WAVE FIELD DECOMPOSITION IN ANISOTROPIC FLUIDS: A SPECTRAL THEORY APPROACH

B. LARS G. JONSSON* AND MAARTEN V. DE HOOP†

Abstract. An extension of directional wave field decomposition in acoustics from heterogenous isotropic media to generic heterogenous anisotropic media is established. We make a connection between the Dirichlet-to-Neumann map for a level plane, the solution to an algebraic Riccati operator equation, and a projector defined via a Dunford-Schwartz type integral over the resolvent of a non-normal, non-compact matrix operator with continuous spectrum.

In the course of the analysis, the spectrum of the Laplace transformed acoustic system's matrix is analyzed and shown to separate into two non-trivial parts. The existence of a projector is established and using a generalized eigenvector procedure, we find the solution to the associated algebraic Riccati operator equation. The solution generates the decomposition of the wave field and is expressed in terms of the elements of a Dunford-Schwartz type integral over the resolvent.

Keywords: directional wave field decomposition, wave splitting, spectral reduction, acoustic anisotropy, generalized eigenvalue problem, algebraic Riccati operator equation, Dirichlet-to-Neumann maps, generalized vertical wave number operators, generalized vertical slowness.

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1. Introduction. Directional wave field decomposition is a tool for analyzing and computing the propagation of waves in configurations characterized by a certain directionality. The method consists of three main steps: (i) decomposition of the field into two constituents, propagating upward or downward along a preferred direction, (ii) computation of the interaction of the counter-propagating constituents and (iii) recombination of the constituents into observables at the positions of interest. The method is useful because it leads to computationally efficient *modeling* algorithms. It can also be used to separate wave-field constituents, which is of importance in the interpretation and *inversion* of wave-field measurements on a boundary.

Over the past few years, methods have been developed which allow us to carry out a decomposition in the case of *heterogeneous, isotropic* media [24]. Both exact theories, based on the calculus of pseudodifferential operators in the time-Laplace domain [5], and approximate theories, based on a uniform asymptotic expansion in the time-Fourier domain [10], have been introduced. It has also been shown that the extension from isotropic media to anisotropic media with up/down symmetry, such as media with uniaxial or biaxial symmetry, is feasible with the 'isotropic' techniques [6].

The calculation of the decomposition of the operator has for the isotropic case given rise to several successful methods for analyzing wave propagation, including uniform asymptotics and normal modes [7, 11]. Fast numerical algorithms for calculations of the fields are another spin off of this decomposition. Among their implementations we have 'rational approximations' and 'generalized screens' [8, 21].

In the present paper, a generalization of the wave field decomposition approach to general *anisotropic* fluids is presented. The approach makes use of functional analysis, through the spectral theory of an unbounded, non-normal operator; pseudodifferential calculus with parameters underlies part of the proofs. A key aspect of the procedure is that modes of propagation cannot and need not be separated. A full spectral reduction of the acoustic system's matrix operator is difficult or impossible to obtain due to its nonstandard form, but this is not necessary in order to decompose the operator. We state a number of propositions in order to prove that the decomposition exists. The procedure gives conditions for the decomposition to

* Department of Electromagnetic Theory, Royal Institute of Technology, Stockholm, Sweden. (lars@tet.kth.se)
†Center for Wave Phenomena, Colorado School of Mines, Golden CO 80401-1887, USA. (mdehoop@dix.Mines.EDU)

exist — the conditions are mostly sufficient and not necessary. For the material parameters there are no constraints that one would not expect from physical considerations, apart from a smoothness requirement.

The paper is structured as a set of four propositions and two corollaries, collecting the results needed in the proof of the fact that the wave field decomposition exists. In §2 we introduce the acoustic equations and rewrite them into a form suitable for wave field decomposition and in §3 we state the problem. In §4, the first proposition appears establishing the (spectral) properties, with the aid of functional analysis, of the acoustic system's matrix; in a corollary it is proved that the spectrum is separated into two parts. This separation is found in the form of bounds for the areas in the complex plane that contain the spectrum. In §5 we introduce a family of operators defined by a Dunford-Schwartz type integral over the resolvent of the acoustic system's matrix. The second proposition states the properties of this family of operators, such as their existence as pseudodifferential operators with parameters, the fact that they commute with the acoustic system's matrix, and that they are idempotent under certain restrictions. The section ends with a corollary defining two non-orthogonal projectors via a particular member of the above mentioned family of operators.

In §6 we state the last two propositions. The first of these introduce the splitting matrix and its properties. The splitting matrix is directly defined in terms of the above obtained projectors. In particular, we show that the splitting matrix has generalized eigenvectors given in terms of the Dirichlet-to-Neumann map. The second proposition shows that these eigenvectors are generalized eigenvectors of the acoustic system's matrix and hence can be used to decompose it. Furthermore it is shown that the Dirichlet-to-Neumann map is the solution to an algebraic Riccati operator equation. In the last section we discuss the results.

In Appendix A we present a short collection of key results for the case of a homogeneous isotropic medium, for parallel reading to compare the general results with well known results. In Appendix B we derive the homogeneous anisotropic medium case in preparation for Appendix C where we show that the principal part of the 'positive' projector projects out only the positive real part of the spectrum. The spectrum of the acoustic system's matrix is divided into at least two sets, the two sets are distinguishable by the sign of the real part. Given the set with positive (negative) real part of the spectrum, we can show that a restriction of this set corresponds to positive (negative) vertical local group velocity.

2. Wave motion in the Laplace domain.

2.1. The reduced equations. We consider linear acoustic wave motion in an anisotropic fluid. In each subdomain of the configuration the acoustic properties may vary continuously with position but they are assumed to be independent of time. The acoustic wave field satisfies the first-order hyperbolic system of partial differential equations [4, §2.3]

$$\partial_k p + \rho_{kj}\partial_t v_j = f_k \ , \tag{2.1}$$

$$\kappa\partial_t p + \partial_j v_j = q \ , \tag{2.2}$$

where p = acoustic pressure [Pa], v_j = particle velocity [m/s], ρ_{kj} = anisotropic symmetric volume density of mass tensor [kg/m^3], κ = compressibility [Pa^{-1}], q = volume source density of injection rate [s^{-1}], and f_k = volume source density of force [N/m^3]. In this paper $\boldsymbol{x} = \{x_1, x_2, x_3\}$ are the right-handed orthogonal Cartesian coordinates, t is the time, and the subscript notation and the summation convention for Cartesian tensors are employed; $j, k \in \{1, 2, 3\}$ are used as indices.

Further, causality of the wave motion is enforced. This implies that if the sources that generate the wave field are switched on at the instant $t = 0$, the wave field quantities satisfy the conditions

$$p(\boldsymbol{x}, t) = 0 \quad \text{for} \quad t < 0 \quad \text{and all} \quad \boldsymbol{x}, \tag{2.3}$$

$$v(\boldsymbol{x}, t) = 0 \quad \text{for} \quad t < 0 \quad \text{and all} \quad \boldsymbol{x}. \tag{2.4}$$

Due to the time invariance of the medium, the causality of the wave motion can also be taken into account by carrying out a one-sided Laplace transform with respect to time and requiring that the transform domain wave quantities are bounded functions of position in all of space when the time Laplace transform parameter s, which is in general complex, lies in the right half plane, that is

$$\mathrm{Re}\{s\} > 0 \,. \tag{2.5}$$

To show the notation, we give the expression for the acoustic pressure,

$$\hat{p}(\boldsymbol{x}, s) = \int_0^\infty \exp(-st)p(\boldsymbol{x}, t)\,\mathrm{d}t \,. \tag{2.6}$$

Under this transformation, assuming zero initial conditions, we have $\partial_t \to s$. The transformed system of first-order equations follows from Eqs.(2.1)-(2.2) as

$$\partial_k\hat{p} + s\,\rho_{kj}\hat{v}_j = \hat{f}_k \,, \tag{2.7}$$

$$s\,\kappa\hat{p} + \partial_j\hat{v}_j = \hat{q} \,. \tag{2.8}$$

The analysis of the wave motion will be carried out in the Laplace domain, and hence we omit the hat, $\hat{}$, in the remainder of the paper.

The 'evolution' of the wave field in space along a direction of preference can be expressed in terms of the changes of the wave field in the directions perpendicular to it. The direction of preference is taken to be along the x_3-axis (or 'vertical' axis) and the remaining ('horizontal') coordinates are denoted by $x_\mu, x_\nu,\ \mu, \nu \in \{1, 2\}$ or $\boldsymbol{x}' = \{x_1, x_2\}$ when convenient. The procedure requires a separate handling of the horizontal components of the particle velocity. Let

$$\alpha_{jk} = (\rho^{-1})_{jk} \,, \tag{2.9}$$

denote the reciprocal density tensor. From Eq.(2.7) we obtain

$$v_\mu = -\alpha_{\mu k}s^{-1}(\partial_k p - f_k) \,, \tag{2.10}$$

leaving, upon substitution, the matrix differential equation

$$(I\partial_3 + \mathcal{A})\,F = N \,, \tag{2.11}$$

in which the elements of the acoustic field matrix are given by

$$F_1 = v_3 \,, \qquad F_2 = p \,. \tag{2.12}$$

The acoustic system's matrix, $\mathcal{A}$, is given by

$$\mathcal{A} = \mathcal{A}(\boldsymbol{x}, \partial_1, \partial_2; s) = \begin{pmatrix} \partial_\nu\,\alpha_{3\nu}\alpha_{33}^{-1} & \kappa s - s^{-1}\partial_\mu\,Q_{\mu\nu}\partial_\nu \\ s\alpha_{33}^{-1} & \alpha_{33}^{-1}\alpha_{3\nu}\partial_\nu \end{pmatrix} \,, \tag{2.13}$$

in which

$$Q_{\mu\nu} = \alpha_{\mu\nu} - \alpha_{3\mu}\alpha_{33}^{-1}\alpha_{3\nu} \,, \tag{2.14}$$

and the elements of the notional source matrix are given by

$$N_1 = q - s^{-1}\partial_\nu[(\alpha_{\nu j} - \alpha_{3\nu}\alpha_{33}^{-1}\alpha_{3j})f_j] \,, \qquad N_2 = \alpha_{33}^{-1}\alpha_{3j}f_j \,. \tag{2.15}$$

The compressibility is positive, bounded from above and bounded away from zero. The density is a positive definite, symmetric and bounded tensor. The anisotropy contained in the density tensor causes the slowness surface (cf. Appendix B) to deform from a sphere to an ellipsoid. The off-diagonal elements of the tensor represent the deviation from up/down symmetry with respect to the direction of preference. The reciprocal density tensor is symmetric, positive definite and bounded also. Let the lower bound $\alpha_0 > 0$ of the upper-left 2×2 matrix of the reciprocal density tensor be defined through

$$u_\mu \alpha_{\mu\nu} \overline{u}_\nu \geq \alpha_0 u_\mu \overline{u}_\mu \tag{2.16}$$

for any (complex) field u and where $\bar{\cdot}$ denotes complex conjugation. The lower bound exists since the density is finite, that is bounded above. It is clear that $Q_{\mu\nu}$ is symmetric. To see that $Q_{\mu\nu}$ is positive definite we use the following identity

$$u_\mu Q_{\mu\nu} \bar{u}_\nu = w_j \alpha_{jk} \bar{w}_k \ , \tag{2.17}$$

where

$$w_j = u_\mu \left(\delta_{j\mu} - \alpha_{33}^{-1} \alpha_{3\mu} \delta_{3j} \right) \tag{2.18}$$

for any (complex) field u. Since α_{jk} is positive definite $Q_{\mu\nu}$ must be positive definite. (Similar considerations holds in elastodynamics [3, p.70 §3.2].) Analogously to α_0, we define the constant $q_1 > 0$ to be the corresponding lower bound for $Q_{\mu\nu}$. The existence of this lower bound follows from the positive definiteness of $Q_{\mu\nu}$ and q_1 can be taken as the reciprocal of the absolute values of the largest eigenvalue of the density matrix.

2.2. Wave fields in Hilbert spaces. In the following we consider the acoustic system's matrix and other operators on Sobolev spaces. The *Hilbert space* $(\mathrm{H}^r, (\cdot,\cdot)_r)$ (cf. e.g. [16, p.281, §5.13]) is a weighted Sobolev space of order r on $\mathbb{R}^2$. The complex inner product $(\cdot,\cdot)_r$ is defined as

$$(F_1, G_1)_r = \int_{\mathbb{R}^2} \mathrm{d}^2\boldsymbol{x}' \sum_{|\beta|\leq r} y_0^{2|\beta|} \partial_{\boldsymbol{x}'}^{\beta} F_1(\boldsymbol{x}) \overline{\partial_{\boldsymbol{x}'}^{\beta} G_1(\boldsymbol{x})} \ . \tag{2.19}$$

The (complex) fields F_1, G_1 depend on $\boldsymbol{x} = \{x_1, x_2, x_3\}$ but the norms and derivatives refer to $\boldsymbol{x}' = \{x_1, x_2\}$. The variable x_3 is treated as a parameter throughout the paper. In (2.19) y_0 is a positive (bounded away from zero), finite number of dimension [length]. It is introduced to make the analysis dimensionally correct. We adopt the multi-index notation of pseudodifferential calculus and use $\beta \in \mathbb{N}_0^2$, where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ throughout the paper, together with

$$|\beta| = \beta_1 + \beta_2 \ . \tag{2.20}$$

The norm corresponding to the inner product is

$$\|F_1\|_r = \sqrt{(F_1, F_1)_r} \ . \tag{2.21}$$

We will use $q, r \in \mathbb{N}$ as Sobolev space order indicators throughout the paper and note that for the case $r = 0$ we recover the Lebesgue space $(\mathrm{L}^2, (\cdot,\cdot)_0)$ of square integrable functions, with the standard complex inner product.

The analysis of unbounded operators — such as the acoustic system's matrix — requires that one specifies the domain of the operators and its embedding space for the operator to be fully defined. In the present paper, all embedding spaces are Sobolev spaces (that is Hilbert spaces), and all domains are dense in the respective Hilbert spaces. Several of the operators occurring in our construction contain derivatives, which are unbounded and discontinuous on $(\mathrm{L}^2, (\cdot,\cdot)_0)$. Note that if we consider the derivative on $(\mathrm{L}^2, (\cdot,\cdot)_0)$ then, even if we restrict the domain of the derivative to the *set* of functions $\mathrm{H}^1 \subset (\mathrm{L}^2, (\cdot,\cdot)_0)$, it is still unbounded on this restricted set, where it is well defined. This is not to be confused with the derivative between the Sobolev *spaces* $(\mathrm{H}^{r+1}, (\cdot,\cdot)_{r+1}) \to (\mathrm{H}^r, (\cdot,\cdot)_r)$, where it is a bounded operator and, hence, a continuous operator. Observe that we use the notation $(\mathrm{H}^r, (\cdot,\cdot)_r)$ to indicate the space and H^r to indicate the set.

We introduce the weighted product Sobolev spaces;

$$\mathrm{H}^q_r = \begin{pmatrix} \mathrm{H}^q \\ \mathrm{H}^r \end{pmatrix} . \tag{2.22}$$

with the complex inner product defined as

$$(F,G)_{[q,r]} = \int_{\mathbb{R}^2} \mathrm{d}^2\boldsymbol{x}' \left\{ Y_0^{-1} \sum_{|\gamma|\le q} y_0^{2|\gamma|} \partial^\gamma_{\boldsymbol{x}'} F_1 \, \overline{\partial^\gamma_{\boldsymbol{x}'} G_1} + Y_0 \sum_{|\beta|\le r} y_0^{2|\beta|} \partial^\beta_{\boldsymbol{x}'} F_2 \, \overline{\partial^\beta_{\boldsymbol{x}'} G_2} \right\} , \tag{2.23}$$

where F, G are complex fields. The norm corresponding to $(\cdot,\cdot)_{[q,r]}$ is defined as

$$\|F\|_{[q,r]} = \sqrt{(F,F)_{[q,r]}} . \tag{2.24}$$

The constant Y_0 of dimension [acoustic admittance] or [$\mathrm{m}^2\mathrm{s/kg}$], is greater than zero and is introduced to make the elements of the acoustic field matrix mutually compatible and the inner product dimensionally correct. In analogy with the set H^r we will use the notation H^q_r for the *set* of functions such that (2.24) is finite. The inner product (2.23) induces a Hilbert space structure on H^q_r since $(\mathrm{H}^r, (\cdot,\cdot)_r)$ is a Hilbert space and $\mathrm{H}^q_r \cong \mathrm{H}^q \times \mathrm{H}^r$ (cf. e.g. [9, p.257 §IV.4.19] [16, p.342 §5.20]), and we will denote the Hilbert space by $(\mathrm{H}^q_r, (\cdot,\cdot)_{[q,r]})$. For the case $q = r = 0$ we recover a weighted Lebesgue space, $((\mathrm{L}^2)^2, (\cdot,\cdot)_{[0,0]})$. The choice of smoothness, that is the choise of r, q, corresponds to a requirement of regularity for the fields that the operator acts upon. The 'unweighted' inner product space, with respect to Y_0, for $q = r = 0$, is also a Hilbert space and is denoted by $((\mathrm{L}^2)^2, (\cdot,\cdot)_{(0,0)})$, with the corresponding norm. We note that the Sobolev sets have the property $\cdots \subset \mathrm{H}^{r+1} \subset \mathrm{H}^r \subset \cdots$ densely, when considered as a chain of subsets of functions with the same norm.

2.3. The acoustic system's matrix and its resolvent. The acoustic system's matrix is an unbounded operator on $((\mathrm{L}^2)^2, (\cdot,\cdot)_{[0,0]})$ in the sense explained above. Following [17] we define the operator $\mathcal{A}$ on $((\mathrm{L}^2)^2, (\cdot,\cdot)_{[0,0]})$ as

$$\mathcal{A} : \mathrm{D}(\mathcal{A}) \to (\mathrm{L}^2)^2 , \tag{2.25}$$

with the requirement that the coefficients of the partial differential operators in each element of the matrix are sufficiently smooth such that we can choose

$$\mathrm{D}(\mathcal{A}) = \mathrm{H}^1_2 \subset \mathrm{L}^2 \tag{2.26}$$

as domain. For every other unbounded operator in this paper, a similar procedure is considered, that is the operator is defined on a dense subset in a Hilbert space.

The domain of the operator on a space is fundamental for considerations of operations such as addition and composition of operators, since the result may be defined only on a subset of the domains of the respective individual operators. Here, we consider operators with dense domains and, when necessary, with restrictions to dense subsets of the their domains for the operation under consideration to be defined. In the case of such a restriction we use the notation $\mathcal{A}|_q$ for the operator restricted to this dense subset of its domain and indicate by q what dense subset is understood to be the restricted domain. In order to avoid cases such that $\mathrm{D}(\mathcal{A}) \cap \mathrm{D}(\mathcal{B}) = \emptyset$ all domains and restrictions are to sets of the kind H^q_r, or embedded between two such sets.

It will follow that the directional decomposition of waves is closely related to the spectral analysis of the operator $\mathcal{A}$. Separation of wave constituents is related to the separability of the spectrum of $\mathcal{A}$ into multiple parts.

To analyze the spectrum of the operator $\mathcal{A}$ for fixed s, we introduce the operator $\mathcal{A}_{s,\lambda}$ defined on $((\mathrm{L}^2)^2, (\cdot,\cdot)_{[0,0]})$ with parameters,

$$\mathcal{A}_{s,\lambda} = \mathcal{A} - \lambda I \;:\; \mathrm{H}^1_2 \to (\mathrm{L}^2)^2 \; . \tag{2.27}$$

Following [20, §5, p.253], [16, §6.5, p.412] and [25, §VIII.1, p.209], we define the spectrum of $\mathcal{A}$ as follows: if the scalar $\lambda \in \mathbb{C}$ is such that the range of $\mathcal{A}_{s,\lambda}$ is dense in $((\mathrm{L}^2)^2, (\cdot,\cdot)_{[0,0]})$ and $\mathcal{A}_{s,\lambda}$ has a bounded inverse, λ is in the *resolvent set*, $\mathrm{P}(\mathcal{A})$, of $\mathcal{A}$, and we denote this inverse by $\mathcal{A}^{-1}_{s,\lambda}$ and call it the *resolvent* (at s, λ) of $\mathcal{A}$. All complex numbers not in the resolvent set form a set $\Sigma(\mathcal{A})$ called the *spectrum* of $\mathcal{A}$.

To extend the analysis of $\mathcal{A}_{s,\lambda}$ to an operator defined on $(\mathrm{H}^r_r, (\cdot,\cdot)_{[r,r]})$, we introduce the following procedure. Let

$$\Upsilon = \begin{pmatrix} (1-\mathbf{\Delta}')^{1/2} & 0 \\ 0 & (1-\mathbf{\Delta}')^{1/2} \end{pmatrix} , \tag{2.28}$$

with $\mathbf{\Delta}' = y_0^2 \partial_\mu \partial_\mu$, then $\Upsilon : (\mathrm{H}^{r+1}_{r+1}, (\cdot,\cdot)_{[r+1,r+1]}) \to (\mathrm{H}^r_r, (\cdot,\cdot)_{[r,r]})$. Then, the conjugation

$$\mathcal{A}_{s,\lambda;r} \equiv \Upsilon^{-r} \mathcal{A}_{s,\lambda} \Upsilon^r \;:\; \mathrm{H}^{r+1}_{r+2} \to (\mathrm{H}^r_r, (\cdot,\cdot)_{[r,r]}) \; , \tag{2.29}$$

can be considered as an acoustic system's matrix. That we can consider this operator as a generalization of $\mathcal{A}_{s,\lambda}$ follows from:

$$(I\partial_3 + \mathcal{A})\, F = N \Leftrightarrow \Upsilon^{-r} \,(I\partial_3 + \mathcal{A})\, F = \Upsilon^{-r} N \; . \tag{2.30}$$

By introducing $F' = \Upsilon^{-r} F$ it follows that

$$(I\partial_3 + \mathcal{A}_{;r})\, F' = N' \; , \tag{2.31}$$

where $N' = \Upsilon^{-r} N$. Thus the operator $\mathcal{A}_{;r}$ is an acoustic system's matrix that acts on 'smoothed' fields. That is $\mathcal{A}_{s,\lambda}$ is 'smoother' that $\mathcal{A}_{s,\lambda;r}$ for $r \in \mathbb{N}_0$.

3. Directional wave field decomposition.

3.1. Formulation of the problem. To be able to solve the scattering process along the vertical direction separately from the scattering process in the horizontal directions, we diagonalize the operator on the left-hand side of Eq.(2.11). This procedure will possibly lead to an additional source term on the right-hand side that accounts for the coupling. To achieve this, we construct an appropriate linear operator $\mathcal{L}$ with

$$F = \mathcal{L}W, \tag{3.1}$$

that, with the aid of the commutation relation

$$(\partial_3 \mathcal{L}) = [\partial_3, \mathcal{L}] \tag{3.2}$$

$([.,.]$ denotes the commutator), transforms Eq.(2.11) into

$$\mathcal{L}\,(I\partial_3 + \mathcal{V})W = -(\partial_3 \mathcal{L})\,W + N \tag{3.3}$$

as to make $\mathcal{V}$, defined by

$$\mathcal{A}\mathcal{L} = \mathcal{L}\mathcal{V}\ , \tag{3.4}$$

a diagonal matrix of operators. We call $\mathcal{L}$ the composition operator, and W the wave matrix. The elements of the wave matrix represent locally the down- and upgoing constituents. The expression in parentheses on the left-hand side of Eq.(3.3) represents the two so-called *one-way* wave operators. The first term on the right-hand side of Eq.(3.3) is a representative for the scattering due to variations of the medium properties in the vertical direction. The scattering due to variations of the medium properties in the horizontal directions is contained in $\mathcal{V}$ and, implicitly, in $\mathcal{L}$ also.

To investigate whether solutions $\{\mathcal{L}, \mathcal{V}\}$ of Eq.(3.4) exist, we introduce the column matrices, or generalized eigenvectors, $\mathcal{L}^{\pm}$, according to

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}^{+} & \mathcal{L}^{-} \end{pmatrix}\ . \tag{3.5}$$

Upon writing the diagonal elements of $\mathcal{V}$ (generalized eigenvalues) as

$$\mathcal{V}_{11} = \mathcal{S}^{+}\ , \qquad \mathcal{V}_{22} = \mathcal{S}^{-}\ , \tag{3.6}$$

Eq.(3.4) decomposes into the two systems of equations

$$\mathcal{A}\mathcal{L}^{\pm} = \mathcal{L}^{\pm}\mathcal{S}^{\pm}\ . \tag{3.7}$$

To show that there exists an operator pair, $\{\mathcal{L}, \mathcal{V}\}$, such that the above operator equation is satisfied, is the central problem that we consider in the present paper.

The difference between $\mathcal{S}^{\pm}$ and the generalized vertical slowness operator introduced in [5] is a factor of s. Note that the first element of the operators $\mathcal{L}^{\pm}$, $\mathcal{L}_1^{\pm}$, composes the vertical particle velocity and that the second element, $\mathcal{L}_2^{\pm}$, composes the acoustic pressure, whereas the elements of W may be physically 'non-observable'.

The fundamental question of this paper is: Does there exist a decomposition operator $\mathcal{L}$ that decomposes $\mathcal{A}$ in the above sense? To show that there exists such a decomposition of $\mathcal{A}$ we derive properties of the spectrum of $\mathcal{A}$ with an associated projector — we use a generalized eigenvalue argument.

3.2. The Dirichlet-to-Neumann maps. The Dirichlet-to-Neumann maps — at the planar surfaces $x_3 =$ constant — follow as $\mathcal{Y}^{\pm}F_2 \mathrm{d}x_1 \mathrm{d}x_2$ where $\mathcal{Y}^{\pm}$ satisfies the algebraic Riccati operator equation, which we will derive at the end of the paper. Upon constraining x_3 to a level surface the Dirichlet-to-Neumann map associated with the up- or downgoing field constituents now follows from the relations

$$F_2 = \mathcal{Z}^{+}F_1 \quad \text{if} \quad W_2 = 0\ , \qquad \mathcal{Z}^{+} = \mathcal{L}_{21}\mathcal{L}_{11}^{-1}\ , \tag{3.8}$$

$$F_2 = \mathcal{Z}^{-}F_1 \quad \text{if} \quad W_1 = 0\ , \qquad \mathcal{Z}^{-} = \mathcal{L}_{22}\mathcal{L}_{12}^{-1}\ , \tag{3.9}$$

where $\mathcal{Z}^{\pm} = (\mathcal{Y}^{\pm})^{-1}$. Note that the 'Dirichlet-to-Neumann' map is a mapping between the pressure and the vertical particle velocity.

4. Properties of the $\mathcal{A}_{s,\lambda}$ operator. It is not necessary for us to have a complete knowledge of the spectral decomposition. However for the problem addressed in this paper, we need to establish a separation of the spectrum into two non-trivial sets. With help of the following proposition we find the spectrum of $\mathcal{A}_{;r}$ as a corollary.

PROPOSITION 1. *Let s, λ be complex numbers, $r \in \mathbb{N}_0$ and let $\mathcal{A}_{s,\lambda;r} = \mathcal{A}_{;r} - \lambda I$, defined through (2.13), (2.27) and (2.29). Furthermore let*

$$\mathrm{Q} = \left\{ \{s,\lambda\} \in \mathbb{C}^2 : \mathrm{Re}\,\{s\} > 0 \quad \textit{and} \quad (\mathrm{Re}\,\{\lambda\})^2 < (\mathrm{Re}\,\{s\})^2 \inf_{\boldsymbol{x}' \in \mathbb{R}^2} \frac{\kappa}{\alpha_{33}} \right\} .$$

Then for $\{s,\lambda\} \in \mathrm{Q}$, $\mathcal{A}_{s,\lambda;r}$

1. *is bounded from below;*
2. *is one-to-one;*
3. *has dense range;*
4. *is closable and the closure fulfills 1-3 above.*

REMARK 1.1. *The conditions on Q are sufficient but not necessary with respect to the λ parameter.*

REMARK 1.2. *The Hilbert identity or resolvent equation,*

$$\mathcal{A}_{s,\lambda}^{-1} - \mathcal{A}_{s,\lambda'}^{-1} = (\lambda - \lambda')\mathcal{A}_{s,\lambda}^{-1}\mathcal{A}_{s,\lambda'}^{-1} ,$$

for $\{s,\lambda\}, \{s,\lambda'\} \in \mathrm{Q}$ is defined for any operator. If the operator is closed the resolvent is defined everywhere *and is a holomorphic function of λ if $\{s,\lambda\} \in \mathrm{Q}$ (cf. [14, pp.172-174, §III.6.1], [25, pp.211-212, § VIII.2] and [2, p.84, §3.7.5]).*

REMARK 1.3. *From part one of the above proposition it directly follows that the inverse is a bounded operator on its range (cf. [16, p.244, theorem 5.7.1]).*

The conclusion of this proposition, as needed in continuation of this paper, is stated in Corollary 1.1 in §4.6.

4.1. Scalarization. The operator $\mathcal{A}_{s,\lambda}$ can be quasi-diagonalized and this diagonalization yields two results that simplify the analysis. Let

$$K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4.1}$$

with $K^{-1} = K$. We find that

$$K\mathcal{A}_{s,\lambda} = \begin{pmatrix} \mathcal{A}_{21} & \mathcal{A}_{22} - \lambda \\ \mathcal{A}_{11} - \lambda & \mathcal{A}_{12} \end{pmatrix} .$$

The operator $\mathcal{A}_{21}$ is the multiplication by $s\alpha_{33}^{-1}$. The Laplace parameter s, with the constraint $\mathrm{Re}\,\{s\} > 0$ and that α_{33}^{-1} is bounded away from zero, ensures that $\mathcal{A}_{21}^{-1}$ exists. The operator $K\mathcal{A}_{s,\lambda}$ is then diagonalized as follows,

$$K\mathcal{A}_{s,\lambda}\mathcal{T}_{2;s,\lambda} = \mathcal{T}_{1;s,\lambda}\mathcal{D}_{s,\lambda} , \tag{4.2}$$

where

$$\mathcal{T}_{1;s,\lambda} = \begin{pmatrix} 1 & 0 \\ (\mathcal{A}_{11} - \lambda)\,\mathcal{A}_{21}^{-1} & 1 \end{pmatrix} \tag{4.3}$$

and

$$\mathcal{T}_{2;s,\lambda} = \begin{pmatrix} 1 & -\mathcal{A}_{21}^{-1}(\mathcal{A}_{22} - \lambda) \\ 0 & 1 \end{pmatrix} , \tag{4.4}$$

while

$$\mathcal{D}_{s,\lambda} = \begin{pmatrix} \mathcal{A}_{21} & 0 \\ 0 & s^{-1}\mathcal{E}_{s,\lambda} \end{pmatrix} , \tag{4.5}$$

with

$$\begin{aligned} \mathcal{E}_{s,\lambda} &= s\left(\mathcal{A}_{12} - (\mathcal{A}_{11} - \lambda)\mathcal{A}_{21}^{-1}(\mathcal{A}_{22} - \lambda)\right) \\ &= -\partial_\mu\, \alpha_{\mu\nu}\partial_\nu + \lambda\left(\partial_\mu\, \alpha_{3\mu} + \alpha_{3\nu}\partial_\nu\right) + \kappa s^2 - \alpha_{33}\lambda^2 \ . \end{aligned} \tag{4.6}$$

The operator $\mathcal{E}_{s,\lambda}$ is the extension of the 'transverse Helmholtz' operator [5] to anisotropic fluids; in §5.2.1 we prove that it is elliptic with parameters for certain restrictions on $\{s, \lambda\}$. For each fixed λ, the operators $\mathcal{T}_{1;s,\lambda}, \mathcal{T}_{2;s,\lambda}$ have the inverses:

$$\mathcal{T}_{1;s,\lambda}^{-1} = \begin{pmatrix} 1 & 0 \\ -(\mathcal{A}_{11} - \lambda)\,\mathcal{A}_{21}^{-1} & 1 \end{pmatrix} \tag{4.7}$$

and

$$\mathcal{T}_{2;s,\lambda}^{-1} = \begin{pmatrix} 1 & \mathcal{A}_{21}^{-1}(\mathcal{A}_{22} - \lambda) \\ 0 & 1 \end{pmatrix} , \tag{4.8}$$

respectively; neither of them are bounded on $((\mathrm{L}^2)^2, (\cdot,\cdot)_{[0,0]})$. From the quasi-diagonalization we obtain an explicit expression for $\mathcal{A}_{s,\lambda}^{-1}$ in terms of $\mathcal{E}_{s,\lambda}^{-1}$. Starting from (4.2) and inverting factors in succession, gives

$$\begin{aligned} &\mathcal{A}_{s,\lambda}^{-1} = \mathcal{T}_{2;s,\lambda}\mathcal{D}_{s,\lambda}^{-1}\mathcal{T}_{1;s,\lambda}^{-1}K = \\ s&\begin{pmatrix} -\mathcal{A}_{21}^{-1}\,(\mathcal{A}_{22} - \lambda)\,\mathcal{E}_{s,\lambda}^{-1} & s^{-1}\mathcal{A}_{21}^{-1} + \mathcal{A}_{21}^{-1}\,(\mathcal{A}_{22} - \lambda)\,\mathcal{E}_{s,\lambda}^{-1}\,(\mathcal{A}_{11} - \lambda)\,\mathcal{A}_{21}^{-1} \\ \mathcal{E}_{s,\lambda}^{-1} & -\mathcal{E}_{s,\lambda}^{-1}\,(\mathcal{A}_{11} - \lambda)\,\mathcal{A}_{21}^{-1} \end{pmatrix} = \\ &\begin{pmatrix} (-\alpha_{3\mu}\partial_\mu + \alpha_{33}\lambda)\,\mathcal{E}_{s,\lambda}^{-1} & s^{-1}\alpha_{33} + (-\alpha_{3\mu}\partial_\mu + \alpha_{33}\lambda)\, s^{-1}\mathcal{E}_{s,\lambda}^{-1}\,(-\partial_\nu\,\alpha_{3\nu} + \alpha_{33}\lambda) \\ s\mathcal{E}_{s,\lambda}^{-1} & \mathcal{E}_{s,\lambda}^{-1}\,(-\partial_\mu\,\alpha_{3\mu} + \lambda\alpha_{33}) \end{pmatrix} . \end{aligned} \tag{4.9}$$

The inverse of the scalar operator $\mathcal{E}_{s,\lambda}$ captures most of the behavior of the matrix operator resolvent, $\mathcal{A}_{s,\lambda}^{-1}$, due to the fact that $\mathcal{A}_{21}^{-1}$ is algebraic.

4.2. Proof of Proposition 1, part 1. An operator that is bounded from below, has an inverse which is bounded on the range of the operator. For the system

$$\mathcal{A}_{s,\lambda}F = N \ , \tag{4.10}$$

this statement implies that for each N of (4.10) in the range of $\mathcal{A}_{s,\lambda}$, there exists a unique solution F that has finite norm.

First we consider the case of $\mathcal{A}_{s,\lambda}$ on $((\mathrm{L}^2)^2, (\cdot,\cdot)_{[0,0]})$ and then extend the analysis to the general case in which the operator $\mathcal{A}_{s,\lambda;r}$ on $(\mathrm{H}_r^r, (\cdot,\cdot)_{[r,r]})$ is considered. The method for obtaining conditions for an elliptic scalar operator to be bounded from below is here used as a pattern for obtaining conditions for $\mathcal{A}_{s,\lambda}$ to be bounded from below.

4.2.1. The $((\mathrm{L}^2)^2, (\cdot,\cdot)_{[0,0]})$ case. To start the proof that the operator $\mathcal{A}_{s,\lambda}$ is bounded from below, we employ Schwartz' inequality,

$$\begin{aligned} \|\mathcal{A}_{s,\lambda}F\|_{[0,0]}\, \|F\|_{[0,0]} &= \|K_{Y_0}\mathcal{A}_{s,\lambda}F\|_{[0,0]}\, \|F\|_{[0,0]} \geq \left|\mathrm{Re}\left\{(F, K_{Y_0}\mathcal{A}_{s,\lambda}F)_{[0,0]}\right\}\right| \\ &= \left|\mathrm{Re}\left\{(F, K\mathcal{A}_{s,\lambda}F)_{(0,0)}\right\}\right| , \end{aligned} \tag{4.11}$$

with $F \in \mathrm{D}(\mathcal{A}_{s,\lambda})$ and where K is unitary in $\|\cdot\|_{(0,0)}$ and is given by Eq.(4.1). Also,

$$K_{Y_0} = \begin{pmatrix} 0 & Y_0 \\ Y_0^{-1} & 0 \end{pmatrix} \tag{4.12}$$

is unitary in $\|\cdot\|_{[0,0]}$. Here Y_0 is the weight used in the inner product $(\cdot,\cdot)_{[0,0]}$ (cf. (2.23)). If there is a constant $C_1(\lambda, s)$ such that $\|\mathcal{A}_{s,\lambda}F\|_{[0,0]} \geq C_1(\lambda,s)\, \|F\|_{[0,0]}$ for all $F \in \mathrm{D}(\mathcal{A}_{s,\lambda})$, then the operator is bounded from below in $((\mathrm{L}^2)^2, (\cdot,\cdot)_{[0,0]})$. Let us evaluate $K\mathcal{A}_{s,\lambda}F$ (cf. Eq.(2.13)),

$$K\mathcal{A}_{s,\lambda}F = \begin{pmatrix} s\alpha_{33}^{-1}F_1 + \alpha_{33}^{-1}\alpha_{3\mu}\partial_\mu F_2 - \lambda F_2 \\ \partial_\nu\, \alpha_{3\nu}\alpha_{33}^{-1}F_1 - \lambda F_1 + \kappa s F_2 - s^{-1}\partial_\mu\, Q_{\mu\nu}\partial_\nu F_2 \end{pmatrix} . \tag{4.13}$$

Then

$$\begin{aligned} \left(\overline{K\mathcal{A}_{s,\lambda}F}\right)_I F_I = &\ \bar{s}\alpha_{33}^{-1}|F_1|^2 + \kappa\bar{s}|F_2|^2 - \bar{\lambda}\,\mathrm{Re}\left\{F_1\overline{F}_2\right\} \\ &+ (\partial_\nu\, \alpha_{3\nu}\alpha_{33}^{-1}\overline{F}_1)\, F_2 + \alpha_{33}^{-1}\alpha_{3\nu}(\partial_\mu\overline{F}_2)\, F_1 - \bar{s}^{-1}F_2\,\partial_\mu\, Q_{\mu\nu}\partial_\nu\overline{F}_2 , \end{aligned} \tag{4.14}$$

where the index I in the first term indicates summation over $I = 1, 2$. This expression is the integrand in the $((\mathrm{L}^2)^2, (\cdot,\cdot)_{(0,0)})$ inner product integral over $\mathbb{R}^2$. By applying integration by parts and taking the real part of Eq.(4.14), we obtain the integrand of $\mathrm{Re}\left\{(F, K\mathcal{A}_{s,\lambda}F)_{(0,0)}\right\}$:

$$\begin{aligned} &s_{\mathrm{r}}(\cos\sigma)\left(\alpha_{33}^{-1}|F_1|^2 + \kappa|F_2|^2\right) - 2\lambda_{\mathrm{R}}\,\mathrm{Re}\left\{F_1\overline{F}_2\right\} \\ &\qquad + s_{\mathrm{r}}^{-1}(\cos\sigma)Q_{\mu\nu}(\partial_\mu F_2)\,(\partial_\nu\overline{F}_2) , \end{aligned} \tag{4.15}$$

where

$$\lambda_{\mathrm{R}} = \mathrm{Re}\{\lambda\} \qquad \text{and} \qquad s = s_{\mathrm{r}}\exp(\mathrm{i}\sigma) . \tag{4.16}$$

Since $Q_{\mu\nu}$ is positive definite and symmetric, we find that

$$\begin{aligned} Q_{\mu\nu}(\partial_\mu F_2)(\partial_\nu\overline{F}_2) &= Q_{11}|\partial_1 F_2|^2 + Q_{22}|\partial_2 F_2|^2 + 2Q_{12}\mathrm{Re}\left\{(\partial_1 F_2)\,\partial_2\overline{F}_2\right\} \\ &\geq q_1\, |\boldsymbol{\nabla}' F_2|^2 , \end{aligned} \tag{4.17}$$

where $\boldsymbol{\nabla}' = \{\partial_1, \partial_2\}$. Hence, this term is real and furthermore positive unless $\boldsymbol{\nabla}' F_2 = 0$. Here $F_2 \in \mathrm{H}^2(\mathbb{R}^2)$ and hence the only function fulfilling $\boldsymbol{\nabla}' F_2 = 0$ is the zero function.

As noted earlier, we constrain the Laplace parameter, s, according to

$$\mathrm{Re}\{s\} > 0 .$$

This, together with the inequality

$$2\lambda_{\mathrm{R}}\mathrm{Re}\left\{F_1\overline{F}_2\right\} \leq \eta\kappa|F_2|^2 s_{\mathrm{r}}\cos\sigma + \frac{\lambda_{\mathrm{R}}^2}{\eta\kappa s_{\mathrm{r}}\cos\sigma}|F_1|^2 \ , \tag{4.18}$$

where η is a real and positive dimensionless parameter gives that expression (4.15) has a lower bound of the form

$$(4.15) \geq \left(s_{\mathrm{r}}(\cos\sigma)\alpha_{33}^{-1} - \frac{\lambda_{\mathrm{R}}^2}{\eta s_{\mathrm{r}}(\cos\sigma)\kappa}\right)|F_1|^2 + (1-\eta)s_{\mathrm{r}}(\cos\sigma)\kappa|F_2|^2 \ . \tag{4.19}$$

Here, η has to be restricted to $\eta \in (0,1)$. The upper bound of η follows from the requirement that the last term on the right hand side of (4.19) should be positive. For each fixed $0 < \eta < 1$ and fixed s satisfying (2.5), there exists λ such that $\mathcal{A}_{s,\lambda}$ is bounded from below. Using this, we find that for each λ such that

$$\lambda_{\mathrm{R}}^2 < (s_{\mathrm{r}}\cos\sigma)^2 \inf_{\boldsymbol{x}'\in\mathbb{R}^2}\frac{\kappa}{\alpha_{33}} = (\mathrm{Re}\{s\})^2 \inf_{\boldsymbol{x}'\in\mathbb{R}^2}\frac{\kappa}{\alpha_{33}} \tag{4.20}$$

and for each fixed s satisfying (2.5) there exists an optimal η such that the approximation of the lower bound of the acoustic system's matrix is maximal. The maximal lower bound, $C_1(\lambda,s)$, for fixed $\{s,\lambda\} \in \mathrm{Q}$, becomes

$$\begin{aligned} C_1(\lambda,s) = \max_{\eta\in(0,1)}\inf_{\boldsymbol{x}'\in\mathbb{R}^2}&\left\{(1-\eta)s_{\mathrm{r}}(\cos\sigma)\kappa Y_0^{-1} \ , \left(s_{\mathrm{r}}(\cos\sigma)\alpha_{33}^{-1} - \frac{\lambda_{\mathrm{R}}^2}{\eta s_{\mathrm{r}}(\cos\sigma)\kappa}\right)Y_0\right\} \\ &> 0 \ . \end{aligned} \tag{4.21}$$

By integrating (4.19) over $\mathbb{R}^2$ with respect to $\boldsymbol{x}'$ we now obtain

$$\begin{aligned} \left|\mathrm{Re}\left\{(F, K\mathcal{A}_{s,\lambda}F)_{(0,0)}\right\}\right| &\geq C_1(\lambda,s)\left(Y_0^{-1}\left\|F_1\right\|_0^2 + Y_0\left\|F_2\right\|_0^2\right) \\ &= C_1(\lambda,s)\left\|F\right\|_{[0,0]}^2 \ . \end{aligned} \tag{4.22}$$

We find that the restriction of the Laplace parameter s such that $\mathcal{A}_{s,\lambda}$ is bounded from below, is that $\mathrm{Re}\{s\} > 0$. Furthermore, for each such s we find that the region described by (4.20) is a sector about the imaginary axis (see figure 4.1). We conclude that for $\{s,\lambda\} \in \mathrm{Q}$, there exists a lower bound for the acoustic system's matrix. Note that the restrictions on s, λ are sufficient, but not necessary. □

4.2.2. The $(\mathrm{H}_r^r, (\cdot,\cdot)_{[r,r]})$ case. To extend the estimate of the lower bound to the case of $\mathcal{A}_{s,\lambda;r}$ for arbitrary $r \in \mathbb{N}_0$, we have to show that

$$\left\|\mathcal{A}_{s,\lambda;r}F\right\|_{[r,r]} = \left\|\Upsilon^{-r}\mathcal{A}_{s,\lambda}\Upsilon^r F\right\|_{[r,r]} \geq C_2(\lambda,s;r)\left\|F\right\|_{[r,r]} \tag{4.23}$$

for some constant $C_2(\lambda,s;r)$ for all $F \in \mathrm{D}(\mathcal{A}_{s,\lambda;r}) \subset ((\mathrm{L}^2)^2, (\cdot,\cdot)_{[0,0]})$. To this end, we need the Fourier transform, $\mathcal{F}$, with parameter

$$(\mathcal{F}F_1)(\xi', x_3) = \int_{\mathbb{R}^2}\mathrm{d}^2\boldsymbol{x}' \ \exp(-\mathrm{i}\langle\boldsymbol{x}',\xi'\rangle)F_1(\boldsymbol{x}) \ , \tag{4.24}$$

where $\boldsymbol{x}' = \{x_1,x_2\}$ and $\xi' = \{\xi_1,\xi_2\}$ are a dual pair of vectors in $\mathbb{R}^2$; $\langle\xi',\boldsymbol{x}'\rangle = \xi_\mu x_\mu$. From Parseval's equality we have $\left\|\mathcal{F}F_1\right\|_0 = (2\pi)^2\left\|F_1\right\|_0$ for all $F_1 \in (\mathrm{L}^2, (\cdot,\cdot)_0)$. The Fourier transform of $1-\boldsymbol{\Delta}'$ is $1+|y_0\xi'|^2$,

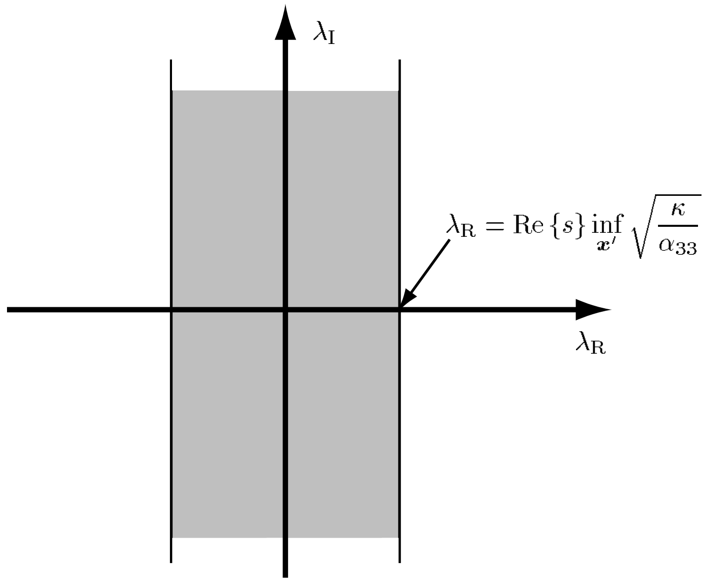

FIG. 4.1. *The gray region represents* $\lambda \in \mathbb{C}$ *for fixed* s *with* $\mathrm{Re}\{s\} > 0$ *where* $\mathcal{A}_{s,\lambda}$ *is bounded from below (cf. (4.20)).*

where y_0 was introduced in (2.19) to make the derivative dimensionless. To simplify the equations, we introduce the notation $\zeta = y_0 \xi'$. With the aid of the binomial theorem we obtain the following inequalities:

$$2^{-r}(1+|\zeta|^2)^r \leq 1 + \sum_{n=1}^{r} |\zeta|^{2n} \leq \left(1+|\zeta|^2\right)^r \ , \tag{4.25}$$

$$\sum_{|\beta|=n} |\zeta^\beta|^2 \leq |\zeta|^{2n} \leq 2^n \sum_{|\beta|=n} |\zeta^\beta|^2 \ , \tag{4.26}$$

where we have used Stirling's formula [1, p.257,(6.1.38)] for the upper limit of the estimate

$$1 \leq \binom{r}{n} = \frac{r!}{n!(r-n)!} \leq 2^r \ , \tag{4.27}$$

for $0 \leq n \leq r$. We consider the action of $(1-\boldsymbol{\Delta}')^{r/2}$ on a complex field F_1 where $F_1 \in \mathrm{H}^r \subset (\mathrm{L}^2, (\cdot,\cdot)_0)$. Then, using Parseval's equality and inequalities (4.25) and (4.26), it follows that

$$\begin{aligned} \left\| (1-\boldsymbol{\Delta}')^{r/2} F_1 \right\|_0^2 &= (2\pi y_0)^{-2} \left\| \left(1+|\zeta|^2\right)^{r/2} \mathcal{F} F_1 \right\|_0^2 \\ &= (2\pi y_0)^{-2} \int_{\mathbb{R}^2} \mathrm{d}^2\zeta \ (1+|\zeta|^2)^r |\mathcal{F}F_1|^2 \\ \geq (2\pi y_0)^{-2} \int_{\mathbb{R}^2} \mathrm{d}^2\zeta \ \left(1 + \sum_{n=1}^{r} |\zeta|^{2n}\right) |\mathcal{F}F_1|^2 &\geq (2\pi y_0)^{-2} \int_{\mathbb{R}^2} \mathrm{d}^2\zeta \ \sum_{|\beta|\leq r} |\zeta^\beta \mathcal{F} F_1|^2 \\ = (2\pi y_0)^{-2} \sum_{|\beta|\leq r} \left\| \zeta^\beta \mathcal{F} F_1 \right\|_0^2 &= \sum_{|\beta|\leq r} \left\| y_0^{|\beta|} \partial_{\boldsymbol{x}'}^\beta F_1 \right\|_0^2 = \|F_1\|_r^2 \ . \end{aligned} \tag{4.28}$$

Analogously, for a function $F_1 \in (\mathrm{L}^2, (\cdot,\cdot)_0)$ we have

$$\left\|(1-\boldsymbol{\Delta}')^{-r/2} F_1\right\|_r^2 = \int_{\mathbb{R}^2} \mathrm{d}^2\boldsymbol{x}' \sum_{|\beta|\le r} \left| y_0^{|\beta|}\partial^\beta_{\boldsymbol{x}'} \left(1-\boldsymbol{\Delta}')^{-r/2}F_1\right)\right|^2 \tag{4.29}$$
$$= \sum_{|\beta|\le r} \left\|(1-\boldsymbol{\Delta}')^{-r/2}\, y_0^{|\beta|}\partial^\beta_{\boldsymbol{x}'} F_1\right\|_0^2 = (2\pi y_0)^{-2} \sum_{|\beta|\le r} \left\|(1+|\zeta|^2)^{-r/2}\zeta^\beta \mathcal{F}F_1\right\|_0^2$$
$$= (2\pi y_0)^{-2} \int_{\mathbb{R}^2} \mathrm{d}^2\zeta \ \left(1+|\zeta|^2\right)^{-r} \sum_{|\beta|\le r} |\zeta^\beta|^2 |\mathcal{F}F_1|^2$$
$$\ge (2\pi y_0)^{-2}\, 2^{-2r}\, \|\mathcal{F}f\|_0^2 = 2^{-2r}\, \|F_1\|_0^2 \ ,$$

using Parseval's equality and inequalities (4.25) and (4.26). Thus for all fields $F \in ((\mathrm{L}^2)^2, (\cdot,\cdot)_{[0,0]})$ it follows from (4.29) that

$$\left\|\Upsilon^{-r}F\right\|_{[r,r]}^2 = Y_0^{-1}\, \left\|(1-\boldsymbol{\Delta}')^{-r}F_1\right\|_r^2 + Y_0\, \left\|(1-\boldsymbol{\Delta}')^{-r}F_2\right\|_r^2 \tag{4.30}$$
$$\ge 2^{-2r}\, \|F\|_{[0,0]}^2 \ ,$$

and analogously for $\|\Upsilon^r F\|_{[0,0]}$. Thus for $F \in \mathrm{D}(\mathcal{A}_{s,\lambda;r}) \subset (\mathrm{H}^r_r, (\cdot,\cdot)_{[r,r]})$

$$\left\|\Upsilon^{-r}\mathcal{A}_{s,\lambda}\Upsilon^r F\right\|_{[r,r]}^2 \ge 2^{-2r}\, \|\mathcal{A}_{s,\lambda}\Upsilon^r F\|_{[0,0]}^2 \ge 2^{-2r} C_1^2(\lambda,s)\, \|\Upsilon^r F\|_{[0,0]}^2 \tag{4.31}$$
$$\ge 2^{-2r} C_1^2(\lambda,s)\, \|F\|_{[r,r]}^2 \ ,$$

where $C_1(\lambda,s)$ is the lower bound obtained for $\mathcal{A}_{s,\lambda}$ in (4.21). Hence $C_2(\lambda,s;r) = 2^{-r}C_1(\lambda,s)$. Thus, $\mathcal{A}_{s,\lambda;r}$ is bounded for $\{s,\lambda\} \in \mathrm{Q}$ also. □

4.3. Proof of Proposition 1, part 2. The inequality

$$\left\|\Upsilon^{-r}\mathcal{A}_{s,\lambda}\Upsilon^r F\right\|_{[r,r]} \ge C_2(\lambda,s;r)\, \|F\|_{[r,r]} \ , \tag{4.32}$$

implies that the null space of $\mathcal{A}_{s,\lambda;r}$ only contains the zero element for $\{s,\lambda\} \in \mathrm{Q}$. By [16, p.171, theorem 4.4.1] an operator with trivial null space is one-to-one (injective). Hence, the operator $\mathcal{A}_{s,\lambda;r}$ is one-to-one for all $r \in \mathbb{N}_0$. □

4.4. Proof of Proposition 1, part 3. To prove that $\mathcal{A}_{s,\lambda}$ have dense range in $((\mathrm{L}^2)^2, (\cdot,\cdot)_{[0,0]})$, we consider the graph of $\mathcal{A}_{s,\lambda}$ and find that it suffices to show that the kernel of $\mathcal{A}^*_{s,\lambda}$, the adjoint to $\mathcal{A}_{s,\lambda}$, is trivial. That is,

$$\mathcal{A}^*_{s,\lambda}G = 0, \ G \in \mathrm{D}(\mathcal{A}^*_{s,\lambda}) \Rightarrow G = 0 \ . \tag{4.33}$$

We introduce the adjoint and formal adjoint (cf. [23, pp.67-68, §4.4]) of operator $\mathcal{A}$. The operator $\mathcal{A}$ is defined on the Hilbert space $((\mathrm{L}^2)^2, (\cdot,\cdot)_{[0,0]})$. Assume that $\mathcal{A}'$ is an operator on $((\mathrm{L}^2)^2, (\cdot,\cdot)_{[0,0]})$, then $\mathcal{A}'$ is a *formal adjoint* of $\mathcal{A}$ if

$$(G, \mathcal{A}F)_{[0,0]} = (\mathcal{A}'G, F)_{[0,0]} \quad \text{for all } F \in \mathrm{D}(\mathcal{A}) \ , \ G \in \mathrm{D}(\mathcal{A}') \ . \tag{4.34}$$

If $\mathcal{A}$ is densely defined then there exists a unique maximal formal adjoint, that is the *adjoint* of $\mathcal{A}$, denoted by $\mathcal{A}^*$, defined on the domain

$$\mathrm{D}(\mathcal{A}^*) = \{G \in (\mathrm{L}^2)^2 : \ \exists H \in (\mathrm{L}^2)^2 \text{ such that } (H,F)_{[0,0]} = (G,\mathcal{A}F)_{[0,0]} \ \forall F \in \mathrm{D}(\mathcal{A})\} \ . \tag{4.35}$$

Due to the fact that the adjoint is maximal, it follows that any other formal adjoint is a restriction of $\mathcal{A}^*$ (see [14, p.167, §III.5.5]).

As before, we first consider the operator $\mathcal{A}_{s,\lambda}$ and then extend the analysis to the operator $\mathcal{A}_{s,\lambda;r}$ where we use that Υ^{-r} is a topological isomorphism and thus preserves topological properties.

4.4.1. The $((\mathrm{L}^2)^2, (\cdot,\cdot)_{[0,0]})$ case. To calculate the null space of $\mathcal{A}^*_{s,\lambda}$ we scalarize the equations and derive conditions such that the scalarized form have no non-trivial solutions.

The acoustic system's matrix is densely defined since its domain, the set H^1_2, is dense in the space $((\mathrm{L}^2)^2, (\cdot,\cdot)_{[0,0]})$; hence its adjoint exists. To obtain the explicit form of the adjoint we apply integration by parts within the inner product. We find

$$\mathcal{A}^*_{s,\lambda} = \begin{pmatrix} -\alpha_{33}^{-1}\alpha_{3\mu}\partial_\mu - \bar{\lambda} & Y_0^2\alpha_{33}^{-1}\bar{s} \\ Y_0^{-2}\left(\kappa\bar{s} - \bar{s}^{-1}\partial_\nu\, Q_{\nu\mu}\partial_\mu\right) & -\partial_\mu\,\alpha_{3\mu}\alpha_{33}^{-1} - \bar{\lambda} \end{pmatrix}, \tag{4.36}$$

defined on the domain $\mathrm{D}(\mathcal{A}^*_{s,\lambda}) \subset ((\mathrm{L}^2)^2, (\cdot,\cdot)_{[0,0]})$. The terms $Y_0^{\pm 2}$ appear corresponding to the weight in the inner product. From the above form of $\mathcal{A}^*_{s,\lambda}$, we observe that the operator maps at least the domain H^2_1 into $((\mathrm{L}^2)^2, (\cdot,\cdot)_{[0,0]})$. This follows from the assumption that the material coefficients are sufficiently smooth. The domain H^2_1 is dense in $((\mathrm{L}^2)^2, (\cdot,\cdot)_{[0,0]})$ and is contained in $\mathrm{D}(\mathcal{A}^*_{s,\lambda})$ since any formal adjoint is a restriction of the adjoint, thus the adjoint is densely defined in $((\mathrm{L}^2)^2, (\cdot,\cdot)_{[0,0]})$. That $\mathcal{A}^*_{s,\lambda}$ is densely defined is used in §4.5.

The null space of $\mathcal{A}^*_{s,\lambda}$ is obtained by solving the equation $\mathcal{A}^*_{s,\lambda}F = 0$. Writing out the components explicitly, with the aid of (4.36) we get

$$\left(-\alpha_{33}^{-1}\alpha_{3\mu}\partial_\mu - \bar{\lambda}\right) F_1 + \bar{s}\alpha_{33}^{-1}Y_0^2 F_2 = 0\,, \tag{4.37}$$

$$Y_0^{-2}\left(\bar{s}\kappa - \bar{s}^{-1}\partial_\nu\, Q_{\nu\mu}\partial_\mu\right) F_1 + \left(-\partial_\mu\,\alpha_{3\mu}\alpha_{33}^{-1} - \bar{\lambda}\right) F_2 = 0\,. \tag{4.38}$$

Since $\bar{s}\alpha_{33}^{-1}Y_0^2 \neq 0$, the first equation enables us to write F_2 explicitly in terms of F_1. Substituting the result into the second equation, gives

$$\left(-\alpha_{33}^{-1}\alpha_{3\mu}\partial_\mu - \bar{\lambda}\right) F_1 + \bar{s}\alpha_{33}^{-1}Y_0^2 F_2 = 0\,, \tag{4.39}$$

$$Y_0^{-2}\bar{s}^{-1}\left(-\partial_\mu\,\alpha_{\mu\nu}\partial_\nu F_1 - \bar{\lambda}\left(\partial_\mu\,\alpha_{3\mu} + \alpha_{3\mu}\partial_\mu\right) + \kappa\bar{s}^2 - \alpha_{33}\bar{\lambda}^2\right) F_1 = 0\,. \tag{4.40}$$

The short form of the last equation is $Y_0^{-2}\bar{s}^{-1}\mathcal{E}^*_{s,\lambda}F_1 = 0$ (cf. (4.6)). If this equation has only the trivial solution it follows directly that the complete system has the trivial solution only. We have obtained a scalarized form of Eqs.(4.37) and (4.38) (cf. §4.1).

To obtain the null space of $\mathcal{E}^*_{s,\lambda}$, we first consider its domain

$$\mathrm{D}(\mathcal{E}^*_{s,\lambda}) = \{g \in \mathrm{L}^2 :\ \exists h \in \mathrm{L}^2 \text{ such that } (h,f)_0 = (g, \mathcal{E}_{s,\lambda}f)_0 \ \ \forall f \in \mathrm{D}(\mathcal{E}_{s,\lambda})\}\,. \tag{4.41}$$

$\mathcal{E}^*_{s,\lambda}$ is a uniformly, strongly elliptic second-order partial differential operator (for fixed $\{s,\lambda\}$) since $\alpha_{\mu\nu}$ is positive definite and symmetric [12, p.2, §1]. Hence, if $h = \mathcal{E}^*_{s,\lambda}g \in \mathrm{L}^2$ then $g \in \mathrm{H}^1$ and thus, at least

$$\mathrm{D}(\mathcal{E}^*_{s,\lambda}) \subset \mathrm{H}^1\,. \tag{4.42}$$

The relation

$$\{F_1 \in \mathrm{D}(\mathcal{E}^*_{s,\lambda}) : \mathcal{E}^*_{s,\lambda}F_1 = 0\} \subset \{F_1 \in \mathrm{D}(\mathcal{E}^*_{s,\lambda}) : (\mathcal{E}^*_{s,\lambda}F_1, F_1)_0 = 0\}\,, \tag{4.43}$$

enable us to consider the restrictions of $\{s,\lambda\}$ for the later form to ensure the a trivial null space of $\mathcal{E}^*_{s,\lambda}$. Thus we set $(\mathcal{E}^*_{s,\lambda}F_1,F_1)_0=0$. After integration by parts the integrand attains the form

$$\alpha_{\mu\nu}(\partial_\mu\overline{F}_1)\partial_\nu F_1-2\bar{\lambda}\mathrm{i}\alpha_{3\mu}\mathrm{Im}\left\{\overline{F}_1\partial_\mu F_1\right\}+\left(\kappa s^2-\alpha_{33}\bar{\lambda}^2\right)|F_1|^2\,. \tag{4.44}$$

We separate the real and the imaginary parts and use the notation introduced in (4.16) together with

$$\lambda_{\mathrm{I}}=\mathrm{Im}\left\{\lambda\right\}\,. \tag{4.45}$$

Then expression (4.44) can be written as

$$\begin{aligned}\alpha_{\mu\nu}(\partial_\mu\overline{F}_1)\partial_\nu F_1&-2\lambda_{\mathrm{I}}\alpha_{3\mu}\mathrm{Im}\left\{\overline{F}_1\partial_\mu F_1\right\}+\left(\kappa s_{\mathrm{r}}^2\cos 2\sigma-\alpha_{33}\lambda_{\mathrm{R}}^2+\alpha_{33}\lambda_{\mathrm{I}}^2\right)|F_1|^2\\&+\mathrm{i}\left(-2\lambda_{\mathrm{R}}\alpha_{3\mu}\mathrm{Im}\left\{\overline{F}_1\partial_\mu F_1\right\}+\left(2\alpha_{33}\lambda_{\mathrm{R}}\lambda_{\mathrm{I}}-\kappa s_{\mathrm{r}}^2\sin 2\sigma\right)|F_1|^2\right)\,.\end{aligned} \tag{4.46}$$

Using the identity

$$\begin{aligned}\alpha_{33}\lambda_{\mathrm{I}}^2|F_1|^2&-2\lambda_{\mathrm{I}}\alpha_{3\mu}\mathrm{Im}\left\{\overline{F}_1\partial_\mu F_1\right\}\\&=\alpha_{33}|F_1\lambda_{\mathrm{I}}+\mathrm{i}\alpha_{33}^{-1}\alpha_{3\mu}\partial_\mu F_1|^2-\alpha_{3\nu}\alpha_{33}^{-1}\alpha_{3\mu}(\partial_\mu\overline{F}_1)\partial_\nu F_1\,,\end{aligned} \tag{4.47}$$

the real part of expression (4.46) becomes (cf. (2.14))

$$\begin{aligned}Q_{\mu\nu}(\partial_\mu\overline{F}_1)\partial_\nu F_1&+\left(\kappa s_{\mathrm{r}}^2\cos 2\sigma-\alpha_{33}\lambda_{\mathrm{R}}^2\right)|F_1|^2\\&+\alpha_{33}|F_1\lambda_{\mathrm{I}}+\mathrm{i}\alpha_{33}^{-1}\alpha_{3\mu}\partial_\mu F_1|^2\,.\end{aligned} \tag{4.48}$$

Due to the fact that $Q_{\mu\nu}$ is positive definite and symmetric, the first term is always positive as long as $F_1\neq 0$, since we are considering functions $F_1\in\mathrm{H}^1$ over the domain $\mathbb{R}^2$ and hence no everywhere defined constant fields exist apart from 0. Upon integrating (4.44) over $\mathbb{R}^2$, using the above representation for the real part, we obtain

$$\begin{aligned}(\mathcal{E}^*_{s,\lambda}F_1,F_1)_0=(Q_{\mu\nu}\partial_\nu F_1,\partial_\mu F_1)_0&+s_{\mathrm{r}}^2\cos 2\sigma\left\|\sqrt{\kappa}F_1\right\|_0^2-\lambda_{\mathrm{R}}^2\left\|\sqrt{\alpha_{33}}F_1\right\|_0^2\\&+\left\|\sqrt{\alpha_{33}}(F_1\lambda_{\mathrm{I}}+\mathrm{i}\alpha_{33}^{-1}\alpha_{3\mu}\partial_\mu F_1)\right\|_0^2+\mathrm{i}\Big(-s_{\mathrm{r}}^2\sin 2\sigma\left\|\sqrt{\kappa}F_1\right\|_0^2\\&+2\lambda_{\mathrm{R}}\lambda_{\mathrm{I}}\left\|\sqrt{\alpha_{33}}F_1\right\|_0^2-2\lambda_{\mathrm{R}}\mathrm{Im}\left\{(\alpha_{3\mu}\partial_\mu F_1,F_1)_0\right\}\Big)=0\,.\end{aligned} \tag{4.49}$$

To find sufficient the conditions on $\{s,\lambda\}\in\mathbb{C}^2$ such that (4.49) only has the trivial solution we assume that $F_1\neq 0$ and obtain necessary conditions such that (4.49) is false. If the imaginary part of the equation is non-zero then (4.49) is false and does not imply any condition. If the imaginary part is zero there are two cases to consider: $\lambda_{\mathrm{R}}=0$ and $\lambda_{\mathrm{R}}\neq 0$. If $\lambda_{\mathrm{R}}=0$ then $\sin 2\sigma=0$ (since $s_{\mathrm{r}}>0$) and thus the real part becomes

$$\begin{aligned}(Q_{\mu\nu}\partial_\nu F_1,\partial_\mu F_1)_0&+(\mathrm{Re}\left\{s\right\})^2\left\|\sqrt{\kappa}F_1\right\|_0^2\\&+\left\|\sqrt{\alpha_{33}}(F_1\lambda_{\mathrm{I}}+\mathrm{i}\alpha_{33}^{-1}\alpha_{3\mu}\partial_\mu F_1)\right\|_0^2>0\,,\end{aligned} \tag{4.50}$$

for $\mathrm{Re}\left\{s\right\}>0$ since the other terms are at least not negative.

If $\lambda_{\mathrm{R}}\neq 0$ the constraint on λ_{I} is

$$\lambda_{\mathrm{I}}\left\|\sqrt{\alpha_{33}}F_1\right\|_0^2=\frac{1}{2\lambda_{\mathrm{R}}}s_{\mathrm{r}}^2\sin 2\sigma\left\|\sqrt{\kappa}F_1\right\|_0^2+\mathrm{Im}\left\{(\alpha_{3\mu}\partial_\mu F_1,F_1)_0\right\} \tag{4.51}$$

for the imaginary part of (4.49) to become zero. Substituting this back into (4.49), upon expanding the term containing λ_{I} we find that

$$
\begin{aligned}
&(Q_{\mu\nu}\partial_\nu F_1, \partial_\mu F_1)_0 + (\alpha_{3\mu}\alpha_{33}^{-1}\alpha_{3\nu}\partial_\nu F_1, \partial_\mu F_1)_0 - \left(\frac{\mathrm{Im}\,\{(\alpha_{3\mu}\partial_\mu F_1, F_1)_0\}}{\left\|\sqrt{\alpha_{33}}F_1\right\|_0}\right)^2 \\
&+ s_{\mathrm{r}}^2 \cos 2\sigma \left\|\sqrt{\kappa}F_1\right\|_0^2 - \lambda_{\mathrm{R}}^2 \left\|\sqrt{\alpha_{33}}F_1\right\|_0^2 + \left(\frac{s_{\mathrm{r}}^2 \sin 2\sigma \left\|\sqrt{\kappa}F_1\right\|_0^2}{2\lambda_{\mathrm{R}}\left\|\sqrt{\alpha_{33}}F_1\right\|_0}\right)^2 = 0\,.
\end{aligned}
\tag{4.52}
$$

This equality can be rewritten in the form

$$
\begin{aligned}
&(Q_{\mu\nu}\partial_\nu F_1, \partial_\mu F_1)_0 + (\alpha_{3\mu}\alpha_{33}^{-1}\alpha_{3\nu}\partial_\nu F_1, \partial_\mu F_1)_0 - \left(\frac{\mathrm{Im}\,\{(\alpha_{3\mu}\partial_\mu F_1, F_1)_0\}}{\left\|\sqrt{\alpha_{33}}F_1\right\|_0}\right)^2 \\
&+ \left(1 + \frac{(\mathrm{Im}\,\{s\})^2}{\lambda_{\mathrm{R}}^2 \left\|\sqrt{\alpha_{33}}F_1\right\|_0^2}\right)\left((\mathrm{Re}\,\{s\})^2 \left\|\sqrt{\kappa}F_1\right\|_0^2 - \lambda_{\mathrm{R}}^2 \left\|\sqrt{\alpha_{33}}F_1\right\|_0^2\right) = 0\,.
\end{aligned}
\tag{4.53}
$$

To obtain conditions for this equation to be false for $F_1 \neq 0$, we note that the first line is non-negative, using the Schwartz' inequality

$$
\begin{aligned}
&(\alpha_{3\mu}\alpha_{33}^{-1}\alpha_{3\nu}\partial_\nu F_1, \partial_\mu F_1)_0 - \left(\frac{\mathrm{Im}\,\{(\alpha_{3\mu}\partial_\mu F_1, F_1)_0\}}{\left\|\sqrt{\alpha_{33}}F_1\right\|_0}\right)^2 \geq \\
&\qquad (\alpha_{3\mu}\alpha_{33}^{-1}\alpha_{3\nu}\partial_\nu F_1, \partial_\mu F_1)_0 - \left\|\alpha_{33}^{-1/2}\alpha_{3\mu}\partial_\mu F_1\right\|_0^2 = 0\,,
\end{aligned}
\tag{4.54}
$$

together with the fact that $Q_{\mu\nu}$ is positive definite and symmetric. The expression in the second line of (4.53) is positive if

$$
|\lambda_{\mathrm{R}}| < |\mathrm{Re}\,\{s\}\,| \inf \sqrt{\kappa\alpha_{33}^{-1}} \leq |\mathrm{Re}\,\{s\}\,| \frac{\left\|\sqrt{\kappa}F_1\right\|_0}{\left\|\sqrt{\alpha_{33}}F_1\right\|_0}\,.
\tag{4.55}
$$

Observe that $\inf \sqrt{\kappa\alpha_{33}^{-1}} = \sqrt{\inf \kappa\alpha_{33}^{-1}}$ since the square root is a smooth monotone function for positive arguments. Hence for $\{s,\lambda\} \in \mathrm{Q}$ only $F_1 = 0$ is in the null space of $\mathcal{E}_{s,\lambda}^*$, thus the null space of $\mathcal{A}_{s,\lambda}^*$ is trivial by the argument above. This implies that the range of $\mathcal{A}_{s,\lambda}$ is dense in $((\mathrm{L}^2)^2, (\cdot,\cdot)_{[0,0]})$ for $\{s,\lambda\} \in \mathrm{Q}$, (see figure 4.1). □

4.4.2. The $(\mathrm{H}_r^r, (\cdot,\cdot)_{[r,r]})$ case. In order to prove that $\mathcal{A}_{s,\lambda;r}$ has dense range in $(\mathrm{H}_r^r, (\cdot,\cdot)_{[r,r]})$, we use the fact that denseness is preserved under a homeomorphism. We note that actually the mapping Υ^{-r} is a topological isomorphism and thus also a homeomorphism. Thus, if the set $\mathrm{D}(\mathcal{A})$ is dense in the first space, then $\Upsilon^{-r}\mathrm{D}(\mathcal{A})$ is dense in the other space.

From (4.28) and (4.30) we have

$$
2^{-r} \|F\|_{[0,0]} \leq \left\|\Upsilon^{-r}F\right\|_{[r,r]} \leq \|F\|_{[0,0]}\,,
\tag{4.56}
$$

and hence the mapping $\Upsilon^{-r} : (\mathrm{H}_r^r, (\cdot,\cdot)_{[r,r]}) \to ((\mathrm{L}^2)^2, (\cdot,\cdot)_{[0,0]})$ is surjective, injective and bounded. Hence using the open mapping principle, there exists a bounded inverse. Υ^{-r} is a topological isomorphism of

$(\mathrm{H}_r^r, (\cdot,\cdot)_{[r,r]})$ onto $((\mathrm{L}^2)^2, (\cdot,\cdot)_{[0,0]})$. Thus $(\mathrm{H}_r^r, (\cdot,\cdot)_{[r,r]})$ and $((\mathrm{L}^2)^2, (\cdot,\cdot)_{[0,0]})$ are topologically isomorphic (cf. [16, pp.257-258, §5.9]) via Υ^{-r}.

In the previous subsections we have shown that $\mathcal{A}_{s,\lambda}$ has dense domain and dense range in $((\mathrm{L}^2)^2, (\cdot,\cdot)_{[0,0]})$ for $\{s,\lambda\} \in \mathrm{Q}$. That the range is dense can be expressed as

$$(G, \mathcal{A}_{s,\lambda} F)_{[0,0]} = 0 \ , \quad \forall F \in \mathrm{D}(\mathcal{A}_{s,\lambda}) \ \text{ implies that } G = 0 \ , \tag{4.57}$$

for complex fields G and F. If we make the change of fields, $F = \Upsilon^r F'$, and substitute this into the above, we obtain

$$(G, \mathcal{A}_{s,\lambda} \Upsilon^r F')_{[0,0]} = 0 \ , \quad \forall F' \in \Upsilon^{-r}\mathrm{D}(\mathcal{A}_{s,\lambda}) \ \text{ implies that } G = 0 \ . \tag{4.58}$$

The above statements show that every field $F \in \mathrm{D}(\mathcal{A}_{s,\lambda})$ has an equivalent field $F' \in (\mathrm{H}_r^r, (\cdot,\cdot)_{[r,r]})$ and that $\mathrm{D}(\mathcal{A}_{s,\lambda}\Upsilon^r) = \Upsilon^{-r}\mathrm{D}(\mathcal{A}_{s,\lambda})$, since Υ^{-r} is a topological isomorphism the domain is dense in $(\mathrm{H}_r^r, (\cdot,\cdot)_{[r,r]})$. Furthermore, the range of $\mathcal{A}_{s,\lambda}\Upsilon^r$, $\mathrm{R}(\mathcal{A}_{s,\lambda}\Upsilon^r)$, is dense in $((\mathrm{L}^2)^2, (\cdot,\cdot)_{[0,0]})$, and if we transform the set $\mathrm{R}(\mathcal{A}_{s,\lambda}\Upsilon^r)$ under the topological isomorphism, the transformed set is also dense in the new space. Thus, $\Upsilon^{-r}\mathcal{A}_{s,\lambda}\Upsilon^r$ has a dense range and domain in $(\mathrm{H}_r^r, (\cdot,\cdot)_{[r,r]})$ for $\{s,\lambda\} \in \mathrm{Q}$. □

4.5. Proof of Proposition 1, part 4. In §5 we consider a Dunford-Schwartz integral over the resolvent of $\mathcal{A}$, which analysis simplifies if $\mathcal{A}$ is closed. We noted above that $\mathcal{A}_{s,\lambda}^*$ is densely defined for $\{s,\lambda\} \in \mathrm{Q}$ and thus, by [14, p.168, §III.5.5], the operator $\mathcal{A}_{s,\lambda}$ is closable. The closure is denoted by $\{\mathcal{A}_{s,\lambda}\}_{\mathrm{cl}}$.

To show that the range is dense and that the operator is bounded from below for $\{s,\lambda\} \in \mathrm{Q}$, we first recall the precise definition of closure. The closure of the operator $\mathcal{A}_{s,\lambda}$, on $((\mathrm{L}^2)^2, (\cdot,\cdot)_{[0,0]})$, is constructed by expanding the original domain to include the limits of all series of all complex fields $\{F_n\}_{n=0}^\infty$ with the property that both $\{F_n\}_{n=0}^\infty$ and $\{\mathcal{A}_{s,\lambda}F_n\}_{n=0}^\infty$ are Cauchy sequences. If F is a limit point of $\{F_n\}_{n=0}^\infty$ then we define $\{\mathcal{A}_{s,\lambda}\}_{\mathrm{cl}} F = \lim_{n\to\infty} \mathcal{A}_{s,\lambda}F_n$.

To show that the closed operator is bounded from below, we rely on Corollary 1.19 [14, pp.316-317, §VI.1.5]. This corollary applies to sesquilinear forms in Hilbert spaces, but due to example 1.23 and example 1.3 in [14] we draw the conclusion that we can employ the sesquilinear form $(\mathcal{A}_{s,\lambda}F, \mathcal{A}_{s,\lambda}G)_{[0,0]}$ in Corollary 1.19 [14, pp.316-317, §VI.1.5] that is only closable when $\mathcal{A}_{s,\lambda}$ is closable. Thus, by the above mentioned corollary, we obtain that the closed form is bounded from below with the same constant and thus, the closed operator is bounded from below for $\{s,\lambda\} \in \mathrm{Q}$.

Concerning the question whether the range is dense in $((\mathrm{L}^2)^2, (\cdot,\cdot)_{[0,0]})$, we note that $((\mathrm{L}^2)^2, (\cdot,\cdot)_{[0,0]})$ is a Hilbert space, and hence complete, thus any Cauchy sequence has a limit in the space. Furthermore,

$$\mathrm{R}(\mathcal{A}_{s,\lambda}) \subset \mathrm{R}(\{\mathcal{A}_{s,\lambda}\}_{\mathrm{cl}}) \subset (\mathrm{L}^2)^2 \ , \tag{4.59}$$

hence if $\mathrm{R}(\mathcal{A}_{s,\lambda})$ is dense then $\mathrm{R}(\{\mathcal{A}_{s,\lambda}\}_{\mathrm{cl}})$ is dense, for the same restrictions on $\{s,\lambda\}$ as for $\mathcal{A}_{s,\lambda}$ before closure. Thus the closed operator $\{\mathcal{A}_{s,\lambda}\}_{\mathrm{cl}}$ is bounded from below and has a dense range for the same restrictions on $\{s,\lambda\}$ as $\mathcal{A}_{s,\lambda}$.

For the operator $\mathcal{A}_{s,\lambda;r}$ essentially the same applies, but we have to prove that the domain of the adjoint is dense in $(\mathrm{H}_r^r, (\cdot,\cdot)_{[r,r]})$, — only then $\mathcal{A}_{s,\lambda;r}$ is closable. We form the adjoint of $\mathcal{A}_{s,\lambda;r}^* = (\Upsilon^{-r}\mathcal{A}_{s,\lambda}\Upsilon^r)^*$ and note that we have an unbounded operators acting on a bounded operator $(\mathcal{A}_{s,\lambda}\Upsilon^r)$. In the case of unbounded operators the adjoint of a 'product' of operators is not the same as the 'product' of the adjoints, in reverse order as follows from [23, p.73, §4.4]. Given that $\mathcal{A}$ and Υ^{-r} are densely defined operators on Hilbert spaces, if $\Upsilon^{-r}\mathcal{A}$ is densely defined then

$$\mathrm{D}(\mathcal{A}^* \left(\Upsilon^{-r}\right)^*) \subset \mathrm{D}(\left(\Upsilon^{-r}\mathcal{A}\right)^*) \tag{4.60}$$

and

$$\mathcal{A}^* \left(\Upsilon^{-r}\right)^* F = \left(\Upsilon^{-r}\mathcal{A}\right)^* F \quad \text{if} \quad F \in \mathrm{D}(\mathcal{A}^* \left(\Upsilon^{-r}\right)^*) \,, \tag{4.61}$$

where D(·) denotes the domain. If Υ^{-r} is a bounded operator then

$$\left(\Upsilon^{-r}\mathcal{A}\right)^* = \mathcal{A}^* \left(\Upsilon^{-r}\right)^* \,, \tag{4.62}$$

defined for all fields $F \in \mathrm{D}(\mathcal{A}^* \left(\Upsilon^{-r}\right)^*) = \mathrm{D}((\Upsilon^{-r}\mathcal{A})^*)$. The domain of $\mathcal{A}_{s,\lambda;r}$ is H^r_{r+1} and thus $\mathcal{A}_{s,\lambda;r}$ is densely defined. Thus Υ^{-r} is a bounded operator between $((\mathrm{L}^2)^2, (\cdot,\cdot)_{[0,0]})$ and $(\mathrm{H}^r_r, (\cdot,\cdot)_{[r,r]})$ and $\mathcal{A}_{s,\lambda;r}$ is densely defined, thus we have from the above that

$$\left(\Upsilon^{-r}\mathcal{A}_{s,\lambda}\Upsilon^{r}\right)^* = \left(\mathcal{A}_{s,\lambda}\Upsilon^{r}\right)^* \left(\Upsilon^{-r}\right)^* \supset \left(\Upsilon^{r}\right)^* \mathcal{A}^*_{s,\lambda} \left(\Upsilon^{-r}\right)^* \,. \tag{4.63}$$

We observe that the right-most operator, $\left(\Upsilon^{-r}\right)^*$, is the adjoint of a bounded operator between the Hilbert spaces $((\mathrm{L}^2)^2, (\cdot,\cdot)_{[0,0]})$ and $(\mathrm{H}^r_r, (\cdot,\cdot)_{[r,r]})$ and from [14, p.154, §III.3.3] it follows that it must be bounded. Furthermore, from the fact that Υ^r is bounded between $((\mathrm{L}^2)^2, (\cdot,\cdot)_{[0,0]})$ and $(\mathrm{H}^r_r, (\cdot,\cdot)_{[r,r]})$, it follows that Υ^{-r} is bounded from below. By [14] this implies that $(\Upsilon^{-r})^*$ is bounded from below and hence $(\Upsilon^{r})^*$ and $\left(\Upsilon^{-r}\right)^*$ are topological isomorphisms, by the same argument as above. Thus the right-most side of equation (4.63) is densely defined and since the adjoint, $\mathcal{A}^*_{s,\lambda;r}$, is an extension, it is also densely defined and hence $\mathcal{A}_{s,\lambda;r}$ is closable. Now we follow the same procedure as for $\mathcal{A}_{s,\lambda}$ to close $\mathcal{A}_{s,\lambda;r}$ and it will be bounded from below, and have dense range in $(\mathrm{H}^r_r, (\cdot,\cdot)_{[r,r]})$ for the same restriction on $\{s,\lambda\}$ as $\mathcal{A}_{s,\lambda}$. □

4.6. Decomposition of the spectrum. We derived the conditions which allow us to find bounds for the spectrum of the acoustic system's matrix. The constraint on the Laplace parameter is, as usual,

$$\mathrm{Re}\,\{s\} > 0 \,.$$

Given a Laplace parameter that satisfies the above constraint, there exists an upper bound for λ_{R} such that λ does not belong to the spectrum of $\mathcal{A}$. A sufficient condition is stated in Proposition 1:

$$\lambda^2_{\mathrm{R}} < (\mathrm{Re}\,\{s\})^2 \inf_{\boldsymbol{x}'\in\mathbb{R}^2} \frac{\kappa}{\alpha_{33}} \,. \tag{4.64}$$

Thus we can state the following corollary to Proposition 1.

COROLLARY 1.1. *For all $r \in \mathbb{N}_0$ and for any fixed $s \in \mathbb{C}$ such that $\mathrm{Re}\,\{s\} > 0$ the spectrum of the acoustic system's matrix is divided into two non-trivial parts separated by the strip of all $\lambda \in \mathbb{C}$ such that*

$$(\mathrm{Re}\,\{\lambda\})^2 < (\mathrm{Re}\,\{s\})^2 \inf_{\boldsymbol{x}'\in\mathbb{R}^2} \frac{\kappa}{\alpha_{33}} \,.$$

That is, the strip belongs to the resolvent set, $\mathrm{P}(\mathcal{A}_{;r})$.

Proof. The separation of the spectrum is a direct consequence of Proposition 1. To show that the spectrum is non-trivial consider the special case $r = 0$ and the point spectrum:

$$\begin{aligned} \|\mathcal{A}_{s,\lambda}F\|^2_{[0,0]} &= Y_0^{-1}\,\|(\mathcal{A}_{11}-\lambda)F_1 + \mathcal{A}_{12}F_2\|^2_0 + Y_0\,\|\mathcal{A}_{21}F_1 + (\mathcal{A}_{22}-\lambda)F_2\|^2_0 \\ &= 0 \,. \end{aligned} \tag{4.65}$$

Since both terms are positive, we can consider them separately. Writing out the equation $\mathcal{A}_{s,\lambda}F = 0$ in its components, expressing F_1 in terms of F_2 from the second equation as in (4.37)-(4.38) and substituting the result into the first equation give

$$s\mathcal{E}_{s,\lambda}F_2 = 0 \,, \tag{4.66}$$

$$\mathcal{A}_{21}F_1 + (\mathcal{A}_{22}-\lambda)F_2 = 0 \,. \tag{4.67}$$

Writing out the terms of the first equation yields (cf. (4.6))

$$\left(-\partial_\mu\, \alpha_{\mu\nu} \partial_\nu + \lambda\, (\partial_\mu\, \alpha_{3\mu} + \alpha_{3\nu} \partial_\nu) + \kappa s^2 - \alpha_{33} \lambda^2\right) F_2 = 0\ . \tag{4.68}$$

This is a strongly elliptic equation for fixed s and for a fixed λ such that $\mathrm{Im}\,\{\lambda\} = 0$; for $|\mathrm{Re}\,\{\lambda\}\,|$ large enough it has non-trivial solutions. This implies that there are both positive and negative values of λ_{R} that belong to the spectrum. Hence neither part of the spectrum is empty. For more details, consider the analogy presented in §4.4. For $r > 0$ we note that the above λ in the point spectrum of $\mathcal{A}_{s,\lambda}$ also belongs to the point spectrum of $\mathcal{A}_{s,\lambda;r}$. □

5. The $\mathcal{P}$ operator. From Corollary 1.1 we know that the spectrum of $\mathcal{A}$ separates into at least two parts. Since the operator is neither normal nor compact, a spectral reduction is difficult or impossible to obtain. But due to the splitting of the spectrum, we accomplish an 'up/down' decomposition of the acoustic system's matrix operator through the construction of a commuting operator (derived in the next section). To obtain this commuting operator, we introduce a non-orthogonal projector defined as a 'Dunford-Schwartz' type integral over the operator's resolvent along a path in the known part of the resolvent set. We prove that the integral is well defined in the sense of pseudodifferential operators with parameter, that it has the idempotent property (under some restrictions), and that it 'commutes' with $\mathcal{A}$ in a generalized sense: it is a projector. In appendices B and C we derive the principal symbol of the projector and show that it projects out the values of the spectrum for the principal part of $\mathcal{A}$ with positive real part.

Given a fixed positive constant $S_{\mathrm{R}} > 0$, let

$$\mathrm{Q}_1 = \Big\{ \{s, \lambda\} \in \mathbb{C}^2 : \mathrm{Re}\,\{s\} > S_{\mathrm{R}},\ |\arg s| < \pi/4 \ \text{ and } \ \mathrm{Re}\,\{\lambda\}^2 < S_{\mathrm{R}}^2 \inf_{\boldsymbol{x}' \in \mathbb{R}^2} \frac{\kappa}{\alpha_{33}} \Big\}\ . \tag{5.1}$$

From Proposition 1 we note that for $\mathrm{Re}\,\{s\} > S_{\mathrm{R}}$, $\mathrm{Q}_1 \subset \mathrm{Q}$, whence by Corollary 1.1 the spectrum of the acoustic system's matrix is decomposed into two parts. We consider the following family of operators, $\mathcal{P}_{;r}(a, s)$, in analogy with the Dunford-Schwartz integrals over operators with discrete spectrum:

$$\mathcal{P}_{;r}(a, s) = \left(a + \frac{1}{2}\right) I + \lim_{n\to\infty} \frac{1}{2\pi \mathrm{i}} \int_{\lambda \in \mathrm{K}_n} \mathrm{d}\lambda\ \left(\{\mathcal{A}_{;r}\}_{\mathrm{cl}} - \lambda\right)^{-1}\ , \tag{5.2}$$

where a is a free parameter. For the special case of $r = 0$ we use the notation $\mathcal{P}$. In §5.5 the value of a for which $\mathcal{P}_{;r}$ is a projector will be derived. The path of integration is parameterized as

$$\mathrm{K}_n = \{\lambda \in \mathbb{C} : \mathrm{Re}\,\{\lambda\} = \tau/2, |\lambda_{\mathrm{I}}| \le n\}\ , \tag{5.3}$$

where

$$\tau = S_{\mathrm{R}} \inf_{\boldsymbol{x}' \in \mathbb{R}^2} \sqrt{\frac{\kappa}{\alpha_{33}}}\ , \tag{5.4}$$

and $S_{\mathrm{R}} > 0$ is the real positive constant defining Q_1. For further convenience, we will use the notation

$$\mathrm{K} = \lim_{n\to\infty} \mathrm{K}_n\ , \tag{5.5}$$

when the limit procedure is not of fundamental importance. If s satisfies appropriate constraints (cf. (5.1)), we have $\{s, \mathrm{K}_n\} \subset \mathrm{Q}_1 \subset \mathrm{Q}$ for each n. Then $\mathrm{K}_n \subset \mathrm{P}(\mathcal{A}_{;r})$, the resolvent set, and hence $\mathcal{A}_{s,\lambda}^{-1}$ is analytic in $\lambda \in \mathrm{K}_n$ (cf. Proposition 1 and Corollary 1.1).

PROPOSITION 2. *Let us assume* $\{s, \mathrm{K}_n\} \in \mathrm{Q}_1$ *and let* $\mathcal{P}_{;r}$ *be defined as in (5.2). Then* $\mathcal{P} : \mathrm{H}_0^0 \to (\mathrm{H}_0^0, (\cdot,\cdot)_{[0,0]})$

1. *is a pseudodifferential operator with parameter of order 0;*
2. *has a restriction,* $\mathcal{P}|_0$, *witch maps* H^0_1 *into* H^0_1*;*
3. *'commutes' with* $\mathcal{A}$ *in the sense that*

$$\mathcal{P}|_0\,\mathcal{A} = \mathcal{A}\,\mathcal{P}|_1 \ ,$$

where $\mathcal{P}|_1$ *is the restriction of* $\mathcal{P}$ *to an operator with domain* H^1_2, $\mathcal{P}|_1(\mathrm{H}^1_2) \subset \mathrm{H}^1_2$*;*

4. *has a restriction,* $\mathcal{P}|_0$, *that is idempotent if and only if* $a = 0$.
5. *The properties above hold for* $r \in \mathbb{N}_0$ *with* $\mathcal{P}_{;r}$ *defined on* $(\mathrm{H}^r_r, (\cdot,\cdot)_{[r,r]})$ *with restrictions in parts 1-4 to domains* H^{r+q}_{r+q+1}, $q = 0, 1$.

REMARK 2.1. *The set* Q_1 *restricting* $\{s, \mathrm{K}_n\}$ *is not maximal. The maximal argument of* s *depends on all occurring variables including* λ *and coefficients* κ, α. *For the particular cases of* $\lambda_\mathrm{R} = 0$ *or* $\lambda_\mathrm{I} \to \infty$ *we obtain the maximal argument as a sector in the complex plane centered around the real* s *axis, that is, any* $\epsilon > 0$ *such that* $|\arg s| < \pi/2 - \epsilon$, *is allowed.*

REMARK 2.2. *Analogous to part 5 of Proposition 2, (cf. §5.6) the extension from a densely defined operator on* $(\mathrm{H}^0_0, (\cdot,\cdot)_{[0,0]})$ *to* $(\mathrm{H}^r_r, (\cdot,\cdot)_{[r,r]})$ *is straightforward. For this reason, there is no need to separate the operators with respect to the embedding Hilbert space, and therefore we omit the index* $;r$ *from §5.7 and onwards.*

5.1. Spectral reflection in the circle. The use of identity (cf. [9, pp.599-604, §VII.9])

$$(\{\mathcal{A}\}_\mathrm{cl} - I\lambda)^{-1} = -\lambda^{-2}(\{\mathcal{A}\}_\mathrm{cl}^{-1} - \lambda^{-1}I)^{-1} - \lambda^{-1}I \ , \tag{5.6}$$

will simplify the later proof that $\mathcal{P}$ is idempotent. Note that since λI is a bounded operator, we have $\{\mathcal{A}_{s,\lambda}\}_\mathrm{cl} = \{\mathcal{A}\}_\mathrm{cl} - \lambda I$. Substituting this back into (5.2) gives, with the change of variables $\varphi = \tau\lambda^{-1}$, and $\mathrm{d}\lambda = -\tau\varphi^{-2}\mathrm{d}\varphi$ where τ is defined in (5.4),

$$\mathcal{P}(a, s) = \left(a + \frac{1}{2}\right) I + \lim_{b\to 0} \frac{1}{2\pi\mathrm{i}} \int_{\varphi\in\mathrm{C}(b,1)} \left(\mathcal{R}_\varphi + \varphi^{-1} I\right) \mathrm{d}\varphi \ . \tag{5.7}$$

Here,

$$\mathcal{R}_\varphi = \left(\tau\,\{\mathcal{A}\}_\mathrm{cl}^{-1} - \varphi I\right)^{-1} \ . \tag{5.8}$$

The curve K_n in (5.2) is transformed into the curve

$$\begin{aligned} \mathrm{C}(b, \rho) = \{\sigma \in \mathbb{C} :\ & \sigma = \rho + \rho \exp(-\mathrm{i}\theta), \\ & \theta \in [-\pi + b, \pi - b],\ \text{fixed } \rho > 1/2\} \ . \end{aligned} \tag{5.9}$$

The freedom in choosing ρ in the above curve, due to the analyticity of $\{\mathcal{A}_{s,\lambda}\}_\mathrm{cl}^{-1}$ with respect to λ, is used in the section below, where it is established that the operator $\mathcal{P}$ is idempotent.

By a modification of Lemma 2 [9, p.600, §VII.9] we find that the spectrum of $\tau\,\{\mathcal{A}\}_\mathrm{cl}^{-1}$ is the set $\Phi(\Sigma\,\{\mathcal{A}\}_\mathrm{cl}) \cup \{\infty\}$, where $\Phi(z) = \tau z^{-1}$ and $\Sigma(\{\mathcal{A}\}_\mathrm{cl})$ denotes the spectrum of $\{\mathcal{A}\}_\mathrm{cl}$. Thus the spectrum of $\tau\,\{\mathcal{A}\}_\mathrm{cl}^{-1}$ is the original spectrum reflected in a scaled circle (see Figure 5.1).

The above expression for $\mathcal{P}$ is simplified by carrying out the integration over $\varphi^{-1}I$ (cf. (5.7))

$$\frac{1}{2\pi\mathrm{i}} \int_{\varphi\in\mathrm{C}(b,1)} \varphi^{-1}\mathrm{d}\varphi = \frac{-1}{2\pi} \int_{-\pi+b}^{\pi-b} \frac{\exp(-\mathrm{i}\theta)\mathrm{d}\theta}{1 + \exp(-\mathrm{i}\theta)} = \frac{-1}{4\pi} \int_{-\pi+b}^{\pi-b} \frac{\exp(-\mathrm{i}\theta/2)\mathrm{d}\theta}{\cos(\theta/2)}$$

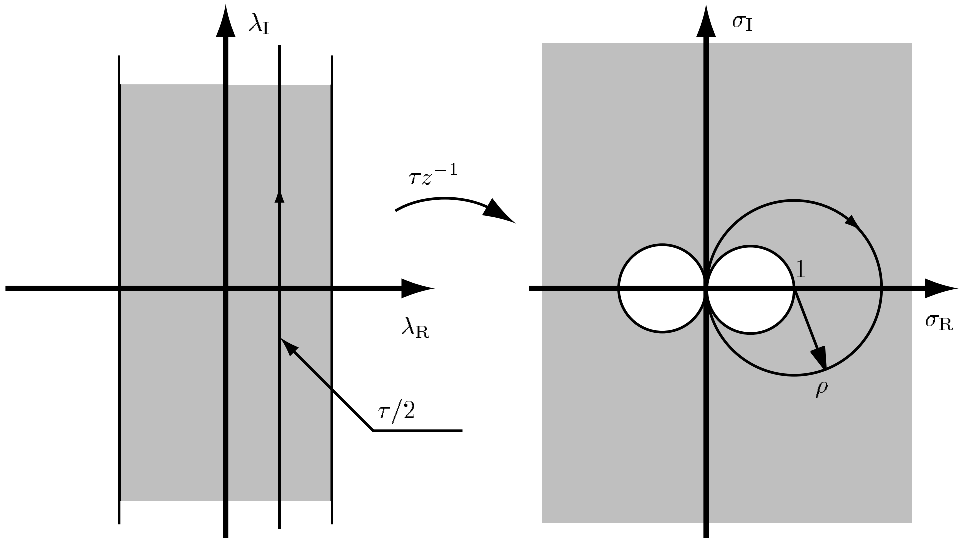

FIG. 5.1. *The integration path* K_n *is marked with an arrow in the left figure, under the transformation* $z \to \tau/z$ *the integration path is deformed into a circle segment with arrow,* $\mathrm{C}(b,\rho)$ *(right figure). The spectrum,* $\Sigma(\mathcal{A})$*, is contained in the white areas of the respective figures.*

$$
\begin{aligned}
&= \frac{-1}{4\pi}\int_{-\pi+b}^{\pi-b} 1 - \mathrm{i}\tan(\theta/2)\mathrm{d}\theta \\
&= -\frac{1}{2} + \frac{b}{2\pi} + 2\mathrm{i}\ln\left|\frac{\cos(\pi/2-b/2)}{\cos(-\pi/2+b/2)}\right| = -\frac{1}{2}+\frac{b}{2\pi}\,. \qquad (5.10)
\end{aligned}
$$

The right-hand side of the above equality does not have a singularity at $b=0$ and hence we can take the limit $b\to 0$ and obtain $-1/2$. Note that the residue of φ^{-1} at zero is 1 and, hence, the integral is minus one half of the residue, as expected. Inserting this into the above expression, yields

$$
\mathcal{P}(a,s) = aI + \lim_{b\to 0}\frac{1}{2\pi\mathrm{i}}\int_{\mathrm{C}(b,1)} \mathrm{d}\varphi\ \mathcal{R}_\varphi\,. \qquad (5.11)
$$

We have obtained a second representation of $\mathcal{P}$. We will proceed with deriving the properties of $\mathcal{P}$ and showing that this operator can be used as an alternative to the classical orthogonal projector for our analysis.

5.2. Proof of Proposition 2, part 1. The operator $\mathcal{P}$ is defined in terms of an improper integral over λ and hence we will have to show that the integral exist pointwise, that is when applied to $F\in \mathrm{H}^0_1$. To show this, we make use of the calculus of pseudodifferential operators and show that the operator exists as a pseudodifferential operator with parameter of order 0. The procedure is to first construct the parametrix for the scalar operator $\mathcal{E}_{s,\lambda}$ and then use this to find the parametrix of the acoustic system's matrix (cf. eq. (4.9)).

5.2.1. The parametrix of $\mathcal{E}_{s,\lambda}$. Pseudodifferential operators are introduced by means of a Fourier transform. For simplicity we use the standard notation for the symbols and their compositions (cf. [13, p.71, §18.1]). The Fourier transform, $\mathcal{F}$, in the plane with respect to the first two variables $\boldsymbol{x}'=\{x_1,x_2\}$

($\boldsymbol{x} = \{\boldsymbol{x}', x_3\}$) has an inverse given by

$$F_1(\boldsymbol{x}, s) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \mathrm{d}^2\xi' \; (\mathcal{F}F_1)(\xi', x_3, s) \exp\left(\mathrm{i}\langle \xi', \boldsymbol{x}'\rangle\right) \tag{5.12}$$

for the complex field $F_1 \in (\mathrm{L}^2, (\cdot,\cdot)_0)$. All pseudodifferential operators in the present paper have parameters. We calculate the symbol of $\mathcal{E}_{s,\lambda}$, denoted by $\boldsymbol{e}$, by letting $\mathcal{E}_{s,\lambda}$ act on $\exp\left(\mathrm{i}\langle \xi', \boldsymbol{x}'\rangle\right)$. We obtain

$$\begin{aligned} \boldsymbol{e}(\boldsymbol{x}, \xi'; s, \lambda) = \alpha_{\mu\nu}\xi_\mu\xi_\nu + 2\mathrm{i}\lambda\alpha_{3\mu}\xi_\mu - \lambda^2\alpha_{33} + s^2\kappa \\ -\mathrm{i}\partial_\mu\alpha_{\mu\nu}\,\xi_\nu + \lambda\,\partial_\mu\alpha_{3\mu}\;. \end{aligned} \tag{5.13}$$

We note that partial differential operators with smooth coefficients are properly supported so are such operators with parameters by a uniformity argument (cf. [19, p.18, §I.3.1 and p.74, §II.9.1]).

To ensure that the parametrix exists, we assume that the coefficients are arbitrarily smooth. We note that both $\mathcal{E}_{s,\lambda}$ and $\mathcal{A}_{s,\lambda}$ are classical (pseudo)differential operators with parameters. Let $\boldsymbol{e}_2$ and $\boldsymbol{e}_1$ be defined by

$$\boldsymbol{e}_2(\boldsymbol{x}, \xi'; s, \lambda) = \alpha_{\mu\nu}\xi_\mu\xi_\nu + 2\mathrm{i}\lambda\alpha_{3\mu}\xi_\mu - \lambda^2\alpha_{33} + s^2\kappa \tag{5.14}$$

and

$$\boldsymbol{e}_1(\boldsymbol{x}, \xi'; s, \lambda) = -\mathrm{i}\partial_\mu\alpha_{\mu\nu}\,\xi_\nu + \lambda\,\partial_\mu\alpha_{3\mu}\;. \tag{5.15}$$

That is, $\boldsymbol{e} = \boldsymbol{e}_2 + \boldsymbol{e}_1$ The symbol $\boldsymbol{e}_2(\boldsymbol{x}, \xi'; s, \lambda)$ is homogeneous of degree 2 in (ξ', s, λ), that is

$$\boldsymbol{e}_2(\boldsymbol{x}, h\xi'; hs, h\lambda) = h^2\boldsymbol{e}_2(\boldsymbol{x}, \xi'; s, \lambda) \quad \text{for } h > 0\,,$$

while $\boldsymbol{e}_1$ is homogeneous of degree 1. This suggests that the time-Laplace parameter s, the transform of a time derivative, is considered to have homogeneity degree 1 to ensure the proper high-frequency behavior. Furthermore, following [19], we let λ have the homogeneity degree 1. Let the parameters s, λ belong to the set Q_1 defined in (5.1). The symbol allows the estimate

$$\left|\partial^\gamma_{\xi'}\partial^\beta_{\boldsymbol{x}'}\boldsymbol{e}(\boldsymbol{x}, \xi'; s, \lambda)\right| \le C_{\gamma,\beta}\left(1 + |\xi'| + |\lambda| + |s|\right)^{2-|\gamma|}\;. \tag{5.16}$$

Furthermore, $\boldsymbol{e}(\boldsymbol{x}, \xi'; s_0, \lambda_0) \in \mathrm{C}^\infty(\mathbb{R}^3 \times \mathbb{R}^2)$ for each fixed $\{s_0, \lambda_0\} \in \mathrm{Q}_1$ since we require the coefficients to be smooth. The symbol $\boldsymbol{e}$ allows an expansion in terms of integer degree of homogeneity and hence it is a *classical* symbol with parameters corresponding to a classical pseudodifferential operator with parameters (cf. [19, p.29, §I.3.7, p.76, §II.9.1]). Thus, $\boldsymbol{e}(\boldsymbol{x}, \xi'; s, \lambda)$ belongs to the class of symbols denoted by $\mathrm{CS}^2_{1,1}(\mathbb{R}^3 \times \mathbb{R}^2, \mathrm{Q}_1)$, a subset of the class $\mathrm{S}^2_{1,1}$. The two lower indices of the classes indicate the highest degree of homogeneity assigned to the parameters s and λ. An operator corresponding to a symbol in the class $\mathrm{CS}^m_{1,1}$ with integer m is denoted by $\mathrm{CL}^m_{1,1}$.

The operator $\mathcal{E}_{s,\lambda}$ is properly supported and belongs to the class $\mathrm{CS}^2_{1,1}$. To show that it is elliptic, we will establish that there exist constants C_3, C_4 and R such that

$$C_3\left(|\xi'| + |\lambda| + |s|\right)^2 \le |\boldsymbol{e}(\boldsymbol{x}, \xi'; s, \lambda)| \le C_4\left(|\xi'| + |\lambda| + |s|\right)^2\;, \tag{5.17}$$

for $|\xi'| + |\lambda| + |s| \ge R$ and $C_3 > 0$ (cf. [19, p.39, Definition 5.2, remark on p.75, §II.9.1]). The upper bound is clear from (5.16) for $\beta = \gamma = 0$ with $C_4 = 2C_{0,0}$ if $|\xi'| + |\lambda| + |s| > 1$. To obtain the lower bound, we use the notation introduced in (4.16) and (4.45), and consider

$$\begin{aligned} \mathrm{Re}\left\{\boldsymbol{e}(\boldsymbol{x}, \xi'; s, \lambda)\right\} = \alpha_{\mu\nu}\xi_\mu\xi_\nu + \alpha_{33}\lambda_\mathrm{I}^2 - 2\lambda_\mathrm{I}\alpha_{33}^{-1}\alpha_{3\mu}\xi_\mu - \alpha_{33}\lambda_\mathrm{R}^2 \\ + \kappa s_\mathrm{r}^2\cos 2\sigma + \lambda_\mathrm{R}\partial_\mu\alpha_{3\mu}\;. \end{aligned} \tag{5.18}$$

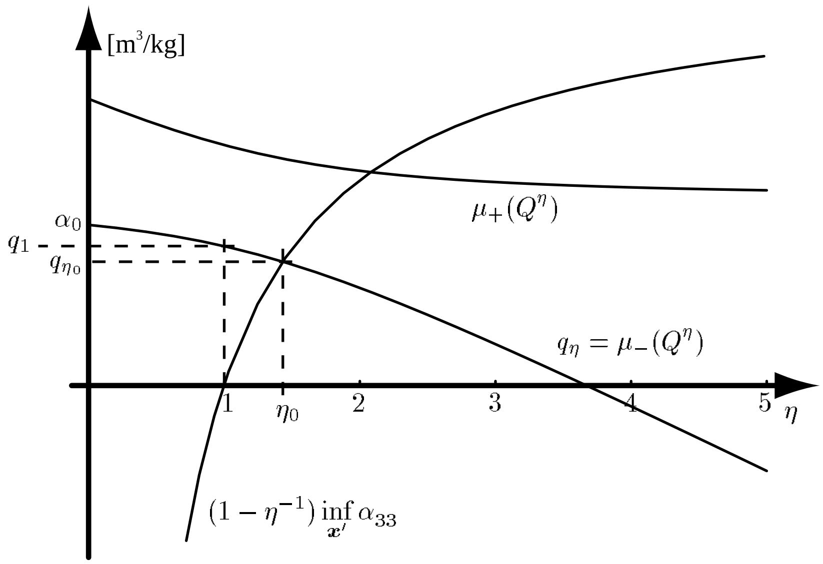

FIG. 5.2. *The optimal q_{η_0}, that is the solution of eq.(5.22) is marked in the figure, here $\mu^{\pm}(Q^{\eta})$ is infimum of the maximal (minimal) eigenvalue of the matrix $Q^{\eta}_{\mu\nu} = \alpha_{\mu\nu} - \eta\alpha_{\mu 3}\alpha_{33}^{-1}\alpha_{3\nu}$. The values for $\mu^{\pm}$ are calculated from an arbitrary fixed, symmetric, positive density tensor.*

With the inequality

(5.19) $$2\lambda_{\mathrm{I}}\alpha_{33}^{-1}\alpha_{3\mu}\xi_{\mu} \leq \eta^{-1}\alpha_{33}\lambda_{\mathrm{I}}^2 + \eta\alpha_{3\mu}\alpha_{33}^{-1}\alpha_{3\nu}\xi_{\mu}\xi_{\nu} \ ,$$

for $\eta > 0$, we obtain

(5.20) $$\begin{aligned}\operatorname{Re}\{\boldsymbol{e}(\boldsymbol{x},\xi';s,\lambda)\} \geq{}& (\alpha_{\mu\nu} - \eta\alpha_{3\mu}\alpha_{33}^{-1}\alpha_{3\nu})\xi_{\mu}\xi_{\nu} + \alpha_{33}(1-\eta^{-1})\lambda_{\mathrm{I}}^2 \\ &+ \kappa s_{\mathrm{r}}^2\cos 2\sigma - \alpha_{33}\lambda_{\mathrm{R}}^2 + \lambda_{\mathrm{R}}\partial_{\mu}\alpha_{3\mu} \ .\end{aligned}$$

Note that the first term contains a symmetric matrix that is positive definite for $\eta = 0$ with lower bound $q_{\eta=0} = \alpha_0$, and for $\eta = 1$ with lower bound $q_{\eta=1} = q_1$. The condition that the first term on the right-hand side of inequality (5.20) is positive definite is linearly dependent on the values of η, but since $q_1 > 0$ there exists an $\eta > 1$ such that $\alpha_{\mu\nu} - \eta\alpha_{3\mu}\alpha_{33}^{-1}\alpha_{3\nu}$ is still positive definite with corresponding lower bound q_{η}. Thus we obtain the following estimate,

(5.21) $$\begin{aligned}\operatorname{Re}\{\boldsymbol{e}(\boldsymbol{x},\xi';s,\lambda)\} \geq{}& q_{\eta}|\xi'|^2 + \alpha_{33}(1-\eta^{-1})\lambda_{\mathrm{I}}^2 \\ &+ \kappa\cos 2\sigma s_{\mathrm{r}}^2 - \alpha_{33}\lambda_{\mathrm{R}}^2 + \partial_{\mu}\alpha_{3\mu}\lambda_{\mathrm{R}} \ .\end{aligned}$$

To include the last two terms of (5.21), a constant that is a lower bound for the first three coefficients is needed. To obtain this bound it is necessary to fix η. Due to the smoothness of the parameters κ, α we can define η_0, (see figure 5.2), the fixed value of η that is the solution of the equation

(5.22) $$\left|\inf_{\boldsymbol{x}'} q_{\eta_0} - (1-\eta_0^{-1})\inf_{\boldsymbol{x}'}\alpha_{33}\right| = \min_{\eta>0,q_{\eta}>0}\left|\inf_{\boldsymbol{x}'} q_{\eta} - (1-\eta^{-1})\inf_{\boldsymbol{x}'}\alpha_{33}\right| \ .$$

Let

(5.23) $$C_{\eta_0} = \inf_{\boldsymbol{x}'}\{q_{\eta_0}, \alpha_{33}(1-\eta_0^{-1}), \kappa\cos 2\sigma\} \ .$$

Observe that in the conditions in the proposition we have stated that $|\sigma| < \pi/4$ and hence $\cos 2\sigma > 0$ so that $C_{\eta_0} > 0$. To obtain the estimate

$$\mathrm{Re}\,\{\boldsymbol{e}(\boldsymbol{x},\xi';s,\lambda)\} > 3C_3(|\xi'|^2+|\lambda|^2+|s|^2) \geq C_3(|\xi'|+|\lambda|+|s|)^2\ , \tag{5.24}$$

for some R such that $|\xi'|+|\lambda|+|s| > R$ we consider the worst case of (5.21). We then obtain the inequality

$$\begin{aligned} C_{\eta_0}(|\xi'|^2+|\lambda|^2+|s|^2) - \left(C_{\eta_0} + \sup_{\boldsymbol{x}'}|\alpha_{33}|\right)\tau^2 - \sup_{\boldsymbol{x}'}|\partial_\mu\alpha_{3\mu}|\,\tau \\ \geq 3C_3(|\xi'|^2+|\lambda|^2+|s|^2)\ , \end{aligned} \tag{5.25}$$

where we have used that $|\lambda_{\mathrm{R}}| < \tau$ and that both α_{33} and $\partial_\mu\alpha_{3\mu}$ are bounded. Rewriting (5.25) enables us to obtain a constant R_0: we find that for any $C_3 < C_{\eta_0}/3$ and

$$\begin{aligned} (|\xi'|+|\lambda|+|s|)^2 &\geq (|\xi'|^2+|\lambda|^2+|s|^2) \\ &\geq \frac{(C_{\eta_0}+\sup_{\boldsymbol{x}'}|\alpha_{33}|)\,\tau^2 + \sup_{\boldsymbol{x}'}|\alpha_{3\mu,\mu}|\,\tau}{C_{\eta_0}-3C_3} = R_0^2\ , \end{aligned} \tag{5.26}$$

hence for $R = \max(1, R_0)$, (5.24) is satisfied. Now, since

$$|\boldsymbol{e}(\boldsymbol{x},\xi';s,\lambda)| \geq |\mathrm{Re}\,\{\boldsymbol{e}(\boldsymbol{x},\xi';s,\lambda)\}\,| \geq C_3(|\xi'|+|\lambda|+|s|)^2 \tag{5.27}$$

for $C_3 < C_{\eta_0}/3$ and for the given R, we have shown that $\boldsymbol{e}(\boldsymbol{x},\xi';s,\lambda)$ is elliptic with parameters of order 2. Observe that through the definition of a fixed s-independent τ we obtained that the constants R, C_3 and C_4 are independent of s, λ, ξ and $\boldsymbol{x}$.

Since $\mathcal{E}_{s,\lambda}$ is elliptic with parameters its parametrix exists and is of order -2. To derive the parametrix of the operator $\mathcal{E}_{s,\lambda}$ we consider the composition of an unknown operator, $\mathcal{G}_{s,\lambda}$ say, with $\mathcal{E}_{s,\lambda}$. We assume the symbol expansion for $\mathcal{G}_{s,\lambda}$,

$$\boldsymbol{g}(\boldsymbol{x},\xi';s,\lambda) = \sum_{m=2}^{\infty}\boldsymbol{g}_{-m}\ , \tag{5.28}$$

and require that $\mathcal{E}_{s,\lambda}\mathcal{G}_{s,\lambda} - I \in \mathrm{CL}^{-\infty}_{1;1}$. Such an operator is called a parametrix of $\mathcal{E}_{s,\lambda}$ and reduces to an expansion of the Green's function in the case of constant κ, α (cf. Appendix A). In the expansion each term $\boldsymbol{g}_{-m}$ is homogeneous of degree $-m$ in (ξ', s, λ). To derive the form of each term $\boldsymbol{g}_{-m}$, we use the composition of $\mathcal{E}_{s,\lambda}$ and $\mathcal{G}_{s,\lambda}$ and require that the symbol of the composition is 1. We expand the composition and collect terms of equal degree of homogeneity, and require that they match each other. The terms of different degrees of homogeneity lead to the following hierarchy of equalities

$$\boldsymbol{e}_2\boldsymbol{g}_{-2} = 1\ ; \quad \text{degree} \quad 0\ , \tag{5.29}$$

$$\boldsymbol{e}_2\boldsymbol{g}_{-3} + \boldsymbol{e}_1\boldsymbol{g}_{-2} + \sum_{|\beta|=1}\partial^\beta_{\xi'}\boldsymbol{e}_2\ D^\beta_{\boldsymbol{x}'}\boldsymbol{g}_{-2} = 0\ ; \quad \text{degree } -1\ , \tag{5.30}$$

$$\begin{aligned} \boldsymbol{e}_2\boldsymbol{g}_{-2-m} + \boldsymbol{e}_1\boldsymbol{g}_{-1-m} + \sum_{|\beta|=1}\Big(\partial^\beta_{\xi'}\boldsymbol{e}_2\ D^\beta_{\boldsymbol{x}'}\boldsymbol{g}_{-1-m} \\ + \partial^\beta_{\xi'}\boldsymbol{e}_1\ \partial^\beta_{\boldsymbol{x}'}\boldsymbol{g}_{-m}\Big) + \sum_{|\beta|=2}\frac{1}{\beta!}\partial^\beta_{\xi'}\boldsymbol{e}_2\ D^\beta_{\boldsymbol{x}'}\boldsymbol{g}_{-m} = 0\ ; \quad \text{degree } -m,\ m \geq 2\ , \end{aligned} \tag{5.31}$$

where

$$D_{\boldsymbol{x}'} = \frac{1}{\mathrm{i}}\partial_{\boldsymbol{x}'} .$$

Solving each equation for the term with highest degree of homogeneity yields

$$\boldsymbol{g}_{-2} = \boldsymbol{e}_2^{-1} , \tag{5.32}$$

$$\boldsymbol{g}_{-3} = -\boldsymbol{e}_2^{-2}\boldsymbol{e}_1 + \boldsymbol{e}_2^{-3} \sum_{|\beta|=1} \partial_{\xi'}^{\beta}\boldsymbol{e}_2\; D_{\boldsymbol{x}'}^{\beta}\boldsymbol{e}_2 , \tag{5.33}$$

and higher order terms are obtained by recursion. Observe that the parametrix of $\mathcal{E}_{s,\lambda}$ differs from $\mathcal{E}_{s,\lambda}^{-1}$ by a smooth operator. [1]

5.2.2. Pseudodifferential representation of $\mathcal{A}_{s,\lambda}^{-1}$. The next step is to find the parametrix of $\mathcal{A}_{s,\lambda}$. Using the concept of scalarization, the symbol of $\mathcal{A}_{s,\lambda}^{-1}$ is expressed as a composition of the symbol of $\mathcal{E}_{s,\lambda}^{-1}$ with the symbols of the partial differential operators occurring in $\mathcal{A}_{s,\lambda}^{-1}$ (cf. (4.9)). Let the symbol of the resolvent, $\mathcal{A}_{s,\lambda}^{-1}$, be denoted by $\boldsymbol{r}$. Since $\mathcal{A}_{s,\lambda}^{-1}$ is a composition of classical pseudodifferential operators with parameters it is also a classical pseudodifferential operator with parameters. The symbol $\boldsymbol{r}$ of the parametrix of $\mathcal{A}_{s,\lambda}$ attains the form

$$\boldsymbol{r} = \begin{pmatrix} \boldsymbol{r}_{11} & \boldsymbol{r}_{12} \\ \boldsymbol{r}_{21} & \boldsymbol{r}_{22} \end{pmatrix} . \tag{5.34}$$

Here the $(1,1)$ term of the above matrix becomes (cf. (4.9))

$$\boldsymbol{r}_{11} \sim (-\mathrm{i}\alpha_{3\mu}\xi_\mu + \lambda\alpha_{33})\boldsymbol{g} - \alpha_{3\mu}\partial_\mu \boldsymbol{g} , \tag{5.35}$$

upon collecting terms of equal degree of homogeneity we obtain

$$\begin{aligned} \boldsymbol{r}_{11} \sim\; & (-\mathrm{i}\alpha_{3\mu}\xi_\mu + \lambda\alpha_{33})\boldsymbol{g}_{-2} \\ & - \alpha_{3\mu}\partial_\mu \boldsymbol{g}_{-2} + (-\mathrm{i}\alpha_{3\mu}\xi_\mu + \lambda\alpha_{33})\boldsymbol{g}_{-3} \\ & + \text{terms of degree } -3 \text{ and lower} . \end{aligned} \tag{5.36}$$

The $(1,2)$ component becomes

$$\begin{aligned} \boldsymbol{r}_{12} \sim\; & s^{-1}\left(Q_{\mu\nu}\xi_\mu\xi_\nu + \kappa s^2\right)\alpha_{33}\boldsymbol{g}_{-2} \\ & - s^{-1}\partial_\mu\left(\alpha_{3\mu}(\alpha_{33}\lambda - \mathrm{i}\alpha_{3\nu}\xi_\nu)\boldsymbol{g}_{-2}\right) + s^{-1}\left(\alpha_{33}\lambda - \mathrm{i}\alpha_{3\mu}\xi_\mu\right) \\ & \times \Big(\left(\alpha_{33}\lambda - \mathrm{i}\alpha_{3\nu}\xi_\nu\right)\boldsymbol{g}_{-3} - \mathrm{i}\partial_{\boldsymbol{x}'}^{\beta}\left(\alpha_{33}\lambda - \mathrm{i}\alpha_{3\mu}\xi_\mu\right)\partial_{\xi'}^{\beta}\boldsymbol{g}_{-2}\Big) \\ & + \text{terms of degree } -3 \text{ and lower} . \end{aligned} \tag{5.37}$$

In the above expression we have used the abbreviated notation:

$$\partial_{\boldsymbol{x}'}^{\beta}\cdot\;\partial_{\xi'}^{\beta}\cdot\; = \sum_{|\beta|=1}\left[\partial_{x_1}^{\beta_1}\cdot\;\partial_{\xi_1}^{\beta_1}\cdot\; + \partial_{x_2}^{\beta_2}\cdot\;\partial_{\xi_2}^{\beta_2}\cdot\right] . \tag{5.38}$$

[1] To take care of this, a procedure corresponding to [18] would have to be used. Notice that [18] cannot be applied directly since our operator is not compactly supported. The set Q of Proposition 1 would need to be enhanced also.

The remaining components have the same structure and become

$$r_{21} \sim s g_{-2} + s g_{-3} + \text{terms of degree } -3 \text{ and lower} \tag{5.39}$$

and

$$\begin{aligned} r_{22} \sim {} & (-\mathrm{i}\alpha_{3\mu}\xi_\mu + \lambda\alpha_{33})\, g_{-2} \\ & + (-\mathrm{i}\alpha_{3\mu}\xi_\mu + \lambda\alpha_{33})\, g_{-3} - \alpha_{3\nu,\nu} g_{-2} \\ & - \mathrm{i}\partial^\beta_{x'} (-\mathrm{i}\alpha_{3\nu}\xi_\nu + \lambda\alpha_{33})\, \partial^\beta_{\xi'} g_{-2} + \text{terms of degree } -3 \text{ and lower}\,. \end{aligned} \tag{5.40}$$

Thus, the principal part of the symbol of $\mathcal{A}^{-1}_{s,\lambda}$ has the form (cf. (5.32))

$$r_{-1} = \begin{pmatrix} -\mathrm{i}\alpha_{33}^{-1}\alpha_{3\mu}\xi_\mu + \lambda & s\kappa + s^{-1} Q_{\mu\nu}\xi_\mu\xi_\nu \\ s\alpha_{33}^{-1} & -\mathrm{i}\alpha_{33}^{-1}\alpha_{3\mu}\xi_\mu + \lambda \end{pmatrix} \alpha_{33} e_2^{-1}\,, \tag{5.41}$$

where the index -1 indicates that the symbol is homogenous of degree -1.

5.2.3. $\mathcal{P}$ is a pseudodifferential operator with parameter of order zero. We have obtained a parametrix of $\mathcal{A}_{s,\lambda}$ and shown that it is a classical pseudodifferential operator with parameter. Here we show that $\mathcal{P}$ has an asymptotic expansion and that each term of the expansion is non-singular, hence $\mathcal{P}$ is well defined as a pseudodifferential operator with parameter s. Before considering each term of the asymptotic expansion, we calculate the principal part of its symbol, that is we consider the integral

$$\frac{1}{2\pi\mathrm{i}} \int_{\lambda\in\mathrm{K}} \mathrm{d}\lambda\; r_{-1}\,. \tag{5.42}$$

The integrand is a linear combination of e_2^{-1} and λe_2^{-1}. We find that

$$\frac{1}{2\pi\mathrm{i}} \int_{\lambda\in\mathrm{K}} \mathrm{d}\lambda\; e_2^{-1} = \frac{\alpha_{33}^{-1/2}}{2\sqrt{s^2\kappa + Q_{\mu\nu}\xi_\mu\xi_\nu}}\,. \tag{5.43}$$

The integral of λe_2^{-1} requires the introduction of a principal value viz.,

$$\begin{aligned} & \lim_{n\to\infty} \frac{1}{2\pi\mathrm{i}} \int_{\lambda\in\mathrm{K}_n} \mathrm{d}\lambda\; \lambda e_2^{-1}(x,\xi';s,\lambda) \\ & = \lim_{n\to\infty} \frac{1}{2\pi} \int_{0\le\lambda_\mathrm{I}\le n,\lambda_\mathrm{R}=\tau/2} \mathrm{d}\lambda_\mathrm{I}\; \left[\lambda e_2^{-1}(x,\xi';s,\lambda) + \bar\lambda e_2^{-1}(x,\xi';s,\bar\lambda)\right] \\ & = \alpha_{33}^{-1/2} \frac{\mathrm{i}\alpha_{33}^{-1}\alpha_{3\mu}\xi_\mu}{2\sqrt{s^2\kappa + Q_{\mu\nu}\xi_\mu\xi_\nu}}\,. \end{aligned} \tag{5.44}$$

Using (5.43), (5.44) in (5.42) with (5.41) yields the symbol p of $\mathcal{P}$,

$$\begin{aligned} p = {} & \left(a + \frac{1}{2}\right) I + \frac{s\sqrt{\alpha_{33}}}{2\sqrt{s^2\kappa + Q_{\mu\nu}\xi_\mu\xi_\nu}} \begin{pmatrix} 0 & \kappa + s^{-2} Q_{\mu\nu}\xi_\mu\xi_\nu \\ \alpha_{33}^{-1} & 0 \end{pmatrix} \\ & + \text{terms of lower degree}\,, \end{aligned} \tag{5.45}$$

where I is the unit matrix. The principal part is homogenous of degree 0. This is what we expect since after integration of a series of homogeneous terms, each term attains a homogeneity of one order higher.

To show that $\mathcal{P}$ is a pseudodifferential operator with parameter we have to analyze its symbol in more detail. Its symbol can be written as a polyhomogeneous expansion each term of which is obtained by integrating over λ a term of the polyhomogeneous expansion of the symbol, $\boldsymbol{r}$, associated with the parametrix of $\mathcal{A}_{s,\lambda}$. That is

$$\boldsymbol{p} = \left(a + \frac{1}{2}\right) + \frac{1}{2\pi \mathrm{i}} \int_{\lambda \in \mathrm{K}} \mathrm{d}\lambda\ \boldsymbol{r}\ . \tag{5.46}$$

We have to understand how degrees of homogeneity carry over from the second expansion to the first. We also have to ensure that each term in the first expansion is bounded. Note that all the elements in the polyhomogeneous expansion of the symbol $\boldsymbol{r}$ have a λ structure of the form

$$\lambda^m \boldsymbol{e}_2^{-n}\ ,$$

where $2n - m \geq 2$, apart from the principal part, due to the degree of homogeneity of each element in the polyhomogeneous expansion of $\boldsymbol{r}$. In §5.2.1, with Eq. (5.27), we showed that $\boldsymbol{e}_2 \neq 0$ for all $\{s, \lambda\} \in \mathrm{Q}_1$, whence

$$\lim_{|\lambda| \to \infty} \left| \frac{\lambda^{m+1}}{\boldsymbol{e}_2^n} \right| < \lim_{|\lambda| \to \infty} |\alpha_{33}^{-n} \lambda|^{-1} = 0\ . \tag{5.47}$$

But then we can apply the residue theorem to

$$\lim_{n \to \infty} \frac{1}{2\pi \mathrm{i}} \int_{\mathrm{K}_n} \mathrm{d}\lambda\ \lambda^m \boldsymbol{e}_2^{-n}\ .$$

That is, in view of (5.47) and the orientation of integration we have

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\lambda_{\mathrm{I}}\ \frac{\lambda^m}{\boldsymbol{e}_2^n} = -\mathrm{Res}\left\{ \frac{\lambda^m}{\boldsymbol{e}_2^n}; \lambda_+ \right\} \tag{5.48}$$

for $\mathrm{Re}\{\lambda\} < \tau$. Here $\lambda_\pm$ are the solutions to $\boldsymbol{e}_2 = 0$ where the sign corresponds to the sign of the real part,

$$\lambda_\pm = \mathrm{i}\alpha_{33}^{-1} \alpha_{3\mu} \xi_\mu \pm \alpha_{33}^{-1/2} \sqrt{s^2 \kappa + Q_{\mu\nu} \xi_\mu \xi_\nu}\ . \tag{5.49}$$

To calculate the above residue, we rewrite $\lambda^m \boldsymbol{e}_2^{-n}$ as a Laurent series about $\lambda = \lambda_+$ with the aid of the identity

$$\frac{(n-1)!}{(1-y)^n} = \frac{\mathrm{d}^{n-1}(1-y)^{-1}}{\mathrm{d}y^{n-1}} = \sum_{j=0}^{\infty} \frac{(n+j-1)!}{j!} y^j\ , \tag{5.50}$$

valid for $|y| < 1$ and $n \in \{1, 2, \ldots\}$. With the additional use of the binomial theorem, we obtain

$$\begin{aligned} \frac{\alpha_{33}^n \lambda^m}{\boldsymbol{e}_2^n} &= \sum_{p=0}^{m} \binom{m}{p} \frac{(\lambda - \lambda_+)^p \lambda_+^{m-p}}{(\lambda_+ - \lambda)^n (\lambda - \lambda_-)^n} \\ &= \sum_{p=0}^{m} \binom{m}{p} \frac{(\lambda - \lambda_+)^p \lambda_+^{m-p}}{(\lambda_+ - \lambda)^n (\lambda_+ - \lambda_-)^n} \left(1 - \frac{\lambda_+ - \lambda}{\lambda_+ - \lambda_-}\right)^{-n} \\ &= \sum_{p=0}^{m} \sum_{j=0}^{\infty} \binom{m}{p} \frac{(-1)^{n-j}(n+j-1)!(\lambda - \lambda_+)^{p+j-n} \lambda_+^{m-p}}{j!(n-1)!(\lambda_+ - \lambda_-)^{n+j}}\ . \end{aligned} \tag{5.51}$$

The residue at λ_+ is the coefficient to the term $(\lambda-\lambda_+)^{-1}$ in the above expression, hence

$$\operatorname{Res}\left\{\frac{\lambda^m}{e_2^n};\lambda_+\right\}=\sum_{p=0}^{n-1}\binom{m}{p}\binom{2n-p-2}{n-1}\frac{(-1)^{p+1}\lambda_+^{m-p}}{\alpha_{33}^n(\lambda_+-\lambda_-)^{2n-p-1}}\,. \tag{5.52}$$

But then the integral (in (5.48)) is homogenous of degree $m-2n+1$. Thus we have shown that the degree of homogeneity of the integrated expression (5.46) exceeds the degree of homogeneity of the original expression before integration, by one. Furthermore,

$$\begin{aligned}2\mathrm{Re}\{\lambda_+\}=\mathrm{Re}\{\lambda_+-\lambda_-\}&=2\alpha_{33}^{-1/2}\mathrm{Re}\left\{\sqrt{s^2\kappa+Q_{\mu\nu}\xi_\mu\xi_\nu}\right\}\\&\geq 2\mathrm{Re}\{s\}\sqrt{\kappa\alpha_{33}^{-1}}\geq 2\tau\end{aligned} \tag{5.53}$$

for $\mathrm{Re}\{s\}>S_{\mathrm{R}}$, and hence, due to the fact that the residue is a finite sum of bounded terms, the integral is bounded. Hence $\mathcal{P}$ allows a polyhomogeneous expansion of its symbol with well defined terms. The principal term has degree of homogeneity 0, so $\mathcal{P}$ must be a pseudodifferential operator with parameter of order 0. Note that there are pointwise convergence at (x,ξ',s) of the improper integral of each term of the polyhomogeneous expansion. □

5.3. Proof of Proposition 2, part 2. At fixed parameter s, for the operator $\mathcal{P}$, with associated principal symbol given by (5.45), $\mathrm{H}_1^0\subset\mathrm{D}(\mathcal{P})$. Furthermore,

$$\text{if } F\in\mathrm{H}_1^0 \text{ then } \mathcal{P}F\in\mathrm{H}_1^0\,, \tag{5.54}$$

as follows directly from (5.45). Note, however that $\mathcal{P}$ considered as an operator in$((\mathrm{L}^2)^2,(\cdot,\cdot)_{[0,0]})$ with domain restricted to H_1^0 is unbounded, see the remark in §2.2 about unbounded operators. The restriction of the domain to H_1^0 is denoted by $\mathcal{P}|_0$. □

5.4. Proof of Proposition 2, part 3. To consider the commutation of the two unbounded operators $\mathcal{A}$ and $\mathcal{P}$, the explicit domains of the operators are of fundamental importance. Using part 2 of Proposition 2 we restrict the operator $\mathcal{P}$ and denote the new operators by

$$\mathcal{P}|_q:\mathrm{H}_{q+1}^q\to\mathrm{H}_{q+1}^q\,, \tag{5.55}$$

where $q=0,1$. Here $\mathcal{P}|_q$ is densely defined on $(\mathrm{L}^2,(\cdot,\cdot)_0)$.

$\mathcal{A}$ is an operator in $((\mathrm{L}^2)^2,(\cdot,\cdot)_{[0,0]})$ with domain H_2^1 and range in H_1^0. We will prove commutation in the sense

$$\mathcal{P}|_0\,\mathcal{A}=\mathcal{A}\,\mathcal{P}|_1\,, \tag{5.56}$$

where the domains of the left- and right-hand sides match.

We have,

$$\mathcal{A}_{s,\lambda}^{-1}(\mathcal{A}-\lambda I)=I_1\,, \tag{5.57}$$

$$(\mathcal{A}-\lambda I)\mathcal{A}_{s,\lambda}^{-1}=I_0\,, \tag{5.58}$$

where $I_q,\ q=0,1$, is the identity mapping on $((\mathrm{L}^2)^2,(\cdot,\cdot)_{[0,0]})$ restricted to the domains H_{q+1}^q. We will show that

$$\mathcal{A}_{s,\lambda}^{-1}\mathcal{A}\subset\mathcal{A}\mathcal{A}_{s,\lambda}^{-1}\,, \tag{5.59}$$

where $\subset$ indicates that $\mathcal{A}\mathcal{A}_{s,\lambda}^{-1} = \mathcal{A}_{s,\lambda}^{-1}\mathcal{A}$ on the domain H_2^1 and that $\mathcal{A}\mathcal{A}_{s,\lambda}^{-1}$ has a domain that is a super set of H_2^1, here H_1^0. To prove (5.59), we rewrite (5.57) and (5.58) in the forms

$$\mathcal{A}_{s,\lambda}^{-1}\Big|_1 \mathcal{A} = I_1 + \lambda\, \mathcal{A}_{s,\lambda}^{-1}\Big|_1 , \tag{5.60}$$

$$\mathcal{A}\, \mathcal{A}_{s,\lambda}^{-1}\Big|_0 = I_0 + \lambda\, \mathcal{A}_{s,\lambda}^{-1}\Big|_0 , \tag{5.61}$$

where $\mathcal{A}_{s,\lambda}^{-1}\Big|_q$ is $\mathcal{A}_{s,\lambda}^{-1}$ with domain restricted to the Sobolev set H_{q+1}^q. We now observe that the right-hand side of (5.61) restricted to H_2^1 equals (5.60), and hence (5.59) is valid. We can write (5.59) in terms of the 'commutation'

$$\mathcal{A}_{s,\lambda}^{-1}\Big|_0 \mathcal{A} = \mathcal{A}\, \mathcal{A}_{s,\lambda}^{-1}\Big|_1 . \tag{5.62}$$

Note that we do not restrict $\mathcal{A}$ since we can allow $\mathcal{A}$ with the larger domain to act on any smaller domain, here the range of $\mathcal{A}_{s,\lambda}^{-1}\Big|_1$.

We now analyze whether $\mathcal{A}$ commutes with the integral over λ. We cannot apply the standard arguments of uniformly converging series directly due to the fact that it is operators that we study, but in weak form we have,

$$(\mathcal{A}_{s,\lambda}^{-1}\Big|_0 \mathcal{A}F, G)_0 = (\mathcal{A}\, \mathcal{A}_{s,\lambda}^{-1}\Big|_1 F, G)_0 , \tag{5.63}$$

where $F \in \mathrm{H}_2^1$ and for all $G \in (\mathrm{S}(\mathbb{R}^2))^2$, where S denotes the Schwartz space. Note in particular that $(\mathrm{C}_0^\infty)^2 \subset \mathrm{S}^2 \subset (\mathrm{C}^\infty)^2 \subset \mathrm{D}(\mathcal{A}_{s,\lambda}^*)$ and hence S^2 is dense in $((\mathrm{L}^2)^2, (\cdot,\cdot)_{[0,0]})$. Integration of the right-hand side of (5.63) and addition of the term

$$\left(a + \frac{1}{2}\right)(\mathcal{A}F, G)_0 \tag{5.64}$$

and with the substitution $\mathcal{A}F = F' \subset \mathrm{H}_1^0$ gives

$$\begin{aligned}\left(a + \frac{1}{2}\right)(F', G)_0 &+ \frac{1}{2\pi \mathrm{i}} \int_{\lambda\in\mathrm{K}} \mathrm{d}\lambda\; (\mathcal{A}_{s,\lambda}^{-1}\Big|_0 F', G)_0 \\ &= \left(a + \frac{1}{2}\right)(F', G)_0 + \frac{1}{2\pi \mathrm{i}}(\int_{\lambda\in\mathrm{K}} \mathrm{d}\lambda\; \mathcal{A}_{s,\lambda}^{-1}\Big|_0 F', G)_0 \\ &= (\mathcal{P}|_0 F', G)_0 = (\mathcal{P}|_0 \mathcal{A}F, G)_0 ,\end{aligned} \tag{5.65}$$

for all $G \in \mathrm{S}^2$. The interchange of the order of integration with respect to λ and $\boldsymbol{x}'$ is valid due to the Fubini-Tonelli theorem [25, p.18, §0.3] and that

$$|(\mathcal{P}|_0 F', G)_0| \le C'(F', s)\, \|G\|_0 , \tag{5.66}$$

where $C'(F', s) > 0$, since $\mathcal{P}$ is well defined on H_1^0, that is we have a pointwise convergence. The implicit interchange of the limit $\lim_{n\to\infty} \mathrm{K}_n = \mathrm{K}$ and the integral in the scalar product, is valid in the weak sense, due to the pointwise convergence of each of the terms in the polyhomogeneous expansion (cf. end of §5.2) and that the choice of $G \in \mathrm{S}^2$ ensure the absolute integrability of each term in the pseudodifferential representation of $\mathcal{P}$. Observe that integration of $\mathcal{A}_{s,\lambda}^{-1}\Big|_q$ with respect to λ changes the range to yield an operator between

H^q_{q+1} and H^q_{q+1} (cf. Proposition 2, part 2). The integral of the left-hand side of (5.63) with the addition of the term (5.64) becomes

$$(a+\frac{1}{2})(\mathcal{A}F,G)_0+\frac{1}{2\pi\mathrm{i}}\int_{\lambda\in\mathrm{K}}\mathrm{d}\lambda\ (\mathcal{A}\,\mathcal{A}^{-1}_{s,\lambda}\Big|_1 F,G)_0$$
$$=(a+\frac{1}{2})(F,\mathcal{A}^*G)_0+\frac{1}{2\pi\mathrm{i}}\int_{\lambda\in\mathrm{K}}\mathrm{d}\lambda\ (\mathcal{A}^{-1}_{s,\lambda}\Big|_1 F,\mathcal{A}^*G)_0$$
$$=(a+\frac{1}{2})(F,G^*)_0+\frac{1}{2\pi\mathrm{i}}\int_{\lambda\in\mathrm{K}}\mathrm{d}\lambda\ (\mathcal{A}^{-1}_{s,\lambda}\Big|_1 F,G^*)_0$$
$$=(\mathcal{P}|_1 F,G^*)_0=(\mathcal{A}\,\mathcal{P}|_1 F,G)_0 \tag{5.67}$$

where $G^*=\mathcal{A}^*G\subset\mathrm{S}^2$. The interchange of integration order is valid analogously to (5.65) and the use of the adjoint is valid due to the given sets of functions for F,G. Thus by (5.63) we obtain

$$(\mathcal{P}|_0\,\mathcal{A}F,G)_0=(\mathcal{A}\,\mathcal{P}|_1 F,G)_0 \tag{5.68}$$

for all G in a dense set of $((\mathrm{L}^2)^2,(\cdot,\cdot)_{[0,0]})$ hence

$$\mathcal{P}|_0\,\mathcal{A}F=\mathcal{A}\,\mathcal{P}|_1 F \tag{5.69}$$

for any $F\in\mathrm{H}^1_2$. That is

$$\mathcal{P}|_0\,\mathcal{A}=\mathcal{A}\,\mathcal{P}|_1 \tag{5.70}$$

with both sides considered as operators with domain H^1_2. Thus, we have established the commutation of $\mathcal{A}$ and $\mathcal{P}$ in the above sense. □

5.5. Proof of Proposition 2, part 4. Given the operator $\mathcal{P}$ in the second representation (cf. (5.11)), we will study the operator $\mathcal{M}$ defined as the composition

$$\mathcal{M}_{\rho,\rho'}(a,s)=\mathcal{P}_\rho(a,s)\mathcal{P}_{\rho'}(a,s)$$
$$=\lim_{b\to0}\lim_{b'\to0}\left(aI+\frac{1}{2\pi\mathrm{i}}\int_{\varphi\in\mathrm{C}(b,\rho)}\mathrm{d}\varphi\ \mathcal{R}_\varphi\right)\left(aI+\frac{1}{2\pi\mathrm{i}}\int_{\varsigma\in\mathrm{C}(b',\rho')}\mathrm{d}\varsigma\ \mathcal{R}_\varsigma\right)\ , \tag{5.71}$$

where $\rho>\rho'=1$ by choice. Here, the intermediate notation $\mathcal{P}_\rho$ is used to indicate the radius of integration path over the resolvent. Furthermore, in this subsection we use the notation $\mathcal{R}_\varphi$ for $(\tau\{\mathcal{A}\}^{-1}_{\mathrm{cl}}-\varphi I)^{-1}$ and $\mathcal{R}_\varsigma$ for $(\tau\{\mathcal{A}\}^{-1}_{\mathrm{cl}}-\varsigma I)^{-1}$. Carrying out the composition gives

$$\mathcal{M}_{\rho,\rho'}=a^2I+\frac{a}{2\pi\mathrm{i}}\lim_{b\to0}\left(\int_{\varphi\in\mathrm{C}(b,\rho)}\mathrm{d}\varphi\ \mathcal{R}_\varphi+\int_{\varsigma\in\mathrm{C}(b,\rho')}\mathrm{d}\varsigma\ \mathcal{R}_\varsigma\right)$$
$$+\lim_{b\to0,b'\to0}\mathcal{J}_s(b,\rho,b',\rho')\ , \tag{5.72}$$

where

$$\mathcal{J}_s(b,\rho,b',\rho')=\frac{1}{(2\pi\mathrm{i})^2}\int_{\varphi\in\mathrm{C}(b,\rho)}\mathrm{d}\varphi\ \int_{\varsigma\in\mathrm{C}(b',\rho')}\mathrm{d}\varsigma\ \mathcal{R}_\varphi\mathcal{R}_\varsigma\ . \tag{5.73}$$

The resolvent equation,

$$\mathcal{R}_\varphi \mathcal{R}_\varsigma = \frac{\mathcal{R}_\varsigma - \mathcal{R}_\varphi}{\varsigma - \varphi} \tag{5.74}$$

inserted into the expression for $\mathcal{J}_s(b,\rho,b',\rho')$ gives

$$\mathcal{J}_s(b,\rho,b',\rho') = \frac{1}{(2\pi\mathrm{i})^2} \int_{\varphi\in\mathrm{C}(b,\rho)} \mathrm{d}\varphi \int_{\varsigma\in\mathrm{C}(b',\rho')} \mathrm{d}\varsigma\, \frac{\mathcal{R}_\varsigma - \mathcal{R}_\varphi}{\varsigma - \varphi}\,. \tag{5.75}$$

By the Fubini-Tonelli theorem [25, p.18, §0.3] we note that the repeated integral in (5.73) is defined, since they separately converge by §5.2. Hence, interchange of the integration order is allowed and we obtain

$$\begin{aligned} \mathcal{J}_s(b,\rho,b',\rho') = \frac{1}{(2\pi\mathrm{i})^2} \Bigg(&\int_{\varsigma\in\mathrm{C}(b',\rho')} \mathrm{d}\varsigma\, \mathcal{R}_\varsigma \int_{\varphi\in\mathrm{C}(b,\rho)} \mathrm{d}\varphi\, \frac{1}{\varsigma-\varphi} \\ &+ \int_{\varphi\in\mathrm{C}(b,\rho)} \mathrm{d}\varphi\, \mathcal{R}_\varphi \int_{\varsigma\in\mathrm{C}(b',\rho')} \mathrm{d}\varsigma\, \frac{1}{\varphi-\varsigma} \Bigg)\,. \end{aligned} \tag{5.76}$$

For the interchange of limits and integrals we use Lebesgue's dominated convergence theorem. We first note that the last integral in both terms can be dominated for each fixed nonzero b, nonzero b' respectively: for each fixed ς we have

$$\max_{\varphi\in\mathrm{C}(b,\rho)} \frac{2\pi}{|\varsigma-\varphi|} < \infty \tag{5.77}$$

since, $\varsigma \neq \varphi$ for $(b,b') \neq (0,0)$ (see figure 5.3); a similar estimate holds if the roles of ς and φ are interchanged. Furthermore, with help of (5.6) we find an upper bound for $\mathcal{R}_\varsigma$:

$$\begin{aligned} \|\mathcal{R}_\varsigma\|_{[0,0]} &= |\varsigma^2\tau|^{-1} \left\| \left(\{\mathcal{A}\}_{\mathrm{cl}} - \tau\varsigma^{-1}\right)^{-1} + \varsigma\tau^{-1} I \right\|_{[0,0]} \\ &\le |\varsigma^2\tau|^{-1} \left(C_2^{-1}(\tau\varsigma^{-1}, s; r) + |\varsigma\tau^{-1}| \right) \end{aligned} \tag{5.78}$$

by (4.31) of Proposition 1, part 1, since $\{s, \tau\varsigma^{-1}\} \in \mathrm{Q}$ and hence $C_2(\tau\varsigma^{-1}, s; r) > 0$. Hence the right-hand side of (5.78) is bounded for any fixed $\varsigma \in \mathrm{C}(b',\rho')$ and $b' \neq 0$. Thus the first term on the right-hand side of (5.76) exists as an operator on functions for $b' \neq 0$ and interchange of this integral with the limit $b \to 0$ is valid by Lebesgue's dominated convergence theorem. Analogous arguments apply for $\mathcal{R}_\varphi$ and the second term. Note that the limit need not exist for the outer integral for interchange of the first limit and integral to be a allowed. Upon interchanging the respective limits we obtain the residue of the respective integrands. We get

$$\lim_{b\to 0, b'\to 0} \mathcal{J}_s(b,\rho,b',\rho') = \lim_{b\to 0} \frac{1}{2\pi\mathrm{i}} \int_{\varphi\in\mathrm{C}(b,\rho')} \mathrm{d}\varsigma\, \mathcal{R}_\varsigma\,. \tag{5.79}$$

Inserting this result into $\mathcal{M}$ yields

$$\mathcal{M}_{\rho,\rho'}(a,s) = a^2 I + \lim_{b\to 0} \frac{1}{2\pi\mathrm{i}} \left(a \int_{\varphi\in\mathrm{C}(b,\rho)} \mathrm{d}\varsigma\, \mathcal{R}_\varsigma + (a+1) \int_{\varphi\in\mathrm{C}(b,\rho')} \mathrm{d}\varsigma\, \mathcal{R}_\varsigma \right)\,. \tag{5.80}$$

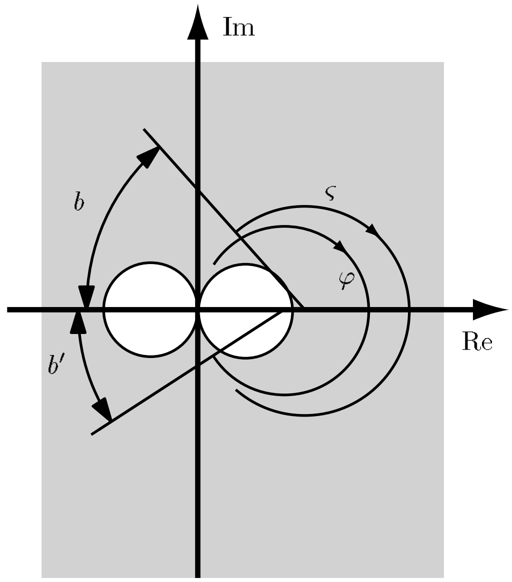

FIG. 5.3. *Integration paths for* φ *and* ς *respectively.*

If we let $\rho \to \rho'$ and choose $a = 0$, we obtain

$$\lim_{\rho \to \rho'} \mathcal{M}_{\rho,\rho'} = \mathcal{P}_{\rho'}(0, s) = \mathcal{P}(0, s) \tag{5.81}$$

and hence $\mathcal{P}$ is idempotent for the above choice of a.

Note that $\mathcal{R}_\varphi$ is analytic in φ (since $\{\mathcal{A}\}_{\mathrm{cl}}$ is closed [14, p.165, §III.5.2]). Hence, $\mathcal{M}_{\rho,\rho'}$ does not depend on the precise path $\mathrm{C}(b, \rho)$ but on the endpoints only. That is, $\mathcal{P}$ is independent of the path and since the limit closes the curve, the endpoints move to the origo in and all curves with $\rho > 1/2$ close in the same point. All curves in representation (5.71) give the same $\mathcal{P}$ and so the limit argument on ρ is unnecessary and $\mathcal{P}$ is idempotent in the strong sense, that is $(\mathcal{P}_{\rho'}(0, s))^2 = \mathcal{P}_{\rho'}(0, s)$ for $\rho' > 1/2$ (that is, outside the spectrum).

We have not specified the domain of definition of $\mathcal{P}^2$; since it is an unbounded operator we have to limit the domain such that the range is included in the domain. This is possible with the aid of a restriction, $\mathcal{P} \to \mathcal{P}|_0$. The later operator has a range contained in H^0_1 and hence for the restricted operator we have shown that

$$\left(\mathcal{P}|_0\,(0, s)\right)^2 = \mathcal{P}|_0\,(0, s)\ , \tag{5.82}$$

valid on the domain H^0_1 as a whole. □

5.6. Proof of Proposition 2, part 5. To extend the analysis above from $((\mathrm{L}^2)^2, (\cdot,\cdot)_{[0,0]})$ to the case when the embedding space is $(\mathrm{H}^r_r, (\cdot,\cdot)_{[r,r]})$, we only have to consider the conjugation

$$\mathcal{P}_{;r} = \frac{1}{2\pi\mathrm{i}} \int_{\lambda\in\mathrm{K}} \mathrm{d}\lambda\ \mathcal{A}^{-1}_{s,\lambda;r} = \Upsilon^{-r} \frac{1}{2\pi\mathrm{i}} \int_{\lambda\in\mathrm{K}} \mathrm{d}\lambda\ \mathcal{A}^{-1}_{s,\lambda} \Upsilon^r = \Upsilon^{-r} \mathcal{P} \Upsilon^r\ . \tag{5.83}$$

Interchange of the integral and the derivatives is allowed (cf. §5.4). It follows directly that $\left(\mathcal{P}_{;r}|_0\right)^2 = \mathcal{P}_{;r}|_0$, and that $\mathcal{P}_{;r}|_0\, \mathcal{A}_{;r} = \mathcal{A}_{;r}\, \mathcal{P}_{;r}|_1$, where the restricted operators $\mathcal{P}_{;r}|_q$ are defined on H^{r+q}_{r+q+1}. Furthermore, $\mathcal{P}_{;r}$ is a pseudodifferential operator with parameter of order zero. □

5.7. The projector. The analysis in the previous sections provided us with the operator $\mathcal{P}(a,s)$. This operator commutes with $\mathcal{A}$ and is idempotent for $a=0$. In Appendices B and C we construct the homogeneous medium case corresponding to the principal part of the operator in the general case, and show that $\mathcal{P}|_0(0,s)\mathcal{A}$ has only spectral points with positive real part, in addition to the point 0. Hence we regard $\mathcal{P}|_0(0,s)$ as a (principal) projector of the spectrum of $\mathcal{A}$ with positive real part. $\mathcal{P}$ cannot be an orthogonal projector since it is not bounded and self adjoint; it is a non-normal operator.

COROLLARY 2.1. *Let $\mathcal{P}(a,s)$ be defined as in Proposition 2, then $\mathcal{P}^{\pm}:\mathrm{H}^r_{r+1}\to\mathrm{H}^r_{r+1}$ defined on $(\mathrm{H}^r_r,(\cdot,\cdot)_{[r,r]})$ by*

$$\mathcal{P}^+=\mathcal{P}|_0(0,s)\;,\quad \mathcal{P}^-=I-\mathcal{P}^+$$

are two projectors such that

$$\mathcal{P}^+\mathcal{P}^+=\mathcal{P}^+\;,\quad \mathcal{P}^-\mathcal{P}^-=\mathcal{P}^-\quad\textit{and}\quad \mathcal{P}^+\mathcal{P}^-=\mathcal{P}^-\mathcal{P}^+=0\;. \tag{5.84}$$

Furthermore, $\mathcal{A}$ and $\mathcal{P}^{\pm}$ commute in the sense of Proposition 2, part 3.

Proof. It is immediate from Proposition 2 that

$$\mathcal{P}^+\mathcal{P}^+=\left(\mathcal{P}|_0(0,s)\right)^2=\mathcal{P}|_0(0,s)=\mathcal{P}^+\;.$$

Analogously,

$$\mathcal{P}^+\mathcal{P}^-=\mathcal{P}|_0(0,s)(I-\mathcal{P}|_0(0,s))=0\quad\text{and}\quad \mathcal{P}^-\mathcal{P}^+=(I-\mathcal{P}|_0(0,s))\,\mathcal{P}|_0(0,s)=0\;.$$

Furthermore

$$\begin{aligned}\mathcal{P}^-\mathcal{P}^-&=(I-\mathcal{P}|_0(0,s))^2=-2\,\mathcal{P}|_0(0,s)+\left(\mathcal{P}|_0(0,s)\right)^2\\&=I-\mathcal{P}|_0(0,s)=\mathcal{P}^-\;.\end{aligned} \tag{5.85}$$

That $\mathcal{P}^{\pm}$ commute with $\mathcal{A}$ follows directly from the definition and part 3 of Proposition 2. The projectors $\mathcal{P}^{\pm}$ induce a decomposition of the set H^r_{r+1}. □

6. Directional spectral decomposition of $\mathcal{A}$. We now have the necessary tools to address the original problem of this paper, that is, the existence of a linear operator, $\mathcal{L}$, such that

$$\mathcal{A}\mathcal{L}=\mathcal{L}\mathcal{V}\;,$$

where $\mathcal{V}$ is a diagonal matrix with operator elements. The columns of $\mathcal{L}$ would be generalized eigenvectors of $\mathcal{A}$. We will construct an operator with a known set of generalized eigenvectors and show that it commutes with $\mathcal{A}$; those eigenvectors can then be associated with the eigenvectors of $\mathcal{A}$. The projectors $\mathcal{P}^{\pm}$ of Corollary 2.1 are used for this purpose. That is,

PROPOSITION 3. *Let $\mathcal{B}:\mathrm{H}^r_{r+1}\to\mathrm{H}^r_{r+1}$ be an operator on $(\mathrm{H}^r_r,(\cdot,\cdot)_{[r,r]})$ defined by*

$$\mathcal{B}=\mathcal{P}^+-\mathcal{P}^-\;,$$

where $\mathcal{P}^{\pm}$ is defined in Corollary 2.1. Then $\mathcal{B}$

1. *is an involution;*
2. *is one-to-one;*
3. *has a dense range;*

4. *commutes with $\mathcal{A}$ in the sense of Proposition 2, part 3, that is*

$$\mathcal{B}|_0 \, \mathcal{A} = \mathcal{A} \, \mathcal{B}|_1$$

where the restriction refers to the set H^{r+q}_{r+q+1} *for* $q = 0, 1$*;*

5. *has for each restriction* $\mathcal{B}|_q$ *to the domain* H^{r+q}_{r+q+1}, $q \in \mathbb{N}_0$, *generalized eigenvalues* ± 1 *with the corresponding generalized eigenvectors*

$$\mathcal{L}^{\pm}\big|_q = \begin{pmatrix} \mathcal{Y}^{\pm}_q \\ I \end{pmatrix} \mathcal{N}^{\pm}_{\mathrm{L}}\big|_q \, ,$$

where the Dirichlet-to-Neumann map (cf. §3.2) $\mathcal{Y}^{\pm}_q$ *is given by*

$$\mathcal{Y}^{\pm}_q = (\mathcal{B}_{21;q})^{-1} \left(\pm I - \mathcal{B}_{22;q} \right) : \mathrm{H}^{r+q+1} \to \mathrm{H}^{r+q} \, ,$$

while

$$\mathcal{N}^{\pm}_{\mathrm{L}}\big|_q : \mathrm{H}^{r+q+1} \to \mathrm{H}^{r+q+1} \, ,$$

are normalization operators that have a densely defined inverse;

6. *has for each restriction* $\mathcal{B}|_q$ *eigenfunctions in* H^{r+q}_{r+q+1} *with corresponding eigenvalue* ± 1 *and each such eigenfunction can be written as* $\mathcal{L}^{\pm}\big|_q f_{\pm;q}$ *for some* $f_{\pm;q} \in \mathrm{H}^{r+q+1}$.

REMARK 3.1. *The operator $\mathcal{B}$ is suggested by similar considerations for wave propagation through layers of homogeneous anisotropic media in elastodynamics (cf. [3, p.33, theorem 2.3]), where the above operators reduce to matrices of functions. The name of the operator is suggested to be the 'splitting matrix' operator.* With this we obtain the final answer to the question about existence of a decomposition.

PROPOSITION 4. *The generalized eigenvectors* $\mathcal{L}^{\pm}|_1$ *of* $\mathcal{B}|_1$ *in Proposition 3 are generalized eigenvectors of* $\mathcal{A}$ *with corresponding generalized eigenvalues* $\mathcal{S}^{\pm}$*, in the sense:*

$$\mathcal{A} \, \mathcal{L}^{\pm}\big|_1 = \mathcal{L}^{\pm}\big|_0 \, \mathcal{S}^{\pm} \, .$$

The composition operator, $\mathcal{L}|_q$*, with* $q = 0, 1$*, is built from the generalized eigenvectors of* $\mathcal{B}$*, with* $\mathcal{L}^{\pm}|_q$*, as columns. The generalized eigenvalues,* $\mathcal{S}^{\pm} : \mathrm{H}^{r+2} \to \mathrm{H}^{r+1}$*, can be expressed as*

$$\mathcal{S}^{\pm} = (\mathcal{N}^{\pm}_{\mathrm{L}}\big|_0)^{-1} \left(\mathcal{A}_{21} \mathcal{Y}^{\pm}_1 + \mathcal{A}_{22} \right) \mathcal{N}^{\pm}_{\mathrm{L}}\big|_1$$

and are the diagonal elements of $\mathcal{V}$ *introduced in §3.*

REMARK 4.1. *The Dirichlet-to-Neumann map satisfies the algebraic Riccati operator equation,*

$$\left(\mathcal{Y}^{\pm} \mathcal{A}_{21} \mathcal{Y}^{\pm} + \mathcal{Y}^{\pm} \mathcal{A}_{22} - \mathcal{A}_{11} \mathcal{Y}^{\pm} - \mathcal{A}_{12} \right) \mathcal{N}^{\pm}_{\mathrm{L}} = 0 \, .$$

This is a direct consequence of the proof of the above theorem. In the up/down symmetric case the terms $\mathcal{A}_{11}$ *and* $\mathcal{A}_{22}$ *vanish and the solution reduces to*

$$\mathcal{Y}^{\pm} = \pm \mathcal{A}_{21}^{-1} (\mathcal{A}_{21} \mathcal{A}_{12})^{1/2} \, .$$

6.1. Proof of Proposition 3, parts 1-4. From the definition of the splitting matrix and (5.84) (Corollary 2.1) it directly follows that it is an involution, that is

$$\mathcal{B}^2 = \left(\mathcal{P}^+ - \mathcal{P}^-\right)\left(\mathcal{P}^+ - \mathcal{P}^-\right) = \mathcal{P}^+ + \mathcal{P}^- = I \ .$$

Hence $\mathcal{B}$ must be one-to-one. If we consider the vector $F = (F_1, 0)^T$ in the equation

$$\mathcal{B}F = \begin{pmatrix} \mathcal{B}_{11}F_1 \\ \mathcal{B}_{21}F_1 \end{pmatrix} = 0$$

we find that the fact that the splitting matrix, $\mathcal{B}$, is one-to-one implies that $\mathcal{B}_{21}$ or $\mathcal{B}_{11}$ is one-to-one; the same is true for $\mathcal{B}_{12}$ or $\mathcal{B}_{22}$. Due to the fact, that we know (cf. (5.45)) that the principal part of the off-diagonal elements are non-trivial, it follows that the off-diagonal elements are certainly one-to-one, thus it has an inverse on its range. An analogous argument for the adjoint of $\mathcal{B}_{21}$ ensure that it has dense range. That the splitting matrix is an involution implies, componentwise, that

$$\mathcal{B}_{21}\mathcal{B}_{11} + \mathcal{B}_{22}\mathcal{B}_{21} = 0 \ ,$$

and

$$\mathcal{B}_{21}\mathcal{B}_{12} + \mathcal{B}_{22}^2 = I \ .$$

Thus, since $\mathcal{B}_{21}$ has an inverse on its (dense) range, $\mathcal{B}_{11}$ and $\mathcal{B}_{12}$ can be expressed in terms of $\mathcal{B}_{21}$ and $\mathcal{B}_{22}$. That is,

$$\mathcal{B}_{11} = -(\mathcal{B}_{21})^{-1}\mathcal{B}_{22}\mathcal{B}_{21} \ , \tag{6.1}$$

$$\mathcal{B}_{12} = (\mathcal{B}_{21})^{-1}\left(I - \mathcal{B}_{22}^2\right) \ . \tag{6.2}$$

Consider the equation

$$\mathcal{B}F = G \in \mathrm{H}^r_{r+1}$$

This equation has a solution for any G in the above set, viz. $F = \mathcal{B}G$. Thus $\mathcal{B}$ has dense range in $(\mathrm{H}^r_r, (\cdot,\cdot)_{[r,r]})$. Due to its construction $\mathcal{B}$ commutes with $\mathcal{A}$ ($\mathcal{P}^\pm$ commutes with $\mathcal{A}$, cf. Corollary 2.1). Hence we have constructed a densely defined operator that commutes with $\mathcal{A}$. □

6.2. Proof of Proposition 3, part 5. To explicitly calculate all the generalized eigenvectors of this operator, we consider the restriction

$$\mathcal{B}|_q : \mathrm{H}^{r+q}_{r+q+1} \to \mathrm{H}^{r+q}_{r+q+1} \tag{6.3}$$

for arbitrary $q, r \in \mathbb{N}_0$. As usual we consider the respective functions above as functions in $(\mathrm{H}^r_r, (\cdot,\cdot)_{[r,r]})$ belonging to the above dense subsets. The operator $\mathcal{B}|_0$, with the largest domain, is exactly $\mathcal{B}$.

An arbitrary operator vector, $\mathcal{L}_\mathrm{v}$, is introduced as a two by one vector of linear operators such that

$$\mathcal{L}_\mathrm{v}|_q = \begin{pmatrix} \mathcal{L}_{\mathrm{v};1;q} \\ \mathcal{L}_{\mathrm{v};2;q} \end{pmatrix} : \mathrm{H}^{r+q+1} \to \mathrm{H}^{r+q}_{r+q+1} \ , \tag{6.4}$$

for any $q, r \in \mathbb{N}_0$. Consider the equation

$$(\mathcal{B}|_q - \lambda I)\, \mathcal{L}_\mathrm{v}|_q = 0 \ . \tag{6.5}$$

Since $\mathcal{B}_{21;q}$ is one-to-one it is invertible on its range, which is dense by Proposition 3 part 2 and 3. The relations (6.1), (6.2) inserted into (6.5) imply after some simplifications

$$\mathcal{B}_{21;q}\mathcal{L}_{\mathrm{v};1;q} + (\mathcal{B}_{22;q} - \lambda)\mathcal{L}_{\mathrm{v};2;q} = 0 \,, \tag{6.6}$$

$$(\mathcal{B}_{21;q})^{-1}\left(1 - \lambda^2\right)\mathcal{L}_{\mathrm{v};2;q} = 0 \,, \tag{6.7}$$

where the index q indicates that $\mathcal{B}$ has the corresponding Sobolev order imposed by (6.3) and (6.4). Hence, the second equation gives that all eigenvalues are $\lambda = \pm 1$ and the first equation gives for fixed λ a relation between the elements of the 'eigenvector'. Analogously to a matrix eigenvalue, eigenvector calculation we introduce a special notation for the solutions to the above equation. The generalized eigenvectors are denoted by $\mathcal{L}^{\pm}|_q$ corresponding to the eigenvalues ± 1 and have the form

$$\mathcal{L}^{\pm}\big|_q = \begin{pmatrix} \mathcal{Y}_q^{\pm} \\ I \end{pmatrix} \mathcal{N}_{\mathrm{L}}^{\pm}\big|_q \,, \tag{6.8}$$

where

$$\mathcal{Y}_q^{\pm} = (\mathcal{B}_{21;q})^{-1}\left(\pm I - \mathcal{B}_{22;q}\right) : \mathrm{H}^{r+q+1} \to \mathrm{H}^{r+q} \,, \tag{6.9}$$

is called the Dirichlet-to-Neumann map (cf. §3.2). Since it is expressed as a composition of two pseudodifferential operators with parameter it itself is a pseudodifferential operator with parameter for the same restriction on s as in Proposition 2. Note that for the sets of (6.4) to agree with (6.8) we must have

$$\mathcal{N}_{\mathrm{L}}^{\pm}\big|_q : \mathrm{H}^{r+q+1} \to \mathrm{H}^{r+q+1} \,. \tag{6.10}$$

We denote $\mathcal{N}_{\mathrm{L}}^{\pm}\big|_q$ as the (scalar) normalization operators; they are chosen to be one-to-one, bounded operators with dense range, that is to be invertible on their range. This choice is allowed, since the normalizations are unspecified except for the function sets they have to be defined between. Note that multiplication with a nonzero constant is an allowed choice of operator.

To explicitly obtain $\mathcal{Y}_{;r}^{\pm}$ with restriction to H^{r+q} in terms of $\mathcal{Y}_{;0}^{\pm}$ we employ the nested conjugation

$$\begin{aligned}
&\mathcal{Y}_{r+q}^{\pm} \\
&= (1 - \mathbf{\Delta}')^{-r/2}\left(\mathcal{B}_{21;0+q}\right)^{-1}(1 - \mathbf{\Delta}')^{r/2}(\pm I - (1 - \mathbf{\Delta}')^{-r/2}\mathcal{B}_{22;0+q}(1 - \mathbf{\Delta}')^{r/2}) \\
&= (1 - \mathbf{\Delta}')^{-r/2}\mathcal{Y}_{0+q}^{\pm}(1 - \mathbf{\Delta}')^{r/2} : \mathrm{H}^{r+q+1} \to \mathrm{H}^{r+q} \,,
\end{aligned}$$

where we have indices $0 + q$ and $r + q$ to indicate the respective spaces. □

6.3. Proof of Proposition 3, part 6. We have constructed generalized eigenvectors (operators) $\mathcal{L}^{\pm}$. Consider any function, $f_{\pm;q} \in \mathrm{H}^{r+q+1}$ to find that $f_{\pm;q}$ generates an eigenvector (field) $F_{\pm;f_{\pm};q} = \mathcal{L}^{\pm}|_q f_{\pm;q} \in \mathrm{H}^{r+q}_{r+q+1}$ with corresponding eigenvalue ± 1. The set of eigenvectors (fields) (denoted by eigenfunctions for convenience) is infinitely large.

To show that all eigenfunctions can be written in the above form, we now consider any field $G \in \mathrm{H}^{r+q}_{r+q+1}$. This field can be written as

$$G = \mathcal{L}^{+}\big|_q f_{+;q} + \mathcal{L}^{-}\big|_q f_{-;q} \,, \tag{6.11}$$

where $f_{+;q}, f_{-;q} \in \mathrm{H}^{r+q+1}$ since we can make the choice,

$$f_{+;q} = \frac{1}{2}\left(\mathcal{N}^{+}\big|_q\right)^{-1}\mathcal{B}_{21;q}\left(G_1 - \mathcal{Y}_q^{-}G_2\right) \,, \tag{6.12}$$

$$f_{-;q} = \frac{1}{2}\left(\mathcal{N}^{-}\big|_q\right)^{-1}\mathcal{B}_{21;q}\left(-G_1 + \mathcal{Y}_q^{+}G_2\right) \,, \tag{6.13}$$

obtained by solving (6.11) componentwise. Thus in some sense the vectors $\mathcal{L}^{\pm}|_q$ span the set H^{r+q}_{r+q+1} for all $q \in \mathbb{N}_0$. Hence eigenfunctions that correspond to $+1$ can be written as $\mathcal{L}^{+}|_q\, f_{+;q}$ and eigenfunctions that corresponds to -1 can be written $\mathcal{L}^{-}|_q\, f_{-;q}$. □

6.4. Proof of Proposition 4. We are given a, with $\mathcal{A}$, commuting operator, $\mathcal{B}$, and its generalized eigenvectors and eigenvalues. To construct the decomposition of $\mathcal{A}$, we consider

$$\mathcal{B}|_0\, \mathcal{A}\, \mathcal{L}^{\pm}\big|_1\, f_{\pm;1} = \mathcal{A}\, \mathcal{B}|_1\, \mathcal{L}^{\pm}\big|_1\, f_{\pm;1} = \pm\mathcal{A}\, \mathcal{L}^{\pm}\big|_1\, f_{\pm;1}\ , \tag{6.14}$$

where $\mathcal{B}|_q$ is defined in (6.3) and $\mathcal{L}^{\pm}|_q$ is defined in (6.8) for $q = 0, 1$. The notation is taken from the proof of Proposition 3, part 6. Here $f_{\pm;1} \in \mathrm{H}^{r+2}$ is an arbitrary function that generates an eigenfunction to $\mathcal{B}|_1$ by the procedure described in Proposition 3, part 6. We consider all such generators. We note that the vectors

$$\mathcal{A}\, \mathcal{L}^{\pm}\big|_1\, f_{\pm;1} \in \mathrm{H}^{r}_{r+1} \tag{6.15}$$

are eigenfunctions of $\mathcal{B}|_0$ with corresponding eigenvalues ± 1 by (6.14). From part 6 of Proposition 3 we know that we can express such eigenfunctions as

$$\mathcal{A}\, \mathcal{L}^{\pm}\big|_1\, f_{\pm;1} = \mathcal{L}^{\pm}\big|_0\, (\pm f_{\pm;0})\ , \tag{6.16}$$

for some, other, generator $f_{\pm;0} \in \mathrm{H}^{r+1}$. We introduce the unknown operators $\mathcal{S}^{\pm} : \mathrm{H}^{r+2} \to \mathrm{H}^{r+1}$ (cf. (3.7)), such that

$$\pm f_{\pm;0} = \mathcal{S}^{\pm} f_{\pm;1}\ . \tag{6.17}$$

Then

$$\mathcal{A}\, \mathcal{L}^{\pm}\big|_1\, f_{\pm;1} = \mathcal{L}^{\pm}\big|_0\, \mathcal{S}^{\pm} f_{\pm;1} \tag{6.18}$$

and thus $\mathcal{L}^{\pm}$ is considered to be a generalized eigenvector to $\mathcal{A}$ with corresponding generalized eigenvalue $\mathcal{S}^{\pm}$. That is, we have found a decomposition of $\mathcal{A}$ from the wave field decomposition operator, $\mathcal{L}|_q$, defined by

$$\mathcal{L}|_q = \left(\mathcal{L}^{+}\big|_q \;\; \mathcal{L}^{-}\big|_q \right) \tag{6.19}$$

for $q = 0, 1$, with corresponding diagonal matrix

$$\mathcal{V} = \begin{pmatrix} \mathcal{S}^{+} & 0 \\ 0 & \mathcal{S}^{-} \end{pmatrix}\ . \tag{6.20}$$

To find, explicitly, the corresponding equations for the eigenvalues of $\mathcal{A}$, we apply $\mathcal{A}$ to the set of nontrivial eigenvectors $\mathcal{L}^{\pm}$ (cf. (6.8)) and obtain

$$\begin{aligned} \left(\mathcal{A}_{11}\mathcal{Y}_1^{\pm} + \mathcal{A}_{12}\right) \mathcal{N}_{\mathrm{L}}^{\pm}\big|_1 &= \mathcal{Y}_0^{\pm}\, \mathcal{N}_{\mathrm{L}}^{\pm}\big|_0\, \mathcal{S}^{\pm}\ , \\ \left(\mathcal{A}_{21}\mathcal{Y}_1^{\pm} + \mathcal{A}_{22}\right) \mathcal{N}_{\mathrm{L}}^{\pm}\big|_1 &= \mathcal{N}_{\mathrm{L}}^{\pm}\big|_0\, \mathcal{S}^{\pm}\ . \end{aligned} \tag{6.21}$$

Note that $\mathcal{S}^{\pm}$ can be written explicitly in terms of the known operators $\mathcal{Y}^{\pm}$ and A_{21}, A_{22} together with the arbitrary normalization $\mathcal{N}_{\mathrm{L}}^{\pm}|_q$, defined by (6.10). The normalizations are invertible on their range from Proposition 3, part 5 and hence

$$\mathcal{S}^{\pm} = (\mathcal{N}_{\mathrm{L}}^{\pm}|_0)^{-1} \left(\mathcal{A}_{21}\mathcal{Y}_1^{\pm} + \mathcal{A}_{22}\right) \mathcal{N}_{\mathrm{L}}^{\pm}\big|_1\ . \tag{6.22}$$

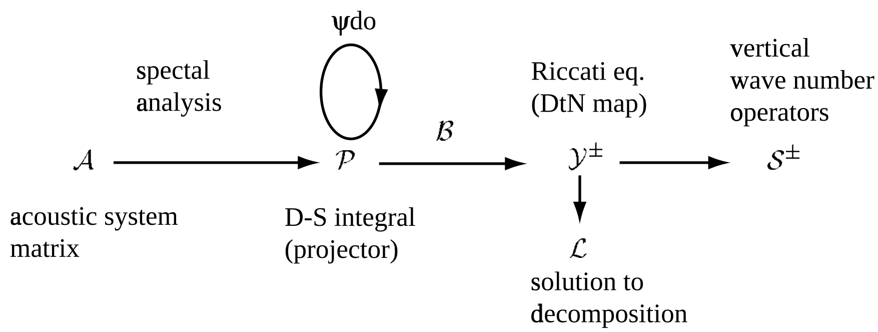

FIG. 7.1. *Wave field decomposition procedure.*

By substituting $\mathcal{N}_{\mathrm{L}}^{\pm}\big|_0 \mathcal{S}^{\pm}$ from the bottom equation in (6.21) into the top one, we obtain the algebraic Riccati operator equation

$$(6.23)\qquad \left(\mathcal{Y}_0^{\pm}\mathcal{A}_{21}\mathcal{Y}_1^{\pm} + \mathcal{Y}_0^{\pm}\mathcal{A}_{22} - \mathcal{A}_{11}\mathcal{Y}_1^{\pm} - \mathcal{A}_{12}\right)\mathcal{N}_{\mathrm{L}}^{\pm}\big|_1 = 0\,.$$

Due to the fact that $\mathcal{N}_{\mathrm{L}}^{\pm}\big|_1$ is invertible and has dense range in $(\mathrm{H}^r, (\cdot,\cdot)_r)$ the expression in parenthesis has to be zero. Two solutions to the above equation are $\mathcal{Y}^{\pm}$. However, we have already constructed these solutions to the algebraic Riccati operator equation (cf. (6.9). We observe that other solutions to this equation might exist.

Note that the orders of $\mathcal{A}_{11},\ldots,\mathcal{A}_{22}$ make the domains of $\mathcal{Y}_1^{\pm}$ and $\mathcal{Y}_0^{\pm}$ to match naturally. Hence we write

$$\left(\mathcal{Y}^{\pm}\mathcal{A}_{21}\mathcal{Y}^{\pm} + \mathcal{Y}^{\pm}\mathcal{A}_{22} - \mathcal{A}_{11}\mathcal{Y}^{\pm} - \mathcal{A}_{12}\right)\mathcal{N}_{\mathrm{L}}^{\pm} = 0\,.$$

The decomposition operator is specified up to a normalization. This freedom is utilized in [5] for the case of isotropic media to obtain different kinds of physical normalizations. □

7. Discussion of the result. In this paper, we have generalized the wave-splitting procedure to configurations with multi-dimensionally varying anisotropic media that deviate from the up/down symmetric case. The procedure developed in this paper is summarized in figure 7.1. The generalization of the procedure from the up/down symmetric case is obtained by employing spectral theory applied to the acoustic system's matrix, a non-normal operator. Given knowledge about the spectrum we can define an associated projector that is a well defined pseudodifferential operator with parameter. The properties of the projector suggest that a splitting matrix could be introduced. The construction of the generalized eigenvectors of the splitting matrix is rater straightforward. Since the splitting matrix, in a particular sense, commutes with the acoustic system's matrix its generalized eigenvectors can be associated with the generalized eigenvectors of the acoustic system's matrix. One element in the generalized eigenvectors of the splitting matrix is the Dirichlet-to-Neumann map. This map is a key part in the construction of the wave field decomposition. It is used both to define the composition operator ($\mathcal{L}$) and the vertical wave number operators ($\mathcal{S}^{\pm}$).

That the Dirichlet-to-Neumann map for a level plane is related to an algebraic Riccati operator equation has been well known for one-dimensional — and isotropic multi-dimensional — problems. Here we extend this observation to the generic anisotropic case. However, we also make the connection via the splitting matrix between the solution to the algebraic Riccati operator equation and the projector, defined by a Dunford-Schwartz type integral over the resolvent of the acoustic system's matrix. The projector enables us to obtain two solutions to the algebraic Riccati operator equation while it also simplifies the analysis of the solution, that is, of the Dirichlet-to-Neumann map for the generic anisotropic case. We have shown that the Dirichlet-to-Neumann operator is a pseudodifferential operator with parameter for a restricted range of values for the Laplace parameter s.

Once we have obtained the wave field decomposition, we can proceed and use the generalized Bremmer coupling series to study direct and inverse scattering problems.

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Appendix A. The isotropic homogeneous medium case. The acoustic system's matrix, $\mathcal{A}$, for the homogeneous isotropic case has the form

$$\mathcal{A} = s \begin{pmatrix} 0 & \kappa - \rho^{-1}s^{-2}\partial_\mu\partial_\mu \\ \rho & 0 \end{pmatrix} \tag{A.1}$$

in the time-Laplace domain. Considered as a partial differential operator with some regularity conditions (that the functions belong to $\mathrm{H}_2^0 \subset ((\mathrm{L}^2)^2, (\cdot,\cdot)_{[0,0]})$ on $\mathbb{R}^2$, the spectrum of $\mathcal{A}$ can be determined by a Fourier transform, $\mathcal{F}$, of the operator, that is by considering $\det\{\boldsymbol{a} - \lambda I\} = 0$, where $\boldsymbol{a}$ is the symbol of $\mathcal{A}$. Expressing λ in terms of the Fourier parameters ξ', gives

$$\lambda^2 = \kappa\rho s^2 + |\xi'|^2 \ . \tag{A.2}$$

The dense range criteria obtained from the kernel of the adjoint $\mathcal{A}^*_{s,\lambda}$, yield the same equation. Here s is a fixed parameter and ξ' is the parameter describing the spectrum, hence all solutions $\lambda(\xi')$ of (A.2) belong to the spectrum of $\mathcal{A}$ for fixed s. For the case when $\lambda_{\mathrm{R}} = 0$ the above equation does not allow a solution for λ_{I} under the restriction that $\mathrm{Re}\{s\} \neq 0$ and hence the spectrum does not cross the imaginary axis. The inverse operator has the explicit form

$$\mathcal{A}^{-1}_{s,\lambda} = (\mathcal{A} + \lambda I)\,\mathcal{E}^{-1}_{s,\lambda} \ , \tag{A.3}$$

where

$$\mathcal{E}_{s,\lambda} = -\partial_\mu\partial_\mu + s^2\kappa\rho - \lambda^2 \ . \tag{A.4}$$

$\mathcal{E}_{s,\lambda}$ is a Helmholtz operator (in dimension 2) and thus we can write the kernel of the inverse as a Green's function $\mathcal{G}_{s,\lambda}$. We get

$$\mathcal{A}^{-1}_{s,\lambda}F = \mathcal{A}_{s,-\lambda}\,\mathcal{G}_{s,\lambda} * F \ , \tag{A.5}$$

where * denotes convolution over $\boldsymbol{x}'$.

To construct the Green's function, we use the rules for composition of operators in the pseudodifferential calculus to obtain

$$\begin{aligned} \mathcal{E}^{-1}_{s,\lambda}F_1 &= (2\pi)^{-2}\int_{\mathbb{R}^2} \mathrm{d}^2\xi' \ \frac{\exp(\mathrm{i}\langle \boldsymbol{x}', \xi'\rangle)}{|\xi'|^2 + k^2}\mathcal{F}(F_1)(\xi') + \text{terms belonging to } \mathrm{CL}^{-\infty}_{1,1} \\ &= \mathcal{F}^{-1}\left(\frac{1}{|\xi'|^2 + k^2}\right) * F_1 + \text{terms belonging to } \mathrm{CL}^{-\infty}_{1,1} \ , \end{aligned} \tag{A.6}$$

where $k^2 = s^2\kappa\rho - \lambda^2$. Upon change of variables to polar coordinates we find

$$\begin{aligned} \mathcal{G}_{s,\lambda} * F_1 = \mathcal{E}^{-1}_{s,\lambda}F_1 &= (2\pi)^{-1}\int_{\mathbb{R}^2} \mathrm{K}_0(k|\boldsymbol{x}' - \boldsymbol{y}'|)F_1(\boldsymbol{y})\mathrm{d}\boldsymbol{y}' \\ &\quad + \text{terms belonging to } \mathrm{CL}^{-\infty}_{1,1} \ , \end{aligned} \tag{A.7}$$

where K_0 is the modified Bessel function, if $\arg k^2 \neq \pi$, as is the case for $\mathrm{Re}\,\{s\} > 0$ and $\lambda_{\mathrm{R}}^2 < \kappa\rho(\mathrm{Re}\,\{s\})^2$. In fact, if we compare (A.7) to [22, p.183, §11.9] we find that there are no extra terms. Thus the pseudodifferential analysis yields the exact Green's function to the Helmholtz operator $\mathcal{E}_{s,\lambda}$.

In the same way that interchange of integration order is allowed for the inhomogeneous case, it is allowed here. Interchange of integration order gives that the principal symbol, $\boldsymbol{p}^{\pm}$, of $\mathcal{P}^{\pm}$ is

$$2\left(\boldsymbol{p}^{\pm} - \frac{1}{2}\right) = \pm\frac{1}{\pi \mathrm{i}}\int_{\lambda\in \mathrm{K}} \mathrm{d}\lambda\ \boldsymbol{a}_{s,-\lambda}\boldsymbol{e}^{-1} = \pm \boldsymbol{a}_{s,0}\frac{1}{\sqrt{|\xi'|^2 + s^2\kappa\rho}}\ , \tag{A.8}$$

where $\boldsymbol{a}_{s,\lambda}$ is the symbol of $\mathcal{A}_{s,\lambda}$:

$$\boldsymbol{a}_{s,\lambda} = \begin{pmatrix} -\lambda & s\kappa + \rho^{-1}s^{-1}|\xi'|^2 \\ s\rho & -\lambda \end{pmatrix}. \tag{A.9}$$

That is, the symbol of $\mathcal{P}^{\pm}$ is

$$\boldsymbol{p}^{\pm} = \frac{1}{2}\begin{pmatrix} 1 & \pm\frac{\sqrt{|\xi'|^2+s^2\kappa\rho}}{s\rho} \\ \pm\frac{s\rho}{\sqrt{|\xi'|^2+s^2\kappa\rho}} & 1 \end{pmatrix}. \tag{A.10}$$

Using the symbol representation to obtain the projector, we find

$$\mathcal{P}^{+}F = \frac{1}{2}\begin{pmatrix} 1 & \rho^{-1}\left(-s^{-2}\partial_\mu\partial_\mu + \kappa\rho\right)^{1/2} \\ \left(\frac{s\rho}{2\pi|\boldsymbol{x}'-\cdot|}\exp\left(-s|\boldsymbol{x}'-\cdot|\sqrt{\kappa\rho}\right)\right)* & 1 \end{pmatrix} F,$$

where we have taken the branch cut for s to be on the negative real axis. We follow the procedure that is presented in this paper and find

$$\begin{aligned}\mathcal{B} = \mathcal{P}^{+} - \mathcal{P}^{-} &= \frac{1}{\pi\mathrm{i}}\int_{\lambda\in\mathrm{K}} \mathrm{d}\lambda\ \mathcal{A}_{s,\lambda}^{-1} \\ &= \begin{pmatrix} 0 & \rho^{-1}\left(-s^{-2}\partial_\mu\partial_\mu + \kappa\rho\right)^{1/2} \\ \left(\frac{s\rho}{2\pi|\boldsymbol{x}'-\cdot|}\exp\left(-s|\boldsymbol{x}'-\cdot|\sqrt{\kappa\rho}\right)\right)* & 0 \end{pmatrix}.\end{aligned} \tag{A.11}$$

Comparing with [3, p.33, theorem 2.3], we see that this agrees with the definition of $\mathcal{B}$ given there. We find that the Dirichlet-to-Neumann map has the form

$$\mathcal{Y}^{\pm} = \pm\mathcal{B}_{21}^{-1} = \pm\rho^{-1}\left(-s^{-2}\partial_\mu\partial_\mu + \kappa\rho\right)^{1/2}\ . \tag{A.12}$$

This result agrees with the definition of the admittance operator of [5, p.3253,(II.53)], for constant coefficients. The inverse, the Neumann-to-Dirichlet map, has the form

$$\mathcal{Z}^{\pm} = \pm\mathcal{B}_{12}^{-1} = \pm\left(-s^{-2}\partial_\mu\partial_\mu + \kappa\rho\right)^{-1/2}\rho\ . \tag{A.13}$$

The corresponding 'eigenvalue' or vertical wave number operators have the form

$$\mathcal{S}^{\pm} = \pm\mathcal{A}_{21}\mathcal{B}_{12}^{-1} = \pm\left(-\partial_\mu\partial_\mu + \kappa\rho s^2\right)^{-1/2}\ . \tag{A.14}$$

This agrees with the $\mathcal{K}_0$ operator of [24, p.1063,(18)].

Appendix B. The anisotropic homogeneous medium case. The acoustic system's matrix, $\mathcal{A}$, for the homogeneous anisotropic case has the form

$$\mathcal{A} = \begin{pmatrix} \alpha_{33}^{-1}\alpha_{3\mu}\partial_\mu & s\kappa - s^{-1}Q_{\mu\nu}\partial_\mu\partial_\nu \\ s\alpha_{33}^{-1} & \alpha_{33}^{-1}\alpha_{3\mu}\partial_\mu \end{pmatrix} . \tag{B.1}$$

Considered as a partial differential operator with only regularity conditions (that the functions belong to $\mathrm{H}_2^1 \subset ((\mathrm{L}^2)^2, (\cdot,\cdot)_{[0,0]})$) on $\mathbb{R}^2$, the spectrum of $\mathcal{A}$ for fixed s can determined by a Fourier transform of the operator, that is by considering

$$\det\{\boldsymbol{a} - \lambda I\} = 0 , \tag{B.2}$$

where $\boldsymbol{a}$ is the symbol of $\mathcal{A}$ and expressing λ in terms of the Fourier parameters ξ', that is

$$\left(\mathrm{i}\alpha_{33}^{-1}\alpha_{3\mu}\xi_\mu - \lambda\right)^2 - s^2\alpha_{33}^{-1}\kappa - \alpha_{33}^{-1}Q_{\mu\nu}\xi_\mu\xi_\nu = 0 , \tag{B.3}$$

with solutions

$$\lambda = \lambda(\xi') = 2\mathrm{i}\alpha_{33}^{-1}\alpha_{3\mu}\xi_\mu \pm \alpha_{33}^{-1/2}\sqrt{\kappa s^2 + Q_{\mu\nu}\xi_\mu\xi_\nu} . \tag{B.4}$$

Note that the dense range criteria for the operator obtained from the kernel of the adjoint $\mathcal{A}_\lambda^*$ yield the same equation. Here s is considered as a fixed parameter and $\xi' = (\xi_1, \xi_2)$ are the parameters describing the spectrum. For the case when $\lambda_\mathrm{R} = 0$ the above equation does not allow a solution for λ_I under the restriction that $\mathrm{Re}\{s\} \neq 0$ and hence the spectrum does not intersect the imaginary axis. The inverse operator is given by

$$\mathcal{A}_{s,\lambda}^{-1} = -\left(\mathcal{A}(\boldsymbol{x}, \partial_1, \partial_2, -s) - \lambda I\right)\alpha_{33}\mathcal{E}_{s,\lambda}^{-1} = -\mathcal{A}_{-s,\lambda}\alpha_{33}\mathcal{E}_{s,\lambda}^{-1} , \tag{B.5}$$

where

$$\mathcal{E}_{s,\lambda} = -\alpha_{\mu\nu}\partial_\mu\partial_\mu + 2\lambda\alpha_{3\mu}\partial_\mu + s^2\kappa - \alpha_{33}\lambda^2 . \tag{B.6}$$

$\mathcal{E}_{s,\lambda}$ is a generalized Helmholtz operator in 2 dimensions.

From the rules for composition of operators in pseudodifferential calculus we obtain

$$\begin{aligned} \mathcal{E}_{s,\lambda}^{-1}F_1 &= (2\pi)^{-2}\int_{\mathbb{R}^2} \mathrm{d}^2\xi' \, \frac{\exp\left(\mathrm{i}\langle \boldsymbol{x}', \xi'\rangle\right)}{\alpha_{\mu\nu}\xi_\mu\xi_\nu + 2\mathrm{i}\lambda\alpha_{3\mu}\xi_\mu + s^2\kappa - \alpha_{33}\lambda^2}\mathcal{F}(F_1)(\xi) \\ &= \mathcal{F}^{-1}\left(\frac{1}{\alpha_{\mu\nu}\xi_\mu\xi_\nu + 2\mathrm{i}\lambda\alpha_{3\mu}\xi_\mu + s^2\kappa - \alpha_{33}\lambda^2}\right) * F_1 . \end{aligned} \tag{B.7}$$

modulo smoother terms.

In the same way that interchange of integration is allowed for the inhomogeneous case, it is allowed here. Interchange of integration order gives that the symbol, $\boldsymbol{p}^\pm$, of $\mathcal{P}^\pm(0, s)$ can be represented as (cf. (B.5))

$$2\left(\boldsymbol{p}^\pm - \frac{1}{2}\right) = \mp\frac{\alpha_{33}}{\pi\mathrm{i}}\int_{\lambda\in\mathrm{K}} \mathrm{d}\lambda \, \boldsymbol{a}_{-s,\lambda}\boldsymbol{e}^{-1} , \tag{B.8}$$

where $\boldsymbol{a}_{s,\lambda}$ is the symbol of $\mathcal{A}_{s,\lambda}$:

$$\boldsymbol{a}_{s,\lambda} = \begin{pmatrix} \alpha_{33}^{-1}\alpha_{3\mu}\xi_\mu - \lambda & s\kappa + s^{-1}Q_{\mu\nu}\xi_\mu\xi_\nu \\ s\alpha_{33}^{-1} & \alpha_{33}^{-1}\alpha_{3\mu}\xi_\mu - \lambda \end{pmatrix} \tag{B.9}$$

Upon integration we obtain the principal symbol, $\boldsymbol{p}^{\pm}$, of $\mathcal{P}^{\pm}$ as

$$\boldsymbol{p}^{\pm} = \frac{1}{2}\begin{pmatrix} 1 & \pm s^{-1}\sqrt{\alpha_{33}}\sqrt{s^2\kappa + Q_{\mu\nu}\xi_\mu\xi_\nu} \\ \pm s\left(\sqrt{\alpha_{33}}\sqrt{s^2\kappa + Q_{\mu\nu}\xi_\mu\xi_\nu}\right)^{-1} & 1 \end{pmatrix} .$$

Using the symbol representation to obtain the projector we find

$$\mathcal{P}^{+}F = \frac{1}{2}\begin{pmatrix} 1 & s^{-1}\sqrt{\alpha_{33}}\left(s^2\kappa - Q_{\mu\nu}\partial_\mu\partial_\nu\right)^{1/2} \\ s\left(\sqrt{\alpha_{33}}\left(s^2\kappa - Q_{\mu\nu}\partial_\mu\partial_\nu\right)^{1/2}\right)^{-1} & 1 \end{pmatrix} F .$$

Following the procedure that is presented in this paper, we find that the splitting matrix $\mathcal{B}$ is given by

(B.10)
$$\mathcal{B} = \mathcal{P}^{+} - \mathcal{P}^{-} = \frac{1}{\pi\mathrm{i}}\int_{\lambda\in\mathrm{K}} \mathrm{d}\lambda\ \mathcal{A}_{s,\lambda}^{-1} =$$
$$\begin{pmatrix} 0 & s^{-1}\sqrt{\alpha_{33}}\left(s^2\kappa - Q_{\mu\nu}\partial_\mu\partial_\nu\right)^{1/2} \\ s\left(\sqrt{\alpha_{33}}\left(s^2\kappa - Q_{\mu\nu}\partial_\mu\partial_\nu\right)^{1/2}\right)^{-1} & 0 \end{pmatrix} .$$

Comparing (B.10) with [3, p.33, theorem 2.3] we see that this agrees with the definition of $\mathcal{B}$ given there. We find that the Dirichlet-to-Neumann map has the form

(B.11)
$$\mathcal{Y}^{\pm} = \pm\mathcal{B}_{21}^{-1} = \pm\alpha_{33}^{1/2}\left(\kappa - s^{-2}Q_{\mu\nu}\partial_\mu\partial_\nu\right)^{1/2} .$$

The inverse, the Neumann-to-Dirichlet map, has the form

(B.12)
$$\mathcal{Z}^{\pm} = \pm\mathcal{B}_{12}^{-1} = \pm\left(\kappa - s^{-2}Q_{\mu\nu}\partial_\mu\partial_\nu\right)^{-1/2}\alpha_{33}^{-1/2} .$$

The corresponding vertical wave number operators have the form

(B.13)
$$\begin{aligned}\mathcal{S}^{\pm} &= \mathcal{A}_{21}\mathcal{Y}^{\pm} + \mathcal{A}_{22} \\ &= \pm\alpha_{33}^{-1/2}\left(s^2\kappa - Q_{\mu\nu}\partial_\mu\partial_\nu\right)^{1/2} + \alpha_{33}^{-1}\alpha_{3\mu}\partial_\mu ,\end{aligned}$$

where the normalizations are the identity operators (cf. Proposition 4).

Appendix C. Local group velocity and the spectrum. To relate the spectral parameter, λ, to a physical quantity, we consider geometrical optics for the wave equation. In geometrical optics, one obtains an eikonal equation for the travel time, τ; the gradient of travel time is identified as the slowness vector γ. Under the restriction that γ is real, the slowness vector lies on an ovoid. This restricted surface is called the slowness surface and with the aid of a Hamilton-Jacobi transformation of the eikonal equation, we can show that the normal to this surface is parallel to the local group velocity, v_{g}.

C.1. The spectrum and the slowness vector. In our case, we have the operator

(C.1)
$$I\partial_3 + \mathcal{A}(\boldsymbol{x}, \partial_1, \partial_2, \partial_t) .$$

We substitute an Ansatz of the form

(C.2)
$$F = F_{(0)}(\boldsymbol{x})\partial_t f(t - \tau(\boldsymbol{x})) + \dots ,$$

where $F_{(0)}(\boldsymbol{x})$ is assumed to be slowly varying compared to f. The choice of $\partial_t f$ comes from the s^{-1} or, in the time domain, $\int^t \mathrm{d}t'$ occurring in $\mathcal{A}_{12}$. The eikonal equation for our system becomes the determinant of the principal order system set to zero [15, pp.316-317, §77]. With the substitution $\gamma_j = \partial_j \tau$ we obtain

$$\det\left(\gamma_3 I + \mathcal{A}(\boldsymbol{x}, -\gamma_1, -\gamma_2, 1)\right) = 0 \; . \tag{C.3}$$

Substituting (B.1) into the determinant gives

$$\alpha_{33}^{-1}\left(\alpha_{\mu\nu}\gamma_\mu\gamma_\nu + 2\gamma_3\gamma_\mu\alpha_{3\mu} + \alpha_{33}\gamma_3^2 - \kappa\right) = 0 \; . \tag{C.4}$$

If we compare the left-hand side of (C.4) to the null space of the principal part of the symbol $\boldsymbol{e}_2$ of $\mathcal{E}_{s,\lambda}$, or (B.3), we find that (cf. (5.14)

$$s^2\left(-\alpha_{\mu\nu}s^{-1}\xi_\mu s^{-1}\xi_\nu - 2\mathrm{i}s^{-1}\lambda s^{-1}\alpha_{3\mu}\xi_\mu + \alpha_{33}s^{-2}\lambda^2 - \kappa\right) = 0 \; . \tag{C.5}$$

upon substituting

$$\gamma_\mu = \frac{\xi_\mu}{\mathrm{i}s} \quad \text{and} \quad \gamma_3 = \frac{\lambda}{s} \; . \tag{C.6}$$

into (C.4) we obtain the equation $\boldsymbol{e}_2 = 0$. We observe that (ξ_1, ξ_2, λ) has the dimension of wave number while $(\gamma_1, \gamma_2, \gamma_3)$ has the dimension of slowness.

We conclude that the spectral parameter λ is proportional to the vertical slowness for each fixed s. Under the assumption that the slowness vector, γ, is real valued, we obtain that for the microlocal contribution to the spectrum

$$\frac{\tilde{\lambda}}{s} = \gamma_3 = -\alpha_{33}^{-1}\alpha_{3\mu}\gamma_\mu \pm \sqrt{\alpha_{33}^{-1}\kappa - \alpha_{33}^{-1}Q_{\mu\nu}\gamma_\mu\gamma_\nu} \; , \tag{C.7}$$

where $\operatorname{Im}\left\{\tilde{\lambda}/s\right\} = 0$, compare with (B.4). Thus the sign in front of the square root corresponding to the positive (negative) sign of the real part of λ does *not* correspond to the sign of γ_3, but rather, for the restricted λ, to the turning points of the ellipse where the local vertical group velocity $v_{\mathrm{g};3} = 0$ (see figure C.1). Hence given the set of the spectrum with positive (negative) real part all points in this set such that $\operatorname{Im}\{\lambda/s\} = 0$ corresponds to the positive (negative) local vertical group velocity. Thus, we have a connection of the local group velocity to a restricted part of the spectrum. Below we show that each of the projectors projects out one root of the above equation. Up to principal parts the projector, in some sense, projects out the up- from the down-going waves at a surface x_3 =constant.

C.2. The projector projects out the spectrum with positive real part. For the homogeneous case, we can apply a Fourier-transform in space and obtain the spectrum of $\mathcal{A}$ explicitly to be the area in the complex plane for λ where (B.3) is fulfilled. The solutions in λ to (B.3) is a function of ξ' for fixed s and can be written in the form (cf. (B.4))

$$\lambda = \lambda(\xi') = 2\mathrm{i}\alpha_{33}^{-1}\alpha_{3\mu}\xi_\mu \pm \alpha_{33}^{-1/2}\sqrt{\kappa s^2 + Q_{\mu\nu}\xi_\mu\xi_\nu} \; .$$

By applying the projector $\mathcal{P}^+$ to $\mathcal{A}$ we obtain an operator with symbol

$$\boldsymbol{p}^+\boldsymbol{a} = \frac{1}{2}\begin{pmatrix} 1 & s^{-1}\alpha_{33}^{1/2}\boldsymbol{d} \\ s\alpha_{33}^{-1/2}\boldsymbol{d}^{-1} & 1 \end{pmatrix}\left(\mathrm{i}\alpha_{33}^{-1}\alpha_{3\mu}\xi_\mu + \alpha_{33}^{-1/2}\boldsymbol{d}\right) \; , \tag{C.8}$$

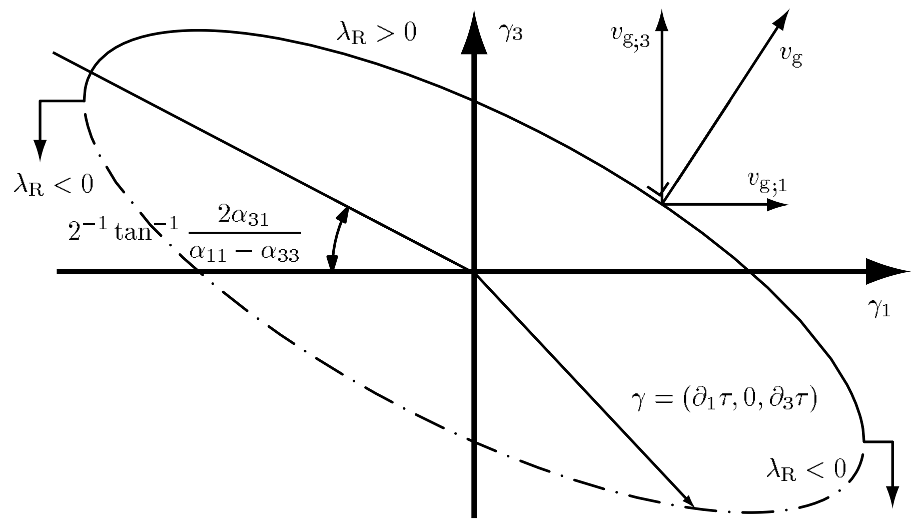

FIG. C.1. *A γ_1, γ_3 projection of the the slowness surface for fixed $\boldsymbol{x}$ and the restriction of γ_1 such that γ_3 remains real. The doted (lower) line corresponds to the negative sign root of the characteristic equation. The local group velocity is in the figure denoted by v_g.*

where

$$d = \sqrt{s^2 \kappa + Q_{\mu\nu} \xi_\mu \xi_\nu} \ . \tag{C.9}$$

The spectrum of the composition is the set of λ' that satisfy the equation

$$\lambda' \left(\lambda' - \mathrm{i}\alpha_{33}^{-1} \alpha_{3\mu} \xi_\mu - \alpha_{33}^{-1/2} \sqrt{s^2 \kappa + Q_{\mu\nu} \xi_\mu \xi_\nu} \right) = 0 \ , \tag{C.10}$$

where λ' is considered as a function of ξ' for fixed s. Apart from $\lambda' = 0$ we retain the subset of the spectrum of $\mathcal{A}$ with *positive* real part, and hence we draw the conclusion that $\mathcal{P}^+$ projects out the elements of the spectrum of $\mathcal{A}$ with positive real part. In inhomogeneous media we note that the above analysis holds microlocally. Hence the operator $\mathcal{P}^+$ projects out wave constituents that travel with positive local vertical group velocity.

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