2.5D downward continuation using data mapping theory (corrected)

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ABSTRACT

Data mapping is a procedure for transforming seismic or acoustic data collected with one set of source/receiver locations and wavespeed model, to data equivalent to that collected in a different source/receiver configuration and wavespeed model. A major application of this general process is downward continuation of the wavefield, or wave-equation datuming. Unlike many previous implementations of the downward continuation process, this study proposes a procedure that is not only kinematically correct, but also dynamically correct in heterogeneous media, at least in a model-consistent sense.

Here the general, "true-amplitude" data mapping platform of Bleistein & Jaramillo (1997) is specialized to the process of downward continuation of receivers given a common source, and to downward continuation of sources given a common receiver. This is done by stationary phase analysis of the general data mapping integral, as specialized to the downward continuation problem for common-source or common-receiver geometries. The asymptotic analysis produces a stationary point for each set of input and output locations, and amplitudes that depend on parameters along the ray paths connecting this stationary point with the source and receiver locations. In general heterogeneous media, a large investment in external ray tracing is required to locate this point and calculate the associated ray quantities. The procedure may be greatly simplified, however, when assumptions about the wavespeed structure of the medium can be made, and analytic results can be used in conjunction with, or in place of, ray tracing results.

Key words: data mapping, downward continuation, wave-equation datuming

Introduction

Reverse propagation of the surface wavefield is an important tool in seismic imaging, processing, and interpretation. In areas of irregular topography or complex near-surface structure, datum correction of seismic surveys via wave-theoretical downward continuation may be preferable to an approximate, vertical static correction (Zhu, et al., 1995, Salinas, 1997). This is especially true when the near surface contains high velocities or other structures that make the assumption of vertical propagation suspect. The ability to perform datuming with accurate amplitudes is also highly desirable for any application involving amplitude-variation-with-offset (AVO) and amplitude-variation-with-angle (AVA) analysis of data collected in geologically challenging areas. More specialized applications have also recently arisen in seismic migration. Most notable is the method of Bevc (1995), which alleviates multipathing problems when first-arrival times are used in Kirchhoff migration by alternating steps of migration and downward continuation.

Expressions for Kirchhoff-type wave-equation datuming in homogeneous media have previously been derived and implemented by Berryhill (1979). More recently, these results have been modified by Bevc (1995) to account for 3D geometrical spreading in 2D media, using stationary phase analysis. Further, kinematic issues in wave-equation datuming have been addressed using these results by Salinas (1997). These previous works, however, are all based on the assumption of constant wavespeed, and are therefore not dynamically accurate in variable media. In this study, a method is proposed to obtain expressions for "true-amplitude" downward continuation in heterogeneous media, at least in the sense that the amplitude mapping is consistent with the assumed wavespeed model.

This method involves the derivation and analysis of integral expressions for 2.5D downward continuation of receivers and sources for offset data based on the 2.5D seismic data mapping theory of Bleistein & Jaramillo (1997). This approach is desirable for several reasons. First, this is a general, dynamically correct theory for the transformation of data from one survey configuration to another. Second, 2.5D theory provides correct amplitudes for data that are truly 3D but collected in a linear survey, provided that the survey line is oriented normal to the dominant structural axis. Finally, the general data mapping platform allows the development of specific methods for repositioning of data to suit subsequent processing or particular applications. More explanation of these issues will be given later.

The result of the basic derivations are formulas for the downward continuation of both receivers and sources, using the data mapping platform as a starting point. The kinematics will be shown to be a function of the traveltime between the input and output locations of the points being datumed. This can be found for general media via ray tracing between the recording surface and the datum, so a kinematically accurate result can be obtained easily in any implementation.

Amplitude, however, depends on the evaluation of various Jacobians on the paths to and from a particular stationary point in the medium. This stationary point must be evaluated for each combination of input and output locations and times. In this paper, stationary phase analysis locates this point at the intersection of an isochron related to a particular input time and the ray connecting the input and output locations. This input isochron has the property that it is tangent to the isochron associated with the desired output time at this intersection point. However, only the times related to travel paths through this point are initially known, not the location of the point itself. In a general heterogeneous medium, the determination of this point and the evaluation of the integrals can be expensive, demanding a great deal of information from external ray tracing. While a specific procedure to accomplish this is not described here, it is evident that the large cost makes extrapolation in a completely general medium somewhat impractical. This arises from the fact that little or nothing can be assumed about the rays and isochrons. In more simple wavespeed models, though, enough generalizations about the isochron geometry may be possible to greatly simplify this process.

So, after the derivation of general expressions for 2.5D downward continuation, the example of a constant wavespeed medium is offered. For constant wavespeed, isochrons are elliptical and rays are straight lines. Therefore, analytic expressions can be developed for the location of the stationary point, given the traveltime. These expressions are derived for both a horizontal recording surface and a general, topographic recording surface. With respect to field data, where the subsurface is usually far from homogenous, the constant wavespeed case is primarily for illustrative purposes. However, given that correct kinematics can be easily installed, one can envision using the fast, constant wavespeed dynamic calculation as an approximation in applications where true-amplitude is not required.

A simple wavespeed model that is usually more representative of the actual subsurface is that of a depth-dependent wavespeed. In many field data problems, assuming a depth-dependent, $v(z)$ medium for the amplitude calculation provides a reasonable approximation of the subsurface wavespeed structure. This important case is left to a future paper.

General Derivations for 2.5D Downward Continuation

In this section, the 2.5D assumption and the general data mapping platform are presented. Then, a common-source geometry is applied to produce a general method for downward continuation of receivers, via stationary phase analysis of the data mapping integral. Following that, the analogous expression for the continuation of the sources given a common-receiver geometry is developed. Results in this section are completely general with respect to velocity heterogeneity, as well as to the shapes of the recording and datum surfaces.

2.5D data mapping formulation

To adequately characterize the 3D nature of the subsurface, 3D data aquisition and processing is generally required. Given the expense of conducting fully 3D surveys, however, it is often still attractive to collect data in linear surveys. The problem with this is that 2D processing is generally applied to such data. Truly 2D methods, however, assume that the source is a line source, and therefore undergoes 2D geometrical spreading. Since the seismic source is generally a point source of waves that spread in 3D, 2D processing will not produce correct amplitudes.

The 2.5D approach is one alternative that is applicable when the linear survey is conducted in the direction of the dominant structural dip of the subsurface. Mathematically, the x_1-direction in a Cartesian coordinate system can be oriented parallel to this survey line, allowing the velocity structure to be assumed constant in the normal, x_2-direction. Under these assumptions, no energy is reflected back into the vertical plane extending below the survey (the survey plane) from any out-of-plane reflectors, making the data independent of x_2. Then, the x_2 dependence of the appropriate 3D integral expression for propagation can be integrated by the method of stationary phase in the x_2-direction to eliminate that dependence. The resulting equation depends on only the in-plane variables, but will preserve amplitude variation due to 3D geometrical spreading. This is known as the two-and-one-half dimensional (2.5D) approximation. A full discussion can be found in Bleistein, et al. (1997).

The seismic data mapping formula of Bleistein & Jaramillo (1997) is a general integral expression that allows sources and receivers of some arbitrary input configuration to be mapped to some other configuration. In their paper, a general 3D mapping integral is derived by cascading an inversion formula and a modeling formula. This is achieved by replacing the spatially dependent reflectivity in the modeling formula with an appropriate expression for data inversion. This is equivalent to inverting the input data for reflectivity and then remodeling through this reflectivity distribution to the output locations, all in one process. Subsequent analysis yields a simplified true-amplitude data mapping platform in 3D. They then derive the 2.5D version by the above procedure, yielding a true-amplitude data mapping platform for parallel, linear input and output survey configurations, under the assumption of constant structure in the direction normal to the data lines.

Following Bleistein & Jaramillo (1997), the general 2.5D data mapping integral is

$$
\begin{aligned}
u_O(\xi_O,\omega_O) \approx{}& \frac{\sqrt{|\omega_O|}}{4\pi^2} e^{-i\pi/4\,\mathrm{sgn}\,(\omega_O)} \int \sqrt{|\omega_I|} \\
&\cdot e^{i\pi/4\,\mathrm{sgn}\,(\omega_I)}\, u_I(\xi_I,\omega_I)\, \frac{a_O(\boldsymbol{x},\xi_O)}{a_I(\boldsymbol{x},\xi_I)} \\
&\cdot \frac{|\nabla_{\boldsymbol{x}}\tau_O(\boldsymbol{x},\xi_O)|}{|\nabla_{\boldsymbol{x}}\tau_I(\boldsymbol{x},\xi_I)|} \frac{\sqrt{\sigma_{IS}+\sigma_{IG}}}{\sqrt{\sigma_{OS}+\sigma_{OG}}} \frac{\sqrt{\sigma_{OS}\,\sigma_{OG}}}{\sqrt{\sigma_{IS}\,\sigma_{IG}}} \\
&\cdot |H(\boldsymbol{x},\xi_I)|\, e^{(i\omega_O\tau_O(\boldsymbol{x},\xi_O)-i\omega_I\tau_I(\boldsymbol{x},\xi_I))} \\
&\cdot d\omega_I d\xi_I d^2x, \qquad (1)
\end{aligned}
$$

where $\boldsymbol{x}$ is the integration variable for the d^2x integral, and represents scattering points in the survey plane. The subscripts I and O denote quantities associated with the input and output configurations, respectively. The subscripts S and G indicate source and geophone, with all vectors evaluated in the survey plane. Source locations in the input data are given by $\boldsymbol{x}_{IS}$, receiver locations by $\boldsymbol{x}_{IG}$, and both are described by a parameter ξ_I. Analogous quantities in the output data are given by $\boldsymbol{x}_{OS}$, $\boldsymbol{x}_{OG}$, and ξ_O. The ratio,

$$
\frac{a_O(\boldsymbol{x},\xi_O)}{a_I(\boldsymbol{x},\xi_I)} = \frac{A(\boldsymbol{x},\boldsymbol{x}_{OS}(\xi_O))\,A(\boldsymbol{x}_{OG}(\xi_O),\boldsymbol{x})}{A(\boldsymbol{x},\boldsymbol{x}_{IS}(\xi_I))\,A(\boldsymbol{x}_{IG}(\xi_I),\boldsymbol{x})}, \qquad (2)
$$

is that of the output to input amplitudes, where each a is the product of the Green's function amplitudes, A, evaluated in the survey plane, as required by the 2.5D theory. The denominator is the product of the amplitudes associated with the paths from $\boldsymbol{x}_{IS}$ to the scattering point $\boldsymbol{x}$, and from $\boldsymbol{x}$ to $\boldsymbol{x}_{IG}$ in the input data. The numerator is the product of amplitudes associated with the paths from $\boldsymbol{x}_{OS}$ to the scattering point $\boldsymbol{x}$, and from $\boldsymbol{x}$ to $\boldsymbol{x}_{OG}$, in the output data. It is important to note that these are ray theoretical Green's functions, and are not valid in the presence of a caustic. The factor,

$$
H(\boldsymbol{x},\xi_I) = \det \left| \begin{matrix} \nabla_{\boldsymbol{x}}\tau_I(\boldsymbol{x},\xi_I) \\ \dfrac{\partial}{\partial\xi_I}\nabla_{\boldsymbol{x}}\tau_I(\boldsymbol{x},\xi_I) \end{matrix} \right|, \qquad (3)
$$

is the 2D Beylkin determinant for the problem, evaluated in-plane as prescribed by the 2.5D result. A full discussion of this can be found in Bleistein, et al. (1997). The symbol σ represents a parameter measured along the raypath from a scatterer to a source or receiver location, as

$$
\left(\frac{\partial\boldsymbol{x}}{\partial\sigma}\right)^2 = \boldsymbol{p}\cdot\boldsymbol{p} = |\nabla\tau|^2 = \frac{1}{c^2(\boldsymbol{x})}, \qquad (4)
$$

$$
d\sigma = c^2(\boldsymbol{x})\, d\tau, \qquad (5)
$$

where $\boldsymbol{p}$ is the ray vector, and τ is travel time along the ray. As before, the subscripts S and G indicate a parameter evaluated on the source-to-scatterer and scatterer-to-geophone paths, respectively.

The data mapping formulation is referred to here as a platform. This choice of terminology reflects the fact that this general integral expression can be modified, and hopefully simplified, for specific data mapping applications. One of the most important is downward extrapolation of the wavefield. This is acheived by asymptotic analysis of the 2.5D data mapping expression under the application of the source and receiver configurations specific to this procedure. Extrapolation of prestack data requires the downward continuation of both receivers and sources, and is approached here as a two-step, cascaded process.

Downward continuation of receivers

Consider the problem of the downward continuation of receivers with a fixed source (common-shot gather). This implies

$$\boldsymbol{x}_{OS} \;=\; \boldsymbol{x}_{IS} \;=\; \boldsymbol{x}_S \;=\; \text{constant}. \tag{6}$$

Assume a Cartesian coordinate system, where x_3 represents depth and x_1 is the direction of the linear survey on the surface. The vertical (x_1, x_3) plane will be referred to as the survey plane. Recall that in the 2.5D assumption, only propagation in this plane is relevant to the input data. In this case, integral (1) is over the survey plane, parameterized by $d^2x = dx_1 dx_3$.

To obtain an expression useful for application of the theory, the spatial integration in equation (1) should be reduced to one over the input data only. Therefore, the integration over the survey plane is performed by the method of stationary phase. This is simplified by a change of variables from (x_1, x_3) to coordinates based on reflection isochrons. In this context, an isochron denotes any surface of scattering points, that, for a given source and receiver, produces reflected arrivals with equal traveltime τ. (For example, in constant wavespeed media these curves are ellipses.)

So, for a given source and receiver pair in the input, and a given receiver location in the output, every depth point $\boldsymbol{x}$ lies on the intersection of two of these isochronal surfaces of constant reflected traveltime. One is associated with the traveltime τ_I from the source to the input receiver location, and the other is associated with the traveltime τ_O from the source to the output receiver location. Therefore, $d^2x = dx_1 dx_3$ can be transformed to $d\gamma d\tau$, where τ is an isochron and γ is a parameter along τ. This geometry is outlined in Figure 1. Since each subsurface point lies on both an input and an output isocrhon, the integration can be performed along either one. For reasons that become obvious in the following analysis, the input isochron, τ_I will be used. This means

$$\boldsymbol{x} \;=\; f\left(\tau_I, \gamma_I\right), \tag{7}$$

$$d^2x \;=\; dx_1 dx_3 \;=\; \left|\frac{\partial\left(\boldsymbol{x}\right)}{\partial\left(\tau_I, \gamma_I\right)}\right| d\gamma_I d\tau_I. \tag{8}$$

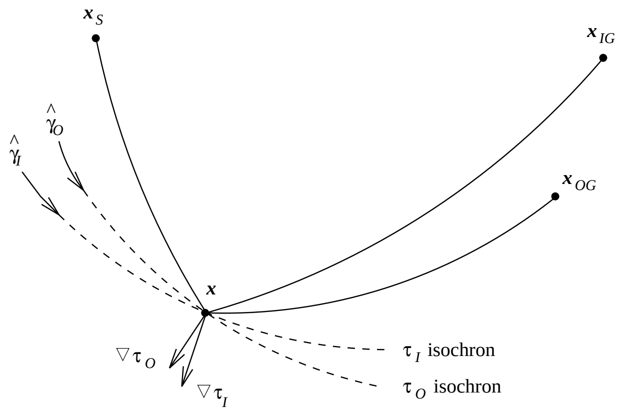

Figure 1. General geometry for downward continuation. Any depth point $\boldsymbol{x}$ can be expressed in coordinates (τ_I, γ_I), referencing the traveltime isochron for the scattered raypath from the source to the input receiver.

In terms of these new variables, equation (1) becomes

$$\begin{aligned} u_O\left(\xi_O, \omega_O\right) \;\approx\; & \frac{\sqrt{|\omega_O|}}{4\pi^2} e^{-i\pi/4\,\mathrm{sgn}\,(\omega_O)} \int \sqrt{|\omega_I|} \\ & \cdot\; e^{i\pi/4\,\mathrm{sgn}\,(\omega_I)} u_I\left(\xi_I, \omega_I\right) \frac{a_O\left(\boldsymbol{x}, \xi_O\right)}{a_I\left(\boldsymbol{x}, \xi_I\right)} \\ & \cdot\; \frac{\left|\nabla_{\boldsymbol{x}} \tau_O\left(\boldsymbol{x}, \xi_O\right)\right|}{\left|\nabla_{\boldsymbol{x}} \tau_I\left(\boldsymbol{x}, \xi_I\right)\right|} \frac{\sqrt{\sigma_{IS} + \sigma_{IG}}}{\sqrt{\sigma_{OS} + \sigma_{OG}}} \frac{\sqrt{\sigma_{OS}\;\sigma_{OG}}}{\sqrt{\sigma_{IS}\;\sigma_{IG}}} \\ & \cdot\; \left|H\left(\boldsymbol{x}, \xi_I\right)\right| e^{\left(i\omega_O \tau_O\left(\boldsymbol{x}, \xi_O\right) - i\omega_I \tau_I\left(\boldsymbol{x}, \xi_I\right)\right)} \\ & \cdot\; \left|\frac{\partial\left(\boldsymbol{x}\right)}{\partial\left(\tau_I, \gamma_I\right)}\right| d\gamma_I d\tau_I d\omega_I d\xi_I. \end{aligned} \tag{9}$$

The next step is to approximate the γ_I integral by the method of stationary phase. The phase in equation (9) is

$$\Phi \;=\; \omega_O \tau_O\left(\boldsymbol{x}, \xi_O\right) - \omega_I \tau_I\left(\boldsymbol{x}, \xi_I\right). \tag{10}$$

The γ_I integration is an integration of $\boldsymbol{x}$ along a particular input isochron, so that τ_I is constant with respect to differentiation over γ_I. Thus, stationarity is defined by the condition

$$\begin{aligned} \frac{\partial\Phi}{\partial\gamma_I} \;&=\; \omega_O \frac{\partial\tau_O\left(\boldsymbol{x}, \xi_O\right)}{\partial\gamma_I} \;=\; \omega_O \frac{\partial\tau_O}{\partial x_i}\frac{\partial x_i}{\partial\gamma_I} \\ &=\; \omega_O\left(\nabla_{\boldsymbol{x}}\tau_O \cdot \frac{\partial\boldsymbol{x}}{\partial\gamma_I}\right) \;=\; 0. \end{aligned} \tag{11}$$

Note that $\nabla_{\boldsymbol{x}}\tau_O$ is a vector normal to the τ_O isochron. Since $\boldsymbol{x}$ is constrained to be on the τ_I isochron, $(\tau_I =$

constant), in this integration, $\partial \boldsymbol{x}/\partial\gamma_I$ is, therefore, a vector that is tangent to the τ_I isochron.

At stationarity, the dot product is zero, requiring that these vectors be orthogonal. This occurs at all $\boldsymbol{x}$ where the tangents of the isochrons τ_O and τ_I are colinear, or equivalently, where the gradients of these isochrons are colinear. For a given source, input, and output locations, assuming no multipathing, there is a unique $\boldsymbol{x}$ on each τ_I isochron where this is true. For a fixed source, both the input and output configurations share the same ray from the source to $\boldsymbol{x}$. When the isochrons are tangent at $\boldsymbol{x}$, the scattered rays to the input and output receivers leave $\boldsymbol{x}$ at the same angle, and therefore the ray from $\boldsymbol{x}$ to $\boldsymbol{x}_{OG}$ overlays the ray from $\boldsymbol{x}$ to $\boldsymbol{x}_{IG}$. This means that stationarity occurs at all $\boldsymbol{x}$ along the ray passing through both $\boldsymbol{x}_{IG}$ and $\boldsymbol{x}_{OG}$, as shown in Figure 2.

An important feature of this result is that both the input and the output paths scatter at the same angle. Therefore, the angularly-dependent reflection coefficient is the same for both the input and output configurations. So, in this case, preservation of the input reflection coefficients in the output correctly maps the amplitude. *

Since the the raypath from the source to the integration point $\boldsymbol{x}$ at stationarity is the same path, and the source location is fixed for both the input and output geophone locations, it follows that

$$\sigma_{OS} \;=\; \sigma_{IS} \;=\; \sigma_S, \tag{12}$$

$$\frac{A\left(\boldsymbol{x}, \boldsymbol{x}_{OS}\left(\xi_O\right)\right)}{A\left(\boldsymbol{x}, \boldsymbol{x}_{IS}\left(\xi_I\right)\right)} \;=\; \frac{\sqrt{\sigma_{OS}}}{\sqrt{\sigma_{IS}}} \;=\; 1. \tag{13}$$

Also, along the stationary ray, the gradients of τ_I and τ_O are equal. So by the stationarity condition,

$$\frac{\left|\nabla_{\boldsymbol{x}}\tau_O\left(\boldsymbol{x}, \xi_O\right)\right|}{\left|\nabla_{\boldsymbol{x}}\tau_I\left(\boldsymbol{x}, \xi_I\right)\right|} \;=\; 1. \tag{14}$$

Evaluation of equation (9) under the stationary phase condition therefore yields

$$u_O\left(\xi_O, \omega_O\right) \;\approx$$

$$\cdot\; \frac{\sqrt{2\pi}}{4\pi^2}\int\sqrt{\frac{|\omega_O|}{|\Phi''|}}e^{-i\pi/4\,\mathrm{sgn}\,(\omega_O)+i\pi/4\,\mathrm{sgn}\,\left(\Phi''\right)}$$

$$\cdot\; \sqrt{|\omega_I|}e^{i\pi/4\,\mathrm{sgn}\,(\omega_I)}u_I\left(\xi_I, \omega_I\right)\frac{A_O\left(\boldsymbol{x}_{OG}\left(\xi_O\right), \boldsymbol{x}\right)}{A_I\left(\boldsymbol{x}_{IG}\left(\xi_I\right), \boldsymbol{x}\right)}$$

* Note that this result is a characteristic of the downward continuation problem, and does not generally occur in data mapping.

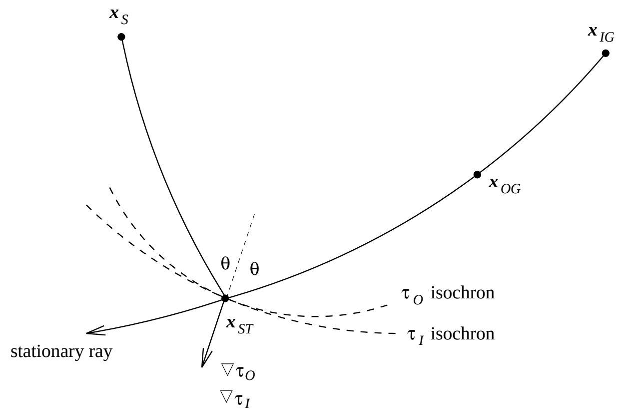

Figure 2. Geometry at stationarity for the γ_I integration. All points along the ray through both $\boldsymbol{x}_{IG}$ and $\boldsymbol{x}_{OG}$ are stationary. The delta function that results from the τ_I integration picks out a single stationary point on this ray, namely where it crosses the $\tau_O = t_O$ isochron. The input and output raypaths are coincident at stationarity.

$$\cdot\; \frac{\sqrt{\sigma_S+\sigma_{IG}}\sqrt{\sigma_{OG}}}{\sqrt{\sigma_S+\sigma_{OG}}\sqrt{\sigma_{IG}}}\left|H\left(\boldsymbol{x}, \xi_I\right)\right|$$

$$\cdot\; e^{\left(i\omega_O\tau_O(\boldsymbol{x},\xi_O)-i\omega_I\tau_I(\boldsymbol{x},\xi_I)\right)}$$

$$\cdot\; \left|\frac{\partial\left(\boldsymbol{x}\right)}{\partial\left(\tau_I, \gamma_I\right)}\right| d\tau_I d\omega_I d\xi_I. \tag{15}$$

The second derivative of the phase and the Jacobian remain to be evaluated. The second derivative calculation is identical in many data mapping applications, and has already been derived in Bleistein [3]. This result is

$$\frac{\partial^2\Phi}{\partial{\gamma_I}^2} \;=\; \omega_O\frac{\cos^2\theta}{c\left(\boldsymbol{x}\right)}\left(\frac{\partial s}{\partial\gamma_I}\right)^2\left(\kappa_{OG}-\kappa_{IG}\right), \tag{16}$$

where κ is the curvature of the wavefront along the stationary ray at the given receiver location due to a source at depth point $\boldsymbol{x}$, as depicted in Figure 3. The angle θ is half of the angle between the incoming ray from the source to $\boldsymbol{x}$, and the stationary ray between $\boldsymbol{x}$ to the receiver locations, shown in Figure 2. Since γ_I is measured along the isochron, but is not necessarily equal to the arclength, s, the derivative $\partial s/\partial\gamma_I$ appears. From equation (16), the sign of the second derivative depends on that of frequency, as well as that of the difference in the curvatures.

In the absence of velocity anomalies that result in lensing, given two wavefronts propagating along the same ray, the maximum wavefront divergence will always be displayed by the wavefront that travels the longest distance along the ray. The downward continuation prob-

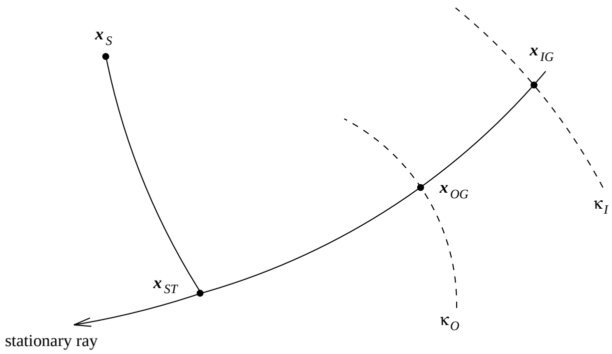

Figure 3. Example of the definitions of the curvatures κ_I and κ_O. These can be viewed as wavefront curvatures at the stationary point $\boldsymbol{x}_{ST}$ due to sources at either $\boldsymbol{x}_{IG}$ or $\boldsymbol{x}_{OG}$.

lem is defined by the condition that $\boldsymbol{x}_{OG}$ be at a greater depth than $\boldsymbol{x}_{IG}$. So, for any point on the stationary ray beyond κ_{OG}, the path length to the datuming surface is always shorter than that to the recording surface, meaning $\kappa_{OG} > \kappa_{IG}$ and the difference of curvatures in (16) is always positive. Since stationary points between the datum and the recording surface generally represent energy scattered from features above the datum that does not belong in the downward continued data, the integral is evaluated only for stationary points below $\boldsymbol{x}_{OG}$, strictly, and the above condition holds. Of course, velocity anomalies can occur along the stationary ray, between the recording and datuming surfaces, that change the sign of the curvature difference. This, however, requires that the ray pass through a caustic. As previously noted, the data mapping expression (1) is based on the use of ray-theoretical Green's functions that assume no multipathing. Therefore, our downward continuation results are subject to the same restriction, and are not valid in the presence of a caustic. Under these limitations, then, $\kappa_{OG} > \kappa_{IG}$, and therefore

$$\operatorname{sgn}\left(\Phi''\right) = \operatorname{sgn}\left(\omega_O\right). \tag{17}$$

Similarly, the expression for the Jacobian at stationarity is identical to that in other data mapping applications, and is derived in Bleistein [3]. This result is

$$\left|\frac{\partial\left(\boldsymbol{x}\right)}{\partial\left(\tau_I,\gamma_I\right)}\right| = \frac{c\left(\boldsymbol{x}\right)}{2\cos\theta}\left(\frac{\partial s}{\partial\gamma_I}\right). \tag{18}$$

So, using equations (16), (17), and (18), and assuming no caustics, the final form of integral (15) at stationarity becomes

$$u_O\left(\xi_O,\omega_O\right) \approx \frac{\sqrt{2\pi}}{8\pi^2}\int\frac{\sqrt{c^3(\boldsymbol{x})\left|\omega_I\right|}}{\cos^2\theta}e^{i\pi/4\,\operatorname{sgn}\left(\omega_I\right)}$$
$$\cdot u_I\left(\xi_I,\omega_I\right)\frac{A_O\left(\boldsymbol{x}_{OG}\left(\xi_O\right),\boldsymbol{x}\right)}{A_I\left(\boldsymbol{x}_{IG}\left(\xi_I\right),\boldsymbol{x}\right)}\frac{\sqrt{\sigma_S+\sigma_{IG}}}{\sqrt{\sigma_S+\sigma_{OG}}}\frac{\sqrt{\sigma_{OG}}}{\sqrt{\sigma_{IG}}}$$
$$\cdot\frac{\left|H\left(\boldsymbol{x},\xi_I\right)\right|}{\sqrt{\left|\kappa_{OG}-\kappa_{IG}\right|}}e^{\left(i\omega_O\tau_O\left(\boldsymbol{x},\xi_O\right)-i\omega_I\tau_I\left(\boldsymbol{x},\xi_I\right)\right)}$$
$$\cdot d\tau_I d\omega_I d\xi_I. \tag{19}$$

Now, take the inverse Fourier transform of equation (19) from ω_O to the output time t_O. Since the only ω_O dependence is in the phase, a delta function results, giving

$$u_O\left(\xi_O,t_O\right) \approx \frac{\sqrt{2\pi}}{8\pi^2}\int\frac{\sqrt{c^3(\boldsymbol{x})\left|\omega_I\right|}}{\cos^2\theta}e^{i\pi/4\,\operatorname{sgn}\left(\omega_I\right)}$$
$$\cdot u_I\left(\xi_I,\omega_I\right)\frac{A_O\left(\boldsymbol{x}_{OG}\left(\xi_O\right),\boldsymbol{x}\right)}{A_I\left(\boldsymbol{x}_{IG}\left(\xi_I\right),\boldsymbol{x}\right)}\frac{\sqrt{\sigma_S+\sigma_{IG}}}{\sqrt{\sigma_S+\sigma_{OG}}}\frac{\sqrt{\sigma_{OG}}}{\sqrt{\sigma_{IG}}}$$
$$\cdot\frac{\left|H\left(\boldsymbol{x},\xi_I\right)\right|}{\sqrt{\left|\kappa_{OG}-\kappa_{IG}\right|}}e^{-i\omega_I\tau_I\left(\boldsymbol{x},\xi_I\right)}\delta\left(t_O-\tau_O\left(\boldsymbol{x},\xi_O\right)\right)$$
$$\cdot d\tau_I d\omega_I d\xi_I. \tag{20}$$

At stationarity, the gradients of the input and output isochrons are equal, so the integration in equation (20) over τ_I can be transformed to an integral over τ_O by simply making the substitution $d\tau_I = d\tau_O$. Using the sifting property of the delta function, the integration is straightforward, yielding the result,

$$u_O\left(\xi_O,t_O\right) \approx \frac{\sqrt{2\pi}}{8\pi^2}\int\frac{\sqrt{c^3(\boldsymbol{x})\left|\omega_I\right|}}{\cos^2\theta}e^{i\pi/4\,\operatorname{sgn}\left(\omega_I\right)}$$
$$\cdot u_I\left(\xi_I,\omega_I\right)\frac{A_O\left(\boldsymbol{x}_{OG}\left(\xi_O\right),\boldsymbol{x}\right)}{A_I\left(\boldsymbol{x}_{IG}\left(\xi_I\right),\boldsymbol{x}\right)}\frac{\sqrt{\sigma_S+\sigma_{IG}}}{\sqrt{\sigma_S+\sigma_{OG}}}\frac{\sqrt{\sigma_{OG}}}{\sqrt{\sigma_{IG}}}$$
$$\cdot\frac{\left|H\left(\boldsymbol{x},\xi_I\right)\right|}{\sqrt{\left|\kappa_{OG}-\kappa_{IG}\right|}}e^{-i\omega_I\tau_I\left(\boldsymbol{x},\xi_I\right)}d\omega_I d\xi_I, \tag{21}$$

evaluated at the point $\boldsymbol{x}$ on the stationary ray where,

$$\tau_O\left(\boldsymbol{x},\xi_O\right) = t_O. \tag{22}$$

Note that integration over the delta function has chosen a single stationary point on what could previously only be constrained to a stationary ray. However, equation (21) still contains the isochron value τ_I in the phase. With the condition (22), though, τ_I can be determined given the traveltime along the stationary ray from $\boldsymbol{x}_{OG}$ to $\boldsymbol{x}_{IG}$, or τ_{IO}. Given this, τ_{IO} can be defined as

$$\tau_I\left(\boldsymbol{x},\xi_I\right) = \tau_O\left(\boldsymbol{x},\xi_O\right)+\tau_{IO}\left(\boldsymbol{x}_{IG},\boldsymbol{x}_{OG}\right)$$

$$= \quad t_O + \tau_{IO}\left(\boldsymbol{x}_{IG}, \boldsymbol{x}_{OG}\right). \qquad (23)$$

Two additional simplifications to equation (21) that are a result of the 2.5D dimensionality are derived in Bleistein, et al. (1997) and are

$$|H(\boldsymbol{x}, \xi_I)| \;=\; \frac{2\cos^2\theta}{c(\boldsymbol{x})}\left|\frac{\partial}{\partial \xi_I}\nabla_{\boldsymbol{x}}\tau_{IG}\right|, \qquad (24)$$

where the gradient is that of the traveltime to the input receiver location, and

$$|A(\boldsymbol{x}_O, \boldsymbol{x})| \;=\; \frac{\text{constant}}{\sqrt{\sigma_O \;\; |J(\boldsymbol{x}_O, \boldsymbol{x})|}}. \qquad (25)$$

Finally, then, the 2.5D downward continuation of receivers can be performed via the integral expression,

$$u_O(\xi_O, t_O) \;\approx\; \frac{1}{\sqrt{2\pi}}\int \frac{\sqrt{c(\boldsymbol{x})}}{\sqrt{|\kappa_{OG}-\kappa_{IG}|}}\frac{\sqrt{|J(\boldsymbol{x}_{IG}, \boldsymbol{x})|}}{\sqrt{|J(\boldsymbol{x}_{OG}, \boldsymbol{x})|}} \cdot \frac{\sqrt{\sigma_S+\sigma_{IG}}}{\sqrt{\sigma_S+\sigma_{OG}}}\left|\frac{\partial \nabla_{\boldsymbol{x}}\tau_{IG}}{\partial \xi_I}\right| D_f\left(\xi_I, \tau_I(\boldsymbol{x}, \xi_I)\right) d\xi_I,$$

$$D_f(\xi_I, t) \;=\; \frac{1}{2\pi}\int \sqrt{|\omega_I|}\; u_I(\xi_I, \omega_I)\, e^{-i\omega_I t + i\pi/4\,\mathrm{sgn}(\omega_I)} d\omega_I. \qquad (26)$$

D_f is a frequency-domain filter on the input data. The entire integration is evaluated at the depth point $\boldsymbol{x}$, which is defined as the point of intersection between the raypath connecting both the input and output receiver locations, and the isochron

$$\tau_I(\boldsymbol{x}, \xi_I) \;=\; t_O + \tau_{IO}\left(\boldsymbol{x}_{IG}, \boldsymbol{x}_{OG}\right). \qquad (27)$$

Downward continuation of sources

The same derivation can be performed for the remapping of source locations in the case of a fixed receiver (common-receiver gather). Recasting the previous derivation in terms of sources, where η_I and η_O are parameters describing the input and output source locations, the expression analogous to equation (26) is

$$u_O(\eta_O, t_O) \;\approx\; \frac{1}{\sqrt{2\pi}}\int \frac{\sqrt{c(\boldsymbol{x})}}{\sqrt{|\kappa_{OS}-\kappa_{IS}|}}\frac{\sqrt{|J(\boldsymbol{x}_{IS}, \boldsymbol{x})|}}{\sqrt{|J(\boldsymbol{x}_{OS}, \boldsymbol{x})|}} \cdot \frac{\sqrt{\sigma_G+\sigma_{IS}}}{\sqrt{\sigma_G+\sigma_{OS}}}\left|\frac{\partial \nabla_{\boldsymbol{x}}\tau_{IS}}{\partial \eta_I}\right| D_s\left(\eta_I, \tau_I(\boldsymbol{x}, \eta_I)\right) d\eta_I,$$

$$D_s(\eta_I, t) \;=\; \frac{1}{2\pi}\int \sqrt{|\omega_I|}\; u_I(\eta_I, \omega_I)\, e^{-i\omega_I t + i\pi/4\,\mathrm{sgn}(\omega_I)} d\omega_I. \qquad (28)$$

D_s is also frequency-domain filter on the input data. Here, the integration is evaluated at the depth point $\boldsymbol{x}$ defined as the point of intersection between the raypath connecting the source locations and the isochron

$$\tau_I(\boldsymbol{x}, \eta_I) \;=\; t_O + \tau_{IO}\left(\boldsymbol{x}_{IS}, \boldsymbol{x}_{OS}\right). \qquad (29)$$

This is a general result, but it is vital to recognize that in practice the receiver and source extrapolations will be a cascaded process, so care must be taken in implementation of the second process to account for the fact that part of the data has already been downward continued. This alters the values of the σ terms in the integral.

Implementation in General Media

In evaluation of the above integrals, only the desired output time, the input and output geometries, and the wavespeed model, are known initially. To obtain the correct kinematics, ray tracing between the recording and datuming surfaces can be used to provide τ_{IO}. and therefore τ_I, giving accurate travel times and phase. However, evaluation of the amplitude factors requires determination of the isochrons and rays so that the stationary points can be located, and the relevant raypaths, Jacobians, curvatures, and ray parameters can be calculated. Given what is known initially, this can only be accomplished in general, heterogenous media by the extensive application of ray tracing. Since these factors must be calculated for every input-output configuration at every output time, this process is extremely expensive, possibly prohibitively. So, further work is required to assess the practicality of implementation for the general case.

Constant Wavespeed Media

Implementation in general media is problematic because nothing can be assumed about the isochron or ray geometries. In constant wavespeed media, these geometries are known, and analytic expressions for the coordinates of the stationary point, as well as for the relevant ray quantities, can be derived. So, in this section, the general expressions for downward continuation are simplified for

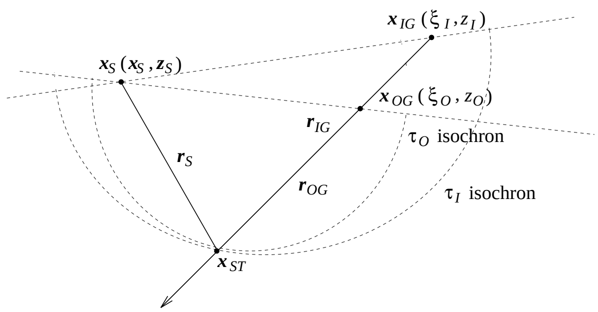

Figure 4. Geometry for receiver continuation in a constant wavespeed medium.

the case of a constant wavespeed medium, with arbitrary recording and datuming surfaces.

Receiver continuation in a constant wavespeed medium

For a constant wavespeed medium, several simplifications to integral (26) can be made. Assuming an arbitrary recording surface, let the source and receiver locations be given by

$$\boldsymbol{x}_{IG} = (\xi_I, z_I)\,, \quad \boldsymbol{x}_{OG} = (\xi_O, z_O)\,, \quad \boldsymbol{x}_S = (x_S, z_S)\,, \tag{30}$$

and define the stationary point as

$$\boldsymbol{x} \;=\; (\overline{x}_1, \overline{x}_3)\,. \tag{31}$$

As shown in Figure 4, rays are straight and isochrons are ellipses, tilted with respect to the coordinate system such that the source and the appropriate receiver location are at the foci. As a result, analytic expressions can be derived for the location of the stationary point that are functions of only the source and receiver locations and the traveltime.

In a constant wavespeed medium, the σ-factors are simply rc, where r is the linear path length, and at stationarity, the Jacobians are simply the path lengths between $\boldsymbol{x}$ and the receiver locations, or

$$|J(\boldsymbol{x}_{IG}, \boldsymbol{x})| \;=\; r_{IG} \;=\; \sqrt{(\overline{x}_1 - \xi_I)^2 + (\overline{x}_3 - z_I)^2} \tag{32}$$

$$|J(\boldsymbol{x}_{OG}, \boldsymbol{x})| \;=\; r_{OG} \;=\; \sqrt{(\overline{x}_1 - \xi_I)^2 + (\overline{x}_3 - z_O)^2}. \tag{33}$$

For homogeneous media, the radii of curvature of the wavefronts emitted from $\boldsymbol{x}$ and arriving at the input and output locations along the stationary ray are simply the distances between the points, and the wavefronts are spherical, so

$$\frac{1}{\kappa_{IG}} \;=\; r_{IG} \;=\; \frac{\sigma_{IG}}{c}, \qquad \frac{1}{\kappa_{OG}} \;=\; r_{OG} \;=\; \frac{\sigma_{OG}}{c}. \tag{34}$$

Finally, the gradient of the scattered traveltime is a vector pointing in the direction of r_{IG} with magnitude $1/c$. In the Appendix, the magnitude of its derivative is shown to be

$$\left|\frac{\partial \nabla_{\boldsymbol{x}} \tau_{IG}}{\partial \xi_I}\right| = \frac{G(\overline{x}_1, \overline{x}_3)}{c\, r_{IG}{}^2}, \tag{35}$$

where G is a factor that contains the influence of topographic variations in the recording surface, and is given by the expression

$$G(\overline{x}_1, \overline{x}_3) \;=\; \left[(\overline{x}_3 - z_I) - (\overline{x}_1 - \xi_I)\,\frac{\partial z_I}{\partial \xi_I}\right]. \tag{36}$$

These results allow equation (26) to be written

$$u_O(\xi_O, t_O) \;\approx\; \frac{1}{\sqrt{2\pi c}} \int d\xi_I \,\frac{G(\overline{x}_1, \overline{x}_3)}{r_{IG}} \cdot \frac{\sqrt{r_S + r_{IG}}}{\sqrt{r_S + r_{OG}}\,\sqrt{|r_{IG} - r_{OG}|}} D_f\left(\xi_I, \tau_I(\boldsymbol{x}, \xi_I)\right).$$

$$D_f(\xi_I, t) = \frac{1}{2\pi} \int \sqrt{|\omega_I|}\; u_I(\xi_I, \omega_I)\, e^{-i\omega_I t + i\pi/4\,\mathrm{sgn}(\omega_I)} d\omega_I. \tag{37}$$

The integral still contains functions of the stationary point, which requires determination of the location of $\boldsymbol{x}$. As previously described, this point lies at the intersection of the ray through $\boldsymbol{x}_{IG}$ and $\boldsymbol{x}_{OG}$ and the output isochron $\tau_O = t_O$, or equivalently, with the input isochron $\tau_I = t_O + \tau_{IO}$, where τ_{IO} is independently determined.

In the general derivation, it was noted that under the assumption of no caustics, downward continuation requires that the stationary point must always be below the datum. In the constant wavespeed case, this is equivalent to the condition

$$r_{IG} > r_{OG}. \tag{38}$$

Since values of t_O exist that violate this condition when evaluating the integral (37) over all output times, requirement (38) must be applied. Not only does this exclude scattered energy associated with features above the datum, but it also avoids the singularity in the denominator. Indeed, for the stationary phase method to be valid here, it is required that $2\omega_I(r_{IG} - r_{OG})/c >> 1$, usually $\geq \pi$.

Location of the stationary point for receiver continuation

Using the geometry shown in Figure 4, analytic expressions can be derived for the location of the stationary point by considering the tilted ellipse of the τ_O isochron and straight ray between the input and output locations. To facilitate the calculation for any general set of source and receiver locations placed on arbitrary recording and datuming surfaces, proceed as follows: For every combination of source and input-output receiver locations,

(i) shift the origin in both coordinate directions to the source location (x_S, z_S);
(ii) rotate the coordinate axes about the new origin so that the new horizontal axis is coincident with the major axis of the τ_O ellipse;
(iii) calculate the location of the stationary point and the required paths in the rotated coordinates.

The shift-rotation can be expressed in terms of the rotation angle and the primed quantities,

$$x_S{}' = 0, \qquad z_S{}' = 0, \qquad z_O{}' = 0, \tag{39}$$

$$\varphi = \tan^{-1}\left(\frac{z_O - z_S}{\xi_O - x_S}\right), \tag{40}$$

$$\xi_O{}' = (\xi_O - x_S)\cos\varphi + (z_O - z_S)\sin\varphi, \tag{41}$$

$$\xi_I{}' = (\xi_I - x_S)\cos\varphi + (z_I - z_S)\sin\varphi, \tag{42}$$

$$z_I{}' = -(\xi_I - x_S)\sin\varphi + (z_I - z_S)\cos\varphi, \tag{43}$$

$$h' = \frac{\xi_O{}'}{2}. \tag{44}$$

Given this change in reference, the major axis of the ellipse representing the τ_O isochron is aligned with the x_1' axis. It is centered at $x_1' = h'$, with h' being the signed half-offset between the source and the output receiver location in the primed coordinate frame. From the general derivation, $\tau_O = t_O$, and using these facts, the ellipse is defined by the equation

$$x_3{}'^2 = Q\left[(ct_O)^2 - 4\left(x_1{}' - h'\right)^2\right]. \tag{45}$$

$$Q = \left(\frac{(ct_O)^2 - 4h'^2}{4(ct_O)^2}\right). \tag{46}$$

The stationary ray is the line through $\boldsymbol{x}_{IG}$ and $\boldsymbol{x}_{OG}$, and is described by

$$x_1{}' = \left(\frac{\xi_I{}' - \xi_O{}'}{z_I{}'}\right) x_3{}' + \xi_O{}'. \tag{47}$$

The stationary point is one of the two intersections of these two curves. In our case of a straight ray that crosses the major axis of the ellipse, these two points are distinguished by the fact that one must lie above the $x_3{}' = 0$ surface, the other below. Because this construction exists only in the $x_3{}' > 0$ halfspace, the simplest approach to the problem is to solve equations (45) and (47) for $x_3{}'$, then choose only the positive solution. After some algebra, this produces

$$\overline{x}_3{}' = \sqrt{S + P^2} - P, \tag{48}$$

where

$$S = \left(\frac{Q\left((ct_O)^2 - \xi_O{}'^2\right) z_I{}'^2}{z_I{}'^2 + 4Q\left(\xi_I{}' - \xi_O{}'\right)^2}\right), \tag{49}$$

and

$$P = \left(\frac{2Q\left(\xi_I{}' - \xi_O{}'\right)\xi_O{}'\, z_I{}'}{z_I{}'^2 + 4Q\left(\xi_I{}' - \xi_O{}'\right)^2}\right), \tag{50}$$

with Q defined in equation (47). Equation (48) produces a value that is both positive and real, since, by the problem geometry, S is always positive.

Now, $\overline{x}_1{}'$ is easily found using this result and either equation (45) or (47). Since rotation of the coordinate system does not change the path lengths between the source and receiver locations and the stationary point, these lengths can be calculated directly in the primed coordinates using

$$r_S = r_S{}' = \sqrt{\overline{x}_1{}'^2 + \overline{x}_3{}'^2}, \tag{51}$$

$$r_{OG} = r_{OG}{}' = \sqrt{\left(\overline{x}_1{}' - \xi_O'\right)^2 + \overline{x}_3{}'^2}, \tag{52}$$

$$r_{IG} = r_{IG}{}' = \sqrt{\left(\overline{x}_1{}' - \xi_I'\right)^2 + \left(\overline{x}_3{}' - z_I'\right)^2}. \tag{53}$$

To evaluate the depth coordinates of the stationary point appearing in the integral, rotate the depth in the primed coordinates back to the unprimed frame, and undo the shift, via

$$\overline{x}_1 = \overline{x}_1{}'\cos\varphi - \overline{x}_3{}'\sin\varphi + x_S, \tag{54}$$

$$\overline{x}_3 = \overline{x}_1{}'\sin\varphi + \overline{x}_3{}'\cos\varphi + z_S. \tag{55}$$

Receiver continuation from a horizontal surface

The previous discussion is applicable to data collected on any arbitrary recording surface. If data are collected on a horizontal recording surface, however, several simplifications can be made. Define the flat surface by letting

$$z_I = \frac{\partial z_I}{\partial \xi_I} = 0. \tag{56}$$

Given this, equation (36) becomes

$$G(\overline{x}_1, \overline{x}_3) = \overline{x}_3, \tag{57}$$

making the integral for the flat surface case,

$$u_O(\xi_O, t_O) \approx \frac{1}{\sqrt{2\pi c}} \int d\xi_I \frac{\overline{x_3}}{r_{IG}} \cdot \frac{\sqrt{r_S + r_{IG}}}{\sqrt{r_S + r_{OG}}\,\sqrt{|r_{IG} - r_{OG}|}} D_f(\xi_I, \tau_I(\boldsymbol{x}, \xi_I)). \tag{58}$$

Location of the stationary point follows the same procedure as for the general case.

Source continuation in a constant wavespeed medium

Now complete the process by implementing a source continuation for constant wavespeed. This derivation parallels that for receiver continuation, where the roles of the source and receiver locations are interchanged. So define

$$\boldsymbol{x}_{IS} = (\eta_I, z_I), \quad \boldsymbol{x}_{OS} = (\eta_O, z_O), \quad \boldsymbol{x}_G = (x_G, z_G). \tag{59}$$

The resulting geometry is shown in Figure 5.

Applying simplifications analogous to those for receiver continuation to the general integral (28), gives the constant wavespeed expression

$$u_O(\eta_O, t_O) \approx \frac{1}{\sqrt{2\pi c}} \int d\eta_I \frac{G(\overline{x}_1, \overline{x}_3)}{r_{IS}} \cdot \frac{\sqrt{r_G + r_{IS}}}{\sqrt{r_G + r_{OS}}\,\sqrt{|r_{IS} - r_{OS}|}} D_s(\eta_I, \tau_I(\boldsymbol{x}, \eta_I)),$$

$$D_s(\eta_I, t) = \frac{1}{2\pi} \int \sqrt{|\omega_I|}\; u_I(\eta_I, \omega_I)\, e^{-i\omega_I t + i\pi/4\,\mathrm{sgn}(\omega_I)} d\omega_I, \tag{60}$$

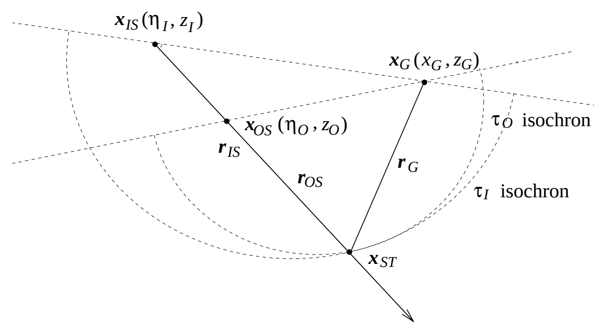

Figure 5. Geometry for source continuation in a constant wavespeed medium.

where the factor containing the topographic variation in the survey surface is

$$G(\overline{x}_1, \overline{x}_3) = \left[(\overline{x}_3 - z_I) - (\overline{x}_1 - \eta_I)\frac{\partial z_I}{\partial \eta_I}\right]. \tag{61}$$

As for the receiver case, the integral contains factors that depend on knowing the location of the stationary point, $\boldsymbol{x}$. Equation (60) is also subject to the same validity conditions as for expression (37), and under the assumption of no caustics, the condition analgous to requirement (38) in this case is

$$r_{IS} > r_{OS}. \tag{62}$$

As before, this condition eliminates scattered energy in the input data that should not appear in the downward continued data. An analogous condition for validity of the stationary phase approximation is also relevant.

Location of the stationary point for source continuation

Analytic expressions for the location of the stationary point in source continuation are analogous to those for the receiver continuation, and follow the geometry outlined in Figure 5. The same shift-rotation is performed for each set of source and receiver locations, into a coordinate system aligned with the axes of the τ_O isochron. This time, however, the origin will be shifted to the common-receiver location, (x_G, z_G). As expected, the results are those of the receiver continuation with the roles of the source and receiver locations interchanged. Using definitions (59), the shift-rotation is expressed in terms of the rotation angle and the primed quantities,

$$x_G{}' = 0, \qquad z_G{}' = 0, \qquad z_O{}' = 0, \tag{63}$$

$$\varphi = \tan^{-1}\left(\frac{z_O - z_G}{\eta_O - x_G}\right), \tag{64}$$

$$\eta_O{}' = (\eta_O - x_G)\cos\varphi + (z_O - z_G)\sin\varphi, \tag{65}$$

$$\eta_I{}' = (\eta_I - x_G)\cos\varphi + (z_I - z_G)\sin\varphi, \tag{66}$$

$$z_I{}' = -(\eta_I - x_G)\sin\varphi + (z_I - z_G)\cos\varphi, \tag{67}$$

$$h' = \frac{\eta_O{}'}{2}, \tag{68}$$

As before, this change in reference means the major axis of the ellipse representing the τ_O isochron is aligned with the x_1' axis. It is centered at $x_1' = h'$, as h' is the signed half-offset between the common-receiver location and the output source location in the primed frame. Again, use the fact that $\tau_O = t_O$. The equation of the ellipse is the same as in the previous case,

$$x_3{}'^2 = Q\left[(c\,t_O)^2 - 4\left(x_1{}' - h'\right)^2\right], \tag{69}$$

$$Q = \left(\frac{(c\,t_O)^2 - 4h'^2}{4(c\,t_O)^2}\right), \tag{70}$$

but interpreted under the source continuation geometry. The stationary ray is the line through $\boldsymbol{x}_{IS}$ and $\boldsymbol{x}_{OS}$, and is described by,

$$x_1{}' = \left(\frac{\eta_I{}' - \eta_O{}'}{z_I{}'}\right) x_3{}' + \eta_O{}'. \tag{71}$$

As before, the stationary point is the intersection of these two curves in the $x_3' > 0$ half-space. This produces the solution

$$\overline{x}_3{}' = \sqrt{S + P^2} - P, \tag{72}$$

where

$$S = \left(\frac{Q\left((c\,t_O)^2 - \eta_O{}'^2\right) z_I{}'^2}{z_I{}'^2 + 4Q\,(\eta_I{}' - \eta_O{}')^2}\right), \tag{73}$$

and

$$P = \left(\frac{2Q\left(\eta_I{}' - \eta_O{}'\right)\eta_O{}' z_I{}'}{z_I{}'^2 + 4Q\,(\eta_I{}' - \eta_O{}')^2}\right). \tag{74}$$

where, again, the result in equation (72) is real and positive since S is always positive. $\overline{x}_1'$ is easily found using this result and either equations (69) or equation (71), and the path lengths are computed in the primed coordinates using

$$r_G = r_G{}' = \sqrt{\overline{x}_1{}'^2 + \overline{x}_3{}'^2}, \tag{75}$$

$$r_{OS} = r_{OS}{}' = \sqrt{\left(\overline{x}_1{}' - \eta_O'\right)^2 + \overline{x}_3{}'^2}, \tag{76}$$

$$r_{IS} = r_{IS}{}' = \sqrt{\left(\overline{x}_1{}' - \eta_I'\right)^2 + \left(\overline{x}_3{}' - z_I'\right)^2}. \tag{77}$$

Rotating back to the unprimed frame and reversing the shift, the depth coordinates of the stationary point appearing in the integral can be determined via

$$\overline{x}_1 = \overline{x}_1{}'\cos\varphi - \overline{x}_3{}'\sin\varphi + x_G, \tag{78}$$

$$\overline{x}_3 = \overline{x}_1{}'\sin\varphi + \overline{x}_3{}'\cos\varphi + z_G. \tag{79}$$

Source continuation from a horizontal surface

Assuming a horizontal recording surface,

$$z_I = \frac{\partial z_I}{\partial \eta_I} = 0, \tag{80}$$

and the topographic factor G becomes

$$G\left(\overline{x}_1, \overline{x}_3\right) = \overline{x}_3, \tag{81}$$

making the integral for the flat suface case

$$u_O\left(\eta_O, t_O\right) \approx \frac{1}{\sqrt{2\pi c}} \int d\eta_I \, \frac{\overline{x}_3}{r_{IS}} \cdot \frac{\sqrt{r_G + r_{IS}}}{\sqrt{r_G + r_{OS}}\sqrt{|r_{IS} - r_{OS}|}} D_s\left(\eta_I, \tau_I\left(\boldsymbol{x}, \eta_I\right)\right). \tag{82}$$

The location of the stationary point can be found by the analytic method used in the topographic case.

Implementation options for constant wavespeed

While the assumption of constant wavespeed is not generally representative of real data, the case is not purely illustrative. Obtaining the correct kinematics in the continuation is simply a matter of having the correct values for τ_{IO}, as previously noted, obtained by ray-tracing between the recording surface and the datum. Given this, an implementation that is kinematically correct, but uses the constant wavespeed assumption for the dynamics, can be developed. This involves using the constant wavespeed form of the integral, approximating all path lengths by straight rays, but using ray-traced values for τ_{IO} to determine τ_I in the phase. In applications where true-amplitudes are not required, this is a fast, kinematically accurate option. The efficiency arises from the ability to calculate the straight raypaths on-the-fly using the analytic expressions developed in this section.

References

N. Bleistein & H. Jaramillo, 1997, *A Platform for Data Mapping in Scalar Models and Data Aquisition.* CWP-267, Colorado School of Mines.

N. Bleistein, J.K. Cohen, J. Stockwell, Jr., 1997, *Mathematics of Multidimensional Seismic Inversion.* Course notes, Colorado School of Mines, in preparation.

N. Bleistein, 1998, *2.5D Data Mappings.* In preparation.

J.R. Berryhill, 1979, *Wave-equation Datuming.* Geophysics, 44, 1329-1344.

D. Bevc, 1995, *Imaging Under Rugged Topography and Complex Velocity Structure.* Ph.D. Thesis, Stanford University.

T. Salinas, 1997, *The Influence of Near-Surface Time Anomalies in the Imaging Process.* M.Sc. Thesis, Colorado School of Mines.

X. Zhu, B.G. Angstman, and D.P. Sixta, 1995, *Overthrust Imaging with Tomo-Datuming.* 65th Annual Internat. Mtg., Soc. Expl. Geophys., Expanded Abstracts, 95, 1397-1400.

APPENDIX A: Derivation of the factor G for constant wavespeed media

This appendix contains the derivation of the topographic factor $G(\overline{x}_1, \overline{x}_3)$, for receiver continuation, as given in equations (35) and (36). First, assume the same definitions for the source and receiver locations as in (30),

$$\boldsymbol{x}_{IG} = (\xi_I, z_I), \quad \boldsymbol{x}_{OG} = (\xi_O, z_O), \quad \boldsymbol{x}_S = (x_S, z_S), \tag{A1}$$

as well as for the stationary point,

$$\boldsymbol{x} = (\overline{x}_1, \overline{x}_3). \tag{A2}$$

Referring to Figure 4, the path r_{IG} is represented by the vector

$$\boldsymbol{r}_{IG} = ((\overline{x}_1 - \xi_I), (\overline{x}_3 - z_I)). \tag{A3}$$

The gradient of the isochron associated with waves propagating along this path is a vector pointing in the direction of $\boldsymbol{r}_{IG}$ with magnitude $1/c$, or

$$\nabla_{\boldsymbol{x}} \tau_{IG} = \frac{\boldsymbol{r}_{IG}}{c\, r_{IG}} = \frac{1}{c\, r_{IG}} ((\overline{x}_1 - \xi_I), (\overline{x}_3 - z_I)). \tag{A4}$$

The derivative of this gradient with respect to the parameter ξ_I is then

$$\frac{\partial \nabla_{\boldsymbol{x}} \tau_{IG}}{\partial \xi_I} = \frac{\partial}{\partial \xi_I} \left(\frac{\overline{x}_1 - \xi_I}{c\, r_{IG}} \right) \hat{x}_1 + \frac{\partial}{\partial \xi_I} \left(\frac{\overline{x}_3 - z_I}{c\, r_{IG}} \right) \hat{x}_3, \tag{A5}$$

where $\hat{x}_1$ and $\hat{x}_3$ are unit vectors along the corresponding coordinate axes. Remembering that z_I is a function of ξ_I, performing the differentiations yields

$$\begin{aligned} \frac{\partial \nabla_{\boldsymbol{x}} \tau_{IG}}{\partial \xi_I} &= \\ & \frac{1}{c\, r_{IG}{}^3} \left[(\overline{x}_1 - \xi_I)(\overline{x}_3 - z_I) \frac{\partial z_I}{\partial \xi_I} - (\overline{x}_3 - z_I)^2 \right] \hat{x}_1 \\ & + \frac{1}{c\, r_{IG}{}^3} \left[(\overline{x}_1 - \xi_I)(\overline{x}_3 - z_I) - (\overline{x}_3 - z_I)^2 \frac{\partial z_I}{\partial \xi_I} \right] \hat{x}_3. \end{aligned} \tag{A6}$$

Calculating the magnitude of the vector produces a perfect square in terms of the derivative, and choosing the negative root gives the result with the correct, positive polarity, as

$$\left| \frac{\partial \nabla_{\boldsymbol{x}} \tau_{IG}}{\partial \xi_I} \right| = \frac{1}{c\, r_{IG}{}^2} \left[(\overline{x}_3 - z_I) - (\overline{x}_1 - \xi_I) \frac{\partial z_I}{\partial \xi_I} \right]. \tag{A7}$$

In the text, the second factor is expressed as G, so that

$$\left| \frac{\partial \nabla_{\boldsymbol{x}} \tau_{IG}}{\partial \xi_I} \right| = \frac{G(\overline{x}_1, \overline{x}_3)}{c\, r_{IG}{}^2}, \tag{A8}$$

where G is defined as in equation (36).

The expression for this factor for source continuation follows an analogous derivation, where only the source and receiver parameters are interchanged, yielding equation (58).