Geometrical spreading in a transversely isotropic medium with vertical symmetry axis

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ABSTRACT

We show that a well known equation for geometrical spreading in horizontally layered elastic media, presented by Ursin, holds in the case of transverse isotropy with a vertical symmetry axis as well. From this equation and the nonhyperbolic traveltime equation of Tsvankin and Thomsen, we derive an explicit analytical formula for geometrical spreading in transversely isotropic media with a vertical symmetry axis. Potential applications of the formula are in prestack anisotropic time migration and amplitude versus offset studies. The analytical formula allows for fast and easy computation of geometrical spreading without the use of ray tracing. Numerical tests show that for P-waves, the analytical geometrical spreading formula compares well with ray tracing results for offset-to-depth ratios up to at least five. For SV-waves the analytical approximation is acceptable for offset-to-depth ratios up to unity.

Key words: Geometrical spreading, anisotropy, nonhyperbolic traveltime.

Introduction

Computation of offset dependent geometrical spreading is needed in order to obtain the weight functions for prestack Kirchhoff migration. Moreover, seismic data must be compensated for geometrical spreading before AVO analysis can be used to study reflection coefficients as a function of offset (or incidence angle).

Ursin (1990) has presented a formula for the offset dependent relative geometrical spreading in a horizontally layered elastic medium. We first show that this formula is valid for any medium with translation and rotation symmetry, including a transversely isotropic medium with vertical symmetry axis (VTI medium). Second, we apply the formula for geometrical spreading together with the nonhyperbolic traveltime equation presented by Tsvankin and Thomsen (1994) in order to derive a closed form equation for geometrical spreading in a VTI medium. Finally, we show numerical examples from three qualitatively different cases of VTI anisotropy.

Potential applications of the results presented here are in anisotropic time-domain processing, including prestack time migration and AVO analysis.

Medium with translation and rotation symmetry

For a horizontally layered elastic isotropic medium Ursin (1990) has shown that the relative geometrical spreading $\mathcal{L}$ can be expressed

$$\mathcal{L} = |\det \boldsymbol{Q}_2|^{1/2}, \tag{1}$$

$$|\det \boldsymbol{Q}_2| = \frac{\cos\theta_s \cos\theta_g}{v_s^2}\left(\frac{x}{p_x}\right)\left(\frac{\partial x}{\partial p_x}\right), \tag{2}$$

where x is the offset, p_x is the horizontal component of slowness, v_s is the velocity at the source position and θ_s and θ_g are the take off and arrival angles. The two factors in parentheses can be interpreted as the out-of-plane and in-plane contributions to geometrical spreading, respectively. Below we show that equation (2) holds in any elastic medium with translation and rotation symmetry.

Tygel et al. (1992) have shown that in general, the offset dependent relative geometrical spreading is given by

$$|\det \boldsymbol{Q}_2| = \frac{\cos\theta_s \cos\theta_g}{v_s^2}\left|\det\left[\frac{\partial^2 T}{\partial x_{ig}\partial x_{js}}\right]\right|^{-1}, \tag{3}$$

where T is the traveltime, v_s is the group velocity at the source, and θ_s and θ_g are take off and arrival group angles. In a medium with translation and rotation symmetry, the traveltime depends on horizontal source and receiver coordinates only through the offset

$$x = [(x_{g1} - x_{s1})^2 + (x_{g1} - x_{s1})^2]^{1/2} = [x_1^2 + x_2^2]^{1/2}. \quad (4)$$

By means of the chain rule and equation (4), the derivatives of the traveltime function can be written

$$\frac{\partial T}{\partial x_{jg}} = \frac{x_j}{x}\frac{\partial T}{\partial x} = -\frac{\partial T}{\partial x_{js}}, \quad (5)$$

$$\frac{\partial^2 T}{\partial x_{ig}\partial x_{js}} = -\left[\left(\frac{\partial^2 T}{\partial x^2}\right)\frac{x_i x_j}{x^2} + \left(\frac{1}{x}\frac{\partial T}{\partial x}\right)\left(\delta_{ij} - \frac{x_i x_j}{x^2}\right)\right]. \quad (6)$$

The determinant of the 2×2 matrix in equation (6) is

$$\det\left[\frac{\partial^2 T}{\partial x_{ig}\partial x_{js}}\right] = \left[\left(\frac{\partial^2 T}{\partial x^2}\right)\frac{x_1^2}{x^2} + \left(\frac{1}{x}\frac{\partial T}{\partial x}\right)\left(1 - \frac{x_1^2}{x^2}\right)\right] \times \left[\left(\frac{\partial^2 T}{\partial x^2}\right)\frac{x_2 x_2^2}{x^2} + \left(\frac{1}{x}\frac{\partial T}{\partial x}\right)\left(1 - \frac{x_2^2}{x^2}\right)\right] - \left[\left(\frac{\partial^2 T}{\partial x^2}\right)\frac{x_1 x_2}{x^2} - \left(\frac{1}{x}\frac{\partial T}{\partial x}\right)\frac{x_1 x_2}{x^2}\right]^2. \quad (7)$$

After some algebra, and a remarkable cancellation of terms, we find

$$\det\left[\frac{\partial^2 T}{\partial x_{ig}\partial x_{js}}\right] = \left(\frac{1}{x}\frac{\partial T}{\partial x}\right)\left(\frac{\partial^2 T}{\partial x^2}\right). \quad (8)$$

Identifying the horizontal slowness by

$$p_x = \frac{\partial T}{\partial x}, \quad (9)$$

and substituting equations (8) and (9) into equation (3), we recover equation (2).

Transversely isotropic medium with vertical symmetry axis

Tsvankin and Thomsen (1994) have shown that the non-hyperbolic traveltime in a VTI medium is given approximately by the formula

$$T^2(x) = T_0^2 + A_2 x^2 + \frac{A_4 x^4}{1 + A_5 x^2}, \quad (10)$$

where T and T_0 are the offset and zero offset traveltimes, which can be interpreted as either one-way or two-way times. The coefficients A_2, A_4 and A_5 are given by Tsvankin and Thomsen (1994). From equation (10) we may calculate the first and second derivatives of the traveltime, appearing on the right-hand side of equation (8). Applying the chain rule, we write

$$p_x = \frac{\partial T}{\partial x} = \frac{1}{2T}\frac{\partial T^2}{\partial x}, \quad (11)$$

$$\frac{\partial p_x}{\partial x} = \frac{\partial^2 T}{\partial x^2} = \frac{1}{2T}\left[\frac{\partial^2 T^2}{\partial x^2} - 2\left(\frac{\partial T}{\partial x}\right)^2\right]. \quad (12)$$

Defining two functions

$$G_4 = \frac{A_4 x^2}{1 + A_5 x^2}, \quad G_5 = \frac{A_5 x^2}{1 + A_5 x^2}, \quad (13)$$

which are zero in the case of hyperbolic moveout, and

$$H = A_2 + G_4(2 - G_5), \quad (14)$$

$$x\frac{\partial H}{\partial x} = 4G_4(1 - G_5)^2, \quad (15)$$

the derivatives of the squared traveltime function can be written

$$\frac{\partial T^2}{\partial x} = 2xH, \quad \frac{\partial^2 T^2}{\partial x^2} = 2(H + x\frac{\partial H}{\partial x}). \quad (16)$$

Substituting equations (11) to (16) into equation (8), we obtain the separate contributions from out-of-plane and in-plane geometrical spreading

$$\frac{1}{x}\frac{\partial T}{\partial x} = \frac{H}{T}, \quad (17)$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{H}{T}\left(1 + \frac{x}{H}\frac{\partial H}{\partial x} - \frac{H}{T^2}x^2\right), \quad (18)$$

and

$$|\det \boldsymbol{Q}_2| = \frac{\cos\theta_s \cos\theta_g}{v_s^2}\left(\frac{T}{H}\right)^2 \times \left[1 + \frac{x}{H}\frac{\partial H}{\partial x} - \frac{H}{T^2}x^2\right]^{-1}. \quad (19)$$

Equation (19) is numerically well behaved for all offsets, including $x = 0$.

For small offsets, expanding equation (19) in a Taylor series and keeping terms to order x^2, we get

$$\frac{1}{x}\frac{\partial T}{\partial x} = \frac{A_2}{T_0}\left[1 + \frac{1}{2}\left(\frac{4A_4}{A_2} - \frac{A_2}{T_0^2}\right)x^2\right], \quad (20)$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{A_2}{T_0}\left[1 + \frac{3}{2}\left(\frac{4A_4}{A_2} - \frac{A_2}{T_0^2}\right)x^2\right], \quad (21)$$

$$|\det \boldsymbol{Q}_2| = \frac{\cos\theta_s \cos\theta_g}{v_s^2}\left(\frac{T_0}{A_2}\right)^2 \times \left[1 - 2\left(\frac{4A_4}{A_2} - \frac{A_2}{T_0^2}\right)x^2\right]. \quad (22)$$

At zero offset, we get

$$|\det \boldsymbol{Q}_2| = \left(\frac{1}{v_s}T_0 V_2^2\right)^2, \quad (23)$$

where we have introduced the NMO velocity

$$V_2^2 = 1/A_2. \quad (24)$$

The NMO velocity for P- and SV- waves are (Thomsen, 1986)

$$V_2^2(P) = V_{P0}^2(1 + 2\delta), \quad (25)$$

$$V_2^2(SV) = V_{S0}^2(1 + 2\sigma), \quad (26)$$

where

$$\sigma = (V_{P0}/V_{S0})^2(\epsilon - \delta), \quad (27)$$

	$V_2(P)$ (m/s)	$V_2(SV)$ (m/s)	ϵ	δ	σ
$\epsilon = \delta$	2191	1000	0.10	0.10	0.00
$\epsilon > \delta$	2098	1183	0.10	0.05	0.20
$\epsilon < \delta$	2280	775	0.10	0.15	-0.20
Isotropy	2000	1000	0.00	0.00	0.00

Table 1. NMO velocities and Thomsen parameters.

V_{P0} and V_{S0} are the vertical P- and S-wave velocities, and ϵ and δ are the Thomsen parameters. Equations (23) to (26) show that anisotropy influence the geometrical spreading even at zero offset. This is not surprising, because anisotropy change the curvature of the wavefront. The SV-mode is not affected by anisotropy if the anisotropy is elliptical ($\epsilon = \delta$). In this case $A_4(SV), A_5(SV)$ and σ are equal to zero.

At large offsets, performing a series expansion in $1/x$ and keeping only lowest order non-vanishing terms we find

$$|\det \boldsymbol{Q}_2| = 2\frac{\cos\theta_s \cos\theta_g}{v_s^2}\left[\frac{x^4}{T_0^2 - A_4/A_5^2}\right], \tag{28}$$

which grows like x^4. This is, however, compensated by $\cos\theta_s, \cos\theta_g \to 0$.

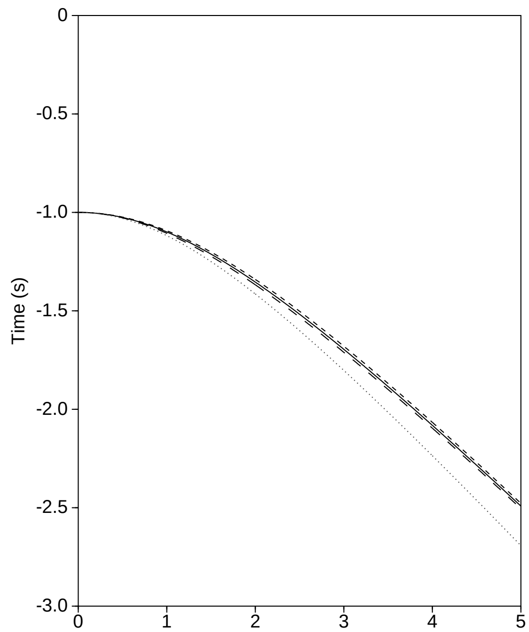

Figure 1. P-wave travel times for $\epsilon = \delta$ (solid line), $\epsilon > \delta$ (long dashes), $\epsilon < \delta$ (short dashes) and isotropic (dotted line).

Numerical examples

We performed a numerical study of the relative geometrical spreading using a model with vertical P- and S-wave velocities $V_{P0} = 2000$ m/s, $V_{S0} = 1000$ m/s, and zero offset traveltimes $T_{P0} = 1.0$ s, $T_{S0} = 2.0$ s. Keeping vertical velocities and zero offset travel times fixed, we studied three qualitatively different cases of anisotropy: $\epsilon = \delta$ (elliptic), $\epsilon > \delta$ and $\epsilon < \delta$. For comparison, we also included the isotropic case, $\epsilon = \delta = 0$. P- and SV-NMO velocities and Thomsen parameters ϵ, δ and σ are given in Table 1.

Figures 1 and 2 show the P- and SV-wave traveltimes computed by equation (10) for offsets to depth ratios in the range 0 - 5. The traveltimes have been computed for all the four different models in Table 1, and can be interpreted as either symmetrical two-way reflection times or as one-way transmission times (measured in a borehole). Figures 3 and 4 show the corresponding inverse relative geometrical spreading $1/\mathcal{L}$, computed using equations (1) and (19).

For P-waves, the anisotropic models give geometrical spreading different from the isotropic case for all offsets. At zero offset, anisotropy gives larger geometrical spreading whenever $\delta > 0$, which is the case for all the examples shown here. This is due to the dependence of the NMO velocity on δ, see equations (23) to (25).

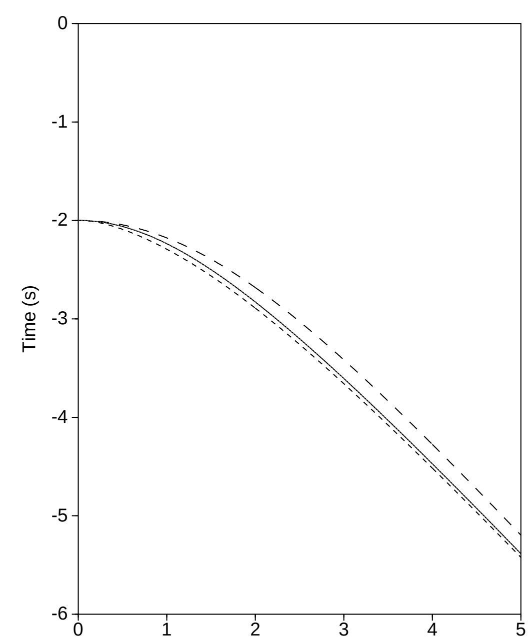

Figure 2. S-wave travel times for $\epsilon = \delta$ (solid line), $\epsilon > \delta$ (long dashes), $\epsilon < \delta$ (short dashes) and isotropic (coincident with $\epsilon = \delta$).

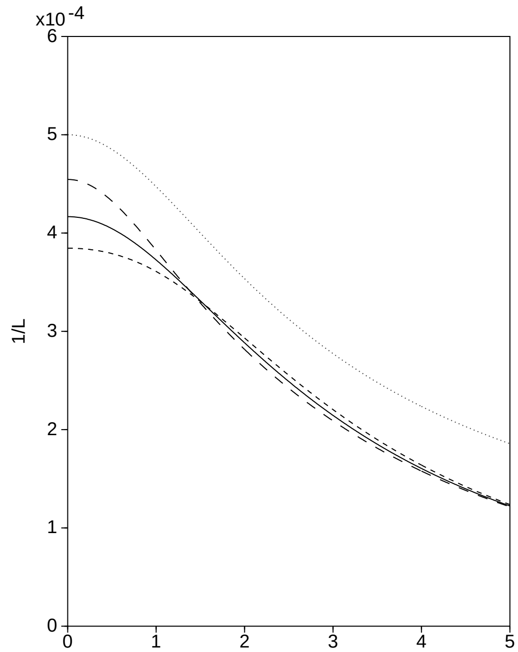

Figure 3. P-wave inverse relative geometrical spreading $1/\mathcal{L}$ for $\epsilon = \delta$ (solid line), $\epsilon > \delta$ (long dashes), $\epsilon < \delta$ (short dashes) and isotropic (dotted line).

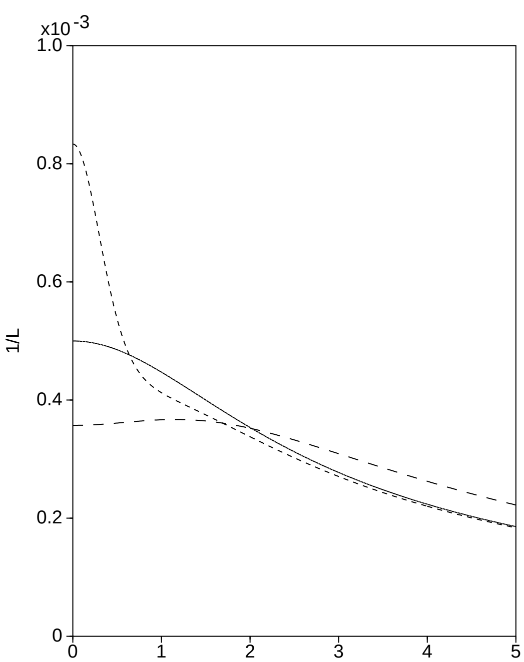

Figure 4. S-wave inverse relative geometrical spreading $1/\mathcal{L}$ for $\epsilon = \delta$ (solid line), $\epsilon > \delta$ (long dashes), $\epsilon < \delta$ (short dashes) and isotropic (coincident with $\epsilon = \delta$).

At large offsets the geometrical spreading for the three anisotropic examples approach asymptotically the same value, which is different from the isotropic case. This is because all anisotropic models used have the same value of ϵ, and consequently the same horizontal P-wave velocity $V_{Ph} = V_{P0}\sqrt{1 + 2\epsilon}$.

For SV-waves, anisotropy depends mainly on the parameter σ, defined in equation (27). Therefore elliptic anisotropy $\epsilon = \delta$ is coincident with the isotropic case. The geometrical spreading at zero offset is determined by the NMO velocity, which depends on σ, see equation (26). Depending on the sign of σ, the zero offset relative geometrical spreading is larger when $\epsilon > \delta$, implying $\sigma > 0$, or smaller when $\epsilon < \delta$, implying $\sigma < 0$, compared to the isotropic case. When anisotropy is anelliptic ($\epsilon \neq \delta$), the geometrical spreading for SV-waves is qualitatively very different from the isotropic case.

For comparison, Figures 5 and 6 show traveltimes and geometrical spreading computed using the nonhyperbolic traveltime formulae together with traveltimes and geometrical spreading computed by ray tracing for the anelliptic model with $\epsilon > \delta$. For P-waves, the nonhyperbolic traveltime results compare well with ray tracing data for all offset-to-depth ratios shown. In the case of SV-waves, we find a significant deviation from ray tracing data at offsets-to-depth ratios greater than unity. The maximum of the inverse relative geometrical spreading is at non-zero offset when $\epsilon > \delta$. In this case the SV phase and group velocities have a maximum in the vicinity of 45 degrees. Consequently, there is a concentration of rays in this direction, which accounts for the larger amplitude and smaller geometrical spreading. The behavior of SV-waves in the vicinity of the velocity maximum is too complex to be modeled accurately by the nonhyperbolic traveltime equation (10).

Discussion and conclusions

We have shown that the equation for offset dependent geometrical spreading in horizontally layered media presented by Ursin (1990) is valid whenever the elastic medium is symmetric with respect to horizontal translation and rotation around the vertical axis. In practice, this means that the formula can be used for plane layered elastic isotropic media and transversely isotropic media with a vertical symmetry axis.

Based on this formula and the nonhyperbolic traveltime formula presented by Tsvankin and Thomsen (1994) we have derived analytical equations for geometrical spreading which can be computed without the use of ray tracing.

Numerical tests show that for P-waves, the analyt-

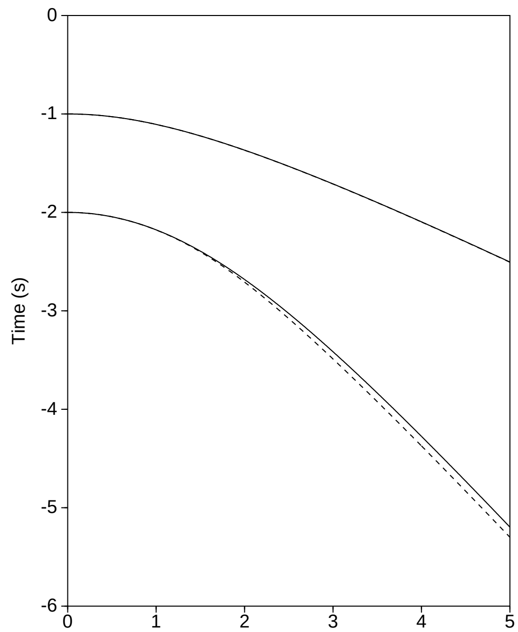

Figure 5. Nonhyperbolic traveltimes (solid line) compared with ray tracing data (dashed line) for the anelliptic model with $\epsilon > \delta$.

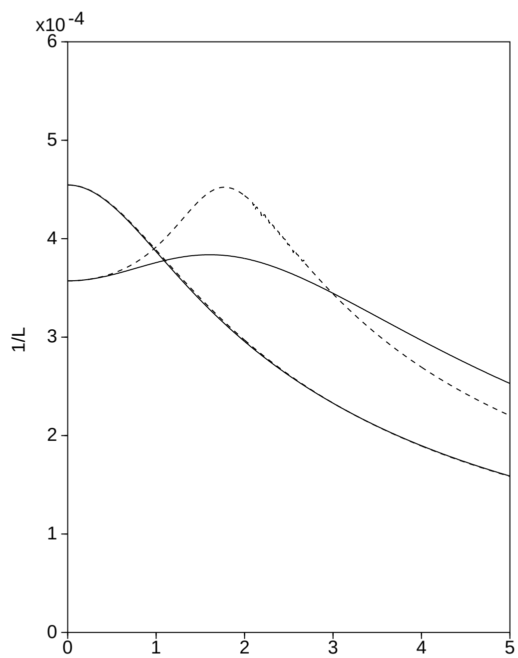

Figure 6. Inverse relative geometrical spreading computed with the nonhyperbolic traveltime formula (solid line) compared with ray tracing data (dashed line) for the anelliptic model with $\epsilon > \delta$.

ical equation for geometrical spreading compares well with ray tracing results up to offset-to-depth ratios of order five. For SV-waves the analytical formula breaks down at an offset-to-depth ratio less than unity. This is due to the velocity maximum of SV-waves in transversely isotropic media around 45 degrees. The numerical results are in agreement with the regime of validity for the nonhyperbolic traveltime equation.

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