9.4) we can substitute

$$(\partial_3\Gamma)\cdot \sim \tfrac{1}{2}\Gamma^{-1}((\partial_3 c^{-2})\cdot) \tag{9.8}$$

which results in the approximation

$$R\cdot \sim \tfrac{1}{4}A^{-1}((\partial_3 c^{-2})\cdot)\ . \tag{9.9}$$

Then, for the kernel of the reflection operator, near its diagonal, we find

$$\mathcal{R}(x_1,x_1') \sim \tfrac{1}{4}\mathcal{A}_{-1}(x_1,x_1')\ (\partial_3 c^{-2})(x_1')\ . \tag{9.10}$$

Note that $\mathcal{A}_{-1}(x_1,x_1') = -\omega^2\tilde{G}(x_1,x_1';0)$, hence, with Eq.(A.1), up to *leading* order near the diagonal we have

$$\mathcal{R}(x_1,x_1') \sim \frac{-\mathrm{i}k_0}{4}\,\frac{\exp[\mathrm{i}k_0\, I_0(x_1,x_1')]}{2\,[n(x_1)n(x_1')]^{1/2}}\ (\partial_3 n^2)(x_1')\ . \tag{9.11}$$

A more refined analysis of Eq.(9.5), valid away from the diagonal, leads to the following finite integral approximation of the reflection operator kernel,

$$\mathcal{R}(x_1,x_1') \sim \frac{\exp[\mathrm{i}k_0\, I_0(x_1,x_1')]}{4\pi\,[n(x_1)n(x_1')]^{1/2}} \int_{x_1''=x_{1<}}^{x_{1>}} \frac{R_0(x_1,x_1';x_1'')}{[I_1(x_1,x_1'')I_1(x_1',x_1'')]^{1/2}}\,\mathrm{d}x_1''\ , \tag{9.12}$$

in which

$$\begin{aligned} R_0(x_1,x_1';x_1'') = &-\frac{\mathrm{i}k_0\,\partial_3 I_0(x_1',x_1'')}{n(x_1'')\,I_1(x_1',x_1'')} + \frac{b_{-1}(x_1,x_1'')\,\partial_3 I_0(x_1',x_1'')}{n(x_1'')\,I_0(x_1,x_1'')I_1(x_1',x_1'')} + \frac{b_{0,3}(x_1,x_1')}{n(x_1'')\,I_1(x_1',x_1'')} \\ &+\Big(I_1(x_1,x_1'') - I_1(x_1',x_1'')\Big)\frac{I_0(x_1,x_1')I_1(x_1,x_1'')\,\partial_3 I_0(x_1',x_1'')}{4\,[I_1(x_1,x_1')]^2 I_0(x_1,x_1'')I_0(x_1',x_1'')} \\ &\times\left(\frac{n(x_1'')}{I_0(x_1,x_1'')} - \frac{n(x_1'')}{I_0(x_1',x_1'')} - \frac{1}{n(x_1'')\,I_1(x_1,x_1'')} + \frac{\partial_3 n(x_1'')}{\partial_3 I_0(x_1',x_1'')}\right)\ . \end{aligned} \tag{9.13}$$

10 The generalized Bremmer coupling series

In this section, finally, we will provide the recipe to generate a full-wave solution with the aid of the kernels derived so far. To this end, we will summarize the generalized Bremmer series solution procedure [1] and then decompose this exact procedure into thin-slab contributions.

The coupled system of integral equations

Applying operators with kernels $\mathcal{G}^{(\pm)}$ (cf. Eq.(3.6)) to Eq.(2.26) we obtain a coupled system of integral equations. In operator form, they are given by

$$(10.1) \qquad (\delta_{IJ} - \mathrm{K}_{IJ}) W_J = W_I^{(0)} \ ,$$

in which $W^{(0)}$ denotes the incident field. In our configuration the domain of heterogeneity will be restricted to the slab $(0, x_3^{\mathrm{exit}}]$ with $x_3^{\mathrm{exit}} > 0$, and the excitation of the waves will be specified through an initial condition at the level $x_3 = 0$, following from source distribution component $(L^{-1})_{1M} N_M$ (cf. Eq.(2.26)) with support $x_3 \leq 0$,

$$(10.2) \qquad W_1^{(0)}(x_1, x_3) = \int_{x_1' \in \mathbb{R}} \mathcal{G}^{(+)}(x_1, x_3; x_1', 0)\, W_1(x_1', 0) \mathrm{d}x_1' \ ,$$

$$(10.3) \qquad W_2^{(0)}(x_1, x_3) = 0 \ ,$$

in the range of interest, $x_3 \in [0, x_3^{\mathrm{exit}}]$; the second equation reflects the assumption that there is no excitation above the heterogeneous slab. The integral operators in Eq.(10.1) are given by

$$(10.4) \qquad (\mathrm{K}_{11} W_1)(x_1, x_3) = \int_{\zeta=0}^{x_3} \int_{x_1' \in \mathbb{R}} \mathcal{G}^{(+)}(x_1, x_3; x_1', \zeta)(T W_1)(x_1', \zeta) \mathrm{d}x_1' \mathrm{d}\zeta \ ,$$

$$(10.5) \qquad (\mathrm{K}_{12} W_2)(x_1, x_3) = \int_{\zeta=0}^{x_3} \int_{x_1' \in \mathbb{R}} \mathcal{G}^{(+)}(x_1, x_3; x_1', \zeta)(R W_2)(x_1', \zeta) \mathrm{d}x_1' \mathrm{d}\zeta \ ,$$

$$(10.6) \qquad (\mathrm{K}_{21} W_1)(x_1, x_3) = \int_{\zeta=x_3}^{x_3^{\mathrm{exit}}} \int_{x_1' \in \mathbb{R}} \mathcal{G}^{(-)}(x_1, x_3; x_1', \zeta)(R W_1)(x_1', \zeta) \mathrm{d}x_1' \mathrm{d}\zeta \ ,$$

$$(10.7) \qquad (\mathrm{K}_{22} W_2)(x_1, x_3) = \int_{\zeta=x_3}^{x_3^{\mathrm{exit}}} \int_{x_1' \in \mathbb{R}} \mathcal{G}^{(-)}(x_1, x_3; x_1', \zeta)(T W_2)(x_1', \zeta) \mathrm{d}x_1' \mathrm{d}\zeta \ .$$

They describe the interaction between the counter-propagating constituent waves.

The Bremmer series

If $\omega = \mathrm{i}s$ (and $p_1 = \mathrm{i}\alpha_1 \in \mathrm{i}\mathbb{R}$, cf. Eq.(4.2)) the uniform asymptotic expansions remain valid. For s real and sufficiently large – which is all we need for causal wavefield representations

– the Neumann series expansion can be employed to invert $(\delta_{IJ} - \mathrm{K}_{IJ})$ in Eq.(10.1), see De Hoop [1]. Such a procedure leads to the Bremmer coupling series,

(10.8) $$W_I = \sum_{j=0}^{\infty} (\mathrm{K}^j)_{IJ} W_J^{(0)} = W_I^{(0)} + \mathrm{K}_{IJ} W_J^{(0)} + (\mathrm{K}^2)_{IJ} W_J^{(0)} + \cdots \ .$$

To emphasize the physical nature of the expansion, we write

(10.9) $$W_I = \sum_{j=0}^{\infty} W_I^{(j)} \ ,$$

in which

(10.10) $$W_I^{(j)} = \mathrm{K}_{IJ} W_J^{(j-1)} \ \text{ for } \ j \geq 1 \ ,$$

can be interpreted as the j-times reflected or scattered wave.

Composition of thin-slab propagators

In general, we can represent the action of the one-sided Green's kernels by a product integral, viz.,

(10.11) $$W_1^{(0)}(.,x_3) = \left\{ \prod_{\zeta'=0}^{x_3} \exp[\mathrm{i}\omega\Gamma^{(+)}(.,\zeta')\mathrm{d}\zeta'] \right\} W_1(.,0) \ .$$

Let us split the interval $[0, x_3^{\mathrm{exit}}]$ into M thin slabs with thickness Δx_3. The propagator, propagating waves through the kth thin slab, can then be written as

(10.12) $$P(.,k) = \left\{ \prod_{\zeta'=(k-1)\Delta x_3}^{k\Delta x_3} \exp[\mathrm{i}\omega\Gamma^{(+)}(.,\zeta')\mathrm{d}\zeta'] \right\} \ ,$$

which implies that Eq.(10.11) can be written in the form

(10.13) $$W_1^{(0)}(.,x_3) = P(.,k)P(.,k-1)\cdots P(.,1)\; W_1(.,0) \ .$$

In the context of our thin-slab decomposition, in accordance with Eq.(3.6), we set $\mathcal{G}^{(+)}(x_1, \Delta x_3; x_1') = \mathcal{G}^{(+)}(x_1, k\Delta x_3; x_1', (k-1)\Delta x_3)$ and we get

(10.14) $$P(.,k)\; W_1(.,(k-1)\Delta x_3) = \int_{x_1' \in \mathbb{R}} \mathcal{G}^{(+)}(x_1, \Delta x_3; x_1')\; W_1(x_1', (k-1)\Delta x_3)\mathrm{d}x_1' \ .$$

Contributions from the thin slabs

To arrive at an iterative scheme for the solution of Eq.(10.1), consider the j-times reflected constituent wave. Set

(10.15) $$W_I^{(j)}(.,k\Delta x_3) = I_{I1}^{(j)}(.,k) + I_{I2}^{(j)}(.,k) \; ,$$

$j = 1, 2, \cdots$ and $k = 0, 1, \cdots, M$, where (cf. Eq.(10.10))

(10.16) $$I_{I1}^{(j)}(.,k) = (\mathrm{K}_{I1} W_1^{(j-1)})(.,k\Delta x_3) \; , \quad I_{I2}^{(j)}(.,k) = (\mathrm{K}_{I2} W_2^{(j-1)})(.,k\Delta x_3) \; .$$

Upon comparison with Eq.(10.4) we find that

(10.17) $$I_{11}^{(j)}(.,k) = \int_{\zeta=0}^{k\Delta x_3} \left\{ \prod_{\zeta'=\zeta}^{k\Delta x_3} \exp[\mathrm{i}\omega\Gamma^{(+)}(.,\zeta')\mathrm{d}\zeta'] \right\} X_{11}^{(j)}(.,\zeta)\mathrm{d}\zeta \; ,$$

with

(10.18) $$X_{11}^{(j)}(.,\zeta) = (T W_1^{(j-1)})(.,\zeta) \; .$$

Similarly,

(10.19) $$I_{12}^{(j)}(.,k) = \int_{\zeta=0}^{k\Delta x_3} \left\{ \prod_{\zeta'=\zeta}^{k\Delta x_3} \exp[\mathrm{i}\omega\Gamma^{(+)}(.,\zeta')\mathrm{d}\zeta'] \right\} X_{12}^{(j)}(.,\zeta)\mathrm{d}\zeta \; ,$$

(10.20) $$I_{21}^{(j)}(.,k) = -\int_{\zeta=k\Delta x_3}^{x_3^{\mathrm{exit}}} \left\{ \prod_{\zeta'=\zeta}^{k\Delta x_3} \exp[\mathrm{i}\omega\Gamma^{(-)}(.,\zeta')\mathrm{d}\zeta'] \right\} X_{21}^{(j)}(.,\zeta)\mathrm{d}\zeta \; ,$$

(10.21) $$I_{22}^{(j)}(.,k) = -\int_{\zeta=k\Delta x_3}^{x_3^{\mathrm{exit}}} \left\{ \prod_{\zeta'=\zeta}^{k\Delta x_3} \exp[\mathrm{i}\omega\Gamma^{(-)}(.,\zeta')\mathrm{d}\zeta'] \right\} X_{22}^{(j)}(.,\zeta)\mathrm{d}\zeta \; ,$$

with

(10.22) $$X_{12}^{(j)}(.,\zeta) = (R W_2^{(j-1)})(.,\zeta) \; ,$$

(10.23) $$X_{21}^{(j)}(.,\zeta) = (R W_1^{(j-1)})(.,\zeta) \; ,$$

(10.24) $$X_{22}^{(j)}(.,\zeta) = (T W_2^{(j-1)})(.,\zeta) \; .$$

In view of Eq.(2.28) we have

$$X_{21}^{(j)} = -X_{11}^{(j)} \;\text{ and }\; X_{12}^{(j)} = -X_{22}^{(j)} \; .$$

To construct the iteration scheme, we carry out the following steps. Using the semi-group property of propagators, we obtain

$$I_{11}^{(j)}(.,k) = \left\{ \prod_{\zeta'=(k-1)\Delta x_3}^{k\Delta x_3} \exp[\mathrm{i}\omega\Gamma^{(+)}(.,\zeta')\mathrm{d}\zeta'] \right\}$$

$$\int_{\zeta=0}^{(k-1)\Delta x_3} \left\{ \prod_{\zeta'=\zeta}^{(k-1)\Delta x_3} \exp[\mathrm{i}\omega\Gamma^{(+)}(.,\zeta')\mathrm{d}\zeta'] \right\} X_{11}^{(j)}(.,\zeta)\mathrm{d}\zeta$$

(10.25)
$$+ \int_{\zeta=(k-1)\Delta x_3}^{k\Delta x_3} \left\{ \prod_{\zeta'=\zeta}^{k\Delta x_3} \exp[\mathrm{i}\omega\Gamma^{(+)}(.,\zeta')\mathrm{d}\zeta'] \right\} X_{11}^{(j)}(.,\zeta)\mathrm{d}\zeta \ ,$$

which can be written as

(10.26)
$$I_{11}^{(j)}(.,k) = P(.,k)\, I_{11}^{(j)}(.,k-1) + Q_{11}^{(j)}(.,k) \ ,$$

where

(10.27)
$$Q_{11}^{(j)}(.,k) = \int_{\zeta=(k-1)\Delta x_3}^{k\Delta x_3} \left\{ \prod_{\zeta'=\zeta}^{k\Delta x_3} \exp[\mathrm{i}\omega\Gamma^{(+)}(.,\zeta')\mathrm{d}\zeta'] \right\} X_{11}^{(j)}(.,\zeta)\mathrm{d}\zeta \ .$$

Recursion relations similar to the one in Eq.(10.26) can be found for the other elements of $I^{(j)}$, viz.,

$$\begin{aligned} I_{1J}^{(j)}(.,k) &= P(.,k)\, I_{1J}^{(j)}(.,k-1) + Q_{1J}^{(j)}(.,k) \quad \text{for} \quad k = 1,2,\cdots,M \ , \\ I_{2J}^{(j)}(.,k) &= P(.,k+1)\, I_{2J}^{(j)}(.,k+1) + Q_{2J}^{(j)}(.,k) \quad \text{for} \quad k = M-1, M-2,\cdots,0 \ . \end{aligned}$$

(10.28)

Here

(10.29)
$$Q_{1J}^{(j)}(.,k) = \int_{\zeta=(k-1)\Delta x_3}^{k\Delta x_3} \left\{ \prod_{\zeta'=\zeta}^{k\Delta x_3} \exp[\mathrm{i}\omega\Gamma^{(+)}(.,\zeta')\mathrm{d}\zeta'] \right\} X_{1J}^{(j)}(.,\zeta)\mathrm{d}\zeta \ ,$$

(10.30)
$$Q_{2J}^{(j)}(.,k) = \int_{\zeta=(k+1)\Delta x_3}^{k\Delta x_3} \left\{ \prod_{\zeta'=k\Delta x_3}^{\zeta} \exp[-\mathrm{i}\omega\Gamma^{(-)}(.,\zeta')\mathrm{d}\zeta'] \right\} X_{2J}^{(j)}(.,\zeta)\mathrm{d}\zeta \ .$$

The initial values for the recursive scheme (10.28) are given by

(10.31)
$$I_{1J}^{(j)}(x_1,0) = 0 \ ,$$

(10.32)
$$I_{2J}^{(j)}(x_1,M) = 0 \ ,$$

again, for $j = 1, 2, \cdots$.

In summary, the generalized Bremmer series is generated through recursion (10.28), with supporting equations (10.14) for the one-way propagation and (10.29)-(10.30) for the interaction (reflection and transmission represented by Eqs.(10.18), (10.22)-(10.24)). All the product integrals have been subjected to substitutions of the type (10.14).

11 Discussion

The Bremmer series expansion of the solution to the multi-dimensional wave equation provides insight in and control over multiple scattering. As such the expansion is a useful tool for analyzing and interpreting (migrating) wave fields in multi-dimensional configurations. In this paper, we have established closed-form uniform asymptotic expansions of the kernels that generate the generalized Bremmer series.

While developing the uniform asymptotic expansion, we have addressed the outstanding problem associated with the continuation of the results of De Hoop [1] based on the calculus of (elliptic) pseudodifferential operators in the time-Laplace domain to the time-Fourier domain. This continuation lies outside the scope of the standard calculus of pseudodifferential operators and may lead to further developments. Also, most numerical schemes generating (some terms of) the generalized Bremmer series solution have been developed in the time-Fourier domain though the theoretical basis for this was missing.

On the one hand, our analysis gave us control over the multi-dimensional scattering process; on the other hand, we have developed a novel scheme for wave propagation and scattering that is accurate in the presence of tranverse medium variations and contains both pre- and post-critical scattering-angle phenomena. Our theory is valid for high-frequencies, and conceptually is an intermediate between asymptotic-ray and full-wave theories. It allows the formation of caustics.

In the uniform asymptotic approximation, the propagator is expanded in a basis derived from Hankel functions, which differs from the Fourier bases often encountered in approximate propagation procedures such as the phase-shift-plus-interpolation [10] (and the McClellan transform approach in three dimensions [11]), the split-step Fourier [12] and the phase screen [13] methods. Hidden in the uniform asymptotic expansion are certain aspects of homogenization: we have introduced an effective index of refraction and an effective metric, which follow from the actual medium variations.

Our uniform asymptotic Bremmer series solution procedure can be applied to the fields of integrated optics, ocean acoustics and exploration seismics. In particular, in seismic imaging – in the search for and analysis of hydro-carbon reservoirs below complex geological structures – where caustics occur and post-critical angle phenomena play a role, our uniform asymptotic approach provides a useful basis for the underlying migration procedure.

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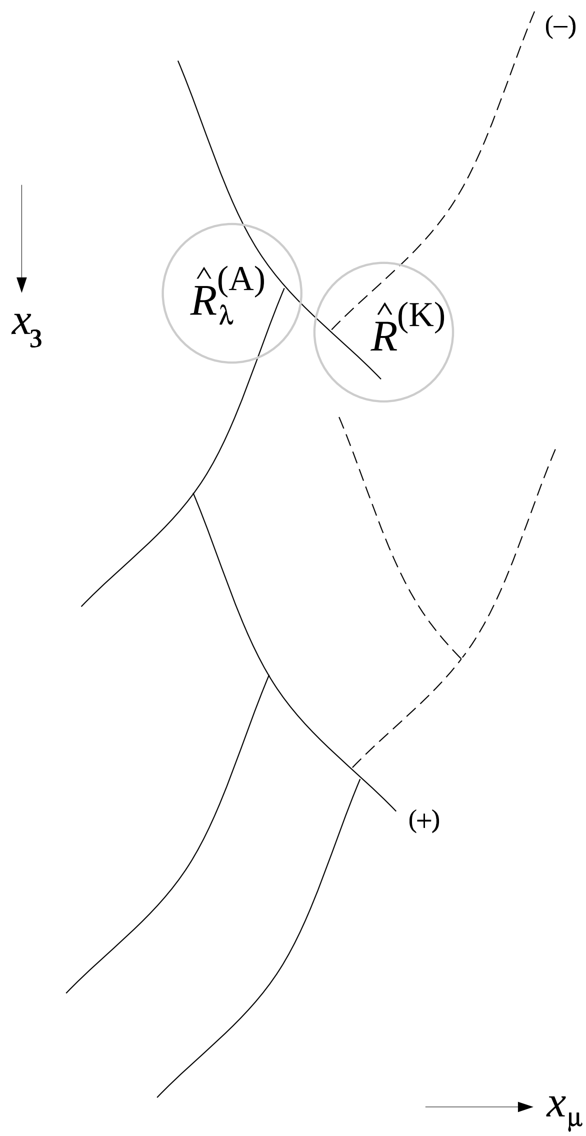

Figure 1.1: Illustration of the Bremmer coupling series. All the solid 'rays' correspond with the lowest-order term; the dashed 'rays' correspond with the next-order term. $\hat{R}^K$ is the resolvent associated with the operator K (Eqs.(10.4)-(10.7)) and generates the up/down scattering; $\hat{R}_\lambda^{(A)}$ is the resolvent associated with the operator A (Eq.(3.3)) and contains the right-left scattering.

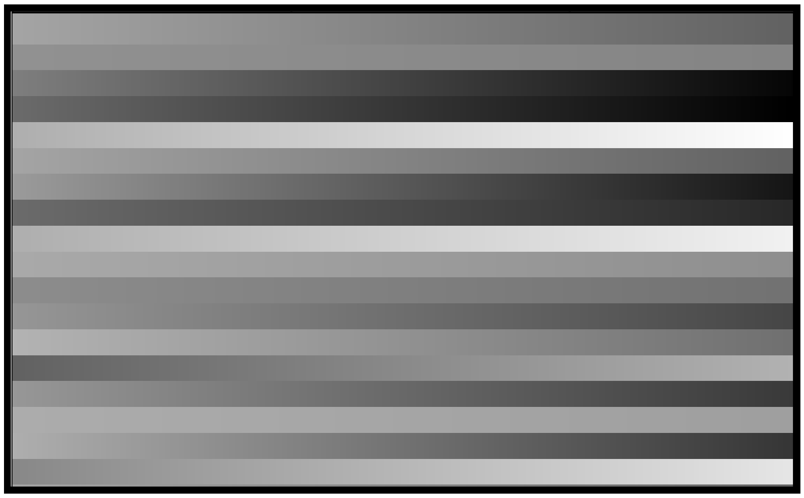

Figure 3.1: Variation of medium properties.

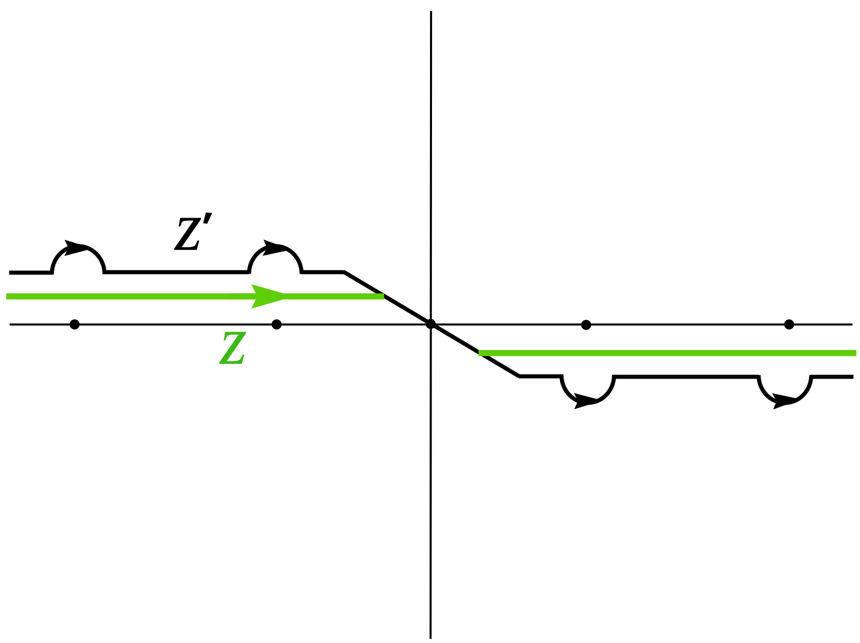

Figure 4.1: Contours of integration in the complex ζ-plane for the Helmholtz-equation Green's function representation.

Appendix A. Derivation of the uniform asymptotic expansion of the characteristic-equation Green's function

We employ the WKB method to obtain the high-frequency asymptotic solution to Eq.(4.11):

$$\tilde{G}(x_1, x_1'; \zeta) = \frac{\mathrm{i}}{k_0} \frac{\exp[\mathrm{i}k_0\, \Phi(x_1, x_1'; \zeta)]}{2\,[\tilde{n}(x_1; \zeta)\tilde{n}(x_1'; \zeta)]^{1/2}} \Bigg\{$$

$$1 + \frac{1}{8\mathrm{i}k_0} \int_{\xi_1 = x_{1<}}^{x_{1>}} F(\xi_1; \zeta)\, \mathrm{d}\xi_1 + \frac{1}{16k_0^2} \left(\frac{F(x_1; \zeta)}{\tilde{n}(x_1; \zeta)} + \frac{F(x_1'; \zeta)}{\tilde{n}(x_1'; \zeta)} + E(x_1, x_1'; \zeta) \right) + \mathcal{O}(k_0^{-3}) \Bigg\} ,$$

(A.1)

where Φ is given by Eq.(6.2),

(A.2) $$\tilde{n}(\xi_1; \zeta) \equiv (n^2(\xi_1) - \zeta^2)^{1/2} ,$$

and

$$F(\xi_1; \zeta) = [\tilde{n}(\xi_1; \zeta)]^{-5} \left\{ 2[\tilde{n}(\xi_1; \zeta)]^2 \left(n(\xi_1)\, \partial_1^2 n(\xi_1) + [\partial_1 n(\xi_1)]^2\right) - 5\,[n(\xi_1)\, \partial_1 n(\xi_1)]^2 \right\} .$$

(A.3)

The term $E(x_1, x_1'; \zeta)$ has the property that $E(x_1, x_1; \zeta) = E(x_1, x_1'; 0) = 0$. Thus an integration by parts in Eq.(4.14) reveals that this term does not contribute to the order in k_0^{-1} considered.

In the following asymptotic analysis we will introduce the average horizontal coordinate,

(A.4) $$\bar{x}_1 = \tfrac{1}{2}(x_1 + x_1') ,$$

the background radial distance,

(A.5) $$r_0(x_1, x_1') = [(x_1 - x_1')^2 + (x_3 - x_3')^2]^{1/2} ,$$

the derivative functions of the index of refraction,

(A.6) $$\delta_2(x_1) = \left(\frac{\partial_1 n(x_1)}{n(x_1)} \right)^2 ,$$

(A.7) $$\delta_1(x_1) = \frac{\partial_1^2 n(x_1)}{n(x_1)} + \delta_2(x_1) ,$$

and the stretched, dimensionless, vertical wavenumber

(A.8) $$\zeta_n \equiv \frac{\zeta}{n(\bar{x}_1)} .$$

Further, we set

(A.9) $$\tilde{n}_0(\zeta_n) = (1-\zeta_n^2)^{1/2} \qquad \text{with} \qquad n(\bar{x}_1)\tilde{n}_0(\zeta_n) = \tilde{n}(\bar{x}_1;\zeta)\ .$$

In the asymptotic analysis we will repeatedly consider the behaviors of the phase $\Phi(x_1, x_1'; \zeta)$, of the denominator $[\tilde{n}(x_1;\zeta)\tilde{n}(x_1';\zeta)]^{1/2}$, of the function $F(\xi_1;\zeta)$, and of the integral of F.

Case 1: $\ell^{-1}|x_1 - x_1'| = \mathcal{O}(1)$ (away from the diagonal). We expand about the stationary point at $\zeta = 0$,

(A.10) $$\Phi(x_1, x_1';\zeta) \sim I_0(x_1,x_1') - \tfrac{1}{2}\zeta^2 I_1(x_1,x_1') - \tfrac{1}{8}\zeta^4 I_2(x_1,x_1') + \cdots\ ,$$

where the I_j are given by Eq.(6.6), while

(A.11) $$[\tilde{n}(x_1;\zeta)]^{-1/2} \sim [n(x_1)]^{-1/2}\left\{1 + \left(\frac{\zeta}{2n(x_1)}\right)^2 + \cdots\right\}$$

and

(A.12) $$F(\xi_1;\zeta) \sim [n(\xi_1)]^{-1}(2\delta_1(\xi_1) - 5\delta_2(\xi_1)) + \cdots\ .$$

Substituting these expansions in the expression (A.1) for $\tilde{G}$, expanding the exponential partially in a Taylor series, yields

$$\tilde{G}(x_1,x_1';\zeta) \sim \frac{\mathrm{i}}{k_0}\,\frac{\exp[\mathrm{i}k_0\,(I_0(x_1,x_1') - \frac{1}{2}\zeta^2 I_1(x_1,x_1'))]}{2\,[n(x_1)n(x_1')]^{1/2}}\,\Bigg\{$$

(A.13) $$1 + \frac{1}{8\mathrm{i}k_0}\int_{\xi_1=x_{1<}}^{x_{1>}} [n(\xi_1)]^{-1}(2\delta_1(\xi_1) - 5\delta_2(\xi_1))\,\mathrm{d}\xi_1 +$$

$$+\frac{\zeta^2}{4}\left(\frac{1}{[n(x_1)]^2} + \frac{1}{[n(x_1')]^2}\right) - \frac{\mathrm{i}k_0\,\zeta^4}{8}\,I_2(x_1,x_1') + \mathcal{O}(k_0^{-2})\Bigg\}\ .$$

Substituting the result into Eq.(4.14) and applying a stationary phase analysis then leads to

$$G(x_1, x_3 - x_3'; x_1') \sim \left[\frac{\mathrm{i}}{2\pi k_0\ I_1(x_1,x_1')}\right]^{1/2}\frac{\exp[\mathrm{i}k_0\,I_0(x_1,x_1')]}{2\,[n(x_1)n(x_1')]^{1/2}}\Bigg\{$$

(A.14) $$1 + \frac{1}{\mathrm{i}k_0\,I_0(x_1,x_1')}\left[b_{-1}(x_1,x_1') + \frac{1}{8} - \frac{[k_0\,(x_3-x_3')]^2 I_0(x_1,x_1')}{2I_1(x_1,x_1')}\right] + \mathcal{O}(k_0^{-2})\Bigg\}\ ,$$

where b_{-1} is given by Eq.(6.19).

Case 2: $k_0|x_1 - x_1'| = \mathcal{O}(1)$ (near the diagonal). We now expand in $|x_1 - x_1'|$,

$$\Phi(x_1,x_1';\zeta) \sim n(\bar{x}_1)\,|x_1 - x_1'|\left\{\tilde{n}_0(\zeta_n) + \frac{(x_1-x_1')^2}{24\tilde{n}_0(\zeta_n)}\left(\delta_1(\bar{x}_1) - \frac{\delta_2(\bar{x}_1)}{[\tilde{n}_0(\zeta_n)]^2}\right) + \cdots\right\}\ ,$$

(A.15)

while

(A.16)
$$[\tilde{n}(x_1;\zeta)\tilde{n}(x_1';\zeta)]^{-1/2} \sim [n(\bar{x}_1)\tilde{n}_0(\zeta_n)]^{-1} \times \left\{1 - \tfrac{1}{8}(x_1 - x_1')^2\Big(\frac{\delta_1(\bar{x}_1)}{[\tilde{n}_0(\zeta_n)]^2} - \frac{2\delta_2(\bar{x}_1)}{[\tilde{n}_0(\zeta_n)]^4}\Big) + \cdots\right\}$$

and

(A.17)
$$\frac{F(\xi_1;\zeta)}{\tilde{n}(\xi_1;\zeta)} \sim [n(\bar{x}_1)]^{-2}\Big(\frac{2\delta_1(\bar{x}_1)}{[\tilde{n}_0(\zeta_n)]^4} - \frac{5\delta_2(\bar{x}_1)}{[\tilde{n}_0(\zeta_n)]^6}\Big) ,$$

so that

(A.18)
$$\int_{\xi_1=x_{1<}}^{x_{1>}} F(\xi_1;\zeta)\,\mathrm{d}\xi_1 \sim [n(\bar{x}_1)]^{-1}|x_1 - x_1'|\left(\frac{2\delta_1(\bar{x}_1)}{[\tilde{n}_0(\zeta_n)]^3} - \frac{5\delta_2(\bar{x}_1)}{[\tilde{n}_0(\zeta_n)]^5}\right) .$$

In anticipation of the asymptotic expansion of the characteristic Green's function represented by Eq.(4.14) in the limiting case under consideration, we introduce integrals $Q_j^{\ll}$ over ζ as,

(A.19)
$$Q_j^{\ll}(x_1, x_3 - x_3'; x_1') = \frac{\mathrm{i}}{4\pi}\int_{\zeta\in\mathcal{Z}'}[\tilde{n}_0(\zeta_n)]^{-j}\exp[\mathrm{i}k_0 n(\bar{x}_1)\left(|x_1 - x_1'|\tilde{n}_0(\zeta_n) + |x_3 - x_3'|\zeta_n\right)]\,\mathrm{d}\zeta_n .$$

Note that

(A.20)
$$Q_1^{\ll}(x_1, x_3 - x_3'; x_1') = \frac{\mathrm{i}}{4}H_0^{(1)}\left(k_0 n(\bar{x}_1)\,[(x_1 - x_1')^2 + (x_3 - x_3')^2]^{1/2}\right) ,$$

while the $Q_j^{\ll}$ satisfy the recursion relation

$$jQ_{j+2}^{\ll} - \mathrm{i}k_0 n(\bar{x}_1)\,|x_1 - x_1'|Q_{j+1}^{\ll} = (j-1)Q_j^{\ll} - [(x_1 - x_1')\partial_1 + |x_3 - x_3'|\partial_3]Q_j^{\ll} .$$

(A.21)

Equation (4.14) now leads to the expansion

(A.22)
$$\begin{aligned} G(x_1, x_3 - x_3'; x_1') \sim{}& Q_1^{\ll} - \\ & \frac{\delta_1(\bar{x}_1)}{24\,[k_0 n(\bar{x}_1)]^2}\Big\{-6\,Q_5^{\ll} + 6\mathrm{i}k_0 n(\bar{x}_1)\,|x_1 - x_1'|\,Q_4^{\ll} \\ & -3[\mathrm{i}k_0 n(\bar{x}_1)\,|x_1 - x_1'|]^2\,Q_3^{\ll} + [\mathrm{i}k_0 n(\bar{x}_1)\,|x_1 - x_1'|]^3\,Q_2^{\ll}\Big\} - \\ & \frac{\delta_2(\bar{x}_1)}{24\,[k_0 n(\bar{x}_1)]^2}\Big\{15\,Q_7^{\ll} - 15\mathrm{i}k_0 n(\bar{x}_1)\,|x_1 - x_1'|\,Q_6^{\ll} \\ & +6[\mathrm{i}k_0 n(\bar{x}_1)\,|x_1 - x_1'|]^2\,Q_5^{\ll} - [\mathrm{i}k_0 n(\bar{x}_1)\,|x_1 - x_1'|]^3\,Q_4^{\ll}\Big\} + \\ & \mathcal{O}(k_0^{-3}) . \end{aligned}$$

Substituting the recursion (A.20)-(A.21) into this equation, simplifies the expansion to

$$G(x_1, x_3 - x_3'; x_1') \sim \frac{\mathrm{i}}{4}\Big\{$$
$$\left[1 + \frac{-\delta_1(\bar{x}_1) + 2\delta_2(\bar{x}_1)}{12}\,(r_0(x_1, x_1'))^2\right] B_0(k_0 n(\bar{x}_1)\, r_0(x_1, x_1')) +$$
$$\left[4(\delta_1(\bar{x}_1) - 2\delta_2(\bar{x}_1)) + (-\delta_1(\bar{x}_1)(x_1 - x_1')^2 + \delta_2(\bar{x}_1)(r_0(x_1, x_1'))^2)(k_0 n(\bar{x}_1))^2\right] \times$$
$$\frac{B_{-1}(k_0 n(\bar{x}_1)\, r_0(x_1, x_1'))}{24\,(k_0 n(\bar{x}_1))^2} + \cdots\Big\}, \tag{A.23}$$

where the B_j are given by Eq.(6.30).

<u>*Case 3:* $(k_0\ell^{-1})^{1/2}|x_1 - x_1'| = \mathcal{O}(1)$</u>. We expand, again, in $|x_1 - x_1'|$ (cf. Eq.(A.9))

$$\Phi(x_1, x_1'; \zeta) = I_0(x_1, x_1')\, \tilde{n}_0(\zeta_n) + \int_{\xi_1 = x_{1<}}^{x_{1>}} \tilde{n}(\xi_1; \zeta)\, \mathrm{d}\xi_1 - I_0(x_1, x_1')\, [n(\bar{x}_1)]^{-1} \tilde{n}(\bar{x}_1; \zeta)$$
$$\sim I_0(x_1, x_1')\, \tilde{n}_0(\zeta_n) + \tfrac{1}{48} n(\bar{x}_1)\, |x_1 - x_1'|^3 \zeta_n^2\, ([2 + \zeta_n^2]\, \delta_1(\bar{x}_1) - 4[1 + \zeta_n^2]\, \delta_2(\bar{x}_1)) + \cdots, \tag{A.24}$$

while

$$[\tilde{n}(x_1; \zeta)\tilde{n}(x_1'; \zeta)]^{-1/2} \sim [n(\bar{x}_1)\tilde{n}_0(\zeta_n)]^{-1} \times$$
$$\Big\{1 - \tfrac{1}{8}(x_1 - x_1')^2\, ([1 + \zeta_n^2]\, \delta_1(\bar{x}_1) - 2[1 + 2\zeta_n^2]\, \delta_2(\bar{x}_1)) + \cdots\Big\} \tag{A.25}$$

and

$$\int_{\xi_1 = x_{1<}}^{x_{1>}} F(\xi_1; \zeta)\, \mathrm{d}\xi_1 \sim F(\bar{x}_1; 0)\, |x_1 - x_1'| \sim [n(\bar{x}_1)]^{-1} |x_1 - x_1'|\, (2\delta_1(\bar{x}_1) - 5\delta_2(\bar{x}_1)). \tag{A.26}$$

Also, note that

$$\zeta\, |x_3 - x_3'| = \zeta_n \nu(x_1, x_1')\, |x_3 - x_3'| + \zeta_n n(\bar{x}_1)[1 - n^{-1}(\bar{x}_1)\nu(x_1, x_1')]\, |x_3 - x_3'|$$
$$\sim \zeta_n \nu(x_1, x_1')\, |x_3 - x_3'| - \tfrac{1}{24}\zeta_n n(\bar{x}_1)(x_1 - x_1')^2(\delta_1(\bar{x}_1) - 2\delta_2(\bar{x}_1))\, |x_3 - x_3'|. \tag{A.27}$$

In anticipation of the asymptotic expansion of the characteristic Green's function in the limiting case under consideration, we introduce integrals $Q^{\sim}_j$ over ζ as,

$$Q^{\sim}_j(x_1, x_3 - x_3'; x_1') = \frac{\mathrm{i}}{4\pi} \int_{\zeta \in \mathcal{Z}'} \zeta_n^{j-1} \exp[\mathrm{i}k_0 \nu(x_1, x_1')\,(\chi_1(x_1, x_1')\tilde{n}_0(\zeta_n) + |x_3 - x_3'|\zeta_n)]\,\mathrm{d}\zeta_n \; . \tag{A.28}$$

As before Eq.(4.14) then leads to the expansion

$$\begin{aligned} &G(x_1, x_3 - x_3'; x_1') \sim \\ &\quad \left\{1 - \tfrac{1}{8}(x_1 - x_1')^2(\delta_1(\bar{x}_1) - 2\delta_2(\bar{x}_1)) - \frac{\mathrm{i}|x_1 - x_1'|}{8\,k_0 n(\bar{x}_1)}\,(2\delta_1(\bar{x}_1) - 5\delta_2(\bar{x}_1))\right\} Q^{\sim}_1 - \\ &\quad \tfrac{1}{24}\mathrm{i}k_0 n(\bar{x}_1)\,(x_1 - x_1')^2\,|x_3 - x_3'|\,(\delta_1(\bar{x}_1) - 2\delta_2(\bar{x}_1))\,Q^{\sim}_2 + \\ &\quad \tfrac{1}{24}\,(x_1 - x_1')^2\left\{\mathrm{i}k_0 n(\bar{x}_1)\,|x_1 - x_1'|\,(\delta_1(\bar{x}_1) - 2\delta_2(\bar{x}_1)) - 3\delta_1(\bar{x}_1) + 12\delta_2(\bar{x}_1)\right\} Q^{\sim}_3 + \\ &\quad \tfrac{1}{48}\mathrm{i}k_0 n(\bar{x}_1)\,|x_1 - x_1'|^3\,(\delta_1(\bar{x}_1) - 4\delta_2(\bar{x}_1))\,Q^{\sim}_5 + \\ &\quad \mathcal{O}(k_0^{-2}) \; . \end{aligned} \tag{A.29}$$

The $Q^{\sim}_j$ in this expansion are known in closed form. However, to the order of approximation considered here, we can replace these integrals by the following expressions,

$$Q^{\sim}_1(x_1, x_3 - x_3'; x_1') = \frac{\mathrm{i}}{4}\,H_0^{(1)}\left(k_0 \nu(x_1, x_1')\,r(x_1, x_1')\right) \; ,$$

$$Q^{\sim}_2(x_1, x_3 - x_3'; x_1') = \left|\frac{x_3 - x_3'}{x_1 - x_1'}\right| \left\{1 + \mathcal{O}(k_0^{-1/2})\right\} Q^{\sim}_1(x_1, x_3 - x_3'; x_1') \; ,$$

$$\begin{aligned} Q^{\sim}_3(x_1, x_3 - x_3'; x_1') = &\left\{\frac{-\mathrm{i}}{k_0 n(\bar{x}_1)|x_1 - x_1'|} + \frac{1}{2\,[k_0 n(\bar{x}_1)|x_1 - x_1'|]^2}\right. \\ &\left. + \left(\frac{x_3 - x_3'}{x_1 - x_1'}\right)^2 + \mathcal{O}(k_0^{-3/2})\right\} Q^{\sim}_1(x_1, x_3 - x_3'; x_1') \; , \end{aligned}$$

$$Q^{\sim}_5(x_1, x_3 - x_3'; x_1') = \frac{-3}{[k_0 n(\bar{x}_1)|x_1 - x_1'|]^2}\left\{1 + \mathcal{O}(k_0^{-1/2})\right\} Q^{\sim}_1(x_1, x_3 - x_3'; x_1') \; ,$$

where r has been replaced by

$$r(x_1, x_1') = |x_1 - x_1'|\left\{1 + \mathcal{O}(k_0^{-1})\right\} \; , \tag{A.30}$$

everywhere except in the arguments of the Hankel functions, and

$$H_1^{(1)}(y) = -\mathrm{i} H_0^{(1)}(y) \left\{ 1 + \frac{\mathrm{i}}{2\,y} + \mathcal{O}(y^{-2}) \right\} . \tag{A.31}$$

With these approximations Eq.(A.29) reduces to

$$G(x_1, x_3 - x_3'; x_1') \sim \left\{ 1 - \tfrac{1}{12}(x_1 - x_1')^2 (\delta_1(\bar{x}_1) - 2\delta_2(\bar{x}_1)) \left(1 + \frac{2\mathrm{i}}{k_0 n(\bar{x}_1)|x_1 - x_1'|} \right) \right\} \times$$

$$\frac{\mathrm{i}}{4} B_0(k_0 \nu(x_1, x_1') r(x_1, x_1')) \ . \tag{A.32}$$

<u>*Uniform expansion*</u>. The uniform expansion is then given by Eq.(6.29). To verify the inner and outer expansions of this expression, we need to consider the following approximations.

Outer expansion: then we have

$$B_0(k_0 \nu(x_1, x_1')\, r(x_1, x_1')) \sim \left(\frac{2}{\pi} \right)^{1/2} (-\mathrm{i})^{1/2} \frac{\exp[k_0\, I_0(x_1, x_1')]}{[k_0\, I_0(x_1, x_1')]^{1/2}} \left\{ 1 + \frac{\mathrm{i} k_0\, (x_3 - x_3')^2}{2\, I_1(x_1, x_1')} - \frac{\mathrm{i}}{8 k_0\, I_0(x_1, x_1')} \right\} , \tag{A.33}$$

$$B_{-1}(k_0 \nu(x_1, x_1')\, r(x_1, x_1')) \sim \left(\frac{2}{\pi} \right)^{1/2} (-\mathrm{i})^{3/2} [k_0\, I_0(x_1, x_1')]^{1/2} \exp[k_0\, I_0(x_1, x_1')] \ , \tag{A.34}$$

with $|x_3 - x_3'| = \mathcal{O}(k_0^{-1})$, and Eq.(6.29) reduces to Eq.(A.14).

Inner expansion: then we have

$$B_0(k_0 \nu(x_1, x_1')\, r(x_1, x_1')) \sim B_0(k_0 n(\bar{x}_1)\, r_0(x_1, x_1')) - \frac{(x_1 - x_1')^2}{24} \left(\delta_1(\bar{x}_1) - 2\delta_2(\bar{x}_1) + \frac{\delta_2(\bar{x}_1)(x_1 - x_1)^2}{(r_0(x_1, x_1'))^2} \right) B_{-1}(k_0 n(\bar{x}_1)\, r_0(x_1, x_1')) \ , \tag{A.35}$$

$$B_{-1}(k_0 \nu(x_1, x_1')\, r(x_1, x_1')) \sim B_{-1}(k_0 n(\bar{x}_1)\, r_0(x_1, x_1')) \ , \tag{A.36}$$

while

$$\frac{\nu(x_1, x_1')}{[n(x_1) n(x_1')]^{1/2}} \sim 1 - \tfrac{1}{12}(\delta_1(\bar{x}_1) - 2\delta_2(\bar{x}_1))(x_1 - x_1')^2 \ , \tag{A.37}$$

$$\frac{\beta_j(x_1, x_1')}{(\chi_1(x_1, x_1'))^2} \sim \beta_j^0(\bar{x}_1) \ , \tag{A.38}$$

$$\frac{b_{-1}(x_1, x_1')}{(\nu(x_1, x_1')\, \chi_1(x_1, x_1'))^2} \sim \frac{1}{6\, n^2(\bar{x}_1)} \left(\delta_1(\bar{x}_1) - 2\delta_2(\bar{x}_1) \right) \ , \tag{A.39}$$

where the β_j^0 are given by Eqs.(6.26)-(6.28). Now Eq.(6.29) reduces to Eq.(A.23).

Overlapping region: use Eqs.(A.37)-(A.39) together with Eqs.(A.30)-(A.31) to reduce Eq.(6.29) to Eq.(A.32).

The error in our expansion is $\mathcal{O}(k_0^{-2})$ uniformly in x_1 and x_1'. More precisely, the error is $\mathcal{O}({k_0}^{-5/2})$ on the outer region, $\mathcal{O}(k_0^{-3})$ on the inner region, and $\mathcal{O}({k_0}^{-9/4})$ on the overlapping region.

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