Transient elastic plane-wave diffraction by a semi-infinite fluid-filled crack in an isotropic, dispersive solid

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ABSTRACT

The canonical problem of the diffraction of a transient elastic plane wave by a semi-infinite fluid-filled crack in an unbounded, homogeneous, isotropic, dispersive solid is considered. The dispersion properties of the solid (and its associated loss mechanism) are accounted for by general, global relaxation functions that meet the causality criterion and encompass the influence of frictional forces and a Maxwell-type visco-elasticity that can be used to describe the temporal dispersion of seismic waves. The problem is solved by a combination of the Wiener-Hopf and modified Cagniard techniques, together with an appropriate correspondence principle that interrelates the transient elastic wave motion in a solid with global relaxation functions and the one in a perfectly elastic solid. Several examples are discussed and numerical results are presented.

Introduction

The detection of cracks in a solid is an issue of technical importance in the non-destructive evaluation of mechanical and civil-engineering structures as well as in the evaluation of oil reservoirs in fossil energy production. In all of these cases, position, extent, orientation and nature of the crack (fluid-filled or not) are to be inferred from scattered elastic wavefield measurements carried out at locations that are often remote from the location of the crack. In such a remote sensing setup, canonical scattering geometries are needed as test cases in feasibility studies that serve a number of purposes. As far as the forward problem is concerned, we mention the evaluation of accuracy and effectiveness of low- and high-frequency asymptotic approximations. With regard to the inverse problem, design and performance of inversion and imaging algorithms, and their associated acquisition of data can be analyzed. In the category of canonical problems, the ones that admit closed-form analytic solutions have the advantage that their answers can be constructed with arbitrary numerical accuracy, while a sensitivity analysis as to the influence of the different parameters can readily be carried out. The model of the present investigation also includes a class of dispersion mechanisms through which wave attenuation and dispersion in the host material can be quantitatively analyzed, while in accordance with the Kramers-Kronig causility relations.

Another area where canonical problems provide a useful tool of analysis is the field of controlled laboratory experiments. In fact, our study was started in direct relationship with a rock physics experiment where the elastic wavefield generated by an ultrasonic transducer and scattered by a fluid-filled crack present in a piece of uniform rock was measured. Here, the model data were used to calibrate the wavefield transmitted by the transducer and to further analyze the wavefield constituents that showed up in the measured data. In this respect, an additional advantage of having the calculated data in an analytic closed form is that each particular wave phenomenon showing up in the scattered wavefield (reflected body waves, transmitted body waves, intercepted body waves, diffracted body waves, surface waves, compressional to shear converted head waves) has a one-to-one relationship with a specific singularity in the analytic answer, a feature that is not available in purely computationally generated model data.

The diffraction of a plane elastic wave by a semi-infinite crack in an unbounded homogeneous, isotropic solid offers such a canonical problem. The problem is analyzed in the two-dimensional approximation where the wave motion takes place in the plane perpendicular to the edge of the crack. The corresponding elastic wave motion then decomposes into a 'horizontally polarized' shear-wave (*SH*-wave) constituent (with an out-of plane particle velocity of the wave motion) and a combined compressional-wave (*P*-wave) and 'vertically polarized' shear-wave (*SV*-wave) constituent (with an in-plane particle velocity) (Achenbach, 1973).

The particle velocity and the dynamic stress of these wave constituents are determined with the aid of a proper combination of the modified Cagniard method for analyzing transient waves in layered media (Achenbach, 1973; Aki & Richards, 1980; Miklowitz, 1978; De Hoop, 1958; de Hoop, 1960; De Hoop & van der Hijden, 1983; De Hoop & van der Hijden, 1984; De Hoop & van der Hijden, 1985; De Hoop, 1988b; De Hoop, 1988a; de Hoop, 1990) and the Wiener-Hopf or factorization method for analyzing 'semi-infinite' diffraction problems (Noble, 1958; Weinstein (Vaynshteyn), 1969). This combination has previously been applied to the analysis of the diffraction of plane elastic waves by a semi-infinite perfectly rigid plane baffle and a semi-infinite plane traction-free crack in a homogeneous, isotropic, perfectly elastic solid (Achenbach, 1973; De Hoop, 1958). Since in the laboratory experiment with regard to which the research initiated, the rock material presumably showed dispersion in its elastodynamic response, we have included in the analysis global elastodynamic relaxation mechanisms of the type that admit the construction of the solution for the dispersive case from the one pertaining to the perfectly elastic counterpart by the application of an appropriate correspondence principle (Chao & Achenbach, 1964; de Hoop, 1995b).

Mathematically, the solution also provides the expressions for the edge diffraction coefficients that are needed for the construction of the asymptotic ray solution to two-dimensional time-domain diffraction problems associated with cracks of the type under consideration (Achenbach *et al.*, 1982).

Position in three-dimensional configuration space $\mathbb{R}^3$ is specified by the coordinates $\{x_1, x_2, x_3\}$ with respect to an orthogonal Cartesian reference frame with the origin $\mathcal{O}$ and the three mutually perpendicular base vectors $\{\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3\}$ of unit length each. The subscript notation for Cartesian vectors and tensors is used and the summation convention applies. The position vector is also indicated as $\boldsymbol{x}$. The time coordinate is t. Partial differentiation with respect to x_m is denoted by ∂_m; ∂_t is a reserved symbol for differentiation with respect to time. The semi-infinite fluid-filled crack is present at $\{\boldsymbol{x} \in \mathbb{R}^3; x_1 > 0, x_3 = 0\}$. The wave propagation takes place in the $\{x_1, x_3\}$-plane; the wave motion has components both parallel to the edge of the crack and in the plane perpendicular to it.

Formulation of the problem

In an unbounded, homogeneous, isotropic, elastic solid with volume density of mass

$$\rho_{kr} = \rho\delta_{kr}, \tag{1}$$

where δ_{kr} is the Kronecker tensor: $\delta_{kr} = 1$ for $k = r$ and $\delta_{kr} = 0$ for $k \neq r$, compliance (de Hoop, 1995a, pp.320)

$$S_{ijpq} = \Lambda\delta_{ij}\delta_{pq} + M(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}), \tag{2}$$

with

$$\Lambda = -\frac{\lambda}{3\lambda + 2\mu}, \quad M = \frac{1}{4\mu}, \tag{3}$$

where λ and μ are the Lamé coefficients, global inertia relaxation function $\phi = \phi(t)$ and global compliance relaxation function $\psi = \psi(t)$, the particle velocity $v_r = v_r(\boldsymbol{x}, t)$ and the dynamic stress $\tau_{pq} = \tau_{pq}(\boldsymbol{x}, t)$ satisfy the first-order coupled elastic wave equations (de Hoop, 1995a, pp.327)

$$-\Delta^+_{kmpq}\partial_m\tau_{pq} + \rho_{kr}\partial_t(\phi \overset{t}{*} v_r) \quad = \quad f_k, \tag{4}$$

$$\Delta^+_{ijnr}\partial_n v_r - S_{ijpq}\partial_t(\psi \overset{t}{*} \tau_{pq}) \quad = \quad h_{ij}, \tag{5}$$

where

$$\Delta^+_{ijpq} = \tfrac{1}{2}(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}) \tag{6}$$

is the symmetrical unit tensor of rank four, $f_k = f_k(\boldsymbol{x}, t)$ is the volume source density of force, $h_{ij} = h_{ij}(\boldsymbol{x}, t)$ is the volume source density of deformation rate and $\overset{t}{*}$ denotes time convolution. The relaxation functions satisfy the causality condition $\phi(t) = 0$ for $t < 0$ and $\psi(t) = 0$ for $t < 0$.

The elastic properties of the fluid-filled crack are characterized by the boundary conditions of the continuity type

$$\lim_{x_3\downarrow 0} v_3 \quad = \quad \lim_{x_3\uparrow 0} v_3 \quad \text{for } 0 < x_1 < \infty,\ -\infty < x_2 < \infty, \tag{7}$$

$$\lim_{x_3\downarrow 0} \tau_{33} \quad = \quad \lim_{x_3\uparrow 0} \tau_{33} \quad \text{for } 0 < x_1 < \infty,\ -\infty < x_2 < \infty, \tag{8}$$

that require the continuity of the normal component of the particle velocity and the normal component of the traction across the crack, and the explicit boundary conditions

$$\lim_{x_3 \downarrow 0} \tau_{13} = \lim_{x_3 \uparrow 0} \tau_{13} = 0 \quad \text{for } 0 < x_1 < \infty,\ -\infty < x_2 < \infty, \tag{9}$$

$$\lim_{x_3 \downarrow 0} \tau_{23} = \lim_{x_3 \uparrow 0} \tau_{23} = 0 \quad \text{for } 0 < x_1 < \infty,\ -\infty < x_2 < \infty, \tag{10}$$

that require the vanishing of the tangential components of the traction upon approaching the crack on either side. Further, all components of the particle velocity and the stress are continuous across the unruptured part of the plane of the crack, i.e., the half-plane $\{\boldsymbol{x} \in \mathbb{R}^3; x_1 < 0, x_3 = 0\}$.

The total wave motion is the superposition of the incident wave (to be indicated by the superscript $^{\mathrm{i}}$), which we shall take to be either a plane *SH* wave, a plane *P* wave, or a plane *SV* wave, all propagating in the $\{x_1, x_3\}$-plane perpendicular to the edge of the crack, and the scattered wave (to be indicated by the superscript $^{\mathrm{s}}$). In view of the uniformity of the medium as well as the crack in the x_2-direction, and the isotropy of the solid, the wave diffraction problem is then two-dimensional in nature and independent of x_2, and the scattered waves, too, decouple into an *SH* wave for which $\boldsymbol{v} = v_2 \boldsymbol{i}_2$ and a combined *P/SV* wave for which $\boldsymbol{v} = v_1 \boldsymbol{i}_1 + v_3 \boldsymbol{i}_3$.

First, the diffraction problem will be solved for a perfectly elastic (i.e. instantaneously reacting) solid, for which $\phi(t) = \delta(t)$ and $\psi(t) = \delta(t)$, where $\delta(t)$ is the Dirac delta distribution. The solution for the solid with relaxation is subsequently determined by invoking the correspondence principle for a medium with global relaxation functions (de Hoop, 1995b).

Spectral representations for the scattered wavefield constituents

The incident plane wave is assumed to hit the edge of the crack at the instant $t = 0$. Under this condition, the scattered wavefield vanishes prior to $t = 0$. The first step towards solving the problem through the combined Wiener-Hopf and modified Cagniard techniques then consists of carrying out a Laplace transformation with respect to time according to

$$\{\hat{v}_r, \hat{\tau}_{pq}\}(\boldsymbol{x}, s) = \int_0^\infty \exp(-st)\{v_r, \tau_{pq}\}(\boldsymbol{x}, t)\mathrm{d}t. \tag{11}$$

The time Laplace-transform parameter (or complex frequency) s is taken to be real and positive, which, in view of Lerch's theorem (Widder, 1946), is a sufficient condition for the uniqueness of the relationship between a causal time function and its time Laplace transform. Next, the spectral representations of the scattered wavefield quantities are introduced according to

$$\{\hat{v}_r, \hat{\tau}_{pq}\}(x_1, x_3, s) = \frac{1}{2\pi\mathrm{i}} \int_{\mathcal{L}} \exp(-spx_1)\{\tilde{v}_r, \tilde{\tau}_{pq}\}(p, x_3, s)\mathrm{d}p, \tag{12}$$

where i is the imaginary unit and p is the complex slowness parameter in the x_1-direction. The (infinite) path of integration $\mathcal{L}$ in the complex p-plane remains to be determined, but a necessary condition for the convergence of the integral is that $\mathcal{L}$ should be parallel to the imaginary p-axis as $|p| \to \infty$. The domain to the left of $\mathcal{L}$ will be denoted by $\mathcal{D}^-$, the domain to the right of $\mathcal{L}$ will be denoted by $\mathcal{D}^+$.

SH wave

The spectral representation to be used for the particle velocity of the scattered *SH* wave is

$$\tilde{v}_2^{\mathrm{s}} = \tilde{v}_2^{\mathrm{s;SH}} = \hat{q}(s) \begin{cases} \tilde{A}^{SH\uparrow}(p)\exp(s\gamma_S x_3) & \text{for} \quad x_3 < 0, \\ \tilde{A}^{SH\downarrow}(p)\exp(-s\gamma_S x_3) & \text{for} \quad x_3 > 0, \end{cases} \tag{13}$$

Here, it has been taken into account that the scattered waves should travel away from the crack, with the shear wave speed $c_S = (\mu/\rho)^{1/2}$, which entails the vertical *S*-wave slowness

$$\gamma_S = (c_S^{-2} - p^2)^{1/2} \quad \text{with } \mathrm{Re}(\gamma_S) \geq 0 \text{ for all } p \in \mathbb{C}, \tag{14}$$

while $\hat{q} = \hat{q}(s)$ is the time Laplace transform of the particle-velocity pulse shape (signature) $q = q(t)$ of the incident *SH* wave. The condition on the definition of the square root in the expression for γ_S is at this point only needed for $p \in \mathcal{L}$; the extension of the condition to the entire complex p-plane is needed in the later application of the modified Cagniard method and implies that branch cuts are introduced along $\{p \in \mathbb{C}; 1/c_S < |\mathrm{Re}(p)| < \infty, \mathrm{Im}(p) = 0\}$. Using the relation

$$s\hat{\tau}_{23}^{\mathrm{s}} = \mu\partial_3\hat{v}_2^{\mathrm{s}} \tag{15}$$

that follows from Equations (2), (3) and (5), and invoking the continuity of $\hat{\tau}^{\rm s}_{23}$ across the entire plane $\{\boldsymbol{x} \in \mathbb{R}^3, x_3 = 0\}$, it follows that

$$-\tilde{A}^{SH\uparrow} = \tilde{A}^{SH\downarrow} = \tilde{A}_\perp. \tag{16}$$

With this, the expression for $\tilde{v}^{\rm s}_2$ becomes

$$\tilde{v}^{\rm s;SH}_2 = \mp\hat{q}(s)\tilde{A}_\perp(p)\exp(-s\gamma_S|x_3|) \quad \text{for } x_3 \lessgtr 0. \tag{17}$$

How to determine $\tilde{A}_\perp = \tilde{A}_\perp(p)$ from the remaining boundary conditions, will be discussed in Section .

P/SV wave

The spectral representations to be used for the particle velocity of the scattered *P* and *SV* waves are

$$\tilde{v}^{\rm s;P}_1 = \hat{q}(s)\begin{cases} p\tilde{A}^{P\uparrow}(p)\exp(s\gamma_P x_3) & \text{for} \quad x_3 < 0, \\ p\tilde{A}^{P\downarrow}(p)\exp(-s\gamma_P x_3) & \text{for} \quad x_3 > 0, \end{cases} \tag{18}$$

$$\tilde{v}^{\rm s;P}_3 = \hat{q}(s)\begin{cases} -\gamma_P\tilde{A}^{P\uparrow}(p)\exp(s\gamma_P x_3) & \text{for} \quad x_3 < 0, \\ \gamma_P\tilde{A}^{P\downarrow}(p)\exp(-s\gamma_P x_3) & \text{for} \quad x_3 > 0, \end{cases} \tag{19}$$

and

$$\tilde{v}^{\rm s;SV}_1 = \hat{q}(s)\begin{cases} \gamma_S\tilde{A}^{SV\uparrow}(p)\exp(s\gamma_S x_3) & \text{for} \quad x_3 < 0, \\ -\gamma_S\tilde{A}^{SV\downarrow}(p)\exp(-s\gamma_S x_3) & \text{for} \quad x_3 > 0, \end{cases} \tag{20}$$

$$\tilde{v}^{\rm s;SV}_3 = \hat{q}(s)\begin{cases} p\tilde{A}^{SV\uparrow}(p)\exp(s\gamma_S x_3) & \text{for} \quad x_3 < 0, \\ p\tilde{A}^{SV\downarrow}(p)\exp(-s\gamma_S x_3) & \text{for} \quad x_3 > 0. \end{cases} \tag{21}$$

Here, it has been taken into account that the scattered waves should travel away from the crack, that the *P*-wave constituent is to be curlfree and travels with the compressional wave speed $c_P = [(\lambda + 2\mu)/\rho]^{1/2}$, which entails the vertical *P*-wave slowness

$$\gamma_P = (c_P^{-2} - p^2)^{1/2} \quad \text{with } \mathrm{Re}(\gamma_{\rm P}) \geq 0 \text{ for all } \mathrm{p} \in \mathbb{C}, \tag{22}$$

that the *SV*-wave constituent is to be divergencefree and travels with the shear wave speed c_S, while $\hat{q} = \hat{q}(s)$ is the time Laplace transform of the particle-velocity pulse shape (signature) $q = q(t)$ of the incident *P* or *SV* wave. The condition on the definition of the square root in the expression for γ_P is at this point only needed for $p \in \mathcal{L}$; the extension of the condition to the entire complex p-plane is needed in the later application of the modified Cagniard method and implies that branch cuts are introduced along $\{p \in \mathbb{C}; 1/c_P < |\mathrm{Re(p)}| < \infty, \mathrm{Im(p)} = 0\}$. The total scattered wave in this case consists of the superposition of the *P*- and *SV*-wave constituents, i.e.

$$\tilde{v}^{\rm s}_1 = \tilde{v}^{\rm s;P}_1 + \tilde{v}^{\rm s;SV}_1, \tag{23}$$

$$\tilde{v}^{\rm s}_3 = \tilde{v}^{\rm s;P}_3 + \tilde{v}^{\rm s;SV}_3. \tag{24}$$

Using the relations

$$s\hat{\tau}^{\rm s}_{13} = \mu(\partial_1\hat{v}^{\rm s}_3 + \partial_3\hat{v}^{\rm s}_1), \tag{25}$$

$$s\hat{\tau}^{\rm s}_{33} = \lambda(\partial_1\hat{v}^{\rm s}_1 + \partial_3\hat{v}^{\rm s}_3) + 2\mu\partial_3\hat{v}^{\rm s}_3, \tag{26}$$

that follow from Equations (2), (3) and (5), and invoking the continuity of $\hat{v}^{\rm s}_3$, $\hat{\tau}^{\rm s}_{13}$ and $\hat{\tau}^{\rm s}_{33}$ across the entire plane $\{\boldsymbol{x} \in \mathbb{R}^3, x_3 = 0\}$, it follows that

$$-\tilde{A}^{P\uparrow} = \tilde{A}^{P\downarrow} = p\gamma_S\tilde{A}_{||}, \tag{27}$$

$$\tilde{A}^{SV\uparrow} = \tilde{A}^{SV\downarrow} = (p^2 - \tfrac{1}{2}c_S^{-2})\tilde{A}_{||}. \tag{28}$$

With this, the expressions for $\tilde{v}^{\rm s;P}_1$, $\tilde{v}^{\rm s;P}_3$, $\tilde{v}^{\rm s;SV}_1$ and $\tilde{v}^{\rm s;SV}_3$ become

$$\tilde{v}^{\rm s;P}_1 = \mp\hat{q}(s)p^2\gamma_S\tilde{A}_{||}(p)\exp(-s\gamma_P|x_3|) \quad \text{for } x_3 \lessgtr 0, \tag{29}$$

$$\tilde{v}^{\rm s;P}_3 = \hat{q}(s)p\gamma_P\gamma_S\tilde{A}_{||}(p)\exp(-s\gamma_P|x_3|) \quad \text{for } x_3 \lessgtr 0, \tag{30}$$

$$\tilde{v}^{\rm s;SV}_1 = \pm\hat{q}(s)\gamma_S(p^2 - \tfrac{1}{2}c_S^{-2})\tilde{A}_{||}(p)\exp(-s\gamma_S|x_3|) \quad \text{for } x_3 \lessgtr 0, \tag{31}$$

$$\tilde{v}^{\rm s;SV}_3 = \hat{q}(s)p(p^2 - \tfrac{1}{2}c_S^{-2})\tilde{A}_{||}(p)\exp(-s\gamma_S|x_3|) \quad \text{for } x_3 \lessgtr 0. \tag{32}$$

How to determine $\tilde{A}_{\|} = \tilde{A}_{\|}(p)$ from the remaining boundary conditions, will be discussed in Section .

Determination of the scattered *SH*-wave spectral amplitude

In this section, the *SH*-wave spectral amplitude $\tilde{A}_{\perp} = \tilde{A}_{\perp}(p)$ will be determined with the aid of the Wiener-Hopf method ('factorization method'). Using Equation (17), the condition that $\hat{v}_2^{\mathrm{s;SH}}$ should be continuous across the half-plane $\{\boldsymbol{x} \in \mathbb{R}^3; x_1 < 0, x_3 = 0\}$ leads to

$$0 = \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{L}} \tilde{A}_{\perp}(p) \exp(-spx_1) \mathrm{dp} \quad \text{for } \mathrm{x}_1 < 0. \tag{33}$$

A sufficient condition for this relation to be satisfied is

$$\tilde{A}_{\perp} = F^{-}(p) \quad \text{for } p \in \mathcal{L}, \tag{34}$$

where $F^{-} = F^{-}(p)$ is some function of p that is regular in $\mathcal{D}^{-}$ and has the property $F^{-}(p) = o(1)$ as $|p| \to \infty$ in $\mathcal{D}^{-}$, uniformly in $\arg(\mathrm{p})$. The proof follows by supplementing $\mathcal{L}$ with a semi-circular arc at infinity in $\mathcal{D}^{-}$ (whose contribution vanishes by virtue of Jordan's lemma) and applying Cauchy's theorem to the resulting closed contour. Finally, the explicit boundary condition

$$\hat{\tau}_{23}^{\mathrm{s}} = -\hat{\tau}_{23}^{\mathrm{i}} \quad \text{for } x_1 > 0,\ x_3 = 0 \tag{35}$$

is invoked. With the incident plane *SH* wave specified by

$$v_2^{\mathrm{i}} = q[t - p_0^S x_1 - \gamma_S(p_0^S) x_3], \tag{36}$$

with $0 \le p_0^S \le c_S^{-1}$ in view of the condition of causality on the scattered wave, the boundary condition (35) leads, with the use of Equation (15) to

$$\frac{1}{2\pi \mathrm{i}} \int_{\mathcal{L}} \gamma_S(p) \tilde{A}_{\perp}(p) \exp(-spx_1) \mathrm{dp} = -\gamma_{\mathrm{S}}(\mathrm{p}_0^{\mathrm{S}}) \exp(-\mathrm{s}\mathrm{p}_0^{\mathrm{S}} \mathrm{x}_1) \quad \text{for } \mathrm{x}_1 > 0. \tag{37}$$

A sufficient condition for this relation to be satisfied is

$$\gamma_S(p) \tilde{A}_{\perp}(p) = \frac{\gamma_S(p_0^S)}{p - p_0^S} + H^{+}(p) \quad \text{for } p \in \mathcal{L}, \tag{38}$$

where $H^{+} = H^{+}(p)$ is some function of p that is regular in $\mathcal{D}^{+}$ and has the property $H^{+}(p) = o(1)$ as $|p| \to \infty$ in $\mathcal{D}^{+}$, uniformly in $\arg(\mathrm{p})$, provided that $\mathcal{L}$ is chosen such that $p_0^S \in \mathcal{D}^{+}$. The proof follows by supplementing $\mathcal{L}$ with a semi-circular arc at infinity in $\mathcal{D}^{+}$ (whose contribution vanishes by virtue of Jordan's lemma) and applying the theorem of residues to the resulting closed contour. Combining Equations (34) and (38), we arrive at

$$\gamma_S(p) F^{-}(p) = \frac{\gamma_S(p_0^S)}{p - p_0^S} + H^{+}(p) \quad \text{for } p \in \mathcal{L}. \tag{39}$$

Equation (39) belongs to the class of Wiener-Hopf relations (Wiener & Hopf, 1931; Titchmarsh, 1948; Baker & Copson, 1950; Noble, 1958; Achenbach, 1973) that are amenable to a solution with the aid of the factorization method (Weinstein (Vaynshteyn), 1969). The procedure goes as follows.

First, observe that ('factorization by inspection')

$$\gamma_S(p) = \gamma_S^{-}(p) \gamma_S^{+}(p) \quad \text{for } p \in \mathcal{L}, \tag{40}$$

where

$$\gamma_S^{-}(p) = (c_S^{-1} - p)^{1/2} \quad \text{with } \mathrm{Re}(\gamma_S^{-}) \ge 0 \tag{41}$$

is regular in $\mathcal{D}^{-}$ and

$$\gamma_S^{+}(p) = (c_S^{-1} + p)^{1/2} \quad \text{with } \mathrm{Re}(\gamma_S^{+}) \ge 0 \tag{42}$$

is regular in $\mathcal{D}^{+}$. With this, Equation (39) is cast into the form

$$\gamma_S^{-}(p) F^{-}(p) - \frac{1}{\gamma_S^{+}(p_0^S)} \frac{\gamma_S(p_0^S)}{p - p_0^S} = \left[\frac{1}{\gamma_S^{+}(p)} - \frac{1}{\gamma_S^{+}(p_0^S)} \right] \frac{\gamma_S(p_0^S)}{p - p_0^S} + \frac{H^{+}(p)}{\gamma_S^{+}(p)} \quad \text{for } p \in \mathcal{L}. \tag{43}$$

By taking $\mathcal{L}$ such that the branch point $p = c_S^{-1}$ of γ_S^{-} and the simple pole $p = p_0^S$ of $(p - p_0^S)^{-1}$ are located in $\mathcal{D}^{+}$ and the branch point $p = -c_S^{-1}$ of γ_S^+ is located in $\mathcal{D}^-$, the left-hand side of this equation is regular in $\mathcal{D}^-$ and the right-hand side is regular in $\mathcal{D}^+$. According to Liouville's first theorem (Titchmarsh, 1950, pp.85), the left-hand side and the right-hand side of Equation (43) then are representations of one and the same entire function that is regular in the entire p-plane. Since the left-hand side is of order $o(p^{1/2})$ as $|p| \to \infty$ in $\mathcal{D}^-$ and the right-hand side is of order $o(1)$ as $|p| \to \infty$ in $\mathcal{D}^+$, this entire function is, by virtue of Liouville's second theorem (Titchmarsh, 1950, pp.85), identically equal to zero. Consequently, by putting the left-hand side equal to zero, we obtain

$$F^- = \frac{1}{\gamma_S^-(p)\gamma_S^+(p_0^S)} \frac{\gamma_S(p_0^S)}{p - p_0^S} \quad \text{for } p \in \{\mathcal{D}^- \cup \mathcal{L}\}. \tag{44}$$

In view of Equation (34), the expression for $\tilde{A}_\perp$ has herewith been determined and the spectral representation of the scattered *SH* wave is known.

To arrive at the corresponding space-time expression for the scattered *SH* wave, the analytic continuation of $\tilde{A}_\perp$ into $\mathcal{D}^+$ is needed. To this end, the definitions of γ_S^- and γ_S^+ as given in Equations (41) and (42) are taken to hold in the entire cut p-plane. In accordance with the condition put on the square roots, the branch cut of $\gamma_S^-(p)$ is along $\{p \in \mathbb{C}; c_S^{-1} < \mathrm{Re(p)} < \infty, \mathrm{Im(p)} = 0\}$ and the branch cut of $\gamma_S^+(p)$ along $\{p \in \mathbb{C}; -\infty < \mathrm{Re(p)} < -\mathrm{c_S^{-1}}, \mathrm{Im(p)} = 0\}$. With this procedure, the expression for $\tilde{A}_\perp$ to be used in the transformation back to the space-time domain becomes

$$\tilde{A}_\perp = \frac{\gamma_S^-(p_0^S)}{\gamma_S^-(p)} \frac{1}{p - p_0^S} \tag{45}$$

in the entire p-plane cut along $\{p \in \mathbb{C}; c_S^{-1} < \mathrm{Re(p)} < \infty, \mathrm{Im(p)} = 0\}$.

Determination of the scattered *P/SV*-wave spectral amplitude

In this section, the *P/SV*-wave spectral amplitude $\tilde{A}_\| = \tilde{A}_\|(p)$ will be determined with the aid of the Wiener-Hopf method ('factorization method'). Using Equations (23), (29) and (31), the condition that $\hat{v}_1^{\mathrm{s}}$ should be continuous across the half-plane $\{\boldsymbol{x} \in \mathbb{R}^3; x_1 < 0, x_3 = 0\}$ leads to

$$0 = \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{L}} \frac{\gamma_S(p)}{2c_S^2} \tilde{A}_\|(p) \exp(-spx_1) \mathrm{dp} \quad \text{for } \mathrm{x_1} < 0. \tag{46}$$

A sufficient condition for this relation to be satisfied is

$$\tilde{A}_\| = \frac{2c_S^2}{\gamma_S(p)} F^-(p) \quad \text{for } p \in \mathcal{L}, \tag{47}$$

where $F^- = F^-(p)$ is some function of p that is regular in $\mathcal{D}^-$ and has the property $F^-(p) = o(1)$ as $|p| \to \infty$ in $\mathcal{D}^-$, uniformly in $\arg(\mathrm{p})$. The proof follows by supplementing $\mathcal{L}$ with a semi-circular arc at infinity in $\mathcal{D}^-$ (whose contribution vanishes by virtue of Jordan's lemma) and applying Cauchy's theorem to the resulting closed contour. Finally, the explicit boundary condition

$$\hat{\tau}_{13}^{\mathrm{s}} = -\hat{\tau}_{13}^{\mathrm{i}} \quad \text{for } x_1 > 0,\ x_3 = 0 \tag{48}$$

is invoked. From here on, the cases of an incident plane *P* wave and an incident plane *SV* wave have to be discussed separately.

Incident plane *P* wave

Let the incident plane *P* wave be specified by

$$v_1^{\mathrm{i;P}} = c_P p_0^P q[t - p_0^P x_1 - \gamma_P(p_0^P) x_3], \tag{49}$$
$$v_3^{\mathrm{i;P}} = c_P \gamma_P(p_0^P) q[t - p_0^P x_1 - \gamma_P(p_0^P) x_3], \tag{50}$$

with $0 \le p_0^P \le c_P^{-1}$ in view of the condition of causality of the scattered wave. Then, the boundary condition (48) leads, with the use of Equation (25) and denoting $\tilde{A}_\|$ for this case by $\tilde{A}_\|^P$, to

$$\frac{1}{2\pi \mathrm{i}} \int_{\mathcal{L}} [p^2 \gamma_P(p) \gamma_S(p) + (p^2 - \tfrac{1}{2} c_S^{-2})^2] \tilde{A}_\|^P(p) \exp(-spx_1) \mathrm{dp} = -c_P p_0^P \gamma_P(p_0^P) \exp(-sp_0^P x_1) \quad \text{for } x_1 > 0. \tag{51}$$

In this expression, we recognize

$$\Delta_R(p) = p^2\gamma_P(p)\gamma_S(p) + (p^2 - \tfrac{1}{2}c_S^{-2})^2, \tag{52}$$

the Rayleigh determinant whose zeros are $p = \pm c_R^{-1}$ where c_R is the wavespeed of Rayleigh surface waves along a tractionfree boundary of a semi-infinite perfectly elastic solid.

A sufficient condition for relation (51) to be satisfied is

$$\Delta_R(p)\tilde{A}_{\|}^P(p) = \frac{c_P p_0^P \gamma_P(p_0^P)}{p - p_0^P} + H^+(p) \quad \text{for } p \in \mathcal{L}, \tag{53}$$

where $H^+ = H^+(p)$ is a function of p that is regular in $\mathcal{D}^+$ and has the property $H^+(p) = o(1)$ as $|p| \to \infty$ in $\mathcal{D}^+$, uniformly in $\arg(\mathrm{p})$, provided that $\mathcal{L}$ is chosen such that $p_0^P \in \mathcal{D}^+$. The proof follows by supplementing $\mathcal{L}$ with a semi-circular arc at infinity in $\mathcal{D}^+$ (whose contribution vanishes by virtue of Jordan's lemma) and applying the theorem of residues to the resulting closed contour. Combining Equations (47) and (53), we arrive at

$$\Delta_R(p)\frac{2c_S^2}{\gamma_S(p)}F^-(p) = \frac{c_P p_0^P \gamma_P(p_0^P)}{p - p_0^P} + H^+(p) \quad \text{for } p \in \mathcal{L}. \tag{54}$$

Equation (54) belongs to the class of Wiener-Hopf relations and is amenable to a solution with the aid of the factorization method. For the factorization of the 'kernel function' (i.e. the function multiplying $F^-(p)$) we now have, however, to rely on the Plemelj formulas (Noble, 1958; Sparenberg, 1958), since a factorization by inspection (as could be carried out for the case of *SH*-wave scattering) is not known. The application of the Plemelj formulas requires a kernel function that approaches the value one as $|p| \to \infty$ along $\mathcal{L}$. To meet this condition, Equation (54) is rewritten as

$$\left(1 - \frac{c_S^2}{c_P^2}\right)\frac{K(p)}{\gamma_S(p)}\left(\frac{1}{c_R^2} - p^2\right)F^-(p) = \frac{c_P p_0^P \gamma_P(p_0^P)}{p - p_0^P} + H^+(p) \quad \text{for } p \in \mathcal{L}, \tag{55}$$

in which

$$K(p) = \frac{2}{c_S^{-2} - c_P^{-2}}\frac{\Delta_R(p)}{c_R^{-2} - p^2} \quad \text{for } p \in \mathcal{L}. \tag{56}$$

A Taylor expansion around $p = \infty$ shows that indeed, $K(p) = 1 + O(p^{-2})$ as $|p| \to \infty$ along $\mathcal{L}$. As Equation (56) shows, $K(p)$ can be continued analytically away from $\mathcal{L}$ into the entire p-plane cut along the common branch cuts of $\gamma_P(p)\gamma_S(p)$, i.e. along $\{p \in \mathbb{C}; c_P^{-1} < |\mathrm{Re(p)}| < \mathrm{c_S^{-1}}, \mathrm{Im(p)} = 0\}$. For this function, the asymptotic relation $K(p) = 1 + O(p^{-2})$ as $|p| \to \infty$ holds uniformly in $\arg(\mathrm{p})$. These properties facilitate the factorization procedure carried out with the aid of the Plemelj formulas (see Appendix A). Let

$$K(p) = K^-(p)K^+(p) \quad \text{for } p \in \mathcal{L}, \tag{57}$$

where $K^-(p)$ as given by Equations (A12) and (A19) is regular in $\mathcal{D}^-$ and $K^+(p)$ as given by Equations (A14) and (A20) is regular in $\mathcal{D}^+$. With the aid of Equations (40)-(42) and (57), Equation (56) is cast into the form

$$\begin{aligned}
&\left(1 - \frac{c_S^2}{c_P^2}\right)\frac{K^-(p)}{\gamma_S^-(p)}\left(\frac{1}{c_R} - p\right)F^-(p) - c_P p_0^P \gamma_P(p_0^P)\frac{\gamma_S^+(p_0^P)}{K^+(p_0^P)}\frac{1}{c_R^{-1} + p_0^P}\frac{1}{p - p_0^P} \\
&= c_P p_0^P \gamma_P(p_0^P)\left[\frac{\gamma_S^+(p)}{K^+(p)}\frac{1}{c_R^{-1} + p} - \frac{\gamma_S^+(p_0^P)}{K^+(p_0^P)}\frac{1}{c_R^{-1} + p_0^P}\right]\frac{1}{p - p_0^P} \\
&\qquad + \frac{\gamma_S^+(p)}{K^+(p)}\frac{1}{c_R^{-1} + p}H^+(p) \quad \text{for } p \in \mathcal{L}.
\end{aligned} \tag{58}$$

By taking $\mathcal{L}$ such that the branch points $p = c_P^{-1}$ and $p = c_S^{-1}$ of $K^-(p)$, the branch point $p = c_S^{-1}$ of $\gamma_S^-(p)$ as well as the simple pole $p = c_R^{-1}$ of $(c_R^{-1} - p)^{-1}$ and the simple pole $p = p_0^P$ of $(p - p_0^P)^{-1}$ are all located in $\mathcal{D}^+$ and the branch points $p = -c_P^{-1}$ and $p = -c_S^{-1}$ of $K^+(p)$, the branch point $p = -c_S^{-1}$ of $\gamma_S^+(p)$ as well as the simple pole $p = -c_R^{-1}$ of $(c_R^{-1} + p)^{-1}$ are all located in $\mathcal{D}^-$, the left-hand side of this equation is regular in $\mathcal{D}^-$ and the right-hand side is regular in $\mathcal{D}^+$. According to Liouville's first theorem (Titchmarsh, 1950, pp.85), the left-hand side and the right-hand side of Equation (58) are then representations of one and the same entire function that is regular in the entire p-plane. Since the left-hand side is of order $o(p^{1/2})$ as $|p| \to \infty$ in $\mathcal{D}^-$ and the right-hand side is of order $o(p^{-1/2})$ as $|p| \to \infty$ in $\mathcal{D}^+$, this entire function is, by virtue of Liouville's second theorem (Titchmarsh, 1950, pp.85), identically equal to zero. Consequently, by putting the left-hand side equal to zero, we obtain

$$F^- = \frac{c_P p_0^P \gamma_P(p_0^P)}{1 - c_S^2/c_P^2}\frac{\gamma_S^-(p)}{K^-(p)(c_R^{-1} - p)}\frac{\gamma_S^+(p_0^P)}{K^+(p_0^P)}\frac{1}{c_R^{-1} + p_0^P}\frac{1}{p - p_0^P} \quad \text{for } p \in \{\mathcal{D}^- \cup \mathcal{L}\}. \tag{59}$$

In view of Equation (47), the expression for $\tilde{A}^P_{\|}$ then becomes

$$\tilde{A}^P_{\|} = \frac{2c_S^2 c_P p_0^P \gamma_P(p_0^P)}{1 - c_S^2/c_P^2} \frac{\gamma_S^-(p)}{\gamma_S(p) K^-(p)(c_R^{-1} - p)} \frac{\gamma_S^+(p_0^P)}{K^+(p_0^P)} \frac{1}{c_R^{-1} + p_0^P} \frac{1}{p - p_0^P} \quad \text{for } p \in \{\mathcal{D}^- \cup \mathcal{L}\}. \tag{60}$$

Substitution of this result into Equations (29)-(32) yields the spectral amplitudes of the scattered *P* and *SV* waves.

To arrive at the space-time expressions for the particle velocities of the scattered *P* and *SV* waves, the analytic continuation of F^- into $\mathcal{D}^+$ is needed. To this end, the definitions of K^- and K^+ as given in Appendix A are taken to hold in the entire cut *p*-plane.

Incident plane *SV* wave

Let the incident plane *SV* wave be specified by

$$v_1^{\text{i;SV}} = -c_S \gamma_S(p_0^S) q[t - p_0^S x_1 - \gamma_S(p_0^S) x_3], \tag{61}$$

$$v_3^{\text{i;SV}} = c_S p_0^S q[t - p_0^S x_1 - \gamma_S(p_0^S) x_3], \tag{62}$$

with $0 \le p_0^S \le c_S^{-1}$ in view of the condition of causality of the scattered wave. Then, the boundary condition (48) leads, with the use of Equation (25) and denoting $\tilde{A}_{\|}$ by $\tilde{A}^S_{\|}$, to

$$\frac{1}{2\pi \mathrm{i}} \int_{\mathcal{L}} \Delta_R(p) \tilde{A}^S_{\|}(p) \exp(-spx_1) \mathrm{d}p = -2c_S[(p_0^S)^2 - \tfrac{1}{2} c_S^{-2}] \exp(-sp_0^S x_1) \quad \text{for } x_1 > 0. \tag{63}$$

A sufficient condition for this relation to be satisfied is

$$\Delta_R(p) \tilde{A}^S_{\|}(p) = \frac{2c_S[(p_0^S)^2 - \frac{1}{2} c_S^{-2}]}{p - p_0^S} + H^+(p) \quad \text{for } p \in \mathcal{L}, \tag{64}$$

where $H^+ = H^+(p)$ is some function of p that is regular in $\mathcal{D}^+$ and has the property $H^+(p) = o(1)$ as $|p| \to \infty$ in $\mathcal{D}^+$, uniformly in $\arg(\mathrm{p})$, provided that $\mathcal{L}$ is chosen such that $p_0^S \in \mathcal{D}^+$. The proof follows by supplementing $\mathcal{L}$ with a semi-circular arc at infinity in $\mathcal{D}^+$ (whose contribution vanishes by virtue of Jordan's lemma) and applying the theorem of residues to the resulting closed contour. Combining Equations (47) and (64), we arrive at

$$\Delta_R(p) \frac{2c_S^2}{\gamma_S(p)} F^-(p) = \frac{2c_S[(p_0^S)^2 - \frac{1}{2} c_S^{-2}]}{p - p_0^S} + H^+(p) \quad \text{for } p \in \mathcal{L}. \tag{65}$$

Equation (65) belongs to the class of Wiener-Hopf relations and is amenable to a solution with the aid of the factorization method. Proceeding as in the case of an incident *P* wave, Equation (65) is rewritten as

$$\left(1 - \frac{c_S^2}{c_P^2}\right) \frac{K(p)}{\gamma_S(p)} \left(\frac{1}{c_R^2} - p^2\right) F^-(p) = \frac{2c_S[(p_0^S)^2 - \frac{1}{2} c_S^{-2}]}{p - p_0^S} + H^+(p) \quad \text{for } p \in \mathcal{L}, \tag{66}$$

in which $K = K(p)$ is again given by Equation (56). With the aid of Equations (40)-(42) and Equation (57), Equation (66) is cast into the form

$$\begin{aligned} &\left(1 - \frac{c_S^2}{c_P^2}\right) \frac{K^-(p)}{\gamma_S^-(p)} \left(\frac{1}{c_R} - p\right) F^-(p) - 2c_S[(p_0^S)^2 - \tfrac{1}{2} c_S^{-2}] \frac{\gamma_S^+(p_0^S)}{K^+(p_0^S)} \frac{1}{c_R^{-1} + p_0^S} \frac{1}{p - p_0^S} \\ &= 2c_S[(p_0^S)^2 - \tfrac{1}{2} c_S^{-2}] \left[\frac{\gamma_S^+(p)}{K^+(p)} \frac{1}{c_R^{-1} + p} - \frac{\gamma_S^+(p_0^S)}{K^+(p_0^S)} \frac{1}{c_R^{-1} + p_0^S}\right] \frac{1}{p - p_0^S} \\ &\qquad + \frac{\gamma_S^+(p)}{K^+(p)} \frac{1}{c_R^{-1} + p} H^+(p) \quad \text{for } p \in \mathcal{L}. \end{aligned} \tag{67}$$

By taking $\mathcal{L}$ such that the branch points $p = c_P^{-1}$ and $p = c_S^{-1}$ of $K^-(p)$, the branch point $p = c_S^{-1}$ of $\gamma_S^-(p)$ as well as the simple pole $p = c_R^{-1}$ of $(c_R^{-1} - p)^{-1}$ and the simple pole $p = p_0^S$ of $(p - p_0^S)^{-1}$ are all located in $\mathcal{D}^+$ and the branch points $p = -c_P^{-1}$ and $p = -c_S^{-1}$ of $K^+(p)$, the branch point $p = -c_S^{-1}$ of $\gamma_S^+(p)$ as well as the simple pole $p = -c_R^{-1}$ of $(c_R^{-1} + p)^{-1}$ are all located in $\mathcal{D}^-$, the left-hand side of this equation is regular in $\mathcal{D}^-$ and the right-hand side is regular in $\mathcal{D}^+$. According to Liouville's first theorem (Titchmarsh, 1950, pp.85), the left-hand side and the right-hand side of Equation (67) are then representations of one and the same entire function that is regular in the entire p-plane. Since the left-hand side is of order $o(p^{1/2})$ as $|p| \to \infty$ in $\mathcal{D}^-$ and the right-hand side is of order $o(p^{-1/2})$ as $|p| \to \infty$ in $\mathcal{D}^+$, this entire function is, by virtue of Liouville's second theorem (Titchmarsh, 1950, pp.85), identically equal to zero. Consequently, by putting the left-hand side equal to zero, we obtain

$$F^- = \frac{2c_S[(p_0^S)^2 - \frac{1}{2}c_S^{-2}]}{1 - c_S^2/c_P^2} \frac{\gamma_S^-(p)}{K^-(p)(c_R^{-1} - p)} \frac{\gamma_S^+(p_0^S)}{K^+(p_0^S)} \frac{1}{c_R^{-1} + p_0^S} \frac{1}{p - p_0^S} \quad \text{for } p \in \{\mathcal{D}^- \cup \mathcal{L}\}. \tag{68}$$

In view of Equation (47), the expression for $\tilde{A}_{\parallel}^S$ then becomes

$$\tilde{A}_{\parallel}^S = \frac{4c_S^3[(p_0^S)^2 - \frac{1}{2}c_S^{-2}]}{1 - c_S^2/c_P^2} \frac{\gamma_S^-(p)}{\gamma_S(p)K^-(p)(c_R^{-1} - p)} \frac{\gamma_S^+(p_0^S)}{K^+(p_0^S)} \frac{1}{c_R^{-1} + p_0^S} \frac{1}{p - p_0^S} \quad \text{for } p \in \{\mathcal{D}^- \cup \mathcal{L}\}. \tag{69}$$

Substitution of this result into Equations (29)-(32) yields the spectral amplitudes of the scattered *P* and *SV* waves.

To arrive at the space-time expressions for the particle velocities of the scattered *P* and *SV* waves, the analytic continuation of F^- into $\mathcal{D}^+$ is needed. To this end, the definitions of K^- and K^+ as given in Appendix A are taken to hold in the entire cut p-plane.

Time-domain expression for the scattered *SH* wave

The transformation of the spectral-domain expressions to the space-time domain is carried out with the aid of the modified Cagniard method. Starting from Equations (12) and (17), it follows that the expression for the time Laplace transform of the particle velocity of the scattered *SH* wave is given by

$$\hat{v}_2^{\mathrm{s;SH}} = \mp\hat{q}(s)\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{L}} \tilde{A}_{\perp}(p)\exp\{-s[px_1 + \gamma_S(p)|x_3|]\}\mathrm{dp} \quad \text{for } \mathrm{x}_3 \gtrless 0, \tag{70}$$

in which $\tilde{A}_{\perp}$ is given by Equation (45). In the representation, the integrand can be continued analytically into the complex p-plane away from the path of integration $\mathcal{L}$. In this process, we meet the singularities of the integrand, viz. the branch points at $p = -c_S^{-1}$ and $p = c_S^{-1}$, and the simple pole at $p = p_0^S$. In the modified Cagniard method, the integration in the right-hand side of Equation (70) is replaced with one along a path where the exponential function is of the form $\exp(-s\tau)$, where τ is real and positive, upon which τ is introduced as the variable of integration. Along such a path, we have

$$px_1 + \gamma_S(p)|x_3| = \tau. \tag{71}$$

Candidates are: the part of the real p-axis in between the branch points $p = -c_S^{-1}$ and $p = c_S^{-1}$ and the hyperbolic path $\{p \in \mathbb{C}; p = p_S\} \cup \{p \in \mathbb{C}; p = p_S^*\}$, with

$$p_S = p_S(x_1, |x_3|, \tau) = \frac{x_1}{r^2}\tau + \mathrm{i}\frac{|\mathrm{x}_3|}{\mathrm{r}^2}(\tau^2 - \mathrm{T}_\mathrm{S}^2)^{1/2} \quad \text{for } \mathrm{T_S} < \tau < \infty, \tag{72}$$

in which

$$r = (x_1^2 + x_3^2)^{1/2} \tag{73}$$

is the distance from the edge of the crack to the point of observation and

$$T_S = r/c_S \tag{74}$$

is the *SH*-wave travel time from the edge of the crack to the point of observation, while * denotes complex conjugate. Further, the modified Cagniard path must originate from $\mathcal{L}$ through a continuous deformation without passing singularities of the integrand in order that Cauchy's theorem can be applied. To this end, circular arcs at infinity are to join $\mathcal{L}$ and the modified Cagniard path. The behavior of $\tilde{A}_{\perp}(p)$ as $|p| \to \infty$ ensures that the contribution from the joining circular arcs at infinity vanishes in view of Jordan's lemma. Hence, the integration along $\mathcal{L}$ can be replaced by an integration along the hyperbolic path $\{p \in \mathbb{C}; p = p_S\} \cup \{p \in \mathbb{C}; p = p_S^*\}$ in the region where its point of intersection with the real p-axis lies to the left of the pole $p = p_0^S$ in the expression for $\tilde{A}_{\perp}$, i.e., for $x_1/r < c_S p_0^S$, whereas in the range $x_1/r \geq c_S p_0^S$ the contribution from the pole $p = p_0^S$ must be taken into account. Taking the contributions from $\{p \in \mathbb{C}; p = p_S\}$ and $\{p \in \mathbb{C}; p = p_S^*\}$ together, applying Schwarz's reflection principle of complex function theory, and introducing τ as the variable of integration, we thus arrive at

$$\hat{v}_2^{\mathrm{s;SH}} = \hat{q}(s)\hat{G}_2^{\mathrm{s;SH}}(x_1, x_3, s), \tag{75}$$

with

$$\hat{G}_2^{\mathrm{s;SH}} = \hat{G}_2^{\mathrm{g;SH}} + \hat{G}_2^{\mathrm{d;SH}}, \tag{76}$$

in which

$$\hat{G}_2^{\mathrm{g;SH}} = \pm\{0, \tfrac{1}{2}, 1\} \ \exp\{-s[p_0^S x_1 + \gamma_S(p_0^S)|x_3|]\}$$
$$\text{for } \{x_1/r < c_S p_0^S, x_1/r = c_S p_0^S, x_1/r > c_S p_0^S\}, \ x_3 \ \lessgtr \ 0, \tag{77}$$

will be identified as the 'geometrical' contribution to $\hat{G}_2^{\mathrm{s;SH}}$ (in the sense of geometrical ray tracing via the crack) and

$$\hat{G}_2^{\mathrm{d;SH}} = \mp\frac{1}{\pi}\int_{T_S}^{\infty} \exp(-s\tau)\mathrm{Im}\left[\tilde{\mathrm{A}}_\perp(\mathrm{p_S})\frac{\partial \mathrm{p_S}}{\partial \tau}\right] \mathrm{d}\tau \quad \text{for } \mathrm{x_3} \ \lessgtr \ 0, \tag{78}$$

will be identified as the 'diffraction' contribution to $\hat{G}_2^{\mathrm{s;SH}}$ (associated with the edge of the crack). In Equation (78),

$$\frac{\partial p_S}{\partial \tau} = \frac{x_1}{r^2} + \mathrm{i}\frac{|\mathrm{x_3}|}{\mathrm{r^2}}\frac{\tau}{(\tau^2 - \mathrm{T_S^2})^{1/2}} \quad \text{for } \mathrm{T_S} < \tau < \infty, \tag{79}$$

while for $x_1/r = c_S p_0^S$ the lower limit T_S of the integration has to be approached from above. In view of Lerch's theorem on the uniqueness of the one-sided Laplace transform, the time-domain values corresponding to Equations (75)-(78) are obtained as

$$v_2^{\mathrm{s;SH}} = q(t) \overset{t}{*} G_2^{\mathrm{s;SH}}(x_1, x_3, t), \tag{80}$$

with

$$G_2^{\mathrm{s;SH}} = G_2^{\mathrm{g;SH}} + G_2^{\mathrm{d;SH}}, \tag{81}$$

in which

$$G_2^{\mathrm{g;SH}} = \pm\{0, \tfrac{1}{2}, 1\} \ \delta[t - p_0^S x_1 - \gamma_S(p_0^S)|x_3|]$$
$$\text{for } \{x_1/r < c_S p_0^S, x_1/r = c_S p_0^S, x_1/r > c_S p_0^S\}, \ x_3 \ \lessgtr \ 0, \tag{82}$$

is the 'geometrical' contribution to $G_2^{\mathrm{s;SH}}$ and

$$G_2^{\mathrm{d;SH}} = \mp\frac{1}{\pi}\mathrm{Im}\left[\tilde{\mathrm{A}}_\perp(\mathrm{p_S})\frac{\partial \mathrm{p_S}}{\partial \tau}\right] \mathrm{H}(\mathrm{t} - \mathrm{T_S}) \quad \text{for } \mathrm{x_3} \ \lessgtr \ 0, \tag{83}$$

where $H(t)$ denotes the Heaviside unit step function ($H(t) = \{0, 1/2, 1\}$ for $\{t < 0, t = 0, t > 0\}$), is the diffraction contribution to $G_2^{\mathrm{s;SH}}$ and, again, for $x_1/r = c_S p_0^S$ the value T_S has to be approached from above.

Equation (80) indicates that the scattered wave motion is the time convolution of the signature of the incident wave and a space-time Green's function given by Equations (81)-(83). The lower sign in Equation (82) leads to the annihilation of the incident plane wave in the 'geometrical shadow region' $\{\boldsymbol{x} \in \mathbb{R}^3; x_1/r > c_S p_0^S, x_3 > 0\}$; the upper sign yields the geometrically reflected plane wave in the 'geometrically illuminated region' $\{\boldsymbol{x} \in \mathbb{R}^3; x_1/r > c_S p_0^S, x_3 < 0\}$. The right-hand side of Equation (83) yields the cylindrical diffraction wave originating at the edge of the crack. The latter's expression is the same as for the diffraction of a plane *SH* wave by a *traction-free* crack, which, in terms of the polar coordinates in the $\{x_1, x_3\}$-plane can be found in A.T. de Hoop (1958) and in Achenbach (1973, pp.378).

Scattered wave in the plane of the crack

In the plane of the crack, the particle velocity of the wave motion takes a particularly simple form. Substituting $x_3 = 0$ in Equation (70), we have

$$\hat{v}_2^{\mathrm{s;SH}}(x_1, \mp 0, s) = \mp\hat{q}(s)\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{L}} \tilde{A}_\perp(p)\exp(-spx_1)\mathrm{dp}. \tag{84}$$

The cases $x_1 < 0$ and $x_1 > 0$ require different treatments and will be discussed separately below.

Unruptured part ($x_1 < 0$)

For the unruptured part of the plane of the crack we have $x_1 < 0$. For this case, the path of integration $\mathcal{L}$ is supplemented with a semi-circular arc at infinity in $\mathcal{D}^-$. Since $\tilde{A}_\perp$ as given by Equation (45) has no singularities in $\mathcal{D}^-$ and satisfies the conditions for the application of Jordan's lemma, it follows that

$$\hat{v}_2^{\mathrm{s;SH}}(x_1, \mp 0, s) = 0 \qquad \text{for } x_1 < 0. \tag{85}$$

Consequently,

$$v_2^{\mathrm{s;SH}}(x_1, \mp 0, t) = 0 \qquad \text{for } x_1 < 0, \tag{86}$$

as it should be in view of the continuity of $v_2^{\mathrm{s;SH}}$ across the unruptured part of the plane of the crack in conjunction with the occurrence of the $\mp$ on the right-hand side of Equation (84).

Ruptured part ($x_1 > 0$)

For the ruptured part of the plane of the crack we have $x_1 > 0$. For this case, the path of integration $\mathcal{L}$ is supplemented with a semi-circular arc at infinity in $\mathcal{D}^+$. Since Jordan's lemma applies, we replace $\mathcal{L}$ by a loop around the branch cut of γ_S^-. Along this loop, we introduce $\tau = px_1$ as the variable of integration, after which the application of Lerch's theorem leads to

$$v_2^{\mathrm{s;SH}}(x_1, \mp 0, t) = q(t) \overset{t}{*} G_2^{\mathrm{s;SH}}(x_1, \mp 0, t), \tag{87}$$

with

$$G_2^{\mathrm{s;SH}}(x_1, \mp 0, t) = G_2^{\mathrm{g;SH}}(x_1, \mp 0, t) + G_2^{\mathrm{d;SH}}(x_1, \mp 0, t), \tag{88}$$

in which

$$G_2^{\mathrm{g;SH}}(x_1, \mp 0, t) = \pm\delta[t - p_0^S x_1] \quad \text{for } x_1 > 0 \tag{89}$$

and

$$G_2^{\mathrm{d;SH}}(x_1, \mp 0, t) = \mp\frac{1}{\pi x_1}\,\mathrm{Im}\left[\tilde{\mathrm{A}}_\perp(\mathrm{p})\right]\Big|_{p=t/x_1} H(t - x_1/c_S) \quad \text{for } x_1 > 0. \tag{90}$$

Time-domain expression for the scattered *P/SV* wave (incident *P* wave)

The transformation of the spectral domain expressions for the *P/SV* wave back to the time domain with the aid of the modified Cagniard method runs, in principle, along the same lines as the one for the *SH* wave, be it that now the spectral domain expressions are more complicated and show more singularities in the complex p-plane. The modified Cagniard path is again dictated by the argument of the exponential functions involved. In view of this, the transformations for the *P*-wave part and the *SV*-wave part have to be carried out separately, after which the two results are to be added. Also, the two cases of an incident *P* wave and an incident *SV* wave have to be considered separately, because of their difference in possible location of the corresponding poles in the complex p-plane. The present section deals with the case of an incident *P* wave; Section deals with the case of an incident *SV* wave.

Scattered *P* wave (incident *P* wave)

Starting from Equations (12), (29) and (30), it follows that the expressions for the time
Laplace transform of the particle velocity components of the scattered *P* wave due to an incident *P* wave are given by

$$\hat{v}_1^{\mathrm{s;P}} = \mp\hat{q}(s)\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{L}} p^2\gamma_S(p)\tilde{A}_\parallel^P(p)\exp\{-s[px_1 + \gamma_P(p)|x_3|]\}\mathrm{dp} \quad \text{for } \mathrm{x}_3 \lessgtr 0, \tag{91}$$

$$\hat{v}_3^{\mathrm{s;P}} = \hat{q}(s)\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{L}} p\gamma_P(p)\gamma_S(p)\tilde{A}_\parallel^P(p)\exp\{-s[px_1 + \gamma_P(p)|x_3|]\}\mathrm{dp} \quad \text{for } \mathrm{x}_3 \lessgtr 0, \tag{92}$$

in which $\tilde{A}_\parallel^P$ is given by Equation (60). In this representation, the integrand can be continued analytically into the complex p-plane away from the path of integration $\mathcal{L}$. In this process, we meet the singularities of the integrand, viz. the branch points $p = -c_P^{-1}$ and $p = c_P^{-1}$, the branch points $p = -c_S^{-1}$ and $p = c_S^{-1}$, the simple pole at $p = c_R^{-1}$ and the simple pole at $p = p_0^P$. Now (de Hoop, 1995a, pp.333), $c_S \leq (\sqrt{3}/2)c_P$ and $c_R \leq c_S$, which implies that the poles $p = \pm c_R^{-1}$ are always more remote from the origin in the complex p-plane than the most remote branch points $p = \pm c_S^{-1}$. Further, for a uniform plane incident wave that hits the crack at the instant $t = 0$, we have $0 \le p_0^P \le c_P^{-1}$.

The modified Cagniard path associated with Equations (91) and (92) must satisfy the equation

$$px_1 + \gamma_P(p)|x_3| = \tau, \tag{93}$$

where τ is real and positive. Candidates are: the part of the real p-axis in between the branch points $p = -c_P^{-1}$ and $p = c_P^{-1}$ of γ_P and the hyperbolic path $\{p \in \mathbb{C}; p = p_P\} \cup \{p \in \mathbb{C}; p = p_P^*\}$, with

$$p_P = p_P(x_1, |x_3|, \tau) = \frac{x_1}{r^2}\tau + \mathrm{i}\frac{|\mathrm{x}_3|}{\mathrm{r}^2}(\tau^2 - \mathrm{T}_\mathrm{P}^2)^{1/2} \quad \text{for } \mathrm{T}_\mathrm{P} < \tau < \infty, \tag{94}$$

in which r is given by Equation (73) and

$$T_P = r/c_P \tag{95}$$

is the P-wave travel time from the edge of the crack to the point of observation. Further, the modified Cagniard path must originate from $\mathcal{L}$ through a continuous deformation without passing singularities of the integrand in order that Cauchy's theorem can be applied. To this end, circular arcs at infinity are to join $\mathcal{L}$ and the modified Cagniard path. In view of the behavior of $\tilde{A}_\parallel^P$ at infinity, viz. $\tilde{A}_\parallel^P = O(p^{-5/2})$ as $|p| \to \infty$, the contribution from the joining circular arcs at infinity vanishes in view of Jordan's lemma as long as $|x_3| > 0$. Under this condition, the integration along $\mathcal{L}$ can be replaced by an integration along the hyperbolic path $\{p \in \mathbb{C}; p = p_P\} \cup \{p \in \mathbb{C}; p = p_P^*\}$ in the region of space where its point of intersection with the real p-axis lies to the left of the pole $p = p_0^P$ in the expression for $\tilde{A}_\parallel^P$, i.e., for $x_1/r < c_P p_0^P$, whereas in the range $x_1/r \ge c_P p_0^P$ the contribution from the pole $p = p_0^P$ must be taken into account. Taking the contributions from $\{p \in \mathbb{C}; p = p_P\}$ and $\{p \in \mathbb{C}; p = p_P^*\}$ together, applying Schwarz's reflection principle of complex function theory and introducing τ as the variable of integration, we thus arrive at

$$\hat{v}_{1,3}^{\mathrm{s;P}} = \hat{q}(s)\hat{G}_{1,3}^{\mathrm{s;P}}(x_1, x_3, s), \tag{96}$$

with

$$\hat{G}_{1,3}^{\mathrm{s;P}} = \hat{G}_{1,3}^{\mathrm{g;P}} + \hat{G}_{1,3}^{\mathrm{d;P}}, \tag{97}$$

in which

$$\hat{G}_1^{\mathrm{g;P}} = \pm\{0, \tfrac{1}{2}, 1\}\,(p_0^P)^2\gamma_S(p_0^P)R^P(p_0^P)\exp\{-s[p_0^P x_1 + \gamma_P(p_0^P)|x_3|]\} \tag{98}$$

$$\hat{G}_3^{\mathrm{g;P}} = -\{0, \tfrac{1}{2}, 1\}\,p_0^P\gamma_P(p_0^P)\gamma_S(p_0^P)R^P(p_0^P)\exp\{-s[p_0^P x_1 + \gamma_P(p_0^P)|x_3|]\} \tag{99}$$

$$\text{for } \{x_1/r < c_P p_0^P, x_1/r = c_P p_0^P, x_1/r > c_P p_0^P\},\ x_3 \lessgtr 0,$$

with

$$R^P(p_0^P) = \mathrm{Res}_{\mathrm{p}=\mathrm{p}_0^\mathrm{P}}\tilde{\mathrm{A}}_\parallel^\mathrm{P}(\mathrm{p}) = \frac{\mathrm{c_P p_0^P}\gamma_\mathrm{P}(\mathrm{p_0^P})}{\Delta_\mathrm{R}(\mathrm{p_0^P})}, \tag{100}$$

will be identified as the 'geometrical' contribution to $\hat{G}_{1,3}^{\mathrm{s;P}}$ (in the sense of geometrical ray tracing via the crack) and

$$\hat{G}_1^{\mathrm{d;P}} = \mp\frac{1}{\pi}\int_{T_P}^{\infty}\exp(-s\tau)\mathrm{Im}\left[\mathrm{p_P^2}\gamma_\mathrm{S}(\mathrm{p_P})\tilde{\mathrm{A}}_\parallel^\mathrm{P}(\mathrm{p_P})\frac{\partial \mathrm{p_P}}{\partial\tau}\right]\mathrm{d}\tau \quad \text{for } \mathrm{x}_3 \lessgtr 0, \tag{101}$$

$$\hat{G}_3^{\mathrm{d;P}} = \frac{1}{\pi}\int_{T_P}^{\infty}\exp(-s\tau)\mathrm{Im}\left[\mathrm{p_P}\gamma_\mathrm{P}(\mathrm{p_P})\gamma_\mathrm{S}(\mathrm{p_P})\tilde{\mathrm{A}}_\parallel^\mathrm{P}(\mathrm{p_P})\frac{\partial \mathrm{p_P}}{\partial\tau}\right]\mathrm{d}\tau \quad \text{for } \mathrm{x}_3 \lessgtr 0, \tag{102}$$

will be identified as the 'diffraction' contribution to $\hat{G}_{1,3}^{\mathrm{s;P}}$ (associated with the edge of the crack). In these expressions

$$\frac{\partial p_P}{\partial\tau} = \frac{x_1}{r^2} + \mathrm{i}\frac{|\mathrm{x}_3|}{\mathrm{r}^2}\frac{\tau}{(\tau^2 - \mathrm{T_P^2})^{1/2}} \quad \text{for } \mathrm{T_P} < \tau < \infty, \tag{103}$$

while for $x_1/r = c_P p_0^P$ the lower limit of integration T_P has to be approached from above. In view of Lerch's theorem on the uniqueness of the one-sided Laplace transform, the time-domain expressions corresponding to Equations (96)-(99) and (101)-(102) are obtained as

$$v_{1,3}^{\mathrm{s;P}} = q(t) \overset{t}{*} G_{1,3}^{\mathrm{s;P}}(x_1, x_3, t), \tag{104}$$

with

$$G_{1,3}^{\mathrm{s;P}} = G_{1,3}^{\mathrm{g;P}} + G_{1,3}^{\mathrm{d;P}}, \tag{105}$$

in which

$$G_1^{\mathrm{g;P}} = \pm\{0, \tfrac{1}{2}, 1\}\,(p_0^P)^2\gamma_S(p_0^P)R^P(p_0^P)\delta[t - p_0^P x_1 - \gamma_P(p_0^P)|x_3|] \quad (106)$$

$$G_3^{\mathrm{g;P}} = -\{0, \tfrac{1}{2}, 1\}\,p_0^P\gamma_P(p_0^P)\gamma_S(p_0^P)R^P(p_0^P)\delta[t - p_0^P x_1 - \gamma_P(p_0^P)|x_3|] \quad (107)$$

$$\text{for } \{x_1/r < c_P p_0^P, x_1/r = c_P p_0^P, x_1/r > c_P p_0^P\},\ x_3 \lessgtr 0$$

and

$$G_1^{\mathrm{d;P}} = \mp\frac{1}{\pi}\mathrm{Im}\left[\mathrm{p_P^2}\gamma_\mathrm{S}(\mathrm{p_P})\tilde{\mathrm{A}}_\|^\mathrm{P}(\mathrm{p_P})\frac{\partial\mathrm{p_P}}{\partial\tau}\right]\ \mathrm{H(t - T_P)}\quad \text{for } \mathrm{x_3} \lessgtr 0, \quad (108)$$

$$G_3^{\mathrm{d;P}} = \frac{1}{\pi}\mathrm{Im}\left[\mathrm{p_P}\gamma_\mathrm{P}(\mathrm{p_P})\gamma_\mathrm{S}(\mathrm{p_P})\tilde{\mathrm{A}}_\|^\mathrm{P}(\mathrm{p_P})\frac{\partial\mathrm{p_P}}{\partial\tau}\right]\ \mathrm{H(t - T_P)}\quad \text{for } \mathrm{x_3} \lessgtr 0, \quad (109)$$

while for $x_1/r = c_P p_0^P$, in the time convolution, the lower limit of integration T_P has to be approached from above.

Scattered *SV* wave (incident *P* wave)

Starting from Equations (12), (31) and (32), it follows that the expressions for the time
Laplace transform of the particle velocity components of the scattered *SV* wave due to an incident *P* wave are given by

$$\hat{v}_1^{\mathrm{s;SV}} = \pm\hat{q}(s)\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{L}}\gamma_S(p)(p^2 - \tfrac{1}{2}c_S^{-2})\tilde{A}_\|^P(p)\exp\{-s[px_1 + \gamma_S(p)|x_3|]\}\mathrm{dp}\quad \text{for } \mathrm{x_3} \lessgtr 0, \quad (110)$$

$$\hat{v}_3^{\mathrm{s;SV}} = \hat{q}(s)\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{L}}p(p^2 - \tfrac{1}{2}c_S^{-2})\tilde{A}_\|^P(p)\exp\{-s[px_1 + \gamma_S(p)|x_3|]\}\mathrm{dp}\quad \text{for } \mathrm{x_3} \lessgtr 0, \quad (111)$$

in which $\tilde{A}_\|^P$ is given by Equation (60).

The modified Cagniard path associated with Equations (110) and (111) must satisfy the equation

$$px_1 + \gamma_S(p)|x_3| = \tau, \quad (112)$$

where τ is real and positive. Candidates are: the part of the real p-axis in between the branch points $p = -c_S^{-1}$ and $p = c_S^{-1}$ of γ_S and the hyperbolic path $\{p \in \mathbb{C}; p = p_S\} \cup \{p \in \mathbb{C}; p = p_S^*\}$, with

$$p_S = p_S(x_1, |x_3|, \tau) = \frac{x_1}{r^2}\tau + \mathrm{i}\frac{|\mathrm{x_3}|}{\mathrm{r^2}}(\tau^2 - \mathrm{T_S^2})^{1/2}\quad \text{for } \mathrm{T_S} < \tau < \infty, \quad (113)$$

in which r is given by Equation (73) and

$$T_S = r/c_S \quad (114)$$

is the *S*-wave travel time from the edge of the crack to the point of observation. Further, the modified Cagniard path must originate from $\mathcal{L}$ through a continuous deformation without passing singularities of the integrand in order that Cauchy's theorem can be applied. To this end, circular arcs at infinity are to join $\mathcal{L}$ and the modified Cagniard path. In view of the behavior of $\tilde{A}_\|^P$ at infinity, viz. $\tilde{A}_\|^P = O(p^{-5/2})$ as $|p| \to \infty$, the contribution from the joining circular arcs at infinity vanishes in view of Jordan's lemma as long as $|x_3| > 0$. Under this condition, the integration along $\mathcal{L}$ can be replaced by an integration along the hyperbolic path $\{p \in \mathbb{C}; p = p_S\} \cup \{p \in \mathbb{C}; p = p_S^*\}$ in the region of space where its point of intersection with the real p-axis lies to the left of the pole $p = p_0^P$ as well as to the left of the branch point $p = c_P^{-1}$, which singularities occur in the expression for $\tilde{A}_\|^P$ (the latter through $K^-(p)$). As a consequence, in the region of space where $x_1/r \geq c_S p_0^P$ the contribution from the pole $p = p_0^P$ must be taken into account, while in the region of space where $x_1/r > c_S/c_P$ the deformation of the path of integration involves a loop integral around the branch cut associated with $K^-(p)$, joining the points of intersection of the hyperbolic part of Equation (112) with the real p-axis on either side of the branch cut. The contribution from this loop integral is associated with the shear head wave and is parametrized by $\{p \in \mathbb{C}; p = p_H\} \cup \{p \in \mathbb{C}; p = p_H^*\}$, with

$$\begin{aligned} p_H &= p_H(x_1, |x_3|, \tau) \\ &= \frac{x_1}{r^2}\tau - \frac{|x_3|}{r^2}(T_S^2 - \tau^2)^{1/2} + \mathrm{i}0\quad \text{for } \mathrm{x_1/r > c_S/c_P},\ \mathrm{T_H} < \tau < \mathrm{T_S}, \end{aligned} \quad (115)$$

with

$$T_H = x_1/c_P + (c_S^{-2} - c_P^{-2})^{1/2}|x_3|. \quad (116)$$

Taking the contributions from $\{p \in \mathbb{C}; p = p_S\}$ and $\{p \in \mathbb{C}; p = p_S^*\}$ as well as from $\{p \in \mathbb{C}; p = p_H\}$ and $\{p \in \mathbb{C}; p = p_H^*\}$ together, applying Schwarz's reflection principle of complex function theory, and introducing τ as the variable of integration, we thus arrive at

$$\hat{v}_{1,3}^{\mathrm{s;SV}} = \hat{q}(s)\hat{G}_{1,3}^{\mathrm{s;SV}}(x_1, x_3, s), \tag{117}$$

with

$$\hat{G}_{1,3}^{\mathrm{s;SV}} = \hat{G}_{1,3}^{\mathrm{g;SV}} + \hat{G}_{1,3}^{\mathrm{d;H}} + \hat{G}_{1,3}^{\mathrm{d;SV}}, \tag{118}$$

in which

$$\hat{G}_1^{\mathrm{g;SV}} = \mp\{0, \tfrac{1}{2}, 1\}\, \gamma_S(p_0^P)[(p_0^P)^2 - \tfrac{1}{2}c_S^{-2}]R^P(p_0^P) \exp\{-s[p_0^P x_1 + \gamma_S(p_0^P)|x_3|]\} \tag{119}$$

$$\hat{G}_3^{\mathrm{g;SV}} = -\{0, \tfrac{1}{2}, 1\}\, p_0^P[(p_0^P)^2 - \tfrac{1}{2}c_S^{-2}]R^P(p_0^P) \exp\{-s[p_0^P x_1 + \gamma_S(p_0^P)|x_3|]\} \tag{120}$$

$$\text{for } \{x_1/r < c_S p_0^P, x_1/r = c_S p_0^P, x_1/r > c_S p_0^P\}, x_3 \lesseqgtr 0,$$

where $R^P(p_0^P)$ is given by Equation (100), will be identified as the 'geometrical' contribution to $\hat{G}_{1,3}^{\mathrm{s;SV}}$ (in the sense of geometrical ray tracing via the crack),

$$\hat{G}_1^{\mathrm{d;H}} = \pm\{0, 1\}\, \frac{1}{\pi}\int_{T_H}^{T_S} \exp(-s\tau)\mathrm{Im}\left[\gamma_{\mathrm{S}}(\mathrm{p_H})(\mathrm{p_H^2} - \tfrac{1}{2}\mathrm{c_S^{-2}})\tilde{\mathrm{A}}_{\parallel}^{\mathrm{P}}(\mathrm{p_H})\frac{\partial \mathrm{p_H}}{\partial\tau}\right]\mathrm{d}\tau \tag{121}$$

$$\hat{G}_3^{\mathrm{d;H}} = \{0, 1\}\, \frac{1}{\pi}\int_{T_H}^{T_S} \exp(-s\tau)\mathrm{Im}\left[\mathrm{p_H}(\mathrm{p_H^2} - \tfrac{1}{2}\mathrm{c_S^{-2}})\tilde{\mathrm{A}}_{\parallel}^{\mathrm{P}}(\mathrm{p_H})\frac{\partial \mathrm{p_H}}{\partial\tau}\right]\mathrm{d}\tau \tag{122}$$

$$\text{for } \{x_1/r < c_S/c_P, x_1/r > c_S/c_P\}, x_3 \lesseqgtr 0,$$

will be identified as the head-wave contribution to $\hat{G}_{1,3}^{\mathrm{s;SV}}$ (associated with the edge of the crack) and

$$\hat{G}_1^{\mathrm{d;SV}} = \pm\frac{1}{\pi}\int_{T_S}^{\infty} \exp(-s\tau)\mathrm{Im}\left[\gamma_{\mathrm{S}}(\mathrm{p_S})(\mathrm{p_S^2} - \tfrac{1}{2}\mathrm{c_S^{-2}})\tilde{\mathrm{A}}_{\parallel}^{\mathrm{P}}(\mathrm{p_S})\frac{\partial \mathrm{p_S}}{\partial\tau}\right]\mathrm{d}\tau \quad \text{for } \mathrm{x_3} \lesseqgtr 0, \tag{123}$$

$$\hat{G}_3^{\mathrm{d;SV}} = \frac{1}{\pi}\int_{T_S}^{\infty} \exp(-s\tau)\mathrm{Im}\left[\mathrm{p_S}(\mathrm{p_S^2} - \tfrac{1}{2}\mathrm{c_S^{-2}})\tilde{\mathrm{A}}_{\parallel}^{\mathrm{P}}(\mathrm{p_S})\frac{\partial \mathrm{p_S}}{\partial\tau}\right]\mathrm{d}\tau \quad \text{for } \mathrm{x_3} \lesseqgtr 0, \tag{124}$$

will be identified as the 'diffraction' contribution to $\hat{G}_{1,3}^{\mathrm{s;SV}}$ (associated with the edge of the crack). In these expressions,

$$\frac{\partial p_H}{\partial\tau} = \frac{x_1}{r^2} + \frac{|x_3|}{r^2}\frac{\tau}{(T_S^2 - \tau^2)^{1/2}} \quad \text{for } x_1/r > c_S/c_P, \qquad T_H < \tau < T_S \tag{125}$$

and

$$\frac{\partial p_S}{\partial\tau} = \frac{x_1}{r^2} + \mathrm{i}\frac{|\mathrm{x_3}|}{\mathrm{r^2}}\frac{\tau}{(\tau^2 - \mathrm{T_S^2})^{1/2}} \quad \text{for } \mathrm{T_S} < \tau < \infty, \tag{126}$$

while for $x_1/r = c_S p_0^P$ the lower limit of integration T_S has to be approached from above. In view of Lerch's theorem on the uniqueness of the one-sided Laplace transform, the time-domain expressions corresponding to Equations (117)-(124) are obtained as

$$v_{1,3}^{\mathrm{s;SV}} = q(t) \overset{t}{*} G_{1,3}^{\mathrm{s;SV}}(x_1, x_3, t), \tag{127}$$

with

$$G_{1,3}^{\mathrm{s;SV}} = G_{1,3}^{\mathrm{g;SV}} + G_{1,3}^{\mathrm{d;H}} + G_{1,3}^{\mathrm{d;SV}}, \tag{128}$$

in which the geometrical part is given by

$$G_1^{\mathrm{g;SV}} = \mp\{0, \tfrac{1}{2}, 1\}\, \gamma_S(p_0^P)[(p_0^P)^2 - \tfrac{1}{2}c_S^{-2}]R^P(p_0^P)\delta[t - p_0^P x_1 - \gamma_S(p_0^P)|x_3|] \tag{129}$$

$$G_3^{\mathrm{g;SV}} = -\{0, \tfrac{1}{2}, 1\}\, p_0^P[(p_0^P)^2 - \tfrac{1}{2}c_S^{-2}]R^P(p_0^P)\delta[t - p_0^P x_1 - \gamma_S(p_0^P)|x_3|] \tag{130}$$

$$\text{for } \{x_1/r < c_S p_0^P, x_1/r = c_S p_0^P, x_1/r > c_S p_0^P\}, x_3 \lesseqgtr 0,$$

the head-wave part by

$$G_1^{\mathrm{d;H}} = \pm\{0,1\}\,\frac{1}{\pi}\mathrm{Im}\left[\gamma_{\mathrm{S}}(\mathrm{p_H})(\mathrm{p_H^2}-\tfrac{1}{2}\mathrm{c_S^{-2}})\tilde{\mathrm{A}}_{\|}^{\mathrm{P}}(\mathrm{p_H})\frac{\partial \mathrm{p_H}}{\partial\tau}\right][\mathrm{H}(\tau-\mathrm{T_H})-\mathrm{H}(\tau-\mathrm{T_S})] \tag{131}$$

$$G_3^{\mathrm{d;H}} = \{0,1\}\,\frac{1}{\pi}\mathrm{Im}\left[\mathrm{p_H}(\mathrm{p_H^2}-\tfrac{1}{2}\mathrm{c_S^{-2}})\tilde{\mathrm{A}}_{\|}^{\mathrm{P}}(\mathrm{p_H})\frac{\partial \mathrm{p_H}}{\partial\tau}\right][\mathrm{H}(\tau-\mathrm{T_H})-\mathrm{H}(\tau-\mathrm{T_S})] \tag{132}$$

$$\text{for } \{x_1/r < c_S/c_P, x_1/r > c_S/c_P\}, x_3 \lesseqgtr 0,$$

and the diffracted part by

$$G_1^{\mathrm{d;SV}} = \pm\frac{1}{\pi}\mathrm{Im}\left[\gamma_{\mathrm{S}}(\mathrm{p_S})(\mathrm{p_S^2}-\tfrac{1}{2}\mathrm{c_S^{-2}})\tilde{\mathrm{A}}_{\|}^{\mathrm{P}}(\mathrm{p_S})\frac{\partial \mathrm{p_S}}{\partial\tau}\right]\mathrm{H}(\tau-\mathrm{T_S}) \quad \text{for } \mathrm{x_3} \lesseqgtr 0, \tag{133}$$

$$G_3^{\mathrm{d;SV}} = \frac{1}{\pi}\mathrm{Im}\left[\mathrm{p_S}(\mathrm{p_S^2}-\tfrac{1}{2}\mathrm{c_S^{-2}})\tilde{\mathrm{A}}_{\|}^{\mathrm{P}}(\mathrm{p_S})\frac{\partial \mathrm{p_S}}{\partial\tau}\right]\mathrm{H}(\tau-\mathrm{T_S}) \quad \text{for } \mathrm{x_3} \lesseqgtr 0. \tag{134}$$

Scattered wave in the plane of the crack

In the plane of the crack, we have $x_3 = 0$ and hence Jordan's lemma fails to apply for the scattered *P* and *SV* waves separately. Upon taking the *P* and *SV* contributions together, we can, however, follow the standard procedures. Using Equations (23), (24) and (29)-(32), and substituting $x_3 = 0$, we obtain

$$\tilde{v}_1^{\mathrm{s;P/SV}}(p,\mp 0,s) = \mp\tfrac{1}{2}\hat{q}(s)\gamma_S(p)c_S^{-2}\tilde{A}_{\|}^{P}(p), \tag{135}$$

$$\tilde{v}_3^{\mathrm{s;P/SV}}(p,\mp 0,s) = \hat{q}(s)[p\gamma_P(p)\gamma_S(p)+p(p^2-\tfrac{1}{2}c_S^{-2})]\tilde{A}_{\|}^{P}(p), \tag{136}$$

in which $\tilde{A}_{\|}^{P}$ is given by Equation (60). Using these expressions in Equation (12), the behavior of $\tilde{A}_{\|}^{P}(p)$ as $|p|\to\infty$ ensures that Jordan's lemma is indeed applicable. The cases $x_1 < 0$ and $x_1 > 0$ require different treatments and will be discussed separately below.

Unruptured part ($x_1 < 0$)

For the unruptured part of the plane of the crack we have $x_1 < 0$. For this case, the path of integration $\mathcal{L}$ is supplemented with a semi-circular arc at infinity in $\mathcal{D}^-$. Since the right-hand side of Equation (135) is free from singularities in $\mathcal{D}^-$, it then follows that

$$\hat{v}_1^{\mathrm{s;P/SV}}(x_1,\mp 0,s) = 0 \qquad \text{for} \qquad x_1 < 0. \tag{137}$$

Consequently,

$$v_1^{\mathrm{s;P/SV}}(x_1,\mp 0,t) = 0 \qquad \text{for} \qquad x_1 < 0, \tag{138}$$

as it should be in view of the continuity of $v_1^{\mathrm{s;P/SV}}$ across the unruptured part of the plane of the crack in conjunction with the occurrence of the $\mp$ on the right-hand side of Equation (135). As far as $\hat{v}_3^{\mathrm{s;P/SV}}$ is concerned, the integration along $\mathcal{L}$ is replaced by one along a loop around the branch cut $\{p\in\mathbb{C}; -\infty < \mathrm{Re(p)} < -1/\mathrm{c_P}, \mathrm{Im(p)} = 0\}$ of the function on the right-hand side of Equation (136). Taking the contributions from the lower and upper parts of this loop together, applying Schwarz's reflection principle of complex function theory, introducing $\tau = px_1$ as the variable of integration, and applying Lerch's theorem, we arrive at

$$v_3^{\mathrm{s;P/SV}}(x_1,\mp 0,t) = q(t) \overset{t}{*} G_3^{\mathrm{s;P/SV}}(x_1,\mp 0,t), \tag{139}$$

with

$$G_3^{\mathrm{s;P/SV}}(x_1,\mp 0,t) = G_3^{\mathrm{d;P/SV}}(x_1,\mp 0,t) \tag{140}$$

consisting of a diffracted part only, given by

$$G_3^{\mathrm{d;P/SV}} = \frac{1}{\pi x_1}\mathrm{Im}\left[[\mathrm{p}\gamma_{\mathrm{P}}(\mathrm{p})\gamma_{\mathrm{S}}(\mathrm{p})+\mathrm{p}(\mathrm{p}^2-\tfrac{1}{2}\mathrm{c_S^{-2}})]\tilde{\mathrm{A}}_{\|}^{\mathrm{P}}(\mathrm{p})\right]\Big|_{\mathrm{p=t/x_1+i0}}\mathrm{H}(\mathrm{t}+\mathrm{x_1/c_P}) \quad \text{for} \quad x_1 < 0. \tag{141}$$

Ruptured part ($x_1 > 0$)

For the ruptured part of the plane of the crack we have $x_1 > 0$. For this case, we replace $\mathcal{L}$ by a loop around the branch cut $\{p \in \mathbb{C}; c_P^{-1} < \mathrm{Re(p)} < \infty, \mathrm{Im(p)} = 0\}$ of the right-hand sides of Equations (135)-(136) and take into account the residue of the simple pole at $p = c_R^{-1}$. Along the loop, we introduce $\tau = px_1$ as the variable of integration, take the contributions from the lower and upper parts together, apply Schwarz's reflection principle of complex function theory, and use Lerch's theorem. Via this procedure we arrive at

$$v_{1,3}^{\mathrm{s;P/SV}}(x_1, \mp 0, t) = q(t) \overset{t}{*} G_{1,3}^{\mathrm{s;P/SV}}(x_1, \mp 0, t), \tag{142}$$

with

$$G_{1,3}^{\mathrm{s;P/SV}}(x_1, \mp 0, t) = G_{1,3}^{\mathrm{g;P/SV}}(x_1, \mp 0, t) + G_{1,3}^{\mathrm{d;P/SV}}(x_1, \mp 0, t), \tag{143}$$

in which the geometrical part is given by

$$G_1^{\mathrm{g;P/SV}}(x_1, \mp 0, t) = \pm \tfrac{1}{2} c_S^{-2} \gamma_S(p_0^P) R^P(p_0^P) \delta(t - p_0^P x_1) \quad \text{for} \quad x_1 > 0, \tag{144}$$

$$G_3^{\mathrm{g;P/SV}}(x_1, \mp 0, t) = -[p_0^P \gamma_P(p_0^P)\gamma_S(p_0^P) + p_0^P[(p_0^P)^2 - \tfrac{1}{2}c_S^{-2}]] R^P(p_0^P) \delta(t - p_0^P x_1) \quad \text{for} \quad x_1 > 0, \tag{145}$$

and the diffracted part by

$$G_1^{\mathrm{d;P/SV}}(x_1, \mp 0, t) = \mp \frac{1}{2\pi x_1 c_S^2} \mathrm{Im} \left[\gamma_{\mathrm{S}}(\mathrm{p}) \tilde{\mathrm{A}}_{\|}^{\mathrm{P}}(\mathrm{p})\right]\Big|_{\mathrm{p=t/x_1}} \mathrm{H(t - x_1/c_P)} \quad \text{for} \quad \mathrm{x_1} > 0, \tag{146}$$

where the value $t = x_1/c_R$ has to be approached from either side, and

$$G_3^{\mathrm{d;P/SV}}(x_1, \mp 0, t) = \frac{1}{\pi x_1} \mathrm{Im} \left[[\mathrm{p}\gamma_{\mathrm{P}}(\mathrm{p})\gamma_{\mathrm{S}}(\mathrm{p}) + \mathrm{p}(\mathrm{p}^2 - \tfrac{1}{2}\mathrm{c_S^{-2}})]\tilde{\mathrm{A}}_{\|}^{\mathrm{P}}(\mathrm{p})\right]\Big|_{\mathrm{p=t/x_1}} \mathrm{H(t - x_1/c_P)}$$
$$- [c_R^{-1}\gamma_P(c_R^{-1})\gamma_S(c_R^{-1}) + c_R^{-1}(c_R^{-2} - \tfrac{1}{2}c_S^{-2})]\, \mathrm{Res}_{\mathrm{p=c_R^{-1}}} \tilde{\mathrm{A}}_{\|}^{\mathrm{P}}(\mathrm{p}) \quad \text{for} \quad \mathrm{x_1} > 0. \tag{147}$$

Time-domain expression for the scattered *P/SV* wave (incident *SV* wave)

In this section, the case of an incident *SV* wave is dealt with. The procedure to be followed is the same as in Section for the incident *P* wave as long as the direction of incidence is precritical, i.e., as long as the line of intersection of the wave front of the incident wave with the plane of the crack travels along the crack with a speed that is greater than c_P. For postcritical incidence, i.e., if this latter speed is less than c_P (though always greater than or equal to c_S), special features show up due to the fact that the simple pole that is indicative for the direction of incidence is now located on the branch cut associated with the diffracted *SV* wave. As a consequence, the residue theorem, that yields the geometrical contribution to the scattered wave motion can no longer be applied in its simple form.

Scattered *P* wave (incident *SV* wave)

Starting from Equations (12), (29) and (30), it follows that the expressions for the time Laplace transform of the particle-velocity components of the scattered *P* wave due to an incident *SV* wave are given by

$$\hat{v}_1^{\mathrm{s;P}} = \mp \hat{q}(s) \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{L}} p^2 \gamma_S(p) \tilde{A}_{\|}^S(p) \exp\{-s[px_1 + \gamma_P(p)|x_3|]\} \mathrm{dp} \quad \text{for } \mathrm{x_3} \lessgtr 0, \tag{148}$$

$$\hat{v}_3^{\mathrm{s;P}} = \hat{q}(s) \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{L}} p \gamma_P(p) \gamma_S(p) \tilde{A}_{\|}^S(p) \exp\{-s[px_1 + \gamma_P(p)|x_3|]\} \mathrm{dp} \quad \text{for } \mathrm{x_3} \lessgtr 0, \tag{149}$$

in which $\tilde{A}_{\|}^S$ is given by Equation (69). Proceeding as in Section we arrive at

$$v_{1,3}^{\mathrm{s;P}} = q(t) \overset{t}{*} G_{1,3}^{\mathrm{s;P}}(x_1, x_3, t), \tag{150}$$

with

$$G_{1,3}^{\mathrm{s;P}} = G_{1,3}^{\mathrm{g;P}} + G_{1,3}^{\mathrm{d;P}}, \tag{151}$$

in which

$$G_1^{\mathrm{g;P}} = \pm\{0, \tfrac{1}{2}, 1\}\,(p_0^S)^2\gamma_S(p_0^S)R^S(p_0^S)\delta[t - p_0^S x_1 - \gamma_P(p_0^S)|x_3|] \tag{152}$$

$$G_3^{\mathrm{g;P}} = \{0, \tfrac{1}{2}, 1\}\,p_0^S\gamma_P(p_0^S)\gamma_S(p_0^S)R^S(p_0^S)\delta[t - p_0^S x_1 - \gamma_P(p_0^S)|x_3|] \tag{153}$$

$$\text{for } \{x_1/r < c_P p_0^S, x_1/r = c_P p_0^S, x_1/r > c_P p_0^S\}, x_3 \lessgtr 0,$$

with

$$R^S(p_0^S) = \mathrm{Res}_{\mathrm{p}=\mathrm{p}_0^\mathrm{S}}\tilde{\mathrm{A}}_{\|}^\mathrm{S}(\mathrm{p}) = \frac{2\mathrm{c_S}[(\mathrm{p}_0^\mathrm{S})^2 - \frac{1}{2}\mathrm{c_S}^{-2}]}{\Delta_\mathrm{R}(\mathrm{p}_0^\mathrm{S})}, \tag{154}$$

is the 'geometrical' contribution to $\hat{G}_{1,3}^{\mathrm{s;P}}$ (in the sense of geometrical ray tracing via the crack) and

$$G_1^{\mathrm{d;P}} = \mp\frac{1}{\pi}\mathrm{Im}\left[\mathrm{p_P^2}\gamma_\mathrm{S}(\mathrm{p_P})\tilde{\mathrm{A}}_{\|}^\mathrm{S}(\mathrm{p_P})\frac{\partial \mathrm{p_P}}{\partial\tau}\right] \quad \text{for } \mathrm{x}_3 \lessgtr 0, \tag{155}$$

$$G_3^{\mathrm{d;P}} = \frac{1}{\pi}\mathrm{Im}\left[\mathrm{p_P}\gamma_P(\mathrm{p_P})\gamma_\mathrm{S}(\mathrm{p_P})\tilde{\mathrm{A}}_{\|}^\mathrm{S}(\mathrm{p_P})\frac{\partial \mathrm{p_P}}{\partial\tau}\right] \quad \text{for } \mathrm{x}_3 \lessgtr 0, \tag{156}$$

is the 'diffraction' contribution to $\hat{G}_{1,3}^{\mathrm{s;P}}$ (associated with the edge of the crack).

Scattered *SV* wave (incident *SV* wave)

Starting from Equations (12), (31) and (32), it follows that the expressions for the time Laplace transform of the particle velocity components of the scattered *SV* wave due to an incident *SV* wave are given by

$$\hat{v}_1^{\mathrm{s;SV}} = \pm\hat{q}(s)\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{L}}\gamma_S(p)(p^2 - \tfrac{1}{2}c_S^{-2})\tilde{A}_{\|}^S(p)\exp\{-s[px_1 + \gamma_S(p)|x_3|]\}\mathrm{dp} \quad \text{for } \mathrm{x}_3 \lessgtr 0, \tag{157}$$

$$\hat{v}_3^{\mathrm{s;SV}} = \hat{q}(s)\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{L}}p(p^2 - \tfrac{1}{2}c_S^{-2})\tilde{A}_{\|}^S(p)\exp\{-s[px_1 + \gamma_S(p)|x_3|]\}\mathrm{dp} \quad \text{for } \mathrm{x}_3 \lessgtr 0, \tag{158}$$

in which $\tilde{A}_{\|}^S$ is given by Equation (69). Proceeding as in Section , we arrive at

$$v_{1,3}^{\mathrm{s;SV}} = q(t) \overset{t}{*} G_{1,3}^{\mathrm{s;SV}}(x_1, x_3, t), \tag{159}$$

with

$$G_{1,3}^{\mathrm{s;SV}} = G_{1,3}^{\mathrm{g;SV}} + G_{1,3}^{\mathrm{d;H}} + G_{1,3}^{\mathrm{d;SV}}, \tag{160}$$

in which the geometrical part is given by

$$G_1^{\mathrm{g;SV}} = \mp\{0, \tfrac{1}{2}, 1\}\,\gamma_S(p_0^S)[(p_0^S)^2 - \tfrac{1}{2}c_S^{-2}]R^S(p_0^S)\delta[t - p_0^S x_1 - \gamma_S(p_0^S)|x_3|] \tag{161}$$

$$G_3^{\mathrm{g;SV}} = -\{0, \tfrac{1}{2}, 1\}\,p_0^S[(p_0^S)^2 - \tfrac{1}{2}c_S^{-2}]R^S(p_0^S)\delta[t - p_0^S x_1 - \gamma_S(p_0^S)|x_3|] \tag{162}$$

$$\text{for } \{x_1/r < c_S p_0^S, x_1/r = c_S p_0^S, x_1/r > c_S p_0^S\}, x_3 \lessgtr 0,$$

with

$$R^S(p_0^S) = \mathrm{Res}_{\mathrm{p}=\mathrm{p}_0^\mathrm{S}}\tilde{\mathrm{A}}_{\|}^\mathrm{S}(\mathrm{p}) = \frac{2\mathrm{c_S}[(\mathrm{p}_0^\mathrm{S})^2 - \frac{1}{2}\mathrm{c_S}^{-2}]}{\Delta_\mathrm{R}(\mathrm{p}_0^\mathrm{S})}, \tag{163}$$

for $0 \le p_0^S < c_P^{-1}$ (i.e. for precritical incidence, for which case $\gamma_P(p_0^S)$ is real-valued) and

$$R^S(p_0^S) = \mathrm{Re}\left[\mathrm{Res}_{\mathrm{p}=\mathrm{p}_0^\mathrm{S}+\mathrm{i}0}\tilde{\mathrm{A}}_{\|}^\mathrm{S}(\mathrm{p})\right] = \mathrm{Re}\left[\frac{2\mathrm{c_S}[(\mathrm{p}_0^\mathrm{S})^2 - \frac{1}{2}\mathrm{c_S}^{-2}]}{\Delta_\mathrm{R}(\mathrm{p}_0^\mathrm{S})}\right], \tag{164}$$

for $c_P^{-1} \le p_0^S < c_S^{-1}$ (i.e. postcritical incidence for which case $\gamma_P(p_0^S + \mathrm{i}0)$ is negative imaginary), the head-wave part by

$$G_1^{\mathrm{d;H}} = \pm\{0, 1\}\,\frac{1}{\pi}\mathrm{Im}\left[\gamma_\mathrm{S}(\mathrm{p_H})(\mathrm{p_H^2} - \tfrac{1}{2}\mathrm{c_S}^{-2})\tilde{\mathrm{A}}_{\|}^\mathrm{S}(\mathrm{p_H})\frac{\partial \mathrm{p_H}}{\partial\tau}\right][\mathrm{H}(\tau - \mathrm{T_H}) - \mathrm{H}(\tau - \mathrm{T_S})] \tag{165}$$

$$G_3^{\mathrm{d;H}} = \{0, 1\}\,\frac{1}{\pi}\mathrm{Im}\left[\mathrm{p_H}(\mathrm{p_H^2} - \tfrac{1}{2}\mathrm{c_S}^{-2})\tilde{\mathrm{A}}_{\|}^\mathrm{S}(\mathrm{p_H})\frac{\partial \mathrm{p_H}}{\partial\tau}\right][\mathrm{H}(\tau - \mathrm{T_H}) - \mathrm{H}(\tau - \mathrm{T_S})] \tag{166}$$

$$\text{for } \{x_1/r < c_S/c_P, x_1/r > c_S/c_P\}, x_3 \lessgtr 0,$$

and the diffracted part by

$$G_1^{\mathrm{d;SV}} = \pm\frac{1}{\pi}\mathrm{Im}\left[\gamma_\mathrm{S}(\mathrm{p_S})(\mathrm{p_S^2} - \tfrac{1}{2}\mathrm{c_S^{-2}})\tilde{\mathrm{A}}_\parallel^\mathrm{S}(\mathrm{p_S})\frac{\partial \mathrm{p_S}}{\partial\tau}\right]\mathrm{H}(\tau - \mathrm{T_S}) \quad \text{for } \mathrm{x_3} \lessgtr 0, \tag{167}$$

$$G_3^{\mathrm{d;SV}} = \frac{1}{\pi}\mathrm{Im}\left[\mathrm{p_S}(\mathrm{p_S^2} - \tfrac{1}{2}\mathrm{c_S^{-2}})\tilde{\mathrm{A}}_\parallel^\mathrm{S}(\mathrm{p_S})\frac{\partial \mathrm{p_S}}{\partial\tau}\right]\mathrm{H}(\tau - \mathrm{T_S}) \quad \text{for } \mathrm{x_3} \lessgtr 0. \tag{168}$$

As far as the contribution from the pole p_0^S, and hence the expression for $R^S(p_0^S)$ is concerned, we have to distinguish between the cases of precritical incidence (for which $0 \le p_0^S < c_P^{-1}$) and postcritical incidence (for which $c_P^{-1} \le p_0^S \le c_S^{-1}$). In the case of precritical incidence the pole p_0^S is situated to the left of the branch point c_P^{-1} and $R^S(p_0^S)$ is given by Equation (163) follows from a straightforward application of the theorem of residues. In the case of postcritical incidence the pole p_0^S is situated on the branch cut leaving c_P^{-1}, and two semi-circular arcs on either side of the branch cut have to be introduced to avoid the pole. These semi-circular arcs symmetrically break up the loop associated with the head-wave contribution. Taking together their contributions, we arrive at Equation (164) as the corresponding expression for $R^S(p_0^S)$.

Scattered wave in the plane of the crack

In the plane of the crack, we have $x_3 = 0$ and hence Jordan's lemma fails to apply for the scattered *P* and *SV* waves separately. Upon taking the *P* and *SV* contributions together, we can, however, follow the standard procedures. Using Equations (23), (24) and (29)-(32), and substituting $x_3 = 0$, we obtain

$$\tilde{v}_1^{\mathrm{s;P/SV}}(p, \mp 0, s) = \mp\tfrac{1}{2}\hat{q}(s)\gamma_S(p)c_S^{-2}\tilde{A}_\parallel^S(p), \tag{169}$$

$$\tilde{v}_3^{\mathrm{s;P/SV}}(p, \mp 0, s) = \hat{q}(s)[p\gamma_P(p)\gamma_S(p) + p(p^2 - \tfrac{1}{2}c_S^{-2})]\tilde{A}_\parallel^S(p), \tag{170}$$

in which $\tilde{A}_\parallel^S$ is given by Equation (69). Using these expressions in Equation (12), the behavior of $\tilde{A}_\parallel^S(p)$ as $|p| \to \infty$ ensures that Jordan's lemma is indeed applicable. The cases $x_1 < 0$ and $x_1 > 0$ require different treatments; their results will be presented separately below.

Unruptured part ($x_1 < 0$)

Proceeding as in Section we arrive for the unruptured part of the plane of the crack at

$$v_1^{\mathrm{s;P/SV}}(x_1, \mp 0, t) = 0 \qquad \text{for} \qquad x_1 < 0, \tag{171}$$

as it should be in view of the continuity of $v_1^{\mathrm{s;P/SV}}$ across the unruptured part of the plane of the crack in conjunction with the occurrence of the $\mp$ on the right-hand side of Equation (169). Furthermore,

$$v_3^{\mathrm{s;P/SV}}(x_1, \mp 0, t) = q(t) \overset{t}{*} G_3^{\mathrm{s;P/SV}}(x_1, \mp 0, t), \tag{172}$$

with

$$G_3^{\mathrm{s;P/SV}}(x_1, \mp 0, t) = G_3^{\mathrm{d;P/SV}}(x_1, \mp 0, t) \tag{173}$$

consisting of a diffracted part only, given by

$$G_3^{\mathrm{d;P/SV}} = \frac{1}{\pi x_1}\mathrm{Im}\left[[\mathrm{p}\gamma_\mathrm{P}(\mathrm{p})\gamma_\mathrm{S}(\mathrm{p}) + \mathrm{p}(\mathrm{p}^2 - \tfrac{1}{2}\mathrm{c_S^{-2}})]\tilde{\mathrm{A}}_\parallel^\mathrm{S}(\mathrm{p})\right]\Big|_{\mathrm{p=t/x_1+i0}}\mathrm{H}(\mathrm{t} + \mathrm{x_1/c_P}) \quad \text{for} \quad x_1 < 0. \tag{174}$$

Ruptured part ($x_1 > 0$)

Proceeding as in Section we arrive for the ruptured part of the plane of the crack at

$$v_{1,3}^{\mathrm{s;P/SV}}(x_1, \mp 0, t) = q(t) \overset{t}{*} G_{1,3}^{\mathrm{s;P/SV}}(x_1, \mp 0, t), \tag{175}$$

with

$$G_{1,3}^{\mathrm{s;P/SV}}(x_1, \mp 0, t) = G_{1,3}^{\mathrm{g;P/SV}}(x_1, \mp 0, t) + G_{1,3}^{\mathrm{d;P/SV}}(x_1, \mp 0, t), \tag{176}$$

in which the geometrical part is given by

$$G_1^{\mathrm{g;P/SV}}(x_1,\mp 0,t) = \pm\tfrac{1}{2}c_S^{-2}\gamma_S(p_0^S)R^S(p_0^S)\delta[t-p_0^S x_1] \quad \text{for} \quad x_1>0, \tag{177}$$

$$G_3^{\mathrm{g;P/SV}}(x_1,\mp 0,t) = -[p_0^S\gamma_P(p_0^S)\gamma_S(p_0^S)+p_0^S[(p_0^S)^2-\tfrac{1}{2}c_S^{-2}]]R^S(p_0^S)\delta[t-p_0^S x_1] \quad \text{for} \quad x_1>0, \tag{178}$$

and the diffracted part by

$$G_1^{\mathrm{d;P/SV}}(x_1,\mp 0,t) = \mp\frac{1}{2\pi x_1 c_S^2}\mathrm{Im}\left[\gamma_{\mathrm{S}}(\mathrm{p})\tilde{\mathrm{A}}_{\parallel}^{\mathrm{S}}(\mathrm{p})\right]\Big|_{\mathrm{p=t/x_1}}\mathrm{H(t-x_1/c_P)} \quad \text{for} \quad x_1>0, \tag{179}$$

$$\begin{aligned} G_3^{\mathrm{d;P/SV}}(x_1,\mp 0,t) = {} & \frac{1}{\pi x_1}\mathrm{Im}\left[[\mathrm{p}\gamma_{\mathrm{P}}(\mathrm{p})\gamma_{\mathrm{S}}(\mathrm{p})+\mathrm{p}(\mathrm{p}^2-\tfrac{1}{2}\mathrm{c_S^{-2}})]\tilde{\mathrm{A}}_{\parallel}^{\mathrm{S}}(\mathrm{p})\right]\Big|_{\mathrm{p=t/x_1}}\mathrm{H(t-x_1/c_P)} \\ & -[c_R^{-1}\gamma_P(c_R^{-1})\gamma_S(c_R^{-1})+c_R^{-1}(c_R^{-2}-\tfrac{1}{2}c_S^{-2})]\mathrm{Res}_{\mathrm{p=c_R^{-1}}}\tilde{\mathrm{A}}_{\parallel}^{\mathrm{S}}(\mathrm{p}) \quad \text{for} \quad x_1>0. \end{aligned} \tag{180}$$

Global relaxation in the solid

In this section, global relaxation in the solid is included in the analysis. The relevant mechanism is represented through the global relaxation functions $\phi=\phi(t)$ for the inertia and $\psi=\psi(t)$ for the compliance as introduced in Section . The consequences of this are most easily investigated in the complex frequency or s-domain. In particular, from the particle velocity elastodynamic wave equation, it is seen that the result of the introduction of the relaxation mechanism is to replace s^2 in this equation by $s^2\hat{\phi}(s)\hat{\psi}(s)$. Since, further, the boundary conditions at the crack are of the homogeneous type, the spectral representations of the scattered wavefield quantities can now profitably taken of the form

$$\{\hat{v}_r,\hat{\tau}_{pq}\}(x_1,x_3,s)=\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{L}}\exp\{-s[\hat{\phi}(s)\hat{\psi}(s)]^{1/2}px_1\}\{\tilde{v}_r,\tilde{\tau}_{pq}\}(p,x_3,s)\mathrm{d}p. \tag{181}$$

In this respect, it is of importance to observe that for linear, causal, dissipative solids, $\hat{\phi}(s)$ and $\hat{\psi}(s)$ are real and positive for positive real values of s and bounded away from 0, while $\hat{\phi}(s)=1+o(1)$ and $\hat{\psi}(s)=1+o(1)$ as $s\to\infty$. Proceeding as in Sections - each scattered wavefield constituent is of the general shape

$$\tilde{w}(p,x_3,s)=\hat{q}(s)\tilde{g}(p,x_3) \tag{182}$$

with

$$\tilde{g}(p,x_3)=\tilde{A}(p)\exp\{-s[\hat{\phi}(s)\hat{\psi}(s)]^{1/2}\gamma_{P,S}(p)|x_3|\}, \tag{183}$$

where $\gamma_{P,S}(p)$ is, still, given by either Equation (22) for *P* waves or Equation (14) for *SH* and *SV* waves. The factorization method remains unaltered, whereas the modified Cagniard method now leads to complex-frequency domain expressions of the type

$$\hat{g}(p,x_3,s)=\frac{1}{\pi}\int_{T_{P,S}}^{\infty}\exp\{-s[\hat{\phi}(s)\hat{\psi}(s)]^{1/2}\tau\}\mathrm{Im}\left[\tilde{\mathrm{A}}(\mathrm{p_{P,S}})\frac{\partial\mathrm{p_{P,S}}}{\partial\tau}\right]\mathrm{d}\tau \tag{184}$$

for diffracted body-wave contributions, and

$$\hat{g}(x_1,x_3,s)=\frac{1}{\pi}\int_{T_H}^{T_S}\exp\{-s[\hat{\phi}(s)\hat{\psi}(s)]^{1/2}\tau\}\mathrm{Im}\left[\tilde{\mathrm{A}}(\mathrm{p_H})\frac{\partial\mathrm{p_H}}{\partial\tau}\right]\mathrm{d}\tau \tag{185}$$

for diffracted head-wave contributions.

Now, for a number of relaxation functions, $\exp\{-s[\hat{\phi}(s)\hat{\psi}(s)]^{1/2}\tau\}$ admits a representation of the type

$$\exp\{-s[\hat{\phi}(s)\hat{\psi}(s)]^{1/2}\tau\}=\int_{\tau}^{\infty}\exp(-st)U(t,\tau)\mathrm{d}t. \tag{186}$$

Using Equation (186) in Equation (185) and employing Lerch's theorem on the uniqueness of the time-Laplace transform for real, positive values of the transform parameter, we end up with

$$g(x_1,x_3,t)=\left[\int_{T_{P,S}}^{t}U(t,\tau)G(x_1,x_3,\tau)\mathrm{d}\tau\right]H(t-T_{P,S}) \tag{187}$$

for diffracted body-wave contributions, and

$$g(x_1, x_3, t) = \left[\int_{T_H}^{\min(t,T_S)} U(t,\tau)G(x_1,x_3,\tau)\mathrm{d}\tau\right][H(t-T_H) - H(t-T_S)] \tag{188}$$

for diffracted head-wave contributions. Here, $G(x_1, x_3, \tau)$ in all cases is scattered wave pertaining to the lossless case for which $\phi(t) = \delta(t)$ and $\psi(t) = \delta(t)$.

Special case: Maxwell-type visco-elasticity

For the special case $\phi(t) = \delta(t) + \alpha H(t)$, $\psi(t) = \delta(t) + \beta H(t)$, we have

$$\hat{\phi}(s) = 1 + \frac{\alpha}{s} \quad \text{for} \quad \mathrm{Re(s)} > 0, \tag{189}$$

$$\hat{\psi}(s) = 1 + \frac{\beta}{s} \quad \text{for} \quad \mathrm{Re(s)} > 0. \tag{190}$$

Here, $\alpha > 0$ and $\beta > 0$. This case includes both the influence of frictional forces and a Maxwell-type visco-elasticity (for the latter see Kolsky (1964, pp.107)).

The relation corresponding to Equation (186) becomes (Abramowitz & Stegun, 1965)

$$\exp[-(s+\alpha)^{1/2}(s+\beta)^{1/2}\tau] = \int_{\tau}^{\infty} \exp(-st)U_1(t,\tau)\mathrm{dt}, \tag{191}$$

in which

$$U_1(t,\tau) = -\partial_\tau U_0(t,\tau), \tag{192}$$

with

$$U_0(t,\tau) = \exp[-(\alpha+\beta)t/2]I_0[(|\beta-\alpha|/2)(t^2-\tau^2)^{1/2}]H(t-\tau), \tag{193}$$

where I_0 denotes the modified Bessel function of the first kind and order 0. This type of relaxation has been used to generate the numerical results for wave diffraction in a dispersive solid.

Numerical results

In this section, some numerical results will be presented. The source signature of the incident plane wave is taken to be the power-exponential pulse,

$$q(t) = A\left(\frac{\alpha t}{\nu}\right)^{\nu} \exp(-\alpha t + \nu)\, H(t). \tag{194}$$

Here, A is the amplitude of the pulse and the parameters ν and α are related to the pulse rise time t_r and the pulse time width t_w via

$$\nu = \frac{t_\mathrm{r}}{\alpha}, \tag{195}$$

$$\alpha = \frac{1}{t_\mathrm{w} - t_\mathrm{r}}. \tag{196}$$

The pulse rise time and the pulse time width must satisfy the condition $t_\mathrm{w} > t_\mathrm{r} > 0$, which entails $\nu > 0$ and $\alpha > 0$.

For any component of the particle velocity, the incident-wave source signature has to be convolved with a Green's function. In the latter, there always occurs a Jacobian of the type $\partial p_{P,S}/\partial\tau$ associated with the modified Cagniard method. This Jacobian has an inverse square-root singularity at the arrival time $T_{P,S}$ of the diffracted body waves, which singularity causes a difficulty in the numerical evaluation of the time-convolution integral. This difficulty can be circumvented by rewriting the integral as a Stieltjes-type integral in which $p_{P,S} = p_{P,S}(x_1, x_3, \tau)$ is incorporated in the differential through $(\partial p_{P,S}/\partial\tau)\,\mathrm{d}\tau = \mathrm{d}[\mathrm{p_{P,S}}(.,.,\tau)]$.

Another difficulty can be caused by the presence of a factor $1/\gamma^{\mp}_{P,S}$ in the expression for the Green's function in case the modified Cagniard path comes close to the pertaining branch point. This factor, too, can be accommodated in the the differential of the Stieltjes-type integral through $[1/\gamma^{\mp}_{P,S}(p_{P,S})]\mathrm{d}[\mathrm{p_{P,S}}(.,.,\tau)] = \mathrm{d}[-2\,\gamma^{\mp}_{\mathrm{P,S}}(\mathrm{p_{P,S}}(.,.,\tau))]$.

Let the generic form of the convolution integral under consideration be given by

$$v^{\rm d} = \int_{T_{P,S}}^{t} q(t-\tau)\,\frac{1}{\pi}{\rm Im}\left[\frac{\tilde{\rm B}({\rm p_{P,S}})}{\gamma^{\mp}_{\rm P,S}({\rm p_{P,S}})}\,\frac{\partial {\rm p_{P,S}}}{\partial \tau}\right]{\rm d}\tau\,{\rm H(t-T_{P,S})}. \tag{197}$$

With the indicated procedure, this expression is rewritten as

$$v^{\rm d} = {\rm Im}\left[\frac{1}{\pi}\int_{\rm T_{P,S}}^{\rm t} {\rm q(t}-\tau)\,\tilde{\rm B}({\rm p_{P,S}}){\rm d}[-2\,\gamma^{\mp}_{\rm P,S}({\rm p_{P,S}}(.,.,\tau))]\right]\,{\rm H(t-T_{P,S})}. \tag{198}$$

This Stieltjes-type integral is, finally, evaluated numerically with the aid the quasi-trapezoidal rule given in Appendix B.

***SH*-wave scattering**

Here, we illustrate the wavefield in the case of *SH*-wave scattering. In Figure 1 a snapshot of the incident plane wave is shown. Figure 2 represents a snapshot associated with Equation (77); Figure 3 represents a snapshot associated with Equation (78). In Figure 4 a snapshot of the total field is shown.

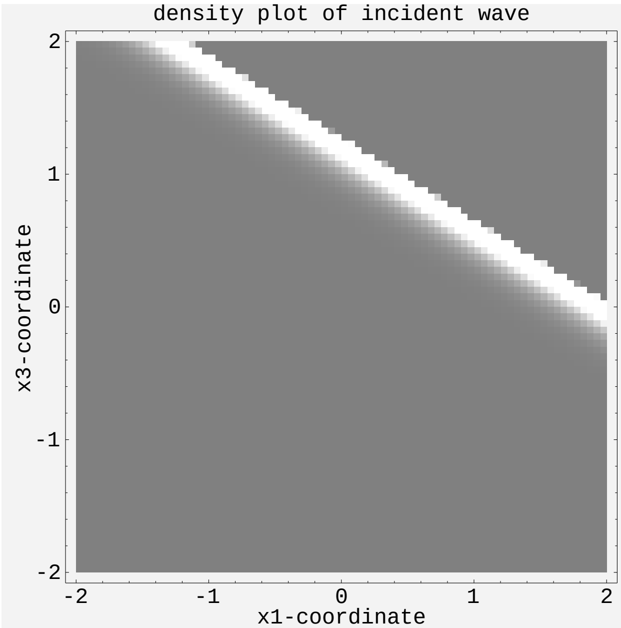

Figure 1. *SH*: snapshot of the incident plane wave.

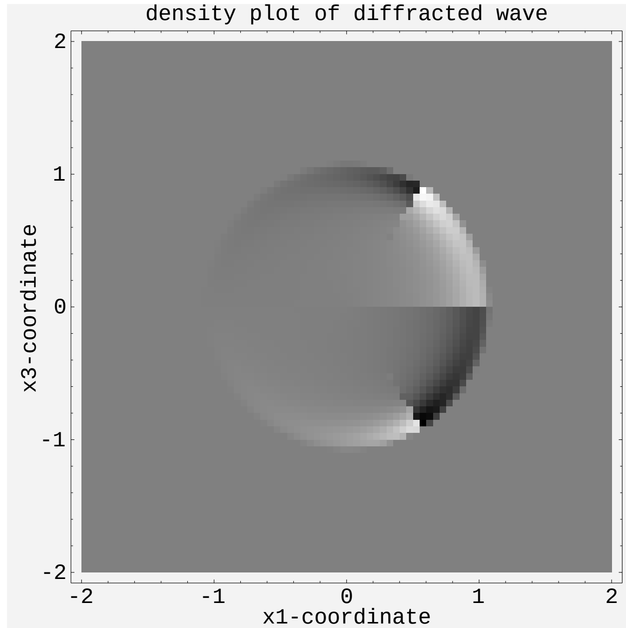

Figure 3. *SH*: snapshot of the diffraction contribution.

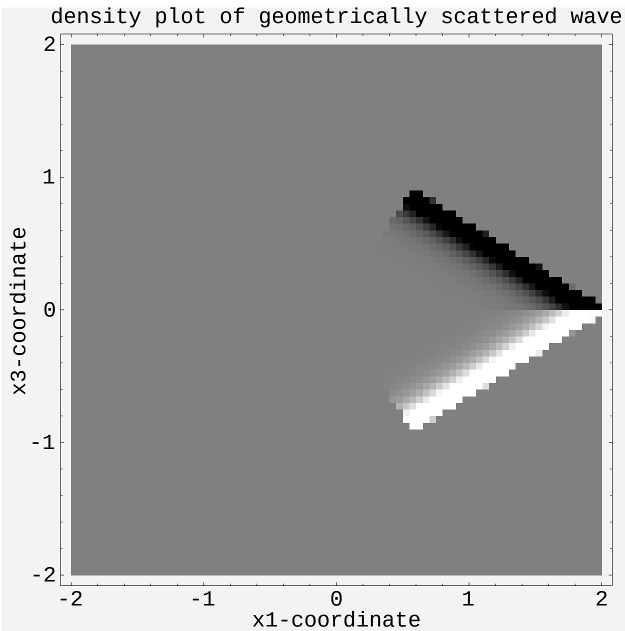

Figure 2. *SH*: snapshot of the geometrical contribution.

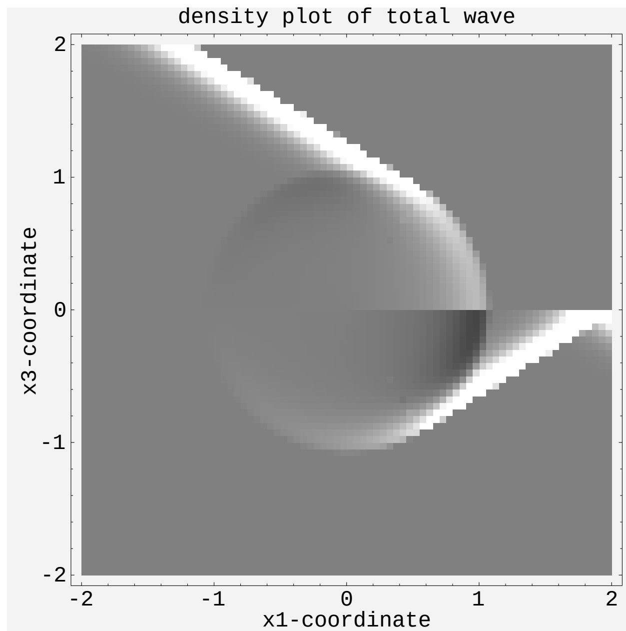

Figure 4. *SH*: snapshot of the total field.

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APPENDIX A: The Plemelj additive decomposition formulas and their application to kernel function factorization

The Plemelj decomposition formulas (Muskhelishvili, 1953) deal with the additive decomposition of a function $Q = Q(p)$ of the type

$$Q(p) = Q^{-}(p) + Q^{+}(p) \quad \text{for } p \in \mathcal{L}, \tag{A1}$$

where $\mathcal{L}$ is an oriented curve in the complex p-plane with the property that it divides the plane into a domain $\mathcal{D}^{-}$ to the left of $\mathcal{L}$ and a domain $\mathcal{D}^{+}$ to the right of $\mathcal{L}$, $Q(p)$ is defined and continuous on $\mathcal{L}$, $Q^{-}(p)$ is regular in $\mathcal{D}^{-}$ and $Q^{+}(p)$ is regular in $\mathcal{D}^{+}$. For our purposes, $\mathcal{L}$ extends to infinity and is the path of integration in the spectral representation of the wave constituents. It will be shown that under certain additional restrictions the decomposition is accomplished by

$$Q^{-}(p) = \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{L}} \frac{Q(w)}{w-p} \mathrm{dw} \quad \text{for } \mathrm{p} \in \mathcal{D}^{-} \tag{A2}$$

and

$$Q^{+}(p) = -\frac{1}{2\pi \mathrm{i}} \int_{\mathcal{L}} \frac{Q(w)}{w-p} \mathrm{dw} \quad \text{for } \mathrm{p} \in \mathcal{D}^{+}. \tag{A3}$$

A sufficient condition for the right-hand sides of Equations (A2) and (A3) to exist is

$$Q(w) = O(w^{-q}) \quad \text{as } |w| \to \infty \text{ along } \mathcal{L}, \ \text{with } \mathrm{II} > /. \tag{A4}$$

Under this condition, Q^{-} is an analytic function of p that is regular in $\mathcal{D}^{-}$ and Q^{+} is an analytic function of p that is regular in $\mathcal{D}^{+}$. Now for an arbitrary point $p = p_0$ on $\mathcal{L}$ define

$$Q^{-}(p_0) = \lim_{\substack{p \to p_0, \\ p \in \mathcal{D}^{-}}} \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{L}} \frac{Q(w)}{w-p} \mathrm{dw} \tag{A5}$$

and

$$Q^{+}(p_0) = -\lim_{\substack{p \to p_0, \\ p \in \mathcal{D}^{+}}} \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{L}} \frac{Q(w)}{w-p} \mathrm{dw}. \tag{A6}$$

A sufficient condition for the limits on the right-hand sides of Equations (A5) and (A6) to exist is that $Q = Q(w)$ satisfies on $\mathcal{L}$ the Hölder condition

$$|Q(w_1) - Q(w_2)| < A|w_1 - w_2|^{\alpha} \quad \text{with } A > 0 \text{ and } \alpha > 0 \tag{A7}$$

for all $w_1 \in \mathcal{L}$ and $w_2 \in \mathcal{L}$. Under this condition (Muskhelishvili, 1953),

$$Q^{-}(p_0) = \frac{1}{2} Q(p_0) + \frac{1}{2\pi \mathrm{i}} \mathrm{PV} \int_{\mathcal{L}} \frac{\mathrm{Q(w)}}{\mathrm{w} - \mathrm{p_0}} \mathrm{dw} \qquad \text{for} \qquad \mathrm{p_0} \in \mathcal{L} \tag{A8}$$

and

$$Q^{+}(p_0) = \frac{1}{2} Q(p_0) - \frac{1}{2\pi \mathrm{i}} \mathrm{PV} \int_{\mathcal{L}} \frac{\mathrm{Q(w)}}{\mathrm{w} - \mathrm{p}} \mathrm{dw}, \qquad \text{for} \qquad \mathrm{p_0} \in \mathcal{L}, \tag{A9}$$

where PV $\int$ denotes the Cauchy principal value of the relevant integral, and the property

$$Q^{-}(p_0) + Q^{+}(p_0) = Q(p_0) \quad \text{for } p_0 \in \mathcal{L} \tag{A10}$$

obviously holds.

To apply these results to the kernel factorization problem of Section , we take

$$Q(p) = \log[K(p)], \tag{A11}$$

where $K(p)$ is given by Equation (56). Through the way in which we have constructed $K(p)$, in particular its behavior as $|p| \to \infty$, $\log[K(p)]$ satisfies all conditions laid on $Q(p)$ and hence we can take

$$K^{-}(p) = \exp[Q^{-}(p)] \quad \text{for } p \in \mathcal{D}^{-}, \tag{A12}$$

with

$$Q^{-}(p) = \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{L}} \frac{\log[K(w)]}{w - p} \mathrm{dw} \quad \text{for p} \in \mathcal{D}^{-} \tag{A13}$$

and

$$K^{+}(p) = \exp[Q^{+}(p)] \quad \text{for } p \in \mathcal{D}^{+}, \tag{A14}$$

with

$$Q^{+}(p) = -\frac{1}{2\pi \mathrm{i}} \int_{\mathcal{L}} \frac{\log[K(w)]}{w - p} \mathrm{dw} \quad \text{for p} \in \mathcal{D}^{+}. \tag{A15}$$

In view of Equations (A8) and (A9) we have

$$Q^{-}(p_0) = \frac{1}{2} \log[K(p_0)] + \frac{1}{2\pi \mathrm{i}} \mathrm{PV} \int_{\mathcal{L}} \frac{\log[\mathrm{K(w)}]}{\mathrm{w - p_0}} \mathrm{dw} \quad \text{for } \mathrm{p_0} \in \mathcal{L} \tag{A16}$$

and

$$Q^{+}(p_0) = \frac{1}{2} \log[K(p_0)] - \frac{1}{2\pi \mathrm{i}} \mathrm{PV} \int_{\mathcal{L}} \frac{\log[\mathrm{K(w)}]}{\mathrm{w - p}} \mathrm{dw} \quad \text{for } \mathrm{p_0} \in \mathcal{L}, \tag{A17}$$

from which it follows that

$$K^{-}(p_0) K^{+}(p_0) = K(p_0) \quad \text{for } p_0 \in \mathcal{L}. \tag{A18}$$

Since through Equation (56) the kernel function $K = K(p)$ is defined everywhere in the complex p-plane cut along the branch cuts $\{p \in \mathcal{C}; \infty/\rfloor_{\mathcal{P}} < |\mathrm{Re(p)}| < 1/c_{\mathrm{S}}, \mathrm{Im(p)} = 0\}$, the right-hand sides of Equations (A13) and (A15) can be transformed into expressions that are more amenable to numerical evaluation. To this end, the path of integration $\mathcal{L}$ is, in the expression for Q^{-}, supplemented by a semi-circular arc of arbitrarily large radius in $\mathcal{D}^{+}$ and in the expression for Q^{+} by a semi-circular arc of arbitrarily large radius in $\mathcal{D}^{-}$. In view of Jordan's lemma, the contribution from these circular arcs vanishes in the limit as their radius goes to infinity. Subsequently, Cauchy's theorem is applied to the domain in between the resulting closed contours and a loop around the brach cut in $\mathcal{D}^{+}$ for Q^{-} and a loop around the brach cut in $\mathcal{D}^{-}$ for Q^{+}. Taking into account that along these branch cuts γ_P is imaginary and γ_S is real, we arrive at the following expressions:

$$\begin{aligned} Q^{-}(p) \quad &= \quad \frac{1}{\pi} \int_{1/c_P}^{1/c_S} \arctan\left[\frac{\mathrm{w}^2 (\mathrm{w}^2 - 1/\mathrm{c_P^2})^{1/2} (1/\mathrm{c_S^2} - \mathrm{w}^2)^{1/2}}{(\mathrm{w}^2 - 1/2\mathrm{c_S^2})^2} \right] \frac{\mathrm{dw}}{\mathrm{w - p}} \\ &\quad\quad \text{for } \mathrm{Re(p)} < 1/\mathrm{c_P} \end{aligned} \tag{A19}$$

and

$$\begin{aligned} Q^{+}(p) \quad &= \quad \frac{1}{\pi} \int_{1/c_P}^{1/c_S} \arctan\left[\frac{\mathrm{w}^2 (\mathrm{w}^2 - 1/\mathrm{c_P^2})^{1/2} (1/\mathrm{c_S^2} - \mathrm{w}^2)^{1/2}}{(\mathrm{w}^2 - 1/2\mathrm{c_S^2})^2} \right] \frac{\mathrm{dw}}{\mathrm{w + p}} \\ &\quad\quad \text{for } \mathrm{Re(p)} > -1/\mathrm{c_P}, \end{aligned} \tag{A20}$$

where for the last result a change of the variable of integration into its opposite has been carried out. From the expressions it follows that $Q^{-}(-p) = Q^{+}(p)$ for all p.

To circumvent possible difficulties in the numerical evaluation of the right-hand sides of Equations (A19) and (A20) due to the occurrence of an algebraic square root behavior near the end points of the integration interval, the variable of integration is replaced by

$$w = [(1/c_P^2)\cos^2(\psi) + (1/c_S^2)\sin^2(\psi)]^{1/2} \quad \text{with } 0 < \psi < \pi/2, \tag{A21}$$

through which

$$(w^2 - 1/c_P^2)^{1/2} \quad = \quad (1/c_S^2 - 1/c_P^2)^{1/2} \sin(\psi), \tag{A22}$$

$$(1/c_S^2 - w^2)^{1/2} \quad = \quad (1/c_S^2 - 1/c_P^2)^{1/2} \cos(\psi), \tag{A23}$$

while the Jacobian of the transformation is

$$\frac{\partial w}{\partial \psi} = \frac{(1/c_S^2 - 1/c_P^2)\cos(\psi)\sin(\psi)}{[(1/c_P^2)\cos^2(\psi) + (1/c_S^2)\sin^2(\psi)]^{1/2}}. \tag{A24}$$

The resulting expressions for $K^{-}(p)$ and $K^{+}(p)$ are used in the main text.

APPENDIX B: Numerical evaluation of Stieltjes-type integral

In this appendix, the quasi-trapezoial rule algorithm for the numerical evaluation of Stieltjes-type integral will be presented. Let the relevant integral be

$$I = \int_{t_1}^{t_2} f(t)\,\mathrm{d}[\mathrm{g(t)}]. \tag{B1}$$

With linear interpolation of f and g on the interval $[t_1, t_2]$, i.e.,

$$f(t) \simeq f(t_1)\,\frac{t - t_1}{t_2 - t_1} + f(t_2)\,\frac{t_2 - t}{t_2 - t_1} \tag{B2}$$

and

$$g(t) \simeq g(t_1)\,\frac{t - t_1}{t_2 - t_1} + g(t_2)\,\frac{t_2 - t}{t_2 - t_1}, \tag{B3}$$

the expression for I is found to be

$$I \simeq \tfrac{1}{2}[f(t_1) + f(t_2)]\,[g(t_2) - g(t_1)]. \tag{B4}$$

This quadrature formula is used in the main text.