Common scattering angle sections

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ABSTRACT
The Generalized Radon Transform (GRT) is employed to carry out a linearized asymptotic inversion of seismic data in a heterogeneous acoustic medium. From the inverse GRT operator, it is natural to define the common scattering-angle (CSA) domain. To obtain a complete pair of operators in the common scattering-angle domain, we introduce both a modified forward operator, which models data from the CSA sections, and an inverse operator. Further we show how the CSA sections can be used in a similar fashion as the common offset sections, i.e. giving a partial reconstruction of the scatterer for each fixed angle. From the forward, and inverse operator in the CSA-domain, we derive the sensitivity transform, which can give a measure of how good the current back-ground model is. This transform can give an update of medium parameters as long as the medium is described using a relatively sparse selection of parameters.

Key words: GRT, common scattering-angle domain, sensitivity transform

Introduction

We present a data domain in which we can do both modeling and imaging/inversion based on a micro-local coordinate representation of the seismic data. We use the approach to micro-local coordinates in inversion introduced by Beylkin et al. (1984). They show how one can use the generalized Radon transform (GRT) to invert seismic data, based on an integral over planes at the image point. However, in their approach the local geometry at the image point is transformed into the more conventional acquisition coordinates by introducing a Jacobian to change the integral variables in the inversion operator. The inverse operator in acquisition coordinates is matched to the volume scattering representation of the forward operator, the forward GRT operator. The forward operator is an integral over isochrons for a fixed source/receiver pair.

We consider a heterogeneous acoustic medium where a remote scattering domain is illuminated with acoustic waves generated by a point source. The scattering is linearized using the familiar perturbation representation, dividing the field into a known smooth background and an unknown singular part. The scattered field is recorded with point receivers on the surface. We invert for the unknown medium parameters using the generalized Radon transform for a common (fixed) scattering angle. This differs from the conventional approach (Miller *et al.*, 1987) in the way we access the data, and the way the actual integration in the GRT is performed. Operating in this domain eliminates any Jacobian to relate the phase directions at the image point to sources and receivers. We do the ray-tracing from the image point and up to the surface, sorting the rays by scattering angle and azimuth and assigning them to be source- or receiver-ray, depending on where they hit the surface. The inversion then becomes an explicit integral over local variables. We also show how the inverse GRT in the common scattering-angle domain can be matched to a forward operator over isochrons for a fixed source-receiver pair.

Once we have established the two operators that can take us between the data and the scatterer domain, we introduce the sensitivity transform. This operator is treated in de Hoop & de Hoop (1997): it is an operator made from a combination of the forward and the inverse operators. The sensitivity transform can be used to analyze the coherency between different seismic experiments, in our case the different common scattering-angle sections. It can also provide an update for the current medium parameters.

Finally, we show an example based on the 2D case of the theory, where we extract common image-point gathers (CIG) from the common scattering-angle data volume. The CIG in scattering angle can be used to do velocity analysis employing the same techniques that are used on CIG's in offset.

Geometry and coordinates

The acquisition coordinates $(\boldsymbol{s}, \boldsymbol{r})$ lie on the surfaces ∂S and ∂R, respectively. The domain in which the scattering takes place is $\mathcal{D} \in \mathbb{R}^3$, and the acquisition surface, $\partial S \times \partial R$ is homeomorphic* to a part of $S^2 \times S^2$, where S^2 is the unit sphere. Measurements are taken in a time interval $[0, \mathcal{T})$. We denote Cartesian coordinates in the configuration as

$$\begin{aligned} \boldsymbol{x} &= (x_1, x_2, x_3) = \text{Cartesian position vector}, \\ \boldsymbol{s} &= (s_1, s_2, s_3) = \text{source point}, \\ \boldsymbol{r} &= (r_1, r_2, r_3) = \text{receiver point}. \end{aligned} \quad (1)$$

Ray geometry

We now define the ray geometry in our configuration. A general ray is a path connecting two points $\boldsymbol{x}$ and $\boldsymbol{x}'$ in space. The travel time between these two points along the ray is denoted by

$$\tau = \tau(\boldsymbol{x}, \boldsymbol{x}'). \quad (2)$$

We define the slowness vector at $\boldsymbol{x}$ associated with a ray emanating from a point source or a point receiver at $\boldsymbol{x}'$ in the surface ∂S or ∂R, to a scattering point $\boldsymbol{x} \in \mathcal{D}$ as

$$\boldsymbol{\gamma}_{\boldsymbol{x}} = \nabla_{\boldsymbol{x}} \tau(\boldsymbol{x}, \boldsymbol{x}'). \quad (3)$$

The unit phase direction associated with the slowness vector at $\boldsymbol{x}$ along the ray is given by

$$\boldsymbol{\alpha}_{\boldsymbol{x}} = c(\boldsymbol{x}) \boldsymbol{\gamma}_{\boldsymbol{x}}(\boldsymbol{x}), \quad (4)$$

where $c(\boldsymbol{x})$ is the medium wave speed. According to equation (3), we can write the slowness vectors for the source ray and the receiver ray at $\boldsymbol{x}$ as

$$\begin{aligned} \boldsymbol{\gamma}^s_{\boldsymbol{x}} &= \nabla_{\boldsymbol{x}} \tau(\boldsymbol{x}, \boldsymbol{s}), \\ \boldsymbol{\gamma}^r_{\boldsymbol{x}} &= \nabla_{\boldsymbol{x}} \tau(\boldsymbol{x}, \boldsymbol{r}), \end{aligned} \quad (5)$$

respectively, and the associated phase directions are

$$\begin{aligned} \boldsymbol{\alpha}^s_{\boldsymbol{x}} &= c(\boldsymbol{x}) \boldsymbol{\gamma}^s_{\boldsymbol{x}}(\boldsymbol{x}), \\ \boldsymbol{\alpha}^r_{\boldsymbol{x}} &= c(\boldsymbol{x}) \boldsymbol{\gamma}^r_{\boldsymbol{x}}(\boldsymbol{x}). \end{aligned} \quad (6)$$

* homeomorphic: has the same shape as

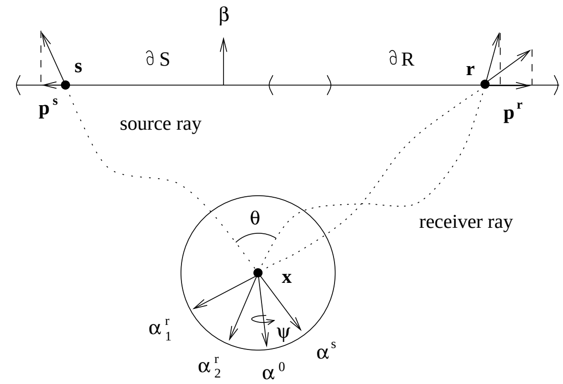

Figure 1. Ray geometry

In *acquisition* coordinates, we define the total or two-way travel time T for the two rays connecting a source $\boldsymbol{s}$ with a receiver $\boldsymbol{r}$ via a scatterer $\boldsymbol{x}$, as

$$T(\boldsymbol{r}, \boldsymbol{x}, \boldsymbol{s}) = \tau(\boldsymbol{r}, \boldsymbol{x}) + \tau(\boldsymbol{x}, \boldsymbol{s}). \quad (7)$$

The mapping

$$\begin{aligned} \boldsymbol{s} : (S^2)_{\boldsymbol{s}} &\to \partial S, \quad \boldsymbol{\alpha}^s_{\boldsymbol{x}} \to \boldsymbol{s}(\boldsymbol{\alpha}^s_{\boldsymbol{x}}) \\ \boldsymbol{r} : (S^2)_{\boldsymbol{r}} &\to \partial R, \quad \boldsymbol{\alpha}^r_{\boldsymbol{x}} \to \boldsymbol{r}(\boldsymbol{\alpha}^r_{\boldsymbol{x}}) \end{aligned} \quad (8)$$

transforms the acquisition variables $(\boldsymbol{s}, \boldsymbol{r})$ into the image point, $\boldsymbol{x}$, related variables $(\boldsymbol{\alpha}^s_{\boldsymbol{x}}, \boldsymbol{\alpha}^r_{\boldsymbol{x}})$. Here the subset $(S^2)_{\boldsymbol{s}}$ of the unit sphere S^2 is the area element which contains $\boldsymbol{\alpha}^s_{\boldsymbol{x}}$, defining a ray that connects $\boldsymbol{x}$ with an actual source $\boldsymbol{s}$. , that is $\boldsymbol{\alpha}^s_{\boldsymbol{x}} \in (S^2)_{\boldsymbol{s}}$. For the receiver we must have $\boldsymbol{\alpha}^r_{\boldsymbol{x}} \in (S^2)_{\boldsymbol{r}}$. Then we write the total travel time T as a function of the two phase directions $(\boldsymbol{\alpha}^r_{\boldsymbol{x}}, \boldsymbol{\alpha}^s_{\boldsymbol{x}})$

$$T(\boldsymbol{\alpha}^r_{\boldsymbol{x}}, \boldsymbol{\alpha}^s_{\boldsymbol{x}}) = \tau(\boldsymbol{\alpha}^s_{\boldsymbol{x}}, \boldsymbol{x}) + \tau(\boldsymbol{\alpha}^r_{\boldsymbol{x}}, \boldsymbol{x}). \quad (9)$$

We also define the gradient of the total travel time

$$\boldsymbol{\gamma}^0_{\boldsymbol{x}} = \boldsymbol{\gamma}^s_{\boldsymbol{x}} + \boldsymbol{\gamma}^r_{\boldsymbol{x}}, \quad (10)$$

which, in acquisition coordinates, corresponds to

$$\boldsymbol{\gamma}^0_{\boldsymbol{x}} = \nabla_{\boldsymbol{x}} T(\boldsymbol{r}, \boldsymbol{x}, \boldsymbol{s}). \quad (11)$$

We introduce a short hand notation, setting $\boldsymbol{\gamma}^0_{\boldsymbol{x}}$ to be the gradient of total travel time at $\boldsymbol{x}$ as defined in (10).

Dip, scattering angle and azimuth

In the framework of generalized Radon transforms (GRT) it is natural to work in local coordinates at the image point. We will use an indexing in migration dip $\boldsymbol{\alpha}^0$, scattering angle θ and azimuth ψ (third Euler angle around $\boldsymbol{\alpha}^0_{\boldsymbol{x}}$). For an image point $\boldsymbol{x} \in \mathcal{D}$ these variables

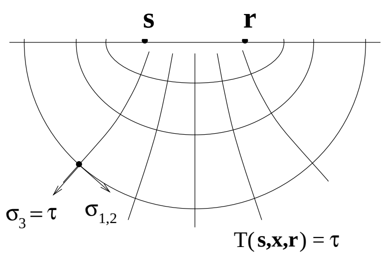

Figure 2. Isochrons and curvi-linear coordinates

are defined as

$$\boldsymbol{\alpha}^0 = \frac{\boldsymbol{\gamma}^0_{\boldsymbol{x}}(\boldsymbol{x})}{|\boldsymbol{\gamma}^0_{\boldsymbol{x}}(\boldsymbol{x})|} \in S^2,$$

$$\cos\theta = \boldsymbol{\alpha}^s_{\boldsymbol{x}} \cdot \boldsymbol{\alpha}^r_{\boldsymbol{x}} \in E_\theta \subset [0,\pi), \tag{12}$$

$$\boldsymbol{\psi} = \frac{(\boldsymbol{\alpha}^s_{\boldsymbol{x}} \cdot \boldsymbol{\alpha}^0_{\boldsymbol{x}})\boldsymbol{\alpha}^r_{\boldsymbol{x}} - (\boldsymbol{\alpha}^r_{\boldsymbol{x}} \cdot \boldsymbol{\alpha}^0_{\boldsymbol{x}})\boldsymbol{\alpha}^s_{\boldsymbol{x}}}{\sin\theta} \in E_\psi \subset S^1 .$$

This gives us a novel coordinate frame: the migration dip vector replaces the midpoint and the combination of the two angles ('directivity' $\boldsymbol{\alpha}^h_{\boldsymbol{x}}$) replaces the offset vector,

$$(\boldsymbol{\alpha}^s_{\boldsymbol{x}}, \boldsymbol{\alpha}^r_{\boldsymbol{x}}) \to (\boldsymbol{\alpha}^0_{\boldsymbol{x}}, \theta, \psi) = (\boldsymbol{\alpha}^0_{\boldsymbol{x}}, \boldsymbol{\alpha}^h_{\boldsymbol{x}}) . \tag{13}$$

For the image point dependent vectors, we use a subscript to indicate which point the vector is related to.

We summarize the different choices of variables by setting up an overview of the change from the surface/acquisition controlled variables to the subsurface/image-point controlled variables. For a given scattering point $\boldsymbol{x} \in \mathcal{D}$, with two rays connecting it with respectively a source $\boldsymbol{s} \in \partial S$ and a receiver $\boldsymbol{r} \in \partial R$, we have

$$\begin{array}{cl} \boldsymbol{x}^0, \boldsymbol{x}^h & \text{midpoint and offset,} \\ \Updownarrow & \\ \boldsymbol{r}, \boldsymbol{s} & \text{source and receiver,} \\ \Updownarrow & \\ \boldsymbol{\alpha}^r_{\boldsymbol{x}} \, , \, \boldsymbol{\alpha}^s_{\boldsymbol{x}} & \text{phase directions,} \\ \Updownarrow & \\ \boldsymbol{\alpha}^0_{\boldsymbol{x}} \, , \, \boldsymbol{\alpha}^h_{\boldsymbol{x}} & \text{migration dip,} \\ & \text{scattering angle and azimuth.} \end{array} \tag{14}$$

These angles and vectors are shown in Figure 1. We can uniquely go from acquisition-surface coordinates to local coordinates in the scattering domain.

Isochron coordinates

We now reconsider the coordinates associated with $\boldsymbol{x} \in \mathcal{D}$. We change those coordinates to curvi-linear ones associated with the acquisition geometry in accordance with the following construction. Let the level surface associated with the total travel time function be defined as

$$\Sigma(\tau) = \{\boldsymbol{x} \in \mathcal{D} : T(\boldsymbol{r}, \boldsymbol{x}, \boldsymbol{s}) = \tau\} , \tag{15}$$

for $\boldsymbol{s} \in \partial S$, $\boldsymbol{r} \in \partial R$ fixed. This surface is the τ isochron for a fixed source and receiver pair $(\boldsymbol{s}, \boldsymbol{r})$. The isochron is made up of all points $\boldsymbol{x} \in \mathcal{D}$ where the travel time from source to scatterer to receiver is equal to the isochron time τ. On the isochron we introduce the curvi-linear coordinates $\boldsymbol{\sigma} = (\sigma_\mu, \sigma_3), \mu = 1, 2$, such that σ_μ are coordinates in the surface of the isochron and σ_3 is the local coordinate normal to the isochron, in the $\boldsymbol{\alpha}^0_{\boldsymbol{x}}$ direction. Thus, σ_3 is the coordinate along the gradient of the total travel time, see Figure 2. If σ_3 represents the actual time of the isochron, we set $\sigma_3 = \tau$. To relate an infinitesimal volume in $\boldsymbol{x}$, i.e. the volume form, to the one in isochron coordinates $\boldsymbol{\sigma}$, we need the Jacobian

$$\begin{array}{cl} \boldsymbol{x} & \\ \downarrow & \dfrac{\partial(\boldsymbol{x})}{\partial(\boldsymbol{\sigma})} = |\partial_{\sigma_3}\boldsymbol{x} \cdot (\partial_{\sigma_1}\boldsymbol{x} \times \partial_{\sigma_2}\boldsymbol{x})| \\ & = \dfrac{|\partial_{\sigma_1}\boldsymbol{x} \times \partial_{\sigma_2}\boldsymbol{x}|}{|\nabla_{\boldsymbol{x}} T(\boldsymbol{r}, \boldsymbol{x}, \boldsymbol{s})|} . \\ \boldsymbol{\sigma} & \end{array} \tag{16}$$

For a homogeneous medium this change would be the one from Cartesian to elliptic coordinates.

To make the mapping from a source or a receiver at the surface to an image point along the rays unique, we introduce the projection of the slowness vectors at the source onto ∂S, and the projection of the slowness vectors at the receiver onto ∂R (de Hoop, 1998) (Figure 1). These projections are given by

$$\begin{aligned} \boldsymbol{p}_{\boldsymbol{s}}(\boldsymbol{x}, \boldsymbol{\alpha}^s_{\boldsymbol{x}}) &= \boldsymbol{\gamma}^s_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{\alpha}^s_{\boldsymbol{x}}) - (\beta(\boldsymbol{s}) \cdot \boldsymbol{\gamma}^s_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{\alpha}^s_{\boldsymbol{x}}))\beta(\boldsymbol{s}) , \\ \boldsymbol{p}_{\boldsymbol{r}}(\boldsymbol{x}, \boldsymbol{\alpha}^r_{\boldsymbol{x}}) &= \boldsymbol{\gamma}^r_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{\alpha}^r_{\boldsymbol{x}}) - (\beta(\boldsymbol{r}) \cdot \boldsymbol{\gamma}^r_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{\alpha}^r_{\boldsymbol{x}}))\beta(\boldsymbol{r}) , \end{aligned} \tag{17}$$

in which

$$\begin{aligned} \beta(\boldsymbol{s}) &= \text{unit normal to } \partial S \text{ at source,} \\ \beta(\boldsymbol{r}) &= \text{unit normal to } \partial R \text{ at receiver.} \end{aligned} \tag{18}$$

$\boldsymbol{p}_{\boldsymbol{s}}$ and $\boldsymbol{p}_{\boldsymbol{r}}$ can be directly estimated from the data as a function of $\boldsymbol{s}$ and $\boldsymbol{r}$ (by slant stacking). Using these, we have the following injective mapping from a source $\boldsymbol{s}$ and a receiver $\boldsymbol{r}$ down to the image point $\boldsymbol{x}$ on the τ isochron

$$(\boldsymbol{s}, \boldsymbol{r}, \tau, \omega\boldsymbol{p}_{\boldsymbol{s}}, \omega\boldsymbol{p}_{\boldsymbol{r}}, -\omega) \to (\boldsymbol{x}, \omega\boldsymbol{\gamma}^0_{\boldsymbol{x}}) , \tag{19}$$

where $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{\sigma})$ and $\sigma_3 = \tau$: for a fixed source-receiver pair and with the two conditions

- the travel time τ is fixed
- the rays intersect.

The mapping given by (19) induces a relation,

$$(\boldsymbol{p_s}, \boldsymbol{p_r}) \to (\boldsymbol{\gamma}^0_{\boldsymbol{x}(\sigma)}) \to (\boldsymbol{\alpha}^0_{\boldsymbol{x}(\sigma)}) . \tag{20}$$

This tells us that we know our position on the isochron from both the curvi-linear coordinates (σ_1, σ_2), and from the gradient of the total travel time $\boldsymbol{\gamma}^0_{\boldsymbol{x}}$. We will later use the change

$$\begin{array}{c} (\sigma_1, \sigma_2) \\ \downarrow \\ (\boldsymbol{\alpha}^0_{\boldsymbol{x}(\sigma_1,\sigma_2,\tau)}) \end{array} \qquad \frac{\partial(\sigma_1, \sigma_2)}{\partial(\boldsymbol{\alpha}^0_{\boldsymbol{x}(\sigma_1,\sigma_2,\tau)})} . \tag{21}$$

When we know our position on the isochron as a function of (σ_1, σ_2) we can use the mapping

$$\boldsymbol{\gamma}^0_{\boldsymbol{x}} \quad \xrightarrow[(\boldsymbol{\alpha}^s_{\boldsymbol{x}}, \boldsymbol{\alpha}^r_{\boldsymbol{x}})]{\boldsymbol{x} \in \Sigma(\tau)} \quad (\theta_{\boldsymbol{x}}, \psi_{\boldsymbol{x}}) \tag{22}$$

to give the scattering angle and azimuth.

For isotropic media, we obtain the migration dip and scattering angle from the gradient of total travel-time using equation (12), and introducing the Jacobian

$$\begin{array}{c} \boldsymbol{\gamma}^0_{\boldsymbol{x}} \\ \downarrow \\ (\boldsymbol{\alpha}^0_{\boldsymbol{x}}, \theta_{\boldsymbol{x}}) \end{array} \qquad \frac{\partial(\boldsymbol{\gamma}^0_{\boldsymbol{x}})}{\partial(\boldsymbol{\alpha}^0_{\boldsymbol{x}}, \theta_{\boldsymbol{x}})} . \tag{23}$$

In the isotropic case the magnitude of the gradient vector is given by

$$|\gamma^0_{\boldsymbol{x}}| = \frac{2\cos\theta/2}{c(\boldsymbol{x})} \tag{24}$$

Associated with $\boldsymbol{\gamma}^0_{\boldsymbol{x}}$ is a ray with take-off direction $\boldsymbol{\alpha}^0_{\boldsymbol{x}}$, that reaches the surface at $\overline{\boldsymbol{x}}^0(\boldsymbol{\alpha}^0_{\boldsymbol{x}}) \in \partial S \cup \partial R$, with slowness vector $\boldsymbol{\gamma}^0_{\overline{\boldsymbol{x}}^0(\boldsymbol{\alpha}^0_{\boldsymbol{x}})}$. Then, let

$$\boldsymbol{p}_0(\overline{\boldsymbol{x}}^0, \boldsymbol{\alpha}^0_{\boldsymbol{x}}) = \boldsymbol{\gamma}^0_{\overline{\boldsymbol{x}}^0(\boldsymbol{\alpha}^0_{\boldsymbol{x}})} - (\beta(\overline{\boldsymbol{x}}^0) \cdot \boldsymbol{\gamma}^0_{\overline{\boldsymbol{x}}^0(\boldsymbol{\alpha}^0_{\boldsymbol{x}})})\beta(\overline{\boldsymbol{x}}^0) , \tag{25}$$

be the projection of this vector onto the acquisition surface.

Source and receiver Green's functions

In the integral representation for the scattered field, we encounter the Green's functions for the rays connecting the source and the receiver with a scattering point. The leading order asymptotic approximation for the Green's function has the following structure

$$G(\boldsymbol{x}, \boldsymbol{x}', \omega) \quad = \quad A(\boldsymbol{x}, \boldsymbol{x}') \exp[i\omega\tau(\boldsymbol{x}, \boldsymbol{x}')]. \tag{26}$$

Here A denotes the pressure amplitude, satisfying the transport equation and originating from a point source at $\boldsymbol{x}'$. We have

$$A(\boldsymbol{x}, \boldsymbol{x}') = \frac{1}{4\pi[\rho(\boldsymbol{x})\rho(\boldsymbol{x}')\mathcal{M}]} ,$$

$$\mathcal{M} = \frac{c(\boldsymbol{x})c(\boldsymbol{x}') \left| \frac{\partial \boldsymbol{x}}{\partial q_1} \times \frac{\partial \boldsymbol{x}}{\partial q_2} \right|_{\boldsymbol{x}}}{\left| \frac{\partial \gamma_{\boldsymbol{x}}}{\partial q_1} \times \frac{\partial \gamma_{\boldsymbol{x}}}{\partial q_2} \right|_{\boldsymbol{x}'}} , \tag{27}$$

where (q_1, q_2) parameterize the rays originating at the source, and can be chosen to be $\boldsymbol{\alpha}^0_{\boldsymbol{x}'}$. Further $c(\boldsymbol{x})$ is the medium wave speed, and $\rho(\boldsymbol{x})$ is density. The source and receiver Green's functions are given as special cases of equation (26), setting $\boldsymbol{x}' = \boldsymbol{s}$ or $\boldsymbol{x}' = \boldsymbol{r}$. The Green's function in time is found by taking the inverse Fourier transform of equation (26). The Green's functions defined above satisfy the scalar wave equation for the medium given by $c(\boldsymbol{x})$.

Forward GRT operator in the CSA domain

In this section, we introduce the volume scattering representation in the Born approximation for the scattered field. Then, we show how this volume integral can be recast into an integral over isochron surfaces. First, we introduce the perturbation representation for the wave speed

$$c^{-2}(\boldsymbol{x}) = c_0^{-2}(\boldsymbol{x}) + c^{(1)}(\boldsymbol{x}) \tag{28}$$

where $c_0^{-2}(\boldsymbol{x})$ is the background model and $c^{(1)}(\boldsymbol{x})$ is the acoustic scattering coefficient. The scattered field is given by the forward generalized Radon transform (Miller *et al.*, 1987)

$$p(\boldsymbol{r}, \boldsymbol{s}, t) \simeq (\boldsymbol{L}[c_0]c^{(1)})(\boldsymbol{r}, \boldsymbol{s}, t) = \\ -\frac{\partial^2}{\partial t^2} \int_{\mathcal{D}} \mathrm{d}\boldsymbol{x} A(\boldsymbol{r}, \boldsymbol{x}, \boldsymbol{s}) c^{(1)}(\boldsymbol{x}) \delta[t - T(\boldsymbol{r}, \boldsymbol{x}, \boldsymbol{s})] \tag{29}$$

This operator is a function of $c_0(\boldsymbol{x})$, and gives the scattered field at a time t for a fixed source-receiver pair, as an integral over the support $\mathcal{D}$ of $c^{(1)}$. The amplitude term is $A(\boldsymbol{r}, \boldsymbol{x}, \boldsymbol{s}) = A(\boldsymbol{s}, \boldsymbol{x})A(\boldsymbol{x}, \boldsymbol{r})$, where the amplitudes from source to scatterer and scatterer to receiver are given as special cases of equation (27).

The hyper-surface over which the integral in the modeling operator (29) is taken is the isochron for a fixed travel time τ and a fixed source-receiver pair $(\boldsymbol{s}, \boldsymbol{r})$ as defined in (15). We will use the same kind of isochron to re-model data from a scattering angle dependent scattering coefficient. On the isochron we introduce curvi-linear coordinates as above. Let $d\Sigma$ denote a surface element on the isochron. Then we can recast the volume integral taken over isochrons to a surface integral taken over isochron coordinates. Using eq.(16) we have

$$dx = \left. \frac{1}{|\nabla T(\boldsymbol{s},\,.\,,\boldsymbol{r})|} \right|_{\boldsymbol{x}(\boldsymbol{\sigma})} \mathrm{d}\tau \mathrm{d}\Sigma\,,$$
$$\text{with } \mathrm{d}\Sigma = |\partial_{\sigma_1}\boldsymbol{x} \times \partial_{\sigma_2}\boldsymbol{x}|\, \mathrm{d}\sigma_1 \mathrm{d}\sigma_2\,. \qquad (30)$$

In carrying out this transformation, we are assuming that there are no singular points along the travel time curve. To rewrite the general expression (29), using eq. (30), we make use of the property,

$$\int_{\mathcal{D}} \ldots \delta(\tau - T(\boldsymbol{r},\boldsymbol{x},\boldsymbol{s}))\mathrm{d}\boldsymbol{x} = \int_{T(\boldsymbol{s},\,.\,,\boldsymbol{r}) = \tau} \cdots \left. \frac{1}{|\nabla T(\boldsymbol{s},\,.\,,\boldsymbol{r})|} \right|_{\boldsymbol{x}(\boldsymbol{\sigma})} \mathrm{d}\Sigma\,. \qquad (31)$$

The forward operator in isochron coordinates then becomes

$$p(\boldsymbol{s},\boldsymbol{r},\tau) \simeq -\frac{\partial^2}{\partial\tau^2} \int_{T(\boldsymbol{s},\,.\,,\boldsymbol{r}) = \tau} A(\boldsymbol{s},\,.\,,\boldsymbol{r}) \cdot c^{(1)}(\,.\,) \left. \frac{\partial(\boldsymbol{x})}{\partial(\boldsymbol{\sigma})} \right|_{\boldsymbol{x}(\boldsymbol{\sigma})} \mathrm{d}\sigma_1\, \mathrm{d}\sigma_2\,. \qquad (32)$$

Since $(\boldsymbol{s},\boldsymbol{r})$ is fixed, $\boldsymbol{\alpha}^0$ is the only degree of freedom along the isochron. By introducing the Jacobian in eq.(21), taking care of the change from isochron coordinates into dip, we can express the forward operator as

$$p(\boldsymbol{s},\boldsymbol{r},\tau) \simeq -\frac{\partial^2}{\partial\tau^2} \int_{(S^2)_0} A(\boldsymbol{s},\,.\,,\boldsymbol{r}) \cdot c^{(1)}(\,.\,) \frac{\partial(\boldsymbol{x})}{\partial(\boldsymbol{\sigma})} \left. \frac{\partial(\sigma_1,\sigma_2)}{\partial(\boldsymbol{\alpha}^0)} \right|_{\boldsymbol{x}(\sigma_{1,2}(\boldsymbol{\alpha}^0),\tau)} \mathrm{d}\boldsymbol{\alpha}^0\,, \qquad (33)$$

where $(S^2)_0$ is the subset of the unit sphere where the dip can vary. This last change of variables is only valid in the absence of singular points on the isochron. From this operator and the mapping in eq.(23), we can find the forward operator in the common scattering angle domain $\boldsymbol{L}^\theta$, where $\theta = \theta(\boldsymbol{\alpha}^0)$.

What then remains is to derive the expression for the two Jacobians. Later in the Appendix, we derive an explicit expression in the case of a homogeneous medium. For the inhomogeneous case, we need to find the expression numerically. Since the input scattering coefficient is a function of scattering angle, we need to relate the scattering angle to the coordinates along the isochron.

The dual GRT operator

The dual generalized Radon transform is

$$\langle c^{(1)}(\boldsymbol{y})\rangle = -\int_{E_\theta}\int_{E_\psi}\int_{E_{\boldsymbol{\alpha}^0}} \cdot[A(\boldsymbol{\alpha}^0_{\boldsymbol{y}},\theta_{\boldsymbol{y}},\psi_{\boldsymbol{y}})]^* p(\boldsymbol{\alpha}^0_{\boldsymbol{y}},\psi_{\boldsymbol{y}},\theta_{\boldsymbol{y}},t)\, \delta''[T(\boldsymbol{\alpha}^0_{\boldsymbol{y}},\theta_{\boldsymbol{y}},\psi_{\boldsymbol{y}}) - t]\mathrm{d}\boldsymbol{\alpha}^0_{\boldsymbol{y}}\, \mathrm{d}\psi_{\boldsymbol{y}}\, \mathrm{d}\theta_{\boldsymbol{y}}\,. \qquad (34)$$

Equation (34) is a weighted Kirchhoff diffraction stack.

The inverse GRT operator

The inverse operator $\boldsymbol{U}^\theta$ for one single scattering angle θ, is given by

$$\langle c^{(1)}(\boldsymbol{y},\theta)\rangle = (\boldsymbol{U}^\theta[c_0]p)(\boldsymbol{y},\theta) = \frac{1}{\pi^2}\int_{E_\psi}\int_{E_{\boldsymbol{\alpha}^0}} \frac{|\gamma^0_{\boldsymbol{y}}(\boldsymbol{\alpha}^0_{\boldsymbol{y}},\theta_{\boldsymbol{y}},\psi_{\boldsymbol{y}})|^3}{A(\boldsymbol{\alpha}^0_{\boldsymbol{y}},\theta_{\boldsymbol{y}},\psi_{\boldsymbol{y}})} \cdot p(\boldsymbol{\alpha}^0_{\boldsymbol{y}},\theta_{\boldsymbol{y}},\psi_{\boldsymbol{y}},t = T(\boldsymbol{\alpha}^0_{\boldsymbol{y}},\theta_{\boldsymbol{y}},\psi_{\boldsymbol{y}}))\, \mathrm{d}\boldsymbol{\alpha}^0_{\boldsymbol{y}}\, \mathrm{d}\psi_{\boldsymbol{y}}\,. \qquad (35)$$

The inversion is performed in as a double integral over azimuth and migration dip. This operator is the inverse generalized Radon transform (Miller *et al.*, 1987). The output from (35) is a partial reconstruction of $c^{(1)}$ for a fixed scattering angle. To accomplish the full inversion we have to integrate over all $\theta \in E_\theta$.

Common scattering-angle sections

A common scattering-angle section (CSA section) is a partial reconstruction of the scattering potential that uses one fixed angle θ between the rays at all the image points. For the fixed scattering angle, we do the inversion by integrating over all the migration dips and azimuths using equation (35). Doing this, we will sweep all possible sources and receivers and we sum the contributions from the rays that hit in the vicinity of any of these data points. The output is a function of scattering angle: this is the angle dependent scattering coefficient or CSA-section. If we compare this process with the more familiar common offset type inversion, we see that the scattering angle plays the same role as offset does there. We have sections representing a fixed angle at every image point, instead of a fixed offset at the surface. In this setting, both the forward and the inverse operator are functions of micro-local coordinates, not of acquisition parameters as in the offset case. The inversion is an explicit integration over the phase directions, or migration dip and scattering angle and azimuth, instead of an implicit integration through sources and receivers. By arranging the operators in this manner, we do not have to introduce any Jacobian for the change from phase directions at the image point to acquisition coordinates on the surface. We access the data based on the ray end-point on the surface by tracing rays from the image point and up to the surface. The end point of each ray will be assigned to be either source or receiver, depending on where it hits the surface. Then, from two rays, one going to a source and one going to a receiver, and the total travel time, we can select the data.

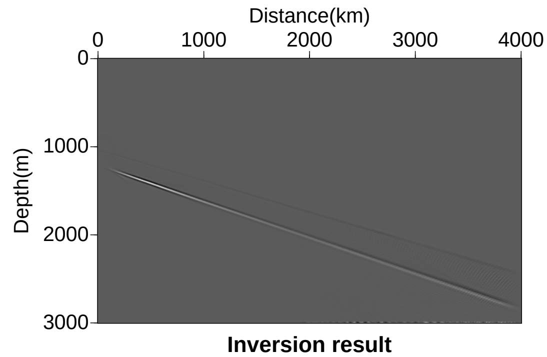

Figure 3. Inversion result, summed over all scattering angles. $v_0 = 900m/s, g_x = 0.3, g_z = 0.7$

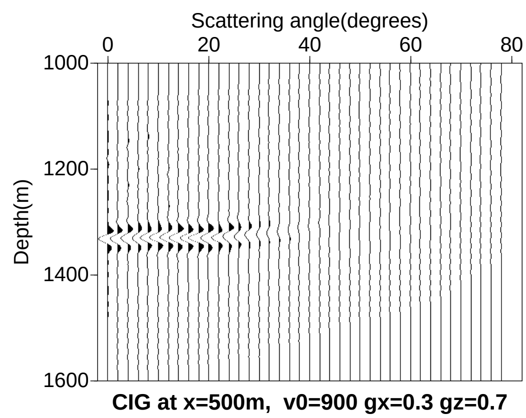

Figure 4. CIG at x=500m, $v_0 = 900m/s, g_x = 0.3, g_z = 0.7$

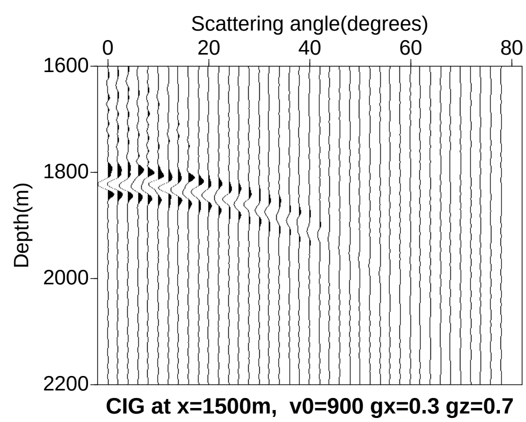

Figure 5. CIG at x=1500m, $v_0 = 900m/s, g_x = 0.3, g_z = 0.7$

Example

When we take a vertical section through a fixed surface position in the CSA data volume, we obtain a common image gather (CIG) in scattering angle and depth. We can perform the same kinds of analysis on these gathers as we usually do using CIG's in offset. The CIG will show where the image front appears in depth for different scattering angles. To illustrate how the CSA-sections can be used in processing, we will perform a short analysis of the common image-point gather in scattering angle, and show it's sensitive to errors in the velocity model. The relation between scattering angle and image depth can be used to do velocity analysis.

To test the concept we use a very simple 2D model. The model consists of two layers separated by a horizontal interface at 1500m. Each of the layers has a constant velocity gradient. The model parameters are for layer1: $v_0 = 1500m/s$, $g_x = 0.0$, $g_z = 0.3$, and layer2: $v_0 = 2500m/s$, $g_x = 0.2$, $g_z = 0.5$. We did an inversion with a background velocity given by $v_0 = 900m/s$, $g_x = 0.3$, $g_z = 0.7$. On the complete image, after summing all the scattering angles, the reflector is blurred and appears to be dipping to the right, see Figure 3. The fact that the reflector appears to be dipping is an effect caused by the lateral gradient introduced in the background model. To analyze the effect of the changing velocity we have extracted three common image gathers at different horizontal locations: $x = 500m$, $x = 1500m$, $x = 2500m$. At 500m the event appears to shallow and the image gather is curving upwards, indicating a too low velocity, Figure 4. As we move further to the right in the model the velocity increase due to the lateral gradient in the background model. We pick up the increasing velocity in the image gathers, where the event appears increasingly deeper as we move to the right. Also the curvature changes from upward to downward, Figure 4 and Figure 5. As we move further to the right the difference between the back-ground velocity model and the correct velocity increases, resulting in an increasing downward curvature in the CIG, see Figure 6.

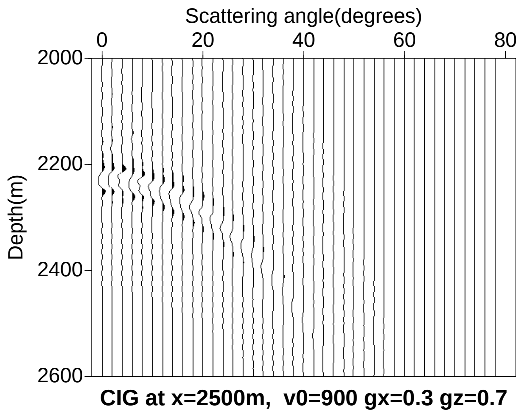

Figure 6. CIG at x=2500m, $v_0 = 900m/s, g_x = 0.3, g_z = 0.7$

Sensitivity transform

By combining several different experiments it is possible, to some extent, to determine spatial medium variations. In our case the different common scattering-angle sections are the different experiments, and we use the degree of coherency between them to derive the medium variations. To reveal the degree of coherency between the separate reconstructions made using eq.(35), we introduce the *differential semblance*

$$\frac{\delta \left\langle c^{(1)}(\boldsymbol{y}, \theta)\right\rangle}{\delta\theta}\,, \qquad \boldsymbol{y} \in \mathcal{D}\,, \tag{36}$$

For a medium update to be meaningful, the differential semblance should be minimized.

In order to get a global measure on how good the current computational state relates to the seismic data, we will do a comparison in the data domain. Then in order to compare the results from the inversion we re-generate data from the output of the inversion, after applying differential semblance. This is equivalent to applying the forward operator in the common scattering-angle domain to (36):

$$\boldsymbol{L}^{\theta}\left[\frac{\delta \left\langle c^{(1)}(\boldsymbol{y}, \theta)\right\rangle}{\delta\theta}\right]\,, \qquad \boldsymbol{y} \in \mathcal{D}\,. \tag{37}$$

Up to leading-order asymptotics, this operator follows

$$(\partial_\theta\, T(\,.\,,\,.\,,\,.\,))\;\partial_t + \partial_\theta \tag{38}$$

Since the medium contrast $c^{(1)}$ is the current output from the inversion process, we can describe the differential semblance as the derivative of the inverse operator (35) with respect to scattering angle θ. Substituting the derivative of the inverse operator into (37) we obtain a new operator, the *sensitivity transform*, that yields a direct measure on how good the current inversion result is. This operator is

$$\boldsymbol{S}^{\theta} \quad = \quad [I - \Delta]^{-1/2}\boldsymbol{L}^{\theta}\,\delta_\theta\boldsymbol{U}^{\theta}\,,$$

$$\Delta = \delta_\theta^2 + c_\triangle^{-2}\partial_\tau^2\,, \tag{39}$$

where we introduce the resolvent of the Laplacian Δ in order to balance the derivative with respect to θ. Let $\langle\,.\,,\,.\,\rangle$ be the L^2 inner product over $(\theta_{\boldsymbol{y}}, \psi_{\boldsymbol{y}}; t)$. Then

$$\epsilon[c_0] = \langle \boldsymbol{S}^{\theta}, \boldsymbol{S}^{\theta}\rangle\,, \tag{40}$$

can be employed as a measure to update the background wave speed c_0 in the zone illuminated by the rays emanating from $\boldsymbol{x}$ to the acquisition surface. If we have the correct background model when doing the inversion, the sensitivity transform of the result will vanish or be close to zero.

Discussion

We have presented a set of coordinate systems that are convenient to use when we work with modeling and inversion using generalized Radon transforms. Since we choose to work in image-point related variables we can control the number of rays that actually contribute to each receiver/source at the surface, hence resolving the problem of multi pathing. Using the phase directions and travel time as index of the rays will give a unique mapping from the image-point and up to either a source or a receiver.

The sensitivity transform is still in a very early development phase. However, used in the framework of the GRT and image-point variables, it has the potential to become a powerful tool for doing velocity analysis and model parameter updates.

Acknowledgments

SBD would like to thank Amoco Norway Oil Company who provided the scholarship that made his stay at CWP possible. Olav I. Barkved and Per Gunnar Folstad with Amoco have provided both motivation for geophysics and promotion of the scholarship at Amoco to make all of this possible. Also several people at CWP have been very helpful and provided great insight into various aspects of this material. The discussions with Norm Bleistein and Vladimir Grechka especially have been of great importance.

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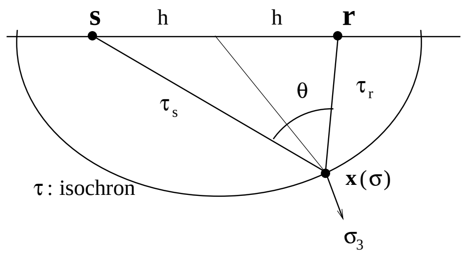

Figure A1. 2D isochron in homogeneous medium

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APPENDIX A: The forward operator in 2D homogeneous medium

In this appendix we present explicit formulas for some of the general expressions introduced in the text. We do this in 2D and a constant background medium. A typical isochron is illustrated in Figure A1. In this setting the isochron is an ellipse, with the source and receiver in each of the two foci, center in the origin, and the principal axis given by the travel time of the isochron τ, source-receiver offset $2h = |\boldsymbol{r} - \boldsymbol{s}|$ and constant wave speed c_0.

First we describe the ellipse parametrically with a convenient variable σ, as follows

$$\boldsymbol{x}(\sigma) = [a\cos(\sigma), b\sin(\sigma)] \qquad \sigma \in [\pi, 2\pi]\,, \tag{A1}$$

where

$$a = \frac{c_0\tau}{2} \quad , \text{and} \qquad b = \sqrt{\left(\frac{c_0\tau}{2}\right)^2 - h^2}\,. \tag{A2}$$

The total travel time is a function of σ_1, and follows as

$$\begin{aligned} \tau &= T(\boldsymbol{x}(\sigma_1)) = T(\boldsymbol{s}, \boldsymbol{x}(\sigma_1), \boldsymbol{r}) \\ &= \tau(\boldsymbol{s}, \boldsymbol{x}(\sigma_1)) + \tau(\boldsymbol{r}, \boldsymbol{x}(\sigma_1))\,. \end{aligned} \tag{A3}$$

Since we only have two spatial dimensions $\boldsymbol{x} = (x_1, x_2)$, we can choose curvi-linear coordinates along the isochron, setting σ_1 as the arc-length, $\sigma_2 = 0$, and σ_3 as the normal direction. We then make use of (30), relating a line segment $\mathrm{d}\boldsymbol{x}$ to the isochron parameter σ

$$\begin{aligned} \frac{\partial \boldsymbol{x}}{\partial(\sigma_1, \tau)} &= \frac{1}{|\nabla T(\boldsymbol{x})|}\,\mathrm{d}\tau\,\mathrm{d}\sigma_1 \\ &= \frac{c_0}{2\cos(\theta/2)}\frac{\partial(\sigma_1)}{\partial(\sigma)}\,\mathrm{d}\tau\,\mathrm{d}\sigma\,. \end{aligned} \tag{A4}$$

Where the additional partial derivative gives the relation between the parametric parameter σ and the arc-length σ_1

$$\frac{\partial(\sigma_1)}{\partial(\sigma)} = \frac{1}{2}\sqrt{-2h^2 + c_0^2\tau^2 - 2h^2\cos(2\sigma)}\,. \tag{A5}$$

Since we have a scattering angle dependent input, we need to relate the scattering angle to the position along the isochron. First we express the scattering angle as a function of σ

$$\theta = \cos^{-1}\left(\frac{-6h^2 + c_0^2\tau^2 + 2h^2\cos(2\sigma)}{c_0^2\tau^2 - 4h^2\cos^2(\sigma)}\right)\,. \tag{A6}$$

Then the Jacobian is given by

$$\frac{\partial(\theta)}{\partial(\sigma)} = \frac{4h\sqrt{c_0^2\tau^2 - 4h^2}\cos^2(\sigma)}{2h^2 - c_0^2\tau^2 + 2h^2\cos(2\sigma)}\,. \tag{A7}$$

In the forward operator, we need the expression for $1/\cos(\theta/2)$, which can be found using (A6)

$$\begin{aligned} \frac{1}{\cos(\theta/2)} &= \pm\sqrt{\frac{2}{1+\cos(\theta)}} \\ &= \pm\sqrt{\frac{c_0^2\tau^2 - 4h^2\cos^2(\sigma)}{-4h^2 + c_0^2\tau^2}} \\ &= -\sqrt{\frac{c_0^2\tau^2 - 4h^2\cos^2(\sigma)}{-4h^2 + c_0^2\tau^2}}\,, \end{aligned} \tag{A8}$$

where we only need to keep the minus sign since $\sigma \in [\pi, 2\pi]$. If we then combine the Jacobians and the term with cosine given above, we obtain

$$\frac{\partial(\sigma_1)}{\partial(\sigma)}\frac{\partial(\theta)}{\partial(\sigma)}\frac{1}{\cos(\theta/2)} = -2\,h\,\cos(\sigma)\,. \tag{A9}$$

The forward operator then becomes

$$\begin{aligned} &p(\boldsymbol{s}, \boldsymbol{r}, \tau) = (\boldsymbol{L}^\theta S)(\boldsymbol{s}, \boldsymbol{r}, \tau) \\ &= \frac{\partial^2}{\partial\tau^2}\int_\pi^{2\pi} A(\boldsymbol{s}, \boldsymbol{x}(\sigma, \tau), \boldsymbol{r}) \\ &\cdot c^{(1)}(\theta(\sigma, \tau), \boldsymbol{x}(\sigma, \tau))c_0\,h\cos(\sigma, \tau)\,\mathrm{d}\sigma \end{aligned} \tag{A10}$$

Where $\theta(\sigma, \tau)$ is given by (A6).

Then, by applying differential semblance to (35), and substituting the whole operator into (A10) we have the sensitivity transform.