

Generalized characteristics and propagation of singularities for a conservative hyperbolic equation with a discontinuous coefficient

Günther Hörmann* and Maarten V. de Hoop†

* *Visiting from Institute of Mathematics, University of Vienna, Vienna, Austria*

† *Center for Wave Phenomena, Colorado School of Mines, Golden, CO 80401*

ABSTRACT

In this paper, we are concerned with developing an analysis to understand wave propagation, in particular propagation of singularities, in complex media. Such analyses have been established in smoothly varying media and media with singularities across isolated interfaces (‘standard’ media). We will introduce a novel theory by means of an example, in which the implications of the theory are more easily understood. In the example, the medium contains a single step (Heaviside) singularity. Our analysis, however, applies to ‘non-standard’ media with a far higher degree of complexity than the step singularity, viz. media described by general Schwartz distributions – including fractals and multifractals. The theory is built on Colombeau algebras of generalized functions. Such algebras are not only appropriate for our analysis of (wave) solutions, but also provide an explicit modeling procedure of complex media

1 INTRODUCTION

In this paper, we are concerned with developing an analysis to understand wave propagation, in particular propagation of singularities, in complex media. Such analyses have been established in smoothly varying media and media with singularities across isolated interfaces (‘standard’ media). However, in various (geo)physical applications ‘non-standard’, such as fractal and multifractal, media occur (for example, sedimentary sequences in the upper crust of the Earth).

We will introduce a novel theory by means of an example, in which the implications of the theory are more easily understood. In the example, the medium contains a single step (Heaviside) singularity. We will introduce a global (wave) solution and introduce an appropriate micolocal analysis to generalize concepts such as (bi)characteristics and wave fronts. However, our analysis should apply to media with a far higher degree of complexity than the step singularity, viz. media described by general Schwartz distributions – including fractals and multifractals.

The theory is built on Colombeau algebras of generalized functions. Such algebras are not only appropriate for our analysis of (wave) solutions, but also provide an explicit modeling procedure

of complex media, exhibiting scaling behavior. Colombeau’s analysis provides a delicate treatment of (preserving and ‘detecting’) singularities.

The wave equation we will employ is a ‘one-way’ wave equation that can be thought of as being obtained after a directional decomposition procedure has been applied to the full-wave equation. Thus, we focus on the transmission of transient acoustic waves. General source distributions and signatures are allowed through the initial value condition. Throughout the analysis, we will ensure that the ‘standard’ results are contained in our ‘non-standard’ ones.

In applications, the propagating singularities contain leading-order information about the medium. Hence, their understanding is of key importance to remote sensing, i.e. in data processing and inverse scattering.

1.1 The one-dimensional problem

We study a simple 1 + 1-dimensional model represented by the following linear initial value problem with a discontinuous coefficient

$$\partial_t u - \partial_x (R(x)u) = 0 \tag{1}$$

$$u|_{t=0} = a. \tag{2}$$

We assume the coefficient to be of the form $R(x) = c_1 H(-x) + c_2 H(x)$ where H is the Heaviside function, i.e., $H(x) = 1$ for $x > 0$ and $H(x) = 0$ for $x < 0$, c_1 and c_2 are real nonnegative constants, and a is some distribution in \mathbb{R} .

1.3 Mathematical reflections on the above problem

The model under investigation falls into a class of mathematical situations that involve all of the following at the same time

- differentiation: here in form of the differential equation as basic model assumption
- singular objects (e.g., measures, discontinuous functions, fractals, stochastic processes ... as typical examples of distributions): here to appear possibly as initial value a and coefficient R
- nonlinear operations: here determined by the mapping of initial value and coefficient (a, R) to the solution u ; in particular, multiplication of the coefficient with the solution is explicitly contained.

Since the above model should only serve as a very simple test case for more realistic models to be considered in the future we refrain from the invention of *ad-hoc methods* for the solution of (1)-(2). Instead we wish to employ a suitable and systematic mathematical strategy which is flexible enough to allow for a wide range of future applications. The following short summary is based on the presentation in (17).

In developing such a mathematical theory the list above could be reformulated systematically using a kind of differential algebra of singular objects. We can formulate our “wish list” in the following way: let Ω be an open subset of \mathbb{R}^n and $(\mathcal{A}(\Omega), +, \circ)$ an associative, commutative algebra with the following properties:

- (i) there is a linear imbedding $\mathcal{D}'(\Omega) \hookrightarrow \mathcal{A}(\Omega)$, and the constant function 1 is the unity in $\mathcal{A}(\Omega)$
- (ii) there are n derivation operators $\partial_1, \dots, \partial_n$ on $\mathcal{A}(\Omega)$, that is, linear maps satisfying the Leibniz rule
- (iii) $\partial_j|_{\mathcal{D}'(\Omega)}$, $j = 1, \dots, n$, coincides with the usual partial derivative
- (iv) \circ restricted to $L_{loc}^\infty \times L_{loc}^\infty$ coincides with the usual product of functions.

It is the content of the famous *impossibility result* of L. Schwartz (19) that no algebra $\mathcal{A}(\Omega)$ can exist having all of the above properties. Even worse it is also impossible to have such an algebra if (iv) is weakened to: \circ restricted to $C^k(\Omega) \times C^k(\Omega)$ coincides with the usual product for some $k \in \mathbb{N}_0$.

Therefore logically there are two possible principal directions to keep consistent with the impossibility result while developing *nonlinear theories of generalized functions*:

- (a) defining products on subspaces of $\mathcal{D}'(\Omega)$ or for certain pairs of distributions only
- (b) imbedding distributions into algebras, but giving up one or the other of the properties in the “wish list”.

A good source for an overview about various approaches for (a) and an organization of a coherent hierarchy of distributional products is (16), Ch. I. In section 2 we will make use of some methods in this context. We present these shortly in the following

Example 1. [Some distributional products]

(i) A powerful and even geometrically invariant form of distributional product is Hörmander’s *wave front set criterion* (introduced in (7), see also (8), Thm. 8.2.10). Let u, v be distributions over an open subset $\Omega \subseteq \mathbb{R}^n$ and denote by $T^*\Omega \setminus 0$ the cotangent bundle over Ω with the zero section removed. Then the product $u \cdot v \in \mathcal{D}'(\Omega)$ can be defined as pullback of the tensor product $u \otimes v \in \mathcal{D}'(\Omega \times \Omega)$ via the diagonal map $x \mapsto (x, x)$, $\Omega \rightarrow \Omega \times \Omega$, unless there is $(x, \xi) \in T^*\Omega \setminus 0$ such that $(x, \xi) \in \text{WF}(u)$ and $(x, -\xi) \in \text{WF}(v)$. For example in one dimension $(x + i0)^{-1} \cdot (x + i0)^{-1}$ exists in this sense but $\text{sgn } x \cdot \text{sgn } x$, $H \cdot \delta$, and $(x + i0)^{-1} \cdot (x - i0)^{-1}$ do not exist.

(ii) A more general strategy than the above one is to define products by localization and \mathcal{S}' -convolution of the Fourier transforms. This *Fourier product* of distributions u and v over Ω is defined if for all test functions $\varphi \in \mathcal{D}(\Omega)$ the Fourier transforms $\mathcal{F}(\varphi u)$ and $\mathcal{F}(\varphi v)$ are \mathcal{S}' -convolvable in the sense of (4) (see also (16), Sect. 6). It is then given locally near $x_0 \in \Omega$ as $(\varphi u)(\varphi v) = \mathcal{F}^{-1}(\mathcal{F}(\varphi u)\mathcal{F}(\varphi v))$ for $\varphi = 1$ near x_0 . If $u \cdot v$ exists in the sense of Hörmander then it exists also as Fourier product and the results are compatible ((16), Prop. 6.3). $\text{sgn } x \cdot \text{sgn } x$ exists as Fourier product but $H \cdot \delta$ and $(x + i0)^{-1} \cdot (x - i0)^{-1}$ do not exist.

(iii) A completely different idea is regularization of one or both factors and passage to the distributional limit if it exists. For example we can choose a net $(u_\varepsilon)_{\varepsilon>0}$ of smooth functions such that $u_\varepsilon \rightarrow u$ in \mathcal{D}' as $\varepsilon \rightarrow 0$ and consider $u_\varepsilon v$. If for all test functions ψ there exists a limit of $\langle u_\varepsilon v, \psi \rangle$ as $\varepsilon \rightarrow 0$ then this defines a distribution which serves as a reasonable candidate for $u \cdot v$. All types of the so-called strict and model products are of this kind with variations concerning the classes of admissible regularizations and the factors to be regularized. These are more general and coherent with the above two products. As an example $H \cdot \delta$ exists as a type-(iv) strict product but $(x + i0)^{-1} \cdot (x - i0)^{-1}$ does not exist in any version of these products.

It will turn out in subsection 2.1 that the method of choice for us will be approach (b), i.e. the framework of algebras of generalized functions. We devote the next subsection to the basic definitions and a short introduction to that subject.

1.4 Algebras of generalized functions

We will describe some aspects of what we called approach (b) within nonlinear theories of generalized functions. In this short presentation we will focus on *Colombeau algebras* which are particularly useful for our purposes because they allow for a refined regularity theory and even microlocal analysis. A very readable short overview for a wide range of different algebra concepts can be found in (17), a more recent proceedings volume on the subject is (6).

The basic idea can be motivated historically by Mikusiński's early sequential approach to distribution theory (12). It is based on the idea to represent a distribution by sequences (or rather nets) of approximating smooth functions. Let E be an infinite index set and consider as a general frame the space of all nets of smooth functions indexed by E (we will often simply refer to its elements as sequences by slight abuse of language)

$$\mathcal{X}(\mathbb{R}^n) = \{u : E \rightarrow C^\infty(\mathbb{R}^n)\} .$$

We will denote the elements in $\mathcal{X}(\mathbb{R}^n)$ by $(u_\varepsilon)_{\varepsilon \in E}$. For the moment let us consider $E = (0, 1)$ as our index set.

Next we single out among all sequences those which are convergent in the distributional sense

$$\mathcal{Y}(\mathbb{R}^n) = \{(u_\varepsilon)_\varepsilon \in \mathcal{X}(\mathbb{R}^n) \mid u_\varepsilon \text{ converges in } \mathcal{D}'(\mathbb{R}^n) \text{ as } \varepsilon \rightarrow 0\} .$$

Clearly $\mathcal{Y}(\mathbb{R}^n)$ is a linear subspace of $\mathcal{X}(\mathbb{R}^n)$. To represent a particular distribution in a unique way we have to identify all sequences which converge to it. This can be expressed by first defining the negligible sequences

$$\mathcal{I}(\mathbb{R}^n) = \{(u_\varepsilon)_\varepsilon \in \mathcal{X}(\mathbb{R}^n) \mid u_\varepsilon \rightarrow 0 \text{ in } \mathcal{D}'(\mathbb{R}^n) \text{ as } \varepsilon \rightarrow 0\}$$

and then passing to the quotient vector space which turns out to be isomorphic to the space of distributions

$$\mathcal{D}'(\mathbb{R}^n) \cong \mathcal{Y}(\mathbb{R}^n) / \mathcal{I}(\mathbb{R}^n) .$$

Therefore on the level of linear distribution theory we have an alternative approach via regularization. This is implicitly used very often in practical computations with distributions.

If we reconsider the space $\mathcal{X}(\mathbb{R}^n)$ we realize that it can easily be equipped with the structure of a differential algebra. We only have to define a multiplication

$$(u_\varepsilon)_{\varepsilon \in E} \circ (v_\varepsilon)_{\varepsilon \in E} := (u_\varepsilon v_\varepsilon)_{\varepsilon \in E}$$

and partial differentiation

$$\partial_j (u_\varepsilon)_{\varepsilon \in E} := (\partial_j u_\varepsilon)_{\varepsilon \in E} .$$

Of course this space will usually be too large and abstract for many applications. But we also note that it is straightforward to construct imbeddings of \mathcal{D}' into $\mathcal{X}(\mathbb{R}^n)$ via choice of a sequence $(\rho_\varepsilon)_{\varepsilon \in E}$ of compactly supported smooth functions which converges to the Dirac measure δ at 0 when $\varepsilon \rightarrow 0$. Then the mapping $w \mapsto (w * \rho_\varepsilon)_{\varepsilon \in E}$ defines an imbedding $\mathcal{D}'(\mathbb{R}^n) \rightarrow \mathcal{X}(\mathbb{R}^n)$.

The art of defining a useful algebra of generalized functions in this way is to choose an appropriate subalgebra of $\mathcal{X}(\mathbb{R}^n)$, typically considerably larger than $\mathcal{Y}(\mathbb{R}^n)$, and a new set of negligible sequences which

happens to be an ideal in the chosen subalgebra and is stable under differentiation. Upon building the corresponding quotient we get a differential algebra which we hope to come as close as possible to the special needs in certain applications.

For our purposes the following version turns out to be most adequate. J.F. Colombeau (2) could show that one can construct an associative and commutative algebra having properties (i)-(iii) of the “wish list” and in addition

(iv)' $\circ : C^\infty(\Omega) \times C^\infty(\Omega)$ coincides with the usual product.

What makes *Colombeau algebras* particularly attractive for the study of models like (1)-(2) are this kind of maximal compatibility with classical operations and their unique features concerning the existence of an intrinsic regularity theory and recently also microlocal analysis (to be discussed in subsection 2.2). Furthermore, there is the remarkable characterization of *Colombeau functions* by evaluation on generalized point values (cf. (18)) which resembles the classical idea of functions in a nice way. Last but not least there are strong existence and uniqueness results for hyperbolic partial differential equations with generalized functions as coefficients which provide the theoretical background of most of the investigations in this paper (see subsection 3.1 for a short discussion).

We recall the basic definitions and notions of Colombeau theory. For a detailed presentations of the general theory, its role in generalized function theory, and a review of applications we refer to (1, 16, 6, 13).

The parameter set for the regularizations defining Colombeau generalized functions is given by the following cascade of normalized test function sets with vanishing moment conditions: for $q \in \mathbb{N}_0$ define

$$\begin{aligned} \mathcal{A}_q(\mathbb{R}) &= \{ \chi \in \mathcal{D}(\mathbb{R}) \mid \int \chi(x) dx = 1, \int x^k \chi(x) dx = 0 \ (1 \leq k \leq q) \} \\ \mathcal{A}_q(\mathbb{R}^n) &= \{ \phi \in \mathcal{D}(\mathbb{R}^n) \mid \exists \chi \in \mathcal{A}_q(\mathbb{R}^n) : \phi(x_1, \dots, x_n) = \chi(x_1) \cdots \chi(x_n) \} \end{aligned}$$

To incorporate real scaling parameters for asymptotic conditions we define for $\phi \in \mathcal{A}_0(\mathbb{R}^n)$

$$\phi_\varepsilon(x) = \varepsilon^{-n} \phi(x/\varepsilon) \quad \varepsilon > 0 .$$

Note that as distributions $\phi_\varepsilon \rightarrow \delta_0$ as $\varepsilon \rightarrow 0$, in particular the support of ϕ_ε shrinks to the single point $\{0\}$. A useful general tool to couple the scaling with the shrinking of the support of ϕ is the *support number*

$$l(\phi) := \sup\{x \in \mathbb{R}^n \mid \phi(x) \neq 0\}$$

which is simply the radius of the smallest closed ball containing $\text{supp}(\phi)$. Note that we have $l(\phi_\varepsilon) = \varepsilon l(\phi)$.

As in the general construction we start out with the set of all maps from the index set, which is now $E = \mathcal{A}_0(\mathbb{R}^n)$, into the smooth functions over an open subset Ω

$$\mathcal{E}[\Omega] = \{ R : \mathcal{A}_0(\mathbb{R}^n) \rightarrow C^\infty(\Omega) \} .$$

This is a differential algebra with component-wise operations, i.e. all operations are reduced to the classical operations at fixed ϕ . We single out certain subalgebras of nets subject to asymptotic conditions uniformly on compact sets, the moderate and the negligible nets:

$$\begin{aligned} \mathcal{E}_M[\Omega] = \left\{ R \in \mathcal{E}[\Omega] \mid \forall K \subset \Omega \text{ compact}, \forall \alpha \in \mathbb{N}_0^n : \exists N \in \mathbb{N} : \right. \\ \left. \forall \phi \in \mathcal{A}_N(\mathbb{R}^n) : \sup_{x \in K} |\partial^\alpha R(\phi_\varepsilon, x)| = O(\varepsilon^{-N}) \quad (\varepsilon \rightarrow 0) \right\} \end{aligned} \tag{3}$$

$$\begin{aligned} \mathcal{N}[\Omega] = \left\{ R \in \mathcal{E}_M[\Omega] \mid \forall K \subset \Omega \text{ compact}, \forall \alpha \in \mathbb{N}_0^n : \exists N \in \mathbb{N} : \right. \\ \left. \forall \phi \in \mathcal{A}_N(\mathbb{R}^n) \forall q \geq N : \sup_{x \in K} |\partial^\alpha R(\phi_\varepsilon, x)| = O(\varepsilon^{q-N}) \quad (\varepsilon \rightarrow 0) \right\} \end{aligned} \tag{4}$$

Equivalently, in the definition of $\mathcal{N}[\Omega]$ one may neglect to check the growth conditions for derivatives of order ≥ 1 (as long as R is known to be an element in $\mathcal{E}_M[\Omega]$; cf. (5)).

$\mathcal{N}[\Omega]$ is an ideal in $\mathcal{E}_M[\Omega]$ hence we may form the quotient algebra

$$\mathcal{G}(\Omega) = \mathcal{E}_M[\Omega] / \mathcal{N}[\Omega]$$

which is called the *Colombeau algebra* over Ω . It is again a differential algebra where operations are reduced

to component-wise operations on representatives. We will use the notation $U = \text{cl}[(u(\phi, \cdot))_\phi]$ to indicate that $U \in \mathcal{G}$ has the representative $u \in \mathcal{E}_M$.

The Colombeau algebra $\mathcal{G}(\Omega)$ has localization properties which guarantees existence of restrictions to open subsets and a meaningful notion of *generalized support*: if U is a Colombeau function then $\text{supp}_g(U)$ is the complement of the largest open subset $X \subseteq \Omega$ such that $U|_X = 0$ in $\mathcal{G}(X)$.

There is a canonical embedding of distributions $\iota : \mathcal{D}'(\Omega) \rightarrow \mathcal{G}(\Omega)$ (as a linear subspace) which in the case of $\Omega = \mathbb{R}^n$ is simply given by

$$\iota(f) = \text{cl}[(f * \phi)_\phi] \quad f \in \mathcal{D}'(\mathbb{R}^n).$$

For Ω arbitrary one first defines this map on the space $\mathcal{E}'(\Omega)$ of distributions with compact support in Ω and then extends by suitable cut-off over any given compact set $K \subset \Omega$ (cf. (16), Sect. III.9).

Restricted to smooth functions this map is also consistent with the “constant” embedding of smooth functions $\sigma : C^\infty(\Omega) \rightarrow \mathcal{G}(\Omega)$,

$$\sigma(f) = \text{cl}[(f)_\phi] \quad f \in C^\infty \quad (\iota(f) = \sigma(f) \text{ in this case}).$$

Some Colombeau functions can be projected to distributions (these are said to have “distributional shadows”). We say that $U \in \mathcal{G}$ is *associated* with $w \in \mathcal{D}'$, notation $U \approx w$, if $U = \text{cl}[(u(\phi, \cdot))_\phi]$ and $\forall \psi \in \mathcal{D} \exists N \in \mathbb{N}$:

$$\lim_{\varepsilon \rightarrow 0} \int u(\phi_\varepsilon, x) \psi(x) dx = \langle w, \psi \rangle \quad \forall \psi \in \mathcal{A}_N.$$

If $U, V \in \mathcal{G}$ then we define $U \approx V$ if $U - V$ is associated with (the distribution) 0. (This is an equivalence relation.) The following properties are immediate: $\iota(w) \approx w$; if $U \approx V$ then $\partial^\alpha U \approx \partial^\alpha V$ and $fU \approx fV$ for $f \in C^\infty$.

Since we want to study Cauchy problems in \mathbb{R}^{m+1} we have to define the restriction of a Colombeau function to a coordinate hyperplane: let $V \in \mathcal{G}(\mathbb{R}^{m+1})$ then if $V = \text{cl}[(v(\phi, \cdot))_\phi]$ and $\phi^{(l)}(x_1, \dots, x_l) := \phi_0(x_1) \cdots \phi_0(x_l)$ for all $\phi_0 \in \mathcal{A}_0(\mathbb{R})$ we define

$$V|_{x_{m+1}=0} = \text{cl}[(v(\phi_0^{(m+1)}, \cdot))_{\phi_0^{(m)}}].$$

2 MODELING PROCEDURE FOR THE HYPERBOLIC EQUATION AND THE MEDIUM PARAMETERS

2.1 Nonexistence of global distributional solutions

A natural classical approach to solve (1)-(2) is to split this into two subproblems (with then constant coefficients) on either side of the jump at $x = 0$ and try to globally adjust the solution afterwards with boundary conditions. However, this is in general impossible within the framework of distribution theory, as already observed in the case $c_1 = 0$, $c_2 = 1$, and $a = 1$ by Hurd and Sattinger (10) with an L_{loc}^∞ -solution concept (see also the short discussion in (16), example 1.7). Essentially, the reason is that combination of the subproblems suggests to consider $u(x, t) = 1 + t\delta(x)$, δ the Dirac delta at 0, as the global solution in $t \geq 0$. But this is not in L_{loc}^∞ and leaves us with the problem of consistently defining $H \cdot \delta$ and then differentiating this object upon checking if u actually solves the above equation. To do so, we want to use one of the notions of \mathcal{D}' -products according to the hierarchy of consistent extensions given in (16), p. 69. Within this hierarchy we find that $H \cdot \delta$ exists only on the most general level of the strict products (and consequently in all higher levels), mentioned already in example 1, (iii), yielding the value $[H \cdot \delta] = \delta/2$ (cf. (16), examples and exercises 7.12). But a simple computation (carefully writing $u = 1 + \delta(x) \otimes t$) shows that $\partial_t u - \partial_x([H \cdot u]) = -t\delta'(x)/2 \neq 0$ and therefore u does not solve the equation globally.

We are going to show that from the point of view of *microlocal analysis* a sort of the above Hurd-Sattinger effect is unavoidable: if $0 \leq c_1 < c_2$ then almost all physically reasonable initial data will lead to inconsistencies of the above type.

In order to simplify the following discussion we will use henceforth the following assumption about the initial or source value:

(o) $a \in \mathcal{D}'(\mathbb{R})$ and a is smooth near 0.

This is valid in particular if the source of the model experiment is concentrated on one side of the medium jump at $x = 0$; this additional assumption enables us to observe more explicitly what is happening at the discontinuity region of the model medium.

To give a precise meaning to the question of existence of distributional solutions for the situation in (1-2) we start with a detailed list of typical mathematical requirements for a distribution $u \in \mathcal{D}'(\mathbb{R}^2)$ to be considered a global solution.

(i) u is continuous in time, i.e. can be considered to be an element of the space $C(\mathbb{R}, \mathcal{D}'(\mathbb{R}))$; in particular, this gives the right to speak of its initial value at $t = 0$ in the distributional sense. This is natural in the framework of hyperbolic Cauchy problems and could be weakened. E.g., we could assume u to be restrictable to $t = 0$ in the sense of (8), Cor. 8.2.7, which would imply the same continuity property of u near the x -axis (cf. the remark in (20), after Prop. 6.11). Furthermore, note that together with requirement (vi) below the continuity with respect to t actually yields $u \in C^\infty(\mathbb{R}, \mathcal{D}'(\mathbb{R}))$.

(ii) $u|_{t=0} := u(0) = a$.

(iii) in the left half space $V_- = \{(x, t) \mid x < 0\}$ we have $\partial_t u = c_1 \partial_x u$ (according to (1) when restricted to V_-). This is actually a consequence of (v) and (vi) below but we prefer to include this redundancy to emphasize coherence with intuition.

(iv) in the right half space $V_+ = \{(x, t) \mid x > 0\}$ we have $\partial_t u = c_2 \partial_x u$ (according to (1) when restricted to V_+). As with (iii) this will also be a consequence of (v)-(vi) below.

(v) the distributional product $(R(x) \otimes 1) \cdot u$ should exist in $\mathcal{D}'(\mathbb{R}^2)$ in the sense of Hörmander ((8), Thm. 8.2.10); in other words the wave front sets of $R \otimes 1$ and u must not contain opposite cotangent directions over the same base point. Note that $\text{WF}((R(x) \otimes 1)) = \{0\} \times \mathbb{R} \times (\mathbb{R} \times \{0\}) \setminus \{(0, 0)\}$ (cf. (8), Thm. 8.2.9, also p. 269) and has therefore exactly horizontal cotangent directions above the t -axis. Therefore we conclude that within the microlocal setting the existence of this product is equivalent to the existence of the restriction $u|_{x=0}$ (cf. (8), Cor. 8.2.7). This in turn enables us to reformulate the transmission at the medium singularity as a boundary value problem. It furthermore implies (as remarked similarly in (i)) that locally near $x = 0$ we may consider u to be continuous in x and distributional in time. (Note that the original Hurd-Sattinger example provides a hint that weakening this requirement within the coherent product hierarchy (16), p. 69, does not lead out of the difficulties, even if the initial values are smooth.)

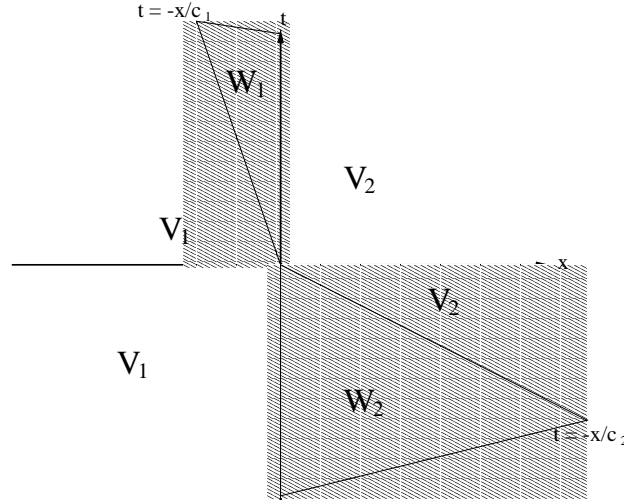
(vi) finally, we require the equation to be satisfied globally in the sense of $\mathcal{D}'(\mathbb{R}^2)$: $\partial_t u - \partial_x((R(x) \otimes 1) \cdot u) = 0$

Theorem 2. [] Assume $0 \leq c_1 < c_2$ and that a satisfies assumption (o) above. Then there is no distribution $u \in \mathcal{D}'(\mathbb{R}^2)$ satisfying all of the requirements (i)-(vi) above unless $a = 0$ or $c_1 = 0$ and $a = 0$ in $(0, \infty)$. In this sense, there is no distributional solution to problem (1)-(2) for nontrivial initial values.

Proof. For the proof of nonexistence we distinguish two cases according to the value of c_1 .

Case $c_1 > 0$: assume that u is a distribution satisfying (i)-(vi). Within the regions $V_1 := \{(x, t) \mid x < 0, x + c_1 t < 0\}$, $V_2 := \{(x, t) \mid x > 0, x + c_2 t > 0\}$ the Cauchy data and the equation determine the solution u to be $a(x + c_1 t)$, $a(x + c_2 t)$ respectively (in the sense of pull-back of a by the functions $(x, t) \mapsto x + c_1 t$, $(x, t) \mapsto x + c_2 t$). These distributions can be considered as smooth maps in the x -variable with distributional values in t -space and vice versa. By (v) the boundary value of u at $x = 0$ exists and can be used to determine u in the open wedges $W_1 := \{(x, t) \mid x < 0, x + c_1 t > 0\}$, $W_2 := \{(x, t) \mid x > 0, x + c_2 t < 0\}$.

We set $b := u|_{x=0}$ which by the local continuity with respect to x may be computed as one-sided limits $\lim_{x \rightarrow 0^+} u(x, \cdot)$ or $\lim_{x \rightarrow 0^-} u(x, \cdot)$ as distributions in the t -variable. For $t > 0$ we take the limit from the right and obtain $b(t) = a(c_2 t)$ as initial value for a Cauchy problem in W_1 where the roles of x and t are interchanged. This yields the formula $u|_{W_1} = a((x + c_1 t)c_2/c_1)$ (again in the sense of the pull-back of a via $(x, t) \mapsto (x + c_1 t)c_2/c_1$). Similarly, we obtain $u|_{W_2} = a((x + c_2 t)c_1/c_2)$ by considering for $t < 0$ the limit from the left.



Making use of the coherence properties in the product hierarchy in (16), Sect. 7, we may compute the product $(R \otimes 1) \cdot u$ as a strict product where only one factor is regularized and then pass to the limit $\varepsilon \rightarrow 0$. We choose a net of smooth functions $(R_\varepsilon)_{\varepsilon > 0}$ such that $R_\varepsilon \rightarrow R$ in $\mathcal{D}'(\mathbb{R})$ as $\varepsilon \rightarrow 0$. We will discuss only the upper half space $t > 0$, the case $t < 0$ is completely analogous.

Since $R \otimes 1$ is constant away from the t -axis we focus on the more interesting part of the upper half space near the axis. Let ψ be a test function on \mathbb{R}^2 having support near the positive t -axis and not intersecting V_1 . Then using the above formulae for u and considering it as a continuous map $x \mapsto u(x, \cdot)$, $\mathbb{R} \rightarrow \mathcal{D}'(\mathbb{R})$, we have

$$\begin{aligned} \langle (R_\varepsilon \otimes 1) \cdot u, \psi \rangle &= \langle u(x, t), R_\varepsilon(x) \psi(x, t) \rangle \\ &= \int_{-\infty}^0 \langle a(\frac{c_2}{c_1}(x + c_1 \cdot)), \psi(x, \cdot) \rangle R_\varepsilon(x) dx + \int_0^\infty \langle a(x + c_2 \cdot), \psi(x, \cdot) \rangle R_\varepsilon(x) dx . \end{aligned}$$

Both integrands are continuous functions (with respect to x), have support within a fixed compact set independent of ε , and have a pointwise limit as $\varepsilon \rightarrow 0$. By dominated convergence we get

$$\langle (R \otimes 1) \cdot u, \psi \rangle = c_1 \int_{-\infty}^0 \langle a(\frac{c_2}{c_1}(x + c_1 \cdot)), \psi(x, \cdot) \rangle dx + c_2 \int_0^\infty \langle a(x + c_2 \cdot), \psi(x, \cdot) \rangle dx .$$

The final check if u is a solution is requirement (vi). For a test function ψ as above we recall that

$$\langle u, \psi \rangle = \int_{-\infty}^0 \langle a(\frac{c_2}{c_1}(x + c_1 \cdot)), \psi(x, \cdot) \rangle dx + \int_0^\infty \langle a(x + c_2 \cdot), \psi(x, \cdot) \rangle dx$$

and therefore (with $\partial_j \psi$ ($j = 1, 2$) denoting the derivative of ψ with respect to its first or second argument)

$$\langle \partial_t u, \psi \rangle = - \int_{-\infty}^0 \langle a(\frac{c_2}{c_1}(x + c_1 \cdot)), \partial_2 \psi(x, \cdot) \rangle dx - \int_0^\infty \langle a(x + c_2 \cdot), \partial_2 \psi(x, \cdot) \rangle dx$$

and

$$\langle \partial_x ((R \otimes 1) \cdot u), \psi \rangle = -c_1 \int_{-\infty}^0 \langle a(\frac{c_2}{c_1}(x + c_1 \cdot)), \partial_1 \psi(x, \cdot) \rangle dx - c_2 \int_0^\infty \langle a(x + c_2 \cdot), \partial_1 \psi(x, \cdot) \rangle dx .$$

We can rewrite the first integrand in $\langle \partial_t u, \psi \rangle$ as follows:

$$\begin{aligned} -\langle a(\frac{c_2}{c_1}(x + c_1 \cdot)), \partial_2 \psi(x, \cdot) \rangle &= c_1 \frac{c_2}{c_1} \langle a'(\frac{c_2}{c_1}(x + c_1 \cdot)), \psi(x, \cdot) \rangle \\ &= c_1 \frac{d}{dx} \langle a(\frac{c_2}{c_1}(x + c_1 \cdot)), \psi(x, \cdot) \rangle - c_1 \langle a(\frac{c_2}{c_1}(x + c_1 \cdot)), \partial_1 \psi(x, \cdot) \rangle . \end{aligned}$$

If we transform the second integrand in $\langle \partial_t u, \psi \rangle$ in the same way and compare with $\langle \partial_x((R \otimes 1) \cdot u), \psi \rangle$ we end up with

$$\langle \partial_t u - \partial_x((R \otimes 1)u), \psi \rangle = c_1 \langle a(c_2 \cdot), \psi(0, \cdot) \rangle - c_2 \langle a(c_2 \cdot), \psi(0, \cdot) \rangle .$$

But this means that

$$\partial_t u - \partial_x((R \otimes 1)u) = (c_1 - c_2) \delta \otimes a(c_2 \cdot)$$

which is not zero unless $a = 0$ in $(0, \infty)$.

The arguments near the negative t -axis are analogous and yield that u cannot satisfy (vi) unless $a = 0$ in $(-\infty, 0)$. Since a is assumed to be smooth near 0 we conclude that this would force $a = 0$ globally.

Case $c_1 = 0$: assume that u is a distribution satisfying (i)-(vi) and without loss of generality that $c_2 = 1$. Set $c(x, t) = x + t$ then $a \otimes 1$, i.e. $a(x)$, and c^*a , i.e. $a(x + t)$, solve the problem in the left half space V_1 and V_2 respectively (notation as in the earlier case; note that now $W_1 = \emptyset$). By smoothness of a near 0 and since $\text{WF}(c^*a)$ cannot contain horizontal cotangent directions we can define the distributions

$$\begin{aligned} u_1 &= (H(-x) \otimes 1) \cdot (a \otimes 1) \\ u_2 &= (H(x) \otimes 1) \cdot c^*a \\ w &= u_1 + u_2 . \end{aligned}$$

We have $u|_{V_j} = u_j|_{V_j}$ ($j = 1, 2$) and therefore $u - w|_{\{t \geq 0, x \neq 0\}} = 0$. Furthermore, the u_j are smooth functions of time with values in \mathcal{D}' over x -space, hence w and $u - w$ are also (using the final remark in requirement (i)).

We can give two variants how to proceed with the nonexistence proof. Variant (a) shows that (i)-(iv) and (v) contradicts (vi) by making use of the product hierarchy. Variant (b) is completely in the spirit of classical microlocal analysis and will show that (i)-(iv) and (vi) contradict (v). Anyway, we see that under assumptions (o) and (i)-(iv) the requirements (v) and (vi) mutually exclude each other.

Variant (a): the arguments are similar to the first case. By assumption (v) u can be considered to be continuous in x and distributional in t near the t -axis. In particular, approaching the positive t -axis from the left this implies that $a(x)$ is a continuous function for small x which clearly is consistent with assumption (o). Let ψ be a test function with support concentrated near the positive t -axis so that u can be considered continuous in x and distributional in t there. Then we have

$$\langle u, \psi \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^0 a(x) \psi(x, t) dx dt + \int_0^{\infty} \langle a(x + \cdot), \psi(x, \cdot) \rangle dx .$$

As in the first case we may compute the product $(R \otimes 1) \cdot u$ as a strict product with one factor regularized. A similar computation as in that case gives

$$\langle (R \otimes 1) \cdot u, \psi \rangle = \int_0^{\infty} \langle a(x + \cdot), \psi(x, \cdot) \rangle dx .$$

Simple computations show that

$$\begin{aligned} \langle \partial_t u, \psi \rangle &= - \int_0^\infty \langle a(x + \cdot), \partial_2 \psi(x, \cdot) \rangle dx \\ \langle \partial_x ((R \otimes 1) \cdot u), \psi \rangle &= - \int_0^\infty \langle a(x + \cdot), \partial_1 \psi(x, \cdot) \rangle dx . \end{aligned}$$

Rewriting

$$-\langle a(x + \cdot), \partial_1 \psi(x, \cdot) \rangle = -\frac{d}{dx} \langle a(x + \cdot), \psi(x, \cdot) \rangle - \langle a(x + \cdot), \partial_2 \psi(x, \cdot) \rangle$$

we arrive at

$$\langle \partial_t u - \partial_x ((R \otimes 1) \cdot u), \psi \rangle = -\langle a, \psi(0, \cdot) \rangle = -\langle \delta \otimes a, \psi \rangle$$

which is nonzero unless $a = 0$ in $(0, \infty)$.

Remark 3. [An observant reader might think that this last local result $\partial_t u - \partial_x (R \cdot u) = -\delta \otimes a$ shows a discrepancy with the discussion of the original Hurd-Sattinger example at the beginning of this section where we obtained $\partial_t u - \partial_x (R \cdot u) = -\delta' \otimes t/2$ if $a = 1$. But note that in the current proof we actually assumed the critical product $R \cdot u$ to exist on a much lower generality level in the product hierarchy: we assumed the WF-condition to hold and then used the simplest type of so-called strict products to conveniently compute it for general a ; in the special Hurd-Sattinger situation we saw that we have to use at least the highest level of strict products which also gives a different result for the product. However, the above proof is much more general with respect to the initial value a .

Variant (b): By construction for all $t > 0$ the distribution $u(t) - w(t) \in \mathcal{D}'(\mathbb{R})$ has support contained in $\{0\}$ and $u(0) - w(0) = 0$. This implies that

$$u(t) - w(t) = \sum_{k=0}^\infty c_k(t) \delta^{(k)} := u_3 \quad \text{if } t \geq 0$$

where c_k are C^∞ functions with $c_k(0) = 0$ and at each point $t \geq 0$ only finitely many c_k are nonzero. In the upper half space $t \geq 0$ we may therefore write $u = u_1 + u_2 + u_3$.

If a is constant then $w = a$ and $u = a + u_3$. If u_3 were 0 the property (vi) would imply $0 = \partial_t u = a \partial_x H(x) = a \delta(x)$, a contradiction since $a \neq 0$ by assumption. On the other hand if $u_3 \neq 0$ then $\text{WF}(u) = \text{WF}(u_3) = \text{WF}(H \otimes 1)$ and hence (v) cannot be valid.

Assume in the sequel that a is not constant. Differentiation of u in the upper half plane with respect to t yields (note that $\partial_t u_1 = 0$)

$$\partial_t u = (H \otimes 1) \cdot (c^* a') + \sum_{k=0}^\infty \delta^{(k)} \otimes c'_k . \tag{5}$$

One more differentiation with $\partial_t - \partial_x$ gives (using Leibniz rule also for the distributional product; cf. (8), p.267, bottom)

$$\begin{aligned} v &:= (\partial_t^2 - \partial_x \partial_t) u \\ &= (H \otimes 1) \cdot (c^* a'') + \sum_{k=0}^\infty \delta^{(k)} \otimes c''_k - (\delta \otimes 1) \cdot (c^* a') - (H \otimes 1) \cdot (c^* a'') - \sum_{k=0}^\infty \delta^{(k+1)} \otimes c'_k \\ &= \sum_{k=0}^\infty (\delta^{(k)} \otimes c''_k - \delta^{(k+1)} \otimes c'_k) - (\delta \otimes 1) \cdot (c^* a') . \end{aligned}$$

We will use the fact that $\text{WF}(v) \subseteq \text{WF}(\partial_t u) \subseteq \text{WF}(u)$ and show that $\text{WF}(v)$ or $\text{WF}(\partial_t u)$ contain horizontal cotangent directions at some point of the positive time axis. This will finally prove that (v) cannot be true and leave us with a contradiction.

First we derive a simple formula for the product $(\delta \otimes 1) \cdot (c^* a')$ appearing as summand in u .

Lemma 4. [If $\chi \in \mathcal{D}'(\mathbb{R})$ and $c(x, t) = x + t$ then in $\mathcal{D}'(\mathbb{R}^2)$

$$(\delta \otimes 1) \cdot c^* \chi = \delta \otimes \chi.$$

Proof of the Lemma: first we note that $\text{WF}(\delta \otimes 1) = \{0\} \times \mathbb{R} \times \mathbb{R} \times \{0\} \setminus 0$ and $\text{WF}(c^* \chi) = \{(x, t, \eta, \eta) \mid (x + t, \eta) \in \text{WF}(\chi)\}$ (apply (8), Thm.8.2.4, to $c^* \chi$ and $d^* c^* \chi$ with $d(x) = (x - d_0, d_0)$ successively) are in favorable position and therefore the product exists in the sense of Hörmander. We compute it directly by convolution on the Fourier transformed side (which is consistent within the product hierarchy by (16), Prop. 6.3): clearly $\widehat{(\delta \otimes 1)} = 2\pi(1 \otimes \delta)$; recall ((8), (6.1.1))

$$\langle c^* \chi, \varphi \rangle = \langle \chi \otimes 1, T^* \varphi \rangle = \langle ({}^t T^{-1})^*(\chi \otimes 1), \varphi \rangle = \langle S^*(\chi \otimes 1), \varphi \rangle$$

where $T(z, r) = (z - r, r)$ and $S = {}^t T^{-1}$; hence we have by simple calculation

$$\widehat{(c^* \chi \otimes 1)} = \widehat{(S^*(\chi \otimes 1))} = T^*(\widehat{(\chi \otimes 1)}) = 2\pi T^*(\widehat{\chi} \otimes \delta) = 2\pi \delta \otimes \widehat{\chi}.$$

Therefore the convolution of these two Fourier transformed distributions is trivial and proves the lemma:

$$(\delta \otimes 1) \cdot c^* \chi = \frac{1}{4\pi^2} \mathcal{F}^{-1}(\widehat{(\delta \otimes 1)} * \widehat{(c^* \chi \otimes 1)}) = \mathcal{F}^{-1}(1 \otimes \widehat{\chi}) = \delta \otimes \chi.$$

Applying the lemma we can rewrite v in the form

$$v = \delta \otimes (c'_0 - a') + \sum_{k=1}^{\infty} \delta^{(k)} \otimes (c'_k - c'_{k-1}). \tag{6}$$

If $v = 0$ for all $t \geq 0$ then $c''_0 = a'$ and $c''_k = c'_{k-1}$ ($k > 0$) there. This implies that a is smooth for positive arguments and hence by assumption (o) smooth on some interval $(-\alpha, \infty)$ with $\alpha > 0$. Let $a'(s_0) \neq 0$ for some $s_0 > 0$ then from (5) we get a local representation of $\partial_t u$ near the point $(0, s_0)$ in the form

$$\partial_t u(x, t) = \psi(x, t)H(x) + \sum_{k \geq 0} c'_k(t)\delta^{(k)}(x)$$

where ψ is smooth with $\psi(0, s_0) \neq 0$ and all c'_k are smooth. All terms in this representation contribute only horizontal cotangent directions in the wave front set. Furthermore by (vi) and the Leibniz rule we get also

$$\partial_t u = \partial_x(H(x)u) = \delta(x)u + H(x)\partial_x u. \tag{7}$$

If u were smooth at $(0, s_0)$ we could therefore derive the identity

$$0 = H(x)(\psi - \partial_x u) + \delta(x)(c'_0(t) - u) + \sum_{k \geq 1} c'_k(t)\delta^{(k)}(x)$$

which in turn implies locally $u = c'_0(t)$ yielding $\partial_x u = 0$, contradicting the necessary identity $\psi = \partial_x u$ since ψ is non-vanishing at $(0, s_0)$.

Therefore it remains to check the case where $v \neq 0$. Again all terms of v in (6) give at most horizontal cotangent directions in $\text{WF}(v)$. Also, the nonzero terms in that representation constitute a linear independent set of distributions, all of them having different order with singularities appearing only on the positive t -axis (caused only by the $\delta^{(k)}(x)$ -factors where the smooth factor is non-vanishing). Smoothness of the sum at all points on the positive axis could only occur by cancellation of all different $\delta^{(k)}$ terms which contradicts the linear independence of these in $\mathcal{D}'(\mathbb{R})$. Hence there is a point $(0, s_0)$ with $s_0 > 0$ where $\text{WF}(v)$ has horizontal cotangent directions.

Finally, for the positive *existence* results: if $a = 0$ or $c_1 = 0$ and a vanishes for positive arguments then $u = a \otimes 1$ is a solution satisfying all properties (i)-(vi). \square

It was shown by Oberguggenberger ((14), see also (16), example 17.6) that there is a remedy for this dissatisfying situation within the framework of Colombeau algebras which also makes the approach to the situation more systematic. The differential equation can be modeled as an equation with coefficient $R(x) = R(x) \otimes 1(t) \approx \Lambda$ in $\mathcal{G}(\mathbb{R}^2)$ and be rewritten in the form

$$\partial_t U - \Lambda \partial_x U = (\partial_x \Lambda)U. \tag{8}$$

Also, a rather general initial condition

$$U|_{t=0} = A \in \mathcal{G}(\mathbb{R}) \tag{9}$$

can then be prescribed. The Colombeau function Λ is given by a representative $(\lambda(\phi))_\phi \in \mathcal{E}_M$, i.e. a family of smooth functions parameterized by mollifiers $\phi \in \mathcal{A}_0$, with the property $\lambda(\phi_\varepsilon) \rightarrow R \otimes 1$ in \mathcal{D}' as $\varepsilon \rightarrow 0$. $\lambda(\phi, x, t)$ is constructed by choosing a (real valued) test function $\chi \in \mathcal{D}(\mathbb{R})$ with $\int \chi = 1$ and using a combined convolution and scaling regularization (recall from the introduction that each $\phi \in \mathcal{A}_0(\mathbb{R}^2)$ is of the form $\phi_0 \otimes \phi_0$):

$$\begin{aligned} \lambda(\phi, x, t) &= R \overset{(x)}{*} (\mu(\phi)\chi(\mu(\phi)\cdot))(x) \\ &= c_1 \mu(\phi) \int_x^\infty \chi(\mu(\phi)y) dy + c_2 \mu(\phi) \int_{-\infty}^x \chi(\mu(\phi)y) dy \end{aligned} \tag{10}$$

where $\mu(\phi_0 \otimes \phi_0) = \log(1/l(\phi_0))$ (here, $l(\phi_0)$ is the support number as defined in the introduction). Note that this gives a scaling factor of $\log(1/\varepsilon) - \log(l(\phi_0))$ when evaluated for ϕ_ε (the reason behind the choice of this scaling will become clear by a short digression into theory in Sect. 3; cf. Rem. 11 and Thm. 12). Clearly $R(x)$ is the weak limit of $\lambda(\phi, x, t)$ as $\varepsilon \rightarrow 0$. Since $\lambda(\phi)$ is actually independent of t we will henceforth often suppress the t -variable (we only have to keep in mind that it is considered to be a Colombeau function in \mathbb{R}^2). Substituting $\mu(\phi)y \rightarrow y$ in the integrals and rewriting $\int_x^\infty \chi = 1 - \int_{-\infty}^x \chi$ we have

$$\lambda(\phi, x) = c_1 + (c_2 - c_1) \int_{-\infty}^{\mu(\phi)x} \chi(y) dy. \tag{11}$$

Furthermore, we will make the following *physical assumptions* about the modeling:

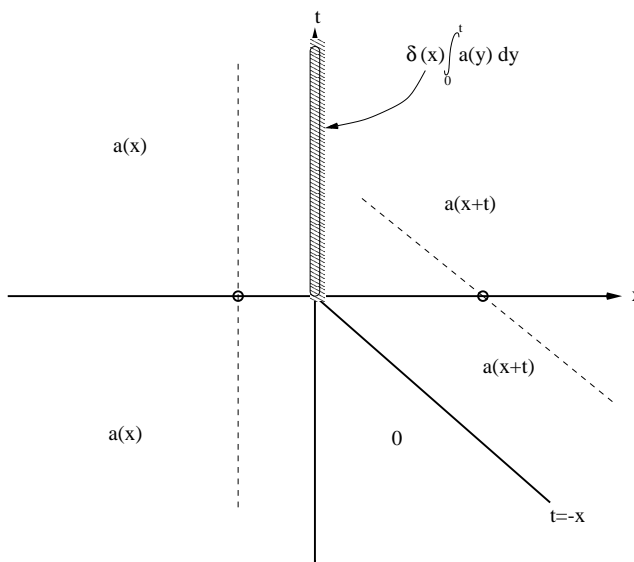
- (i) $c_2 > c_1 \geq 0$ or in terms of the refraction index $n = c_1/c_2$ we have $0 \leq n < 1$
- (ii) all regularized medium approximations have non-negative sound velocities $\lambda(\phi)$, i.e. we have

$$\psi(z) := c_1 + (c_2 - c_1) \int_{-\infty}^z \chi(y) dy \geq 0$$

(this is guaranteed for example if χ is non-negative).

By the results in (15) the Cauchy problem (8)-(9) always has a unique solution U in the Colombeau algebra $\mathcal{G}(\mathbb{R}^2)$. We will actually give a short account of the more general results from (11) in subsection 3.1 below.

In particular, in the case $c_1 = 0$ the following is shown in (16), ex. 17.6: whenever A is (the canonical image of) a locally integrable function a then U has a distributional shadow which can be computed explicitly and has the properties (i)-(iv) above (formula on p. 163 in (16); note that the last term tends to 0 as a distribution in x when $t \rightarrow 0$). It can be nicely illustrated by the following figure (redrawn from (16), Fig. 4.3, p. 164).



Focusing on the possible singularity structure of the Colombeau solution U in the general case $c_1 \geq 0$ a kind of extrapolation from this picture of the distributional limit might suggest what to expect or to investigate in more detail in different regions of the space-time domain:

- (i) in the region V_1 the characteristic flow propagates singularities of the initial data a along the lines parallel to $x + c_1 t = 0$ (vertical if $c_1 = 0$); this should be reflected in spectral (Fourier, cotangent) components of the wave front set of U being parallel to $c_1 \xi - \tau = 0$ (or horizontal if $c_1 = 0$) over this region.
- (ii) in the region V_2 the characteristic flow propagates initial singularities of a along the lines $t = -x/c_2 + t_0$ ($t_0 > 0$) with cotangent components of the wave front set being perpendicular; note that eventually, these singularities will hit the positive half of the axis $x = 0$.
- (iii) the boundaries of the two wedges W_2 and W_1 (which is empty if $c_1 = 0$) will be part of the singular support as long as a does not vanish of infinite order at 0; e.g., in the case $c_1 = 0$ and if a is continuous at 0 and $a(0) \neq 0$ there will be jumps across the half lines $\{0\} \times \mathbb{R}_-$ and $\{(s, -s/c_2) | s > 0\}$ (with cotangent wave front set coordinates typically perpendicular to it).
- (iv) the most striking non-classical features seem to appear along the positive t -axis: for $c_1 = 0$ we saw that there piles up an additional delta-like singularity; this seems to be caused by the coefficient singularity and not (only) by the initial data or a singular source term in the equation (note that even if a is smooth this singularity along $x = 0, t > 0$ appears!); in general, i.e., for arbitrary $0 \leq c_1 < c_2$ and initial value $a \in \mathcal{D}'$, it is not easy to guess what effect on the microlocal properties of U to expect by the interaction of the medium singularity at $x = 0$ with the “arriving” initial singularities propagating in from the right.

The aim of the current paper is to use Colombeau theory as a unifying approach and a systematic tool rather than employing further ad-hoc methods and special product formulas in this problem. In the unifying analysis we also wish to allow for $c_1 = 0$, i.e. the original Hurd-Sattinger situation. In particular, we will study the set of generalized (bi)characteristics and the nonlinear interaction of the singularities at the medium discontinuity from a microlocal point of view.

2.2 Modeling of coefficients with equivalent microlocal properties

Before studying microlocal properties of a Colombeau solution to initial value problems like (8)-(9) we first have to carefully inspect if a transfer or modeling process from given distributional data to appropriate Colombeau objects respects the properties we are interested in. Thereby we want to keep enough flexibility in the modeling methods and also ensure later applicability of more general solvability results to equations of the above type. To this end we not only consider the canonical embedding $\mathcal{D}' \hookrightarrow \mathcal{G}$ but will allow a wider

class of related mappings constructed via more general combinations of convolution and scaling, as already encountered in (10).

The foundation of an intrinsic regularity theory within Colombeau algebras was laid in (16), Sect. 25, via the definition of the subalgebra $\mathcal{G}^\infty \subseteq \mathcal{G}$ of *regular Colombeau functions*. Its elements are exactly those generalized functions having representatives with the same power of ε -growth in each derivative on compact sets. This is motivated by the classical result that a distribution all whose derivatives are measures, and are therefore of order 0, is a smooth function. The algebra of regular functions has the property

$$\mathcal{G}^\infty \cap \mathcal{D}' = \mathcal{C}^\infty,$$

hence the notion of (smooth) regularity is consistent within the subspace of distributions.

By locality of the \mathcal{G}^∞ -property we have a consistent extension of the notion of *singular support* (singsupp_g) of a Colombeau function, defined as the complement of the largest open set on which a generalized function is regular in the above sense. We illustrate this regularity notion in two simple situations.

Example 5. []

(i) $\text{singsupp}_g \iota(\delta) = \{0\} = \text{singsupp } \delta$: since $\iota(\delta)|_{\mathbb{R}^n \setminus \{0\}} = 0$ we only have to check at $x = 0$; the typical representative of $\iota(\delta)$ is $w(\phi, x) = \phi(x)$ and

$$|\partial^\alpha w(\phi_\varepsilon, 0)| = \varepsilon^{-k-1} |\partial^\alpha \phi(0)|;$$

this is a lower bound for the supremum taken over any compact subset containing 0. Whatever $N \in \mathbb{N}_0$ we choose there is a $\phi \in \mathcal{A}_N$ with $\partial^\alpha \phi(0) = c \neq 0$ for infinitely many α . (By the tensor product structure of elements in $\mathcal{A}_N(\mathbb{R}^n)$ for $n > 1$ it suffices to prove this for $\mathcal{A}_N(\mathbb{R})$; one may start with $\psi \in \mathcal{A}_0(\mathbb{R})$, near 0 of the form $\psi(x) = (e^x + e^{-x})/2$ and proceed as in the second part of the proof of Lemma 9.0 in (16).)

(ii) we show that Λ , as defined in (10), is in $\mathcal{G}^\infty(\mathbb{R}^2)$ — this is essentially included already in the remarks preceding (16), Thm. 25.2; we give some details because for the applications we have in mind this also points out the need for (an obvious) refinement of regularity theory which we are going to sketch below. Any t -derivative of order ≥ 1 of the representative λ in (10) gives 0 and $\lambda(\phi_\varepsilon, x, t)$ is bounded uniformly for $0 < \varepsilon < 1$. Therefore it is sufficient to check x -derivatives of order $k \geq 1$. Using the short notation $\mu_\varepsilon := \mu(\phi_\varepsilon) = O(\log(1/\varepsilon))$ we easily obtain the estimate

$$|\partial_x^k \lambda(\phi_\varepsilon, x, t)| = |\partial_x^k \int_{-\infty}^x \mu_\varepsilon \chi(\mu_\varepsilon y) dy| = \mu_\varepsilon^k |\chi^{(k-1)}(\mu_\varepsilon x)| \leq \mu_\varepsilon^k \|\chi^{(k-1)}\|_{L^\infty} = O((\log(1/\varepsilon))^k) = O(1/\varepsilon), \quad (12)$$

which tells us that the \mathcal{G}^∞ -property is satisfied with uniform ε -power -1 .

Localization of a Colombeau function U near a point x_0 can be achieved by using cutoff functions $\varphi \in \mathcal{D}$ with $\varphi(x_0) = 1$. Then φU has compact support and a natural extension of Fourier transform is available to analyze its singularity spectrum. This was initiated in (3, 9, 13) extending many results from distribution theory in terms of the wave front sets ((8), Ch. 8). The *generalized wave front set* WF_g of Colombeau functions is also a consistent extension because we have $\text{WF}_g(\iota(f)) = \text{WF}(f)$ if $f \in \mathcal{D}'$ (cf. (13), Thm. 3.8, and (9), Cor. 24 and Thm. 25).

In example 5, (ii), above we saw that $\text{singsupp}_g \Lambda = \emptyset$ while $\Lambda \approx H \otimes 1$ and $\text{singsupp} H \otimes 1 = \{0\} \times \mathbb{R}$ (the whole t -axis). The logarithmic rescaling μ_ε in the modeling Colombeau function Λ assures solvability of the equations of interest but also suppresses the original singularity structure. But a closer look at the estimate also shows how this information could be conserved in singsupp_g if measuring regularity in terms of uniform powers of μ_ε in this case. It turns out indeed that we only need to adapt all basic notions of microlocal analysis to more generally scaled ε -growth to be able to preserve the microlocal information upon modeling the distributional coefficients as above.

The refined (i.e., rescaled) regularity theory for Colombeau functions can be developed by obvious modification of the already existing theory in (3, 9, 13, 16). However, for convenience of the reader we give detailed definitions and sketch the proof of “microlocal invariance” below.

First, we specify admissible scaling functions γ and give an appropriate definition of regularity in terms of a subalgebra $\mathcal{G}_\gamma^\infty(\Omega)$ of $\mathcal{G}(\Omega)$, Ω an open subset of \mathbb{R}^m .

Definition 6. []

(i) An *admissible scaling* γ is a continuous function $\gamma : (0, 1) \rightarrow \mathbb{R}_+$ with the following properties:

- (a) $\gamma(r) \rightarrow \infty$ and $\gamma(r) = O(1/r)$ as $r \rightarrow 0$
- (b) for any $s > 0$: $\gamma(sr) = O(\gamma(r))$ as $r \rightarrow 0$

(ii) Let γ be an admissible scaling, the algebra $\mathcal{G}_\gamma^\infty(\Omega)$ of γ -regular Colombeau functions is the set of all $U \in \mathcal{G}(\Omega)$ which have a representative $u \in \mathcal{E}_M(\Omega)$ with the property: for all compact subsets $K \subset \Omega$ there is (a uniform growth order) $N \in \mathbb{N}$ such that for all (derivative orders) $\alpha \in \mathbb{N}_0^m$ there is $M \in \mathbb{N}$ so that for all mollifiers $\phi \in \mathcal{A}_M$ there are constants $C > 0, \eta > 0$ such that it holds

$$\sup_{x \in K} |\partial^\alpha u(\phi_\varepsilon, x)| \leq C \gamma(\varepsilon)^N \quad 0 < \varepsilon < \eta. \tag{13}$$

Remark 7. [] The continuity requirement for scalings in Def. 6, (i), is just a matter of technical convenience (for proofs) and not essential for the regularity property itself; Def. 6, (ii), is exactly Def. 25.1 from (16) if $\gamma(r) = 1/r$.

The *generalized γ -singular support* ($\text{singsupp}_\gamma^\gamma$) of a Colombeau function is then defined as the complement of the largest open set where the function is γ -regular in the sense of Def. 6, (ii).

As in the classical situation we can test γ -regularity of a function with compact support by establishing appropriate decay properties of its Fourier transform. Note that a compactly supported Colombeau function U always has a representative $u(\phi, \cdot)$ with support contained in a fixed compact set (use appropriate cut-off by multiplication with a smooth function) and its generalized Fourier transform may be consistently computed by application of the classical formula to the representative (cf. (9), Rem. 19, (i)). The proof of the following theorem is exactly the same as in (9), Thm. 18, with ε^{-1} substituted by $\gamma(\varepsilon)$ in the final estimates.

Theorem 8. [] If U is a Colombeau function of compact support and $\mathcal{F}U$ denotes its generalized Fourier transform then the following statements are equivalent:

- (i) U is in $\mathcal{G}_\gamma^\infty$
- (ii) $\mathcal{F}U$ is γ -rapidly decreasing in \mathbb{R}^m , that means its representative $\widehat{u(\phi, \cdot)}$ (and therefore any representative) having the following property: there is $N \in \mathbb{N}$ such that for all $p \in \mathbb{N}_0$ we can choose $M \in \mathbb{N}_0$ so that for all $\phi \in \mathcal{A}_M$ there are positive constants C, η such that

$$|\widehat{u(\phi_\varepsilon, \cdot)}(\xi)| \leq C \gamma(\varepsilon)^N (1 + |\xi|)^{-p} \tag{14}$$

holds $\forall \xi \in \mathbb{R}^m, 0 < \varepsilon < \eta$.

To put this result into a microlocal context is straightforward.

Definition 9. [] Let V be a Colombeau function over Ω . A pair $(x_0, \xi_0) \in \Omega \times \mathbb{R}^m \setminus 0$ (this means $\Omega \times \mathbb{R}^m$ with the zero section removed) is called *microlocally γ -regular for V* if there is $\varphi \in \mathcal{D}, \varphi(x_0) = 1$, and a conic neighborhood Γ of ξ_0 such that for $U = \varphi V$ the estimate (14) holds $\forall \xi \in \Gamma$, i.e., $\mathcal{F}(\varphi V)$ is γ -rapidly decreasing in the cone Γ . The generalized wave front set $\text{WF}_\gamma^\gamma(V)$ is the complement (in $T^*\Omega \setminus 0$) of all microlocally γ -regular pairs for V .

As in (9), Sect. 5, the basic properties of WF_γ^γ are easily obtained with the classical procedure. In particular, it is a closed conic subset of $\Omega \times \mathbb{R}^m \setminus 0$ and its projection to Ω gives exactly $\text{singsupp}_\gamma^\gamma$.

Now we are in a position to investigate the microlocal properties of modeling Colombeau functions like Λ given in (10). For example, observe that Λ is obtained from the original distribution $H \otimes 1$ by a (space and time) convolution with the scaled mollifier $(x, t) \mapsto \mu(\phi)^2 \chi(\mu(\phi)x) \chi(\mu(\phi)t)$ making use of the identity

$$1 \stackrel{(t)}{*} \mu(\phi) \chi(\mu(\phi)\cdot) = \int \mu(\phi) \chi(\mu(\phi)t) dt = \int \chi = 1.$$

If we choose the admissible scaling $\gamma(r) = \log(1/r)$ then the first line in (12) shows that along $x = 0$ we will be able to detect a γ -singularity since $\mu_\varepsilon = \gamma(\varepsilon) - \text{const}$, as noted after (10). In fact, the following stronger statement is true in general for modeling processes of this type.

Theorem 10. \square Let γ be an admissible scaling and choose $\chi \in \mathcal{D}(\mathbb{R}^m)$ with $\int \chi = 1$. Define the modeling map $\iota_\chi^\gamma : \mathcal{D}' \rightarrow \mathcal{G}$ by setting $\iota_\chi^\gamma(w)$ equal to the Colombeau class of the $\mathcal{E}_M(\mathbb{R}^m)$ function

$$(\phi, x) \mapsto (w * \chi^\gamma(\phi, \cdot))(x) := \gamma(l(\phi_0))^m (w * \chi(\gamma(l(\phi_0)) \cdot))(x) \quad (15)$$

where $\phi = \phi_0 \otimes \cdots \otimes \phi_0$ with $\phi_0 \in \mathcal{A}_0(\mathbb{R})$. Then we have for any distribution w invariance of the microlocal properties in the following sense

$$\text{WF}_g^\gamma(\iota_\chi^\gamma(w)) = \text{WF}(w). \quad (16)$$

In particular, this includes equality of the singular supports

$$\text{singsupp}_g^\gamma \iota_\chi^\gamma(w) = \text{singsupp } w.$$

Proof. This is essentially an adaption of the proof of (13), Thm. 3.8.

$$\text{Step 1: } (x_0, \xi_0) \notin \text{WF}(w) \implies (x_0, \xi_0) \notin \text{WF}_g^\gamma(\iota_\chi^\gamma(w))$$

Let $\varphi \in \mathcal{D}$ with $\varphi(x_0) = 1$ and $\psi \in \mathcal{D}$ with $\psi = 1$ in a neighborhood of $\text{supp } \varphi$. Using the notation $\chi^\gamma(\phi, x) = \gamma(l(\phi_0))^m \chi(\gamma(l(\phi_0))x)$ the Fourier transform of $\varphi \cdot \iota_\chi^\gamma(w)$ has the representative (with \mathcal{F} denoting the classical Fourier transform)

$$(\phi, \xi) \mapsto \mathcal{F}(\varphi \cdot (w * \chi^\gamma(\phi, \cdot)))(\xi).$$

If ε is small enough the support of $\chi^\gamma(\phi_\varepsilon, \cdot)$ will be so small that for x in the support of φ we may rewrite $(w * \chi^\gamma(\phi_\varepsilon, \cdot))(x)$ as $((\psi w) * \chi^\gamma(\phi_\varepsilon, \cdot))(x)$. Hence the above representative evaluated at (ϕ_ε, ξ) can be written in the form

$$\left(\widehat{\varphi} * \left(\widehat{(\psi w)} \cdot \widehat{\chi^\gamma(\phi_\varepsilon, \cdot)} \right) \right)(\xi) = \int \widehat{\varphi}(\xi - \eta) \widehat{(\psi w)}(\eta) \widehat{\chi}\left(\frac{\eta}{\gamma_\varepsilon}\right) d\eta \quad (17)$$

where we have used the short notation $\gamma_\varepsilon := \gamma(\varepsilon l(\phi_0)) = O(\gamma(\varepsilon))$. By assumption there exists a conic neighborhood Γ of ξ_0 such that for supports of φ, ψ small enough the function $\widehat{(\psi w)}$ is rapidly decreasing in Γ .

As in the proof of (8), Lemma 8.1.1, we can find a closed conic neighborhood $\Gamma_1 \subset \Gamma \cup \{0\}$ and a constant $c > 0$ such that

$$\xi \in \Gamma_1, \eta \notin \Gamma \implies |\xi - \eta| \geq c|\xi|.$$

Then we split the estimation of the integral in (17) at $\xi \in \Gamma_1$ into two parts

$$\begin{aligned} \left| \int \widehat{\varphi}(\xi - \eta) \widehat{(\psi w)}(\eta) \widehat{\chi}\left(\frac{\eta}{\gamma_\varepsilon}\right) d\eta \right| &\leq \int_{\Gamma} |\widehat{\varphi}(\xi - \eta)| |\widehat{(\psi w)}(\eta)| |\widehat{\chi}\left(\frac{\eta}{\gamma_\varepsilon}\right)| d\eta + \\ &+ \int_{\mathbb{R}^m \setminus \Gamma} |\widehat{\varphi}(\xi - \eta)| |\widehat{(\psi w)}(\eta)| |\widehat{\chi}\left(\frac{\eta}{\gamma_\varepsilon}\right)| d\eta =: I_1(\xi) + I_2(\xi). \end{aligned}$$

$I_1(\xi)$ contains only rapidly decreasing integrands, the first and second having bounds of the form $(1 + |\xi - \eta|^2)^{-l}$ and $(1 + |\eta|^2)^{-l}$ times some constant. A routine application of Peetre's inequality yields a bound $(1 + |\xi|^2)^{-l}$ times some other constant, l an arbitrary, positive integer. The remaining integral $\int |\widehat{\psi}(\eta/\gamma_\varepsilon)| d\eta$ is bounded by a constant times $\gamma_\varepsilon^k \int d\eta / (1 + |\eta|)^k$, k large enough but fixed.

In $I_2(\xi)$ we first estimate $|\widehat{\varphi}(\xi - \eta)|$ by (a constant times) $(1 + |\xi - \eta|)^{-l}$, l an arbitrary, positive integer. Using the above stated property of Γ_1 this is bounded by $(1 + c|\xi|)^{-l}$. The remaining integral involves the polynomially bounded factor $|\widehat{(\psi w)}(\eta)|$ (since ψw is smooth and has compact support) which together with the last rapidly decreasing factor gives again a bound of the form γ^k times some constant, k large enough but fixed.

To summarize, the generalized Fourier transform of $\varphi \cdot \iota_\chi^\gamma$ has a representative which can be dominated by $C_{l,\phi,k} \gamma_\varepsilon^k (1 + |\xi|)^{-l}$ for l arbitrarily large, ε small, k large enough but fixed, and $\xi \in \Gamma_1$. This finishes the first step.

$$\text{Step 2: } (x_0, \xi_0) \notin \text{WF}_g^\gamma(\iota_\chi^\gamma(w)) \implies (x_0, \xi_0) \notin \text{WF}(w)$$

From the assumption there is a conic neighborhood Γ of ξ_0 and $\varphi \in \mathcal{D}$, $\varphi(x_0) = 1$, and $N \in \mathbb{N}_0$ such

that for arbitrary $p \in \mathbb{N}_0$ we can find $M \in \mathbb{N}_0$ so that for all mollifiers $\phi \in \mathcal{A}_M$ with appropriate positive constants C and ε_0 it holds

$$|\mathcal{F}(\varphi \cdot (w * \chi^\gamma(\phi_\varepsilon, \cdot)))(\xi)| \leq C \gamma_\varepsilon^N (1 + |\xi|)^{-p} \quad \forall \xi \in \Gamma, 0 < \varepsilon < \varepsilon_0.$$

Let $\psi \in \mathcal{D}$ with $\psi = 1$ in a neighborhood of $\text{supp } \varphi$. We have

$$|\widehat{(\varphi w)}(\xi)| \leq |\mathcal{F}(\varphi \cdot (w - w * \chi^\gamma(\phi_\varepsilon, \cdot)))(\xi)| + |\mathcal{F}(\varphi \cdot (w * \chi^\gamma(\phi_\varepsilon, \cdot)))(\xi)|$$

where we can estimate the second term on the right hand side for $\xi \in \Gamma$ as above. As noted in the first step of the proof for ε small we may insert ψ as additional factor for w in the above convolutions and can therefore rewrite the first term in the form

$$\begin{aligned} |\mathcal{F}(\varphi \cdot ((\psi w) * (\delta_0 - \chi^\gamma(\phi_\varepsilon, \cdot)))(\xi)| &= |\widehat{\varphi} * (\widehat{(\psi w)} \cdot (1 - \widehat{\chi^\gamma(\phi_\varepsilon, \cdot)}))(\xi)| \leq \\ &\leq \int |\widehat{\varphi}(\eta)| |\widehat{(\psi w)}(\xi - \eta)| |1 - \widehat{\chi}(\frac{\xi - \eta}{\gamma_\varepsilon})| d\eta. \end{aligned}$$

We can use a polynomial bound (in $\xi - \eta$) for the second factor and by Taylor expansion a bound $|\xi - \eta|/\gamma_\varepsilon$ times a constant for the third factor (note that $\widehat{\chi}$ is real analytic and $\widehat{\chi}(0) = 1$). Altogether the first two factors can be bounded by $C \gamma_\varepsilon^{-1} (1 + |\xi - \eta|^2)^L$, $L \in \mathbb{N}$ fixed, which is in turn bounded by $C' \gamma_\varepsilon^{-1} (1 + |\xi|^2)^L (1 + |\eta|^2)^L$. Since $\widehat{\varphi}$ is rapidly decreasing $\int |\widehat{\varphi}(\eta)| (1 + |\eta|^2)^L d\eta$ is bounded by some constant which yields finally a bound of the form $\gamma_\varepsilon^{-1} (1 + |\xi|^2)^L$ times some constant.

To summarize, we may state that there is $N \in \mathbb{N}$ and $M \in \mathbb{N}$ such that for all $p \in \mathbb{N}$ there is some positive constant C so that

$$|\widehat{(\varphi w)}(\xi)| \leq C \left(\frac{(1 + |\xi|)^{M+1}}{\gamma_\varepsilon} + \frac{\gamma_\varepsilon^N}{(1 + |\xi|)^p} \right)$$

is valid for all $\xi \in \Gamma$. For $|\xi| \geq 1$ we may further rewrite this with some positive constants c', c'' in the form

$$|\xi|^p \gamma_\varepsilon^{-N} |\widehat{(\varphi w)}(\xi)| \leq c' \gamma_\varepsilon^{-N-1} |\xi|^{p+M+1} + c''. \tag{18}$$

We can now proceed using exactly the idea at the end of the proof of (16), Thm. 25.2. We assert that

$$|\xi|^{\frac{p-(M+1)N}{N}} |\widehat{(\varphi w)}(\xi)| \quad \text{is bounded uniformly for } \xi \in \Gamma, |\xi| \geq 1.$$

Since p may be chosen arbitrary large this will finish step 2.

We prove the above assertion by contradiction assuming that one can find a sequence ξ_j ($j \in \mathbb{N}$) within Γ such that $|\xi_j| \rightarrow \infty$ and

$$|\xi_j|^{\frac{p-(M+1)N}{N}} |\widehat{(\varphi w)}(\xi_j)| \rightarrow \infty$$

as $j \rightarrow \infty$. For each j we can choose ε_j such that $\gamma_{\varepsilon_j} = |\xi_j|^{\frac{p-(M+1)N}{N}}$; we may assume that $\varepsilon_j \rightarrow 0$ as $j \rightarrow \infty$. If we apply the inequality (18) to ξ_j and ε_j successively we arrive at the contradiction that the right hand side of (18) stays bounded while the left hand side tends to ∞ . \square

As an immediate application we may now state that for $\Lambda \in \mathcal{G}(\mathbb{R}^2)$, as constructed in (10), and $\gamma(r) = \log(1/r)$ we have

$$\text{WF}_g^\gamma(\Lambda) = \text{WF}(R \otimes 1) = \{0\} \times \mathbb{R} \times (\mathbb{R} \times \{0\} \setminus \{(0, 0)\}).$$

In this short development we focused on the application to embedding or modeling distributions within Colombeau theory. This extended scope of microlocal analysis goes even beyond distribution theory.

3 UNIQUE SOLVABILITY AND ANALYSIS OF THE GENERALIZED CHARACTERISTIC SET

This section is devoted to the analysis of the differential operator in equation (8). Subsection 3.1. serves as a short introduction to relevant results about symmetric hyperbolic systems within Colombeau theory in general and subsection 3.2 collects useful formulae for the solution in our example in particular. In subsection

3.3 we investigate in detail the generalized characteristic set of the operator which in the classical case is the key to the study of propagation of singularities.

3.1 Unique solvability in Colombeau algebras

As an orientation and to point out the rich potential in further applications we present a more general framework and theoretical results behind our model problem.

Consider the following hyperbolic Cauchy problem in \mathbb{R}^{m+1} .

$$\partial_t U - \sum_{j=1}^m A_j(x, t) \partial_{x_j} U - B(x, t) U = F(x, t) \tag{19}$$

$$U(x, 0) = G(x) \tag{20}$$

where A_j ($j = 1, \dots, m$), B are real valued generalized functions in $\mathcal{G}(\mathbb{R}^{m+1})$ (in the sense that all representatives are real valued smooth functions) and initial value $G \in \mathcal{G}(\mathbb{R}^m)$. We mention that all statements in this subsection are valid for symmetric hyperbolic systems but we here we will only use the scalar case.

The coefficients will be subject to some restriction on the allowed divergence in terms of ε -dependence. A Colombeau function $V \in \mathcal{G}(\mathbb{R}^n)$ is said to be of *logarithmic type* if it has a representative $v(\phi)$ with the following property: there is $N \in \mathbb{N}$ such that for every $\phi \in \mathcal{A}_N$ there exist $C > 0, \eta > 0$ with

$$\sup_{y \in \mathbb{R}^n} |v(\phi_\varepsilon, y)| \leq N \log\left(\frac{C}{\varepsilon}\right) \quad 0 < \varepsilon < \eta. \tag{21}$$

(This property then holds for any representative.)

Remark 11. \square We want to emphasize (as mentioned already in (11), p. 99) that a slight modification of (15), Prop. 1.5., provides us with the following important result for modeling distributional coefficients: for any w in $W^{-k, \infty}(\mathbb{R}^{m+1})$ ($k \in \mathbb{Z}$) one can construct a Colombeau function \tilde{W} associated with w and being of logarithmic type. For example, we have used exactly this construction in equation (10) to model the discontinuous coefficient $R \otimes 1$ in a way that its derivative is of logarithmic type. The general modeling procedure falls into the class discussed in Thm. 10.

Theorem 12. [Lafon-Oberguggenberger (11)] Assume that A_j and B are constant for large $|x|$ and that $\partial_{x_k} A_j$ ($k = 1, \dots, m$) as well as B is of logarithmic type. Then given initial data $G \in \mathcal{G}(\mathbb{R}^m)$, problem (19)-(20) has a unique solution $U \in \mathcal{G}(\mathbb{R}^{m+1})$.

The proof is a skillful combination of classical methods for hyperbolic systems with careful estimation techniques for Colombeau objects. We will give a rough sketch of the main ideas in proving the existence. As usual one investigates the regularized problem where smooth representatives of the coefficients and the Cauchy data for a fixed parameter ϕ are inserted. (All equations have to hold up to some null ideal element as ϕ varies.)

Since the coefficients of (19) are constant for large $|x|$ the problem can essentially be split into a constant coefficient problem and some variable coefficient problems over compact supports (in space). The patching up of the solutions is possible by the finite propagation speed property of hyperbolic constant coefficient operators.

Focusing then on the case where G has compact support and the projection of $\text{supp}(F)$ onto the hyperplane $t = 0$ is compact — which implies the same properties for their representatives $g(\phi), f(\phi)$ — one starts with the well-known energy estimates for the candidate $u(\phi)$ for (a representative of) the solution U on a finite time strip $[-T, T]$:

$$\frac{d}{dt} \|u(\phi, \cdot, t)\|_2^2 \leq C \left(2\|b(\phi)\|_\infty + \left\| \sum_{j=1}^m \partial_{x_j} a_j(\phi) \right\|_\infty + 1 \right) \|u(\phi, \cdot, t)\|_2^2 + \|f(\phi, \cdot, t)\|_2^2$$

where small letters systematically denote representatives of corresponding Colombeau objects. Here, the 2-norms are with respect to space \mathbb{R}^m and the ∞ -norms are with respect to $\mathbb{R}^m \times [-T, T]$. Using Gronwall's lemma this implies immediately

$$\|u(\phi, \cdot, t)\|_2^2 \leq \left(\|u(\phi, \cdot, 0)\|_2^2 + \int_0^t \|f(\phi, \cdot, s)\|_2^2 ds \right) \exp \left(CT \left(2\|b(\phi)\|_\infty + \left\| \sum_{j=1}^m \partial_{x_j} a_j(\phi) \right\|_\infty + 1 \right) \right)$$

for $-T \leq t \leq T$. This is the point where the logarithmic growth properties of $\partial_{x_j} A_j$ and B enter which together with the moderate growth of all other objects yield the Colombeau estimate in order 0.

The next task is to prove that all derivatives of U are moderate (cf. Def. 3 in the introduction). Instead of employing the classical higher order estimates this is done rather by direct inspection of the equations solved by the derivatives of U . A simple observation shows that once all x -derivatives are estimated, the derivatives with respect to t and of mixed type are estimated successively by differentiating the equation (19). As for the x -derivatives, one can prove by induction that if $v(\phi)$ is the vector of all x -derivatives of $u(\phi)$ of order r then it satisfies a symmetric hyperbolic equation of the form

$$\partial_t v(\phi) - \sum_{j=1}^m \tilde{a}(\phi)_j \partial_{x_j} v(\phi) - \tilde{b}(\phi) v(\phi) = R(u(\phi))$$

where \tilde{a}_j are block diagonal matrices with copies of a_j along the diagonal, \tilde{b} is a matrix built up solely from $\partial_{x_k} a_j$ and b , and $R(u)$ involves only derivatives of u of order less than r . The same type of energy estimate as above applied to this equation gives now the moderateness property at order r if this is true for all orders less than r . This proves then the existence of a Colombeau solution.

Note that the structure of the last induction proof is the key to the fact that logarithmic growth is only needed for $\partial_{x_k} A_j$ and B and not for higher orders. This insight widens the range of applicability of the theorem significantly by allowing for a large class of distributions to be modeled.

Finally, we mention the following consistency result which shows that Colombeau theory actually includes the classically solvable situations.

Proposition 13. [Lafon-Oberguggenberger (11)] In the above Theorem, assume additionally that the coefficients A_j, B are smooth.

(i) If F and G are smooth then the generalized solution $U \in \mathcal{G}(\mathbb{R}^{m+1})$ is equal (in \mathcal{G}) to the classical smooth solution.

(ii) If $F \in L^2(\mathbb{R}, H^s(\mathbb{R}^m))$ and $G \in H^s(\mathbb{R}^m)$ for some $s \in \mathbb{R}$, then the generalized solution $U \in \mathcal{G}(\mathbb{R}^{m+1})$ is associated to the classical solution belonging to $C(\mathbb{R}, H^s(\mathbb{R}^m))$.

As shown in (16), Ex. 17.1-3, the assumptions on the coefficients in the above theorem cannot be dropped in general: issues like nonuniqueness, nonexistence, and infinite propagation may occur.

3.2 Representations of the generalized solution

In our simple example (8)-(9) we can make use of the special structure to obtain even rather explicit formulae for a representative of the unique Colombeau solution U (along the lines of (16), Ex. 17.6).

Consider the representative $\lambda(\phi)$ of Λ given by equation (10). Let $a(\phi)$ be a representative of $A \in \mathcal{G}(\mathbb{R})$. We want to specify a convenient representative $u(\phi)$ of $U \in \mathcal{G}(\mathbb{R}^2)$ given for fixed ϕ as the unique solution of

$$\partial_t u(\phi) - \lambda(\phi) \partial_x u(\phi) = \lambda'(\phi) u(\phi) \tag{22}$$

$$u(\phi, x, 0) = a(\phi, x) . \tag{23}$$

Since $\lambda(\phi)$ is real valued and smooth we can employ the method of characteristics to determine $u(\phi)$: for ϕ fixed denote by $\sigma(\phi, x, t; s)$ the unique smooth and (by boundedness of $\lambda(\phi)$) global solution of the initial value problem ($\dot{\sigma}$ denoting $\frac{d}{ds} \sigma$)

$$\dot{\sigma}(\phi, x, t; s) = -\lambda(\phi, \sigma(\phi, x, t; s)) \tag{24}$$

$$\sigma(\phi, x, t; t) = x . \tag{25}$$

Then $u(\phi)$ of is given by (note again that $\phi \in \mathcal{A}_0(\mathbb{R}^2)$ is $\phi_0 \otimes \phi_0$ for some $\phi_0 \in \mathcal{A}_0(\mathbb{R})$)

$$u(\phi, x, t) = \underbrace{a(\phi_0, \sigma(\phi, x, t; 0))}_{a_\sigma(\phi, x, t)} \exp \left(\underbrace{\int_0^t \lambda'(\phi, \sigma(\phi, x, t; s)) ds}_{e(\phi, x, t)} \right) \tag{26}$$

which we can interpret as the product of the two Colombeau functions $A_\Sigma = \text{cl}[(a_\sigma(\phi, x, t))_\phi]$ and $E = \text{cl}[(e(\phi, x, t))_\phi]$, i.e., $U = A_\Sigma E$.

If we strengthen the physical non-negativity condition on $\lambda(\phi)$ to $\chi \geq 0$ and use (24) we deduce that $\dot{\sigma}(\phi, x, t; s) < 0$ unless $\lambda(\phi, \sigma(\phi, x, t; s))$ vanishes and is stationary in which case the integrand in the definition of e vanishes. Hence $s \mapsto \sigma(\phi, x, t; s)$ is strictly monotone for the relevant values. If $c_1 > 0$ then $\lambda'(\phi, r)/\lambda(\phi, r)$ is always well-defined, smooth, and integrable with respect to r . This allows for the substitution $r = \sigma(\phi, x, t; s)$, $dr = -\lambda(\phi, r)ds$, in the integral and yields

$$e(\phi, x, t) = \exp \left(\int_{\sigma(\phi, x, t; 0)}^x \frac{\lambda'(\phi, r)}{\lambda(\phi, r)} dr \right) = \frac{\lambda(\phi, \sigma(\phi, x, t; 0))}{\lambda(\phi, x)} \quad \text{if } c_1 > 0. \tag{27}$$

In doing estimates upon inserting ϕ_ε later on it is convenient to introduce the short notations $\lambda^\varepsilon = \lambda(\phi_\varepsilon)$ and $a^\varepsilon, \sigma^\varepsilon, u^\varepsilon, a_\sigma^\varepsilon, e^\varepsilon$, in the same way. Then the solution formula for the representative reads

$$u^\varepsilon(x, t) = a^\varepsilon(\sigma^\varepsilon(x, t; 0)) \exp \left(\int_0^t \lambda^{\varepsilon'}(\sigma^\varepsilon(x, t; s)) ds \right) = a_\sigma^\varepsilon(x, t) e^\varepsilon(x, t). \tag{28}$$

The characteristic coordinates $\sigma(\phi, x, t; s)$ depend smoothly on (x, t) . For example, differentiating the equations (24)-(25) with respect to x gives

$$\begin{aligned} \partial_x \dot{\sigma}^\varepsilon(x, t; s) &= -\lambda^{\varepsilon'}(\sigma^\varepsilon(x, t; s)) \partial_x \sigma^\varepsilon(x, t; s) \\ \partial_x \sigma^\varepsilon(x, t; t) &= 1. \end{aligned}$$

This in turn yields

$$\partial_x \sigma^\varepsilon(x, t; s) = \exp \left(\int_s^t \lambda^{\varepsilon'}(\sigma^\varepsilon(x, t; z)) dz \right) \tag{29}$$

which in the situation $c_1 > 0, \chi \geq 0$, can be rewritten as above into the simple expression $\partial_x \sigma^\varepsilon(x, t; s) = \lambda^\varepsilon(x)/\lambda^\varepsilon(\sigma^\varepsilon(x, t; s))$. If we apply ∂_t to (24)-(25) then the initial condition becomes

$$\partial_t \sigma^\varepsilon(x, t; t) = -\dot{\sigma}^\varepsilon(x, t; t) = \lambda^\varepsilon(\sigma^\varepsilon(x, t; t)) = \lambda^\varepsilon(x)$$

yielding

$$\partial_t \sigma^\varepsilon(x, t; s) = \lambda^\varepsilon(x) \exp \left(\int_s^t \lambda^{\varepsilon'}(\sigma^\varepsilon(x, t; z)) dz \right) \tag{30}$$

or if $c_1 > 0, \chi \geq 0$, then $\partial_t \sigma^\varepsilon(x, t; s) = \lambda^{\varepsilon^2}(x)/\lambda^\varepsilon(\sigma^\varepsilon(x, t; s))$.

We make use of the above observations to derive a simple formula for the distributional action of u^ε (for ε fixed) on an arbitrary test function $\psi \in \mathcal{D}(\mathbb{R}^2)$. As a smooth function u^ε acts on ψ via the usual integral formula where we insert equation (28)

$$\langle u^\varepsilon, \psi \rangle = \iint a^\varepsilon(\sigma^\varepsilon(x, t; 0)) e^{\int_0^t \lambda^{\varepsilon'}(\sigma^\varepsilon(x, t; s)) ds} \psi(x, t) dx dt. \tag{31}$$

We change the coordinates $(x, t) \mapsto (y, t)$ with $y = \sigma^\varepsilon(x, t; 0)$, equivalently $x = \sigma^\varepsilon(y, 0; t)$, and substituting (29) into $dx = \partial_y \sigma^\varepsilon(y, 0; t) dy$ arrive at the simple expression

$$\langle u^\varepsilon, \psi \rangle = \iint a^\varepsilon(y) \psi(\sigma^\varepsilon(y, 0; t), t) dy dt \tag{32}$$

where we also used the flow property $\sigma^\varepsilon(y, 0; s) = \sigma^\varepsilon(x, t; s)$ of the characteristic lines.

Alternatively, mimicking the constructions in Thm. 2 (i), we can give a formula analogous to (26) by tracing the characteristic flow back to the boundary values of U at $x = 0$. From (26) we may directly compute a representative of $B := U|_{x=0} \in \mathcal{G}(\mathbb{R})$ (cf. the restriction formula in the introduction)

$$b(\phi_0, t) = a(\phi_0, \sigma(\phi, 0, t; 0)) \exp\left(\int_0^t \lambda'(\phi, \sigma(\phi, 0, t; s)) ds\right). \tag{33}$$

In the integral (31) above we can also change the coordinates by $x = \sigma^\varepsilon(0, r; t)$ with $\sigma^\varepsilon(x, t, r) = 0$, which means to trace back x to the boundary point $(0, r)$, and use (30). By the definition of r and the flow property of σ^ε we have $\sigma^\varepsilon(x, t; s) = \sigma^\varepsilon(0, r; s)$ and therefore

$$\begin{aligned} \langle u^\varepsilon, \psi \rangle &= \lambda^\varepsilon(0) \iint a^\varepsilon(\sigma^\varepsilon(0, r; 0)) e^{\int_0^r \lambda^{\varepsilon'}(\sigma^\varepsilon(0, r; s)) ds} \psi(\sigma^\varepsilon(0, r; t), t) dr dt \\ &= \lambda^\varepsilon(0) \iint b^\varepsilon(r) \psi(\sigma^\varepsilon(0, r; t), t) dr dt. \end{aligned} \tag{34}$$

3.3 The generalized characteristic set and the characteristic flow

Referring to (3), Def. 3, and adapting it to the full Colombeau algebra we will restate the definition of the *generalized characteristic set* for a differential operator with Colombeau functions as coefficients. Assume that $\Omega \subseteq \mathbb{R}^n$ open and that the operator $P : \mathcal{G}(\Omega) \rightarrow \mathcal{G}(\Omega)$ is given by (notation: for $\alpha \in \mathbb{N}_0^n$ we write $|\alpha| = \alpha_1 + \dots + \alpha_n$ and $D^\alpha = (-i\partial_{x_1})^{\alpha_1} \dots (-i\partial_{x_n})^{\alpha_n}$)

$$P(x, D) = \sum_{|\alpha| \leq m} A_\alpha(x) D^\alpha$$

where $A_\alpha \in \mathcal{G}(\Omega)$ and $A_{\alpha_0} \neq 0$ for some α_0 with $|\alpha_0| = m$. If we denote by $\Xi = (\Xi_1, \dots, \Xi_n)$ the Colombeau class of the smooth coordinate functions (ξ_1, \dots, ξ_n) and set $\Xi^\alpha = \Xi_1^{\alpha_1} \dots \Xi_n^{\alpha_n}$ then the *principal symbol* $P_m \in \mathcal{G}(\Omega \times \mathbb{R}^n)$ can be defined by

$$P_m = \sum_{|\alpha|=m} A_\alpha \Xi^\alpha$$

which in terms of representatives $a_\alpha(\phi)$ for A_α reads more familiarly as

$$p_m(\phi, x, \xi) = \sum_{|\alpha|=m} a_\alpha(\phi, x) \xi^\alpha.$$

Here, $p_m(\phi)$ is a representative of P_m .

The following definition describes an ellipticity condition for P and the characteristic set as the regions of non-ellipticity.

Definition 14. [A point $(x_0, \xi_0) \in T^*\Omega \setminus 0$ is called *non-characteristic* for P if there exist a neighborhood V of x_0 and a conic neighborhood W of ξ_0 such that for some $r \geq 0$ there is $N \in \mathbb{N}_0$ such that for all $\phi \in \mathcal{A}_N$ there are $C > 0$, and $1 > \eta > 0$ yielding the estimate

$$|p_m(\phi_\varepsilon, x, \xi)| \geq C\varepsilon^r |\xi|^m \quad \text{for all } x \in V, \xi \in W, 0 < \varepsilon < \eta. \tag{35}$$

The complement (in $T^*\Omega \setminus 0$) of all non-characteristic points defines the *generalized characteristic set* $\text{Char}_g P$.

Remark 15. [In case the coefficients A_α are smooth functions, e.g., are represented by $a_\alpha(\phi) = a_\alpha \in C^\infty$, then $\text{Char}_g P$ reproduces exactly the classical definition of $\text{Char} P$ as the zero set of the principal symbol. One can try to restate this in the spirit of generalized point values, as described in (18). There it is proved that Colombeau functions over an open set are characterized by their evaluations on so-called compactly supported generalized points within the open set — to put in an other way, a Colombeau function can be identified with its graph in this sense. By the above definition we can observe the following: assume that

a generalized pointvalue $(\tilde{x}_0, \tilde{\xi}_0)$ (in the notation of (18), Def. 2.2) is a zero of the principal symbol P_m , i.e., $P_m(\tilde{x}_0, \tilde{\xi}_0) = 0$ as a generalized Colombeau number; if it also happens to be the class of a classical point $(x_0, \xi_0) \in T^*\Omega \setminus 0$ then we must have $(x_0, \xi_0) \in \text{Char}_g P$. In other words, the classical shadow of the generalized zero set of the principal symbol is contained in $\text{Char}_g P$. A further investigation of this relation could lead to a more geometric description or even provide alternative definitions for the notion of a generalized characteristic set.

We will compute the generalized characteristic set for our example operator in (x, t) -space with Colombeau-coefficients

$$P(x, t, D_x, D_t) = i(D_t - \Lambda D_x) - \partial_x \Lambda. \tag{36}$$

First, we note that the principal symbol $P_1(x, t, \xi, \tau)$ has the representative

$$p_1(\phi, x, t, \xi, \tau) = i(\tau - \lambda(\phi, x, t)\xi), \tag{37}$$

where λ is given by (10). Then a point $(x_0, t_0; \xi_0, \tau_0) \in T^*\mathbb{R}^2 \setminus 0$ is not in $\text{Char}_g P$ if there exist a neighborhood V of (x_0, t_0) and a conic neighborhood W of (ξ_0, τ_0) such that for some $r \geq 0$ there is $N \in \mathbb{N}_0$ such that for all $\phi \in \mathcal{A}_N$ there are $C > 0$, and $1 > \eta > 0$ yielding the estimate

$$|\tau - \lambda(\phi_\varepsilon, x, t)\xi| \geq C\varepsilon^r (|\xi| + |\tau|) \quad \forall (x, t) \in V, (\xi, \tau) \in W, 0 < \varepsilon < \eta. \tag{38}$$

Using (11) we have

$$|p_1(\phi_\varepsilon, x, t, \xi, \tau)| = \left| \tau - \xi \left(c_1 + (c_2 - c_1) \int_{-\infty}^{\mu_\varepsilon x} \chi(y) dy \right) \right|$$

with the notation $\mu_\varepsilon = \mu(\phi_\varepsilon)$ as in example 5, (ii). This is independent of t so we will only distinguish the cases $x_0 < 0$, $x_0 = 0$, and $x_0 > 0$ in the following computations.

As expected we reproduce the classical cases on either side of the medium discontinuity: if $x_0 < 0$ then $\int_{-\infty}^{\mu_\varepsilon x} \chi = 0$ if ε is small for x near x_0 giving $|P_1(\phi_\varepsilon, x, t, \xi, \tau)| = |\tau - c_1 \xi|$; similarly because of $\int_{-\infty}^{\mu_\varepsilon x} \chi = 1$ near $x_0 > 0$ and small ε we obtain $|P_1(\phi_\varepsilon, x, t, \xi, \tau)| = |\tau - c_2 \xi|$.

For the case $x_0 = 0$ we choose an interval $[-\alpha, \alpha]$ ($\alpha > 0$) as neighborhood and have to estimate

$$\min_{|x| \leq \alpha} \left| \tau - \xi \left(c_1 + (c_2 - c_1) \int_{-\infty}^{\mu_\varepsilon x} \chi(y) dy \right) \right| = \min_{|x| \leq \alpha} |\tau - \xi \psi(\mu_\varepsilon x)| \tag{39}$$

from below. By homogeneity in (ξ, τ) it is sufficient to restrict to the situation $|\xi| + |\tau| = 1$ and estimate the above expression by some constant times some power of ε to detect non-characteristic directions.

We divide the investigation of the case $x_0 = 0$ in further sub-cases concerning the (ξ, τ) -directions in the cotangent part of $\text{Char}_g P$:

- if $\xi = 0$ we have $|\tau| = 1$ and $|\tau - \xi\psi| = 1 > 0$; hence the directions $(0, \pm 1)$ are non-characteristic
- if $\tau\xi < 0$ we have $|\tau - \xi\psi| = (-\tau\xi + \xi^2\psi)/|\xi| \geq |\tau| > 0$ (independent of x and ε) so the second and fourth open quadrants in the (ξ, τ) -plane consist of non-characteristic directions only
- if $\xi\tau \geq 0$ we have $\xi = \pm 1 - \tau$ and can rewrite

$$|\tau - \xi\psi| = |\tau(1 + \psi) \mp \psi| = \underbrace{(1 + \psi)}_{1 \leq \leq C_\chi} |\tau \mp \frac{\psi}{1 + \psi}|$$

where $C_\chi = c_1 + (c_2 - c_1)\|\chi\|_{L^1}$. This is bounded from below by $C\varepsilon^r$ if and only if $\tau \neq 0$ and $|\tau| \geq C\varepsilon^r + \frac{\psi}{1 + \psi}$ or $\tau = 0$ and $\psi \geq C\varepsilon^r$ for $|x| \leq \alpha$ and ε small enough. If we define the χ -dependent quantities (note that χ has compact support and $\int \chi = 1$)

$$-\|\chi\|_{L^1} \leq \chi_0 := \min_z \int_z^z \chi(y) dy \leq 0, \quad 1 \leq \chi_1 := \max_z \int_{-\infty}^z \chi(y) dy \leq \|\chi\|_{L^1}$$

then the corresponding quantities $\psi_0 := \inf \psi$ and $\psi_1 := \sup \psi$ can be bounded as follows

$$0 \leq \psi_0 = c_1 + (c_2 - c_1)\chi_0 \leq c_1 < c_2 \leq \psi_1 = c_1 + (c_2 - c_1)\chi_1 \leq C_\chi .$$

Hence we can be sure that $\tau \neq 0$ defines a non-characteristic direction if $|\tau| > \frac{\psi_1}{1+\psi_1}$ (note that $\psi_1/(1+\psi_1) < 1$) and that $(\pm 1, 0)$ is non-characteristic if and only if $\psi_0 > 0$.

On the other hand if we assume that for $|\tau| < 1$ the equation

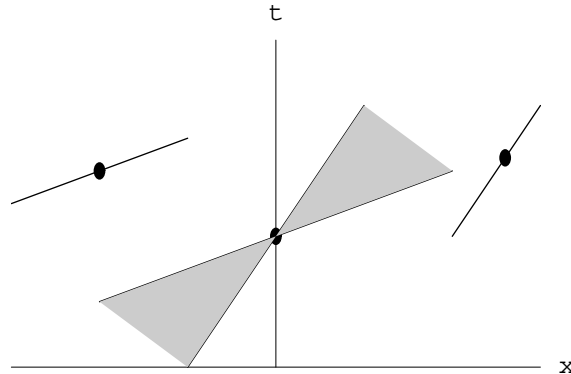
$$\psi(z) = c_1 + (c_2 - c_1) \int_{-\infty}^z \chi(y) dy = \frac{|\tau|}{1 - |\tau|}$$

is solvable for some $z \in \mathbb{R}$ then $p_1(\phi_\varepsilon, x, t, \pm 1 - \tau, \pm |\tau|)$ vanishes identically on the set $\{(x, t, \varepsilon) \mid t \in \mathbb{R}, |x| \leq \alpha, \varepsilon > 0 : \mu_\varepsilon x = z\}$ which includes x arbitrary close to $x_0 = 0$ and ε arbitrary small. Hence in this case the directions $(1 - |\tau|, |\tau|)$ in the first quadrant and $(-1 + |\tau|, -|\tau|)$ in the third quadrant are characteristic. This situation appears if $\frac{\psi_0}{1+\psi_0} \leq |\tau| \leq \frac{\psi_1}{1+\psi_1}$.

To summarize in $\text{Char}_g P$ we have the following pictures for the cotangent directions (ξ, τ) over the base point (x, t) .

Proposition 16. \square Consider P as given in (36) and assume that $(x, t, \xi, \tau) \in \text{Char}_g P$. Then if $x < 0$ then $\tau = c_1 \xi$ and if $x > 0$ then $\tau = c_2 \xi$, if $x = 0$ the characteristic directions cover the cones $\psi_0 \xi \leq \tau \leq \psi_1 \xi$ with $\xi > 0$ and $\psi_0 \xi \geq \tau \geq \psi_1 \xi$ with $\xi < 0$. In particular, this is also true for the case $c_1 = 0$.

Example 17. \square If the modeling mollifier χ is chosen to be nonnegative we have $\psi_0 = c_1$ and $\psi_1 = c_2$ and we see that the characteristic cotangent directions at points $(0, t)$ “interpolate” between the characteristic directions on either side of the axis $x = 0$.



Note that by inspection of the possible values of ψ_0 and ψ_1 one easily verifies that these are actually the *minimal cones* appearing as χ varies in the set of real valued test functions with integral 1.

In the situation of the last example — that is $\chi \geq 0$ — we will also give a more detailed picture about the behavior of the characteristic flow $(x, t, s) \mapsto (\sigma^\varepsilon(x, t; s), s)$ in the plane as $\varepsilon \rightarrow 0$. For $\varepsilon > 0$ fixed this represents the global flow according to the smooth and bounded vector field $(-\lambda^\varepsilon(x), 1)$ in \mathbb{R}^2 . By $-c_2 \leq -\lambda^\varepsilon \leq -c_1 \leq 0$ the space component $\sigma^\varepsilon(x, t; s)$ is non-increasing with respect to the flow parameter s , i.e., the flow never turns to the right. Since the characteristics are given globally as $s \mapsto (\sigma^\varepsilon(x, t; s), s)$ and never intersect we also have monotonicity properties in x and t separately (at fixed ε)

$$\sigma^\varepsilon(x', t'; s) \leq \sigma^\varepsilon(x, t; s) \quad \text{if } x' \leq x \text{ and } t' \leq t . \tag{40}$$

Consider any compact set K contained in the open left half plane V_- . K is contained in some closed box $[x'_0, x_0] \times [t'_0, t_0]$ with $x_0 < 0$. We clearly have $\sigma^\varepsilon(x, t; s) \leq x_0$ for all $(x, t) \in K$ and s such that $(\sigma^\varepsilon(x, t; s), s) \in K$. This implies that uniformly for all such $(x, t; s)$

$$\dot{\sigma}^\varepsilon(x, t; s) = -\lambda^\varepsilon(\sigma^\varepsilon(x, t; s)) = -c_1 - (c_2 - c_1) \int_0^{\mu_\varepsilon \sigma^\varepsilon(x, t; s)} \chi(y) dy \geq -c_1 - (c_2 - c_1) \int_0^{\mu_\varepsilon x_0} \chi(y) dy = -c_1$$

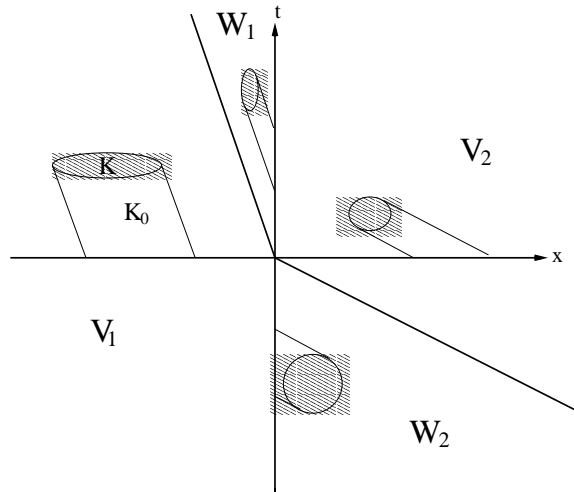
as soon as ε is small enough, say $\varepsilon < \varepsilon_0$. On the other hand since $\chi \geq 0$ the opposite estimate $\sigma^\varepsilon(x, t; s) \leq -c_1$ is always true which implies that for these (x, t) , s , and $0 < \varepsilon < \varepsilon_0$ we have $\sigma^\varepsilon(x, t; s) = -c_1$ and $\sigma^\varepsilon(x, t; t) = x$. This implies that on any compact set $K \subset V_-$ we have eventually (if ε is small enough, e.g., if $|\mu_\varepsilon x_0| > l(\chi)$) $\sigma^\varepsilon(x, t; s) = x + c_1(t - s)$ as long as $(\sigma^\varepsilon(x, t; s), s)$ stays in K .

The case that K is contained in the open right half space V_+ is completely analogous. Therefore we have proved

Lemma 18. \square Assume $\chi \geq 0$ and let K_-, K_+ be compact subsets of V_-, V_+ respectively. Then there is $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$

$$\begin{aligned} \sigma^\varepsilon(x, t; s) &= x + c_1(t - s) && \text{on } \{(x, t, s) \in K \times \mathbb{R} \mid (\sigma^\varepsilon(x, t; s), s) \in K\} \\ \sigma^\varepsilon(x, t; s) &= x + c_2(t - s) && \text{on } \{(x, t, s) \in K \times \mathbb{R} \mid (\sigma^\varepsilon(x, t; s), s) \in K\}. \end{aligned}$$

From this we can easily identify the *domains of dependence* on initial or boundary values for compact subsets within the various open regions defined in the proof of Thm. 2. For example, if K is a compact subset of V_1 then the tubular set $K_0 = \{(x + c_1(t - s), s) \mid st \geq 0, |s| \leq |t|\}$ is also a compact subset of $V_1 \subseteq V_-$. The lemma implies that eventually all characteristic lines joining K with the x -axis will be lines of slope $-1/c_1$ and stay within K_0 . Hence in the solution formula (28) only strictly negative arguments in a^ε and $\lambda^{\varepsilon'}$ will occur if (x, t) varies in a compact set in V_1 . Similarly, for compact subsets of W_1 the characteristic flow eventually will only trace back to boundary values on the positive t -axis, bounded away from $(0, 0)$. We summarize this in the following figure. Note that all this is also valid for $c_1 = 0$ with the only change that W_1 does not appear — the characteristic lines in the left half space are vertical then.



We now come to the most interesting part of the characteristic flow: what happens when the propagating signals cross the t -axis? Since for all ε the characteristic curves cross the x -axis at a certain point we may simply restrict to initial points of the form $(x, 0)$ if all values of s are considered. The case $c_1 = 0$ is already discussed in (17) and we summarize it in the following

Proposition 19. \square Assume $\chi \geq 0$ then the family of smooth functions $(x, s) \mapsto \sigma^\varepsilon(x, 0; s)$ ($\varepsilon > 0$) on \mathbb{R}^2 converges almost everywhere to a continuous function $(x, s) \mapsto \sigma(x, 0; s)$. If $c_1 > 0$ then

$$\sigma(x, 0; s) = \begin{cases} x - c_1 s & \text{if } c_1 s \geq x \text{ and } x < 0 \\ x - c_2 s & \text{if } c_2 s \leq x \text{ and } x \geq 0 \\ \frac{c_2}{c_1} x - c_2 s & \text{if } c_1 s \leq x \text{ and } x \leq 0 \\ \frac{c_1}{c_2} x - c_1 s & \text{if } c_2 s \geq x \text{ and } x \geq 0 \end{cases} \tag{41}$$

and if $c_1 = 0$ then

$$\sigma(x, 0; s) = \begin{cases} x & \text{if } x \leq 0 \\ x - c_2 s & \text{if } c_2 s \leq x \text{ and } x \geq 0 \\ 0 & \text{if } 0 \leq x \leq c_2 s \end{cases} . \tag{42}$$

Proof. The detailed prove for the case $c_1 = 0$ is given in (17), Prop. 3. Note that it does not even assume that χ is nonnegative.

We assume $c_1 > 0$. Following the idea in (17), p. 263, we define $\eta_\varepsilon = \max\{|x| \mid \chi(\mu_\varepsilon x) \neq 0\}$ which is bounded by $0 \leq \eta_\varepsilon \leq l(\chi)/\mu_\varepsilon$ (where $l(\chi)$ is the support number, cf. subsect. 1.3). Therefore $\eta_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$. Note that $\lambda^\varepsilon(x) = c_1$ if $x \leq -\eta_\varepsilon$ and $\lambda^\varepsilon(x) = c_2$ if $x \geq \eta_\varepsilon$ and therefore

$$\sigma^\varepsilon(x, 0; s) = \begin{cases} x - c_1 s & x \leq -\eta_\varepsilon \text{ and } s \geq \frac{x + \eta_\varepsilon}{c_1} \\ x - c_2 s & x \geq \eta_\varepsilon \text{ and } s \leq \frac{x - \eta_\varepsilon}{c_2} \end{cases} .$$

As $\varepsilon \rightarrow 0$ this shows pointwise convergence in the regions $x < 0, s > x/c_1$ and $x > 0, s < x/c_2$ as stated in (41).

Next we consider $x \leq -\eta_\varepsilon$ and follow the characteristic flow through $(x, 0)$ backwards, i.e., for decreasing values of s . Clearly $\sigma^\varepsilon(x, 0; s) = x - c_1 s$ as long as $s \geq (x + \eta_\varepsilon)/c_1$. On the other hand if s_ε marks the entrance point into the region at distance η_ε to the right of the t -axis, i.e., $\sigma^\varepsilon(x, 0; s_\varepsilon) = \eta_\varepsilon$ then we have by the above lemma

$$\sigma^\varepsilon(x, 0; s) = \sigma^\varepsilon(\eta_\varepsilon, s_\varepsilon; s) = \eta_\varepsilon - c_2(s - s_\varepsilon) \quad \forall s \leq s_\varepsilon .$$

Clearly $s_\varepsilon \leq (x + \eta_\varepsilon)/c_1$ because $(\sigma^\varepsilon(x, 0; s), s)$ always moves to the lower right when s is decreasing since $-c_2 \leq \sigma^\varepsilon \leq -c_1 < 0$. By the same estimate for σ^ε we can also estimate s_ε from below by the intersection time s^* of the line $(x - c_1 s, s)$ with the vertical line $(-\eta_\varepsilon, s)$. We have $s^* = (x - \eta_\varepsilon)/c_1$ which shows that $s_\varepsilon \rightarrow x/c_1 =: s_1$ as $\varepsilon \rightarrow 0$. Hence if $x < 0$ and $s < x/c_1$ we have that $\sigma^\varepsilon(x, 0; s) \rightarrow 0 - c_2(s - s_1)$ as $\varepsilon \rightarrow 0$ which proves the assertion in the third line of (41).

The subcase $x \geq \eta_\varepsilon$ is similar but not just a reflection at the t -axis. We now follow the flow through $(x, 0)$ to the right, i.e. for increasing values of s . Let $s_\varepsilon \geq (x - \eta_\varepsilon)/c_2 =: \bar{s}_\varepsilon$ mark the event $\sigma^\varepsilon(x, 0; s_\varepsilon) = -\eta_\varepsilon$ then

$$\sigma^\varepsilon(x, 0; s) = \sigma^\varepsilon(-\eta_\varepsilon, \bar{s}_\varepsilon; s) = -\eta_\varepsilon - c_1(s - \bar{s}_\varepsilon) \quad \forall s \geq s_\varepsilon .$$

We obtain an upper bound for s_ε by considering the time s^* when the line $(\eta_\varepsilon - c_1(s - \bar{s}_\varepsilon), s)$ (this is the line with slope $-1/c_1$ issuing from the entrance point of the flow into the vertical strip of width $2\eta_\varepsilon$ around the t -axis) intersects the vertical $(-\eta_\varepsilon, s)$. Here, we used again the fact that $\sigma^\varepsilon \leq -c_1$. We have $s^* = (x - \eta_\varepsilon)/c_2 + 2\eta_\varepsilon/c_1$ which proves that $s_\varepsilon \rightarrow x/c_2 =: s_2$ as $\varepsilon \rightarrow 0$. Therefore if $x > 0, s > x/c_2$ then $\sigma^\varepsilon(x, 0; s)$ tends to $-0 - c_1(s - s_2)$ which proves the assertion in the fourth line of (41). \square

We thus see that in the limit the characteristic flow produces a kink according to the change of velocity upon transmission through the medium jump. In case $c_1 = 0$ the flow becomes trapped in the singularity as already seen from the figure in subsection 2.1.

4 MICROLOCAL PROPERTIES OF THE GENERALIZED SOLUTION – PROPAGATION OF SINGULARITIES

First we note some consequences concerning the \mathcal{G}^∞ -regularity theory which can be drawn from already existing results. We recall from example 5, (ii), that the Colombeau model coefficient Λ given by equation (10) is in $\mathcal{G}^\infty(\mathbb{R}^2)$ — we remind that this does not mean that it is γ -regular if $\gamma(r) = \log(1/r)$ — and hence the operator P in (36) is a differential operator with \mathcal{G}^∞ coefficients.

Theorem 20. \square Let $A \in \mathcal{G}(\mathbb{R})$ and U be the unique Colombeau solution to problem (8)-(9). Then we have

- (i) $\text{WF}_g(U) \subseteq \text{Char}_g P$.
- (ii) If A is in $\mathcal{G}^\infty(\mathbb{R})$ then U is in $\mathcal{G}^\infty(\mathbb{R}^2)$.

Proof. ad (i): This follows directly from the general theorem about propagation of singularities for linear operators with \mathcal{G}^∞ -coefficients proved by Pilipović' et.al. in (3), Thm. 4. Although the proof is given in detail there for the so-called 'simplified' version of Colombeau algebras (where only ε -parameterization is used) an inspection of the arguments shows that the proof transfers to the 'full' version we are dealing with.

ad (ii): This follows actually from a more general observation, namely that we have \mathcal{G}^∞ -regularity of the solution U to the general hyperbolic problem (19)-(20) if A is \mathcal{G}^∞ . This can be seen by inspection of the estimates in the existence proof from (11) as we presented it in subsection 3.1. \square

However, we are mainly interested in the influence of the original coefficient singularity corresponding to R , which we modeled with γ -scaling, on that of the solution U . Therefore led by Thm. 10 we expect to get more refined information through the analysis of the generalized wave front set $\text{WF}_g^\gamma(U)$ for $\gamma(r) = \log(1/r)$. Note that we clearly have

$$\text{WF}_g^\gamma(U) \supseteq \text{WF}_g(U) ;$$

a particular question is if generalized wave front set can still be bounded by the characteristic set of the operator P . Note that the determination of $\text{WF}_g^\gamma(U)$ also means that according to (26) we have to deal with the product of two γ -singular generalized functions since $U = A_\Sigma E$ and have to give estimates for $\text{WF}_g^\gamma(A_\Sigma E)$ in terms of their respective wave front sets.

To obtain sensible results we also want to exclude unnatural pathologies concerning mixtures of regularity scales. For example, taking the generalized constant A_0 given by $(1/l(\phi))_\phi$ as initial value A would always produce an (x, t) -independent factor $1/l(\phi_\varepsilon) = 1/\varepsilon l(\phi)$ in the expressions for U . But $1/\varepsilon$ can never be dominated by powers of $\log(1/\varepsilon)$ and would thus "simulate" an unwanted kind of singular behavior everywhere in the solution. Of course this is mathematically correct because A_0 is an element in $\mathcal{G}^\infty \setminus \mathcal{G}_\gamma^\infty$ but it is completely misleading if we want to observe propagating singularities caused by distributional initial values. Therefore we assume the initial value A to be of γ -type (in order 0):

(A) Any representative $(a(\phi))_\phi$ of $A \in \mathcal{G}(\mathbb{R})$ has the property that for every compact set K there is $N \in \mathbb{N}_0$ such that $\sup_{x \in K} |a(\phi_\varepsilon, x)| = O(\gamma(\varepsilon)^N) \quad \varepsilon \rightarrow 0$.

$\text{WF}_g^\gamma(U)$ is local with respect to the base space: the cones of irregular directions over a certain base point (x_0, t_0) are detected according to Def. 9 by choosing test functions φ with support near (x_0, t_0) and $\varphi(x_0, t_0) = 1$ and then investigating the decay properties of the Fourier transform $\widehat{(\varphi u^\varepsilon)}(\xi, \tau)$ where as usual $u^\varepsilon = u(\phi_\varepsilon)$ denotes an arbitrary representative of U .

If we use the representative obtained by the method of characteristics in subsect. 3.1 then by (28) an explicit expression for $\widehat{(\varphi u^\varepsilon)}$ is given by

$$\widehat{(\varphi u^\varepsilon)}(\xi, \tau) = \iint e^{-i(\xi x + \tau t) + \int_0^t \lambda^{\varepsilon'}(\sigma^\varepsilon(x, t; s)) ds} a^\varepsilon(\sigma^\varepsilon(x, t; 0)) \varphi(x, t) dx dt . \tag{43}$$

Using equation (32) with $e^{-i(\xi x + \tau t)} \varphi(x, t)$ in place of $\psi(x, t)$ this can be written with characteristic coordinates in the alternative form

$$\widehat{(\varphi u^\varepsilon)}(\xi, \tau) = \iint e^{-i(\xi \sigma^\varepsilon(y, 0; t) + \tau t)} a^\varepsilon(y) \varphi(\sigma^\varepsilon(y, 0; t), t) dy dt . \tag{44}$$

As with the computation of $\text{Char}_g P$ in further investigation we divide \mathbb{R}^2 into several domains according to the geometry of the generalized characteristic flow. For the sake of brevity we will focus now on that part of the forward time domain which includes the transmission from one medium into the other. Thus we will investigate the singular behavior at points $(x_0, t_0) \in V_2 \cup W_2$. The exact microlocal properties at the discontinuity $x = 0$ will follow from a generalized stationary phase analysis to be published in full generality elsewhere.

$(x_0, t_0) \in V_2$ If (x, t) varies in a small neighborhood of (x_0, t_0) then by the results of subsect. 3.3 we have $\sigma^\varepsilon(x, t; s) = x + c_2(t - s)$ for small ε as long as $(\sigma^\varepsilon(x, t; s), s)$ stays within a compact subset of V_2 . Hence if $\text{supp}(\varphi)$ is concentrated in such a neighborhood of (x_0, t_0) we have for small $\varepsilon > 0$

$$\widehat{(\varphi u^\varepsilon)}(\xi, \tau) = \iint e^{-i(\xi x + \tau t)} a^\varepsilon(x + c_2 t) \varphi(x, t) dx dt .$$

It is admissible to assume that φ is of the form $\varphi_1(x + c_2t)\varphi_2(t)$ where φ_1 , resp. φ_2 , is concentrated near $x_0 + c_2t_0$, resp. t_0 . This is sufficient by appealing to property (13) (or rather its γ -analogue) in (9) and noting that $(x + c_2t, t)$ are coordinates in \mathbb{R}^2 . Then we change coordinates according to $y = x + c_2t$ in the integral and obtain

$$\begin{aligned} & \iint e^{-i(\xi y + t(\tau - c_2\xi))} a^\varepsilon(y) \varphi_1(y) \varphi_2(t) dy dt \\ &= \int e^{-i\xi y} a^\varepsilon(y) \varphi_1(y) dy \int e^{-it(\tau - c_2\xi)} \varphi_2(t) dt = \widehat{(\varphi_1 a^\varepsilon)}(\xi) \widehat{\varphi_2}(\tau - c_2\xi). \end{aligned}$$

Here, by assumption (A) the first factor is bounded by

$$\|\varphi_1 a^\varepsilon\|_{L^1} \leq \|\varphi_1\|_{L^1} \sup_{y \in \text{supp}(\varphi_1)} |a^\varepsilon(y)| = O(\gamma(\varepsilon)^N) \text{ for some } N \in \mathbb{N}_0$$

and the second factor is rapidly decreasing in (ξ, τ) if $\tau \neq c_2\xi$. On the other hand if $\tau = c_2\xi$ then since φ_1 is concentrated near $x_0 + c_2t_0$ the whole expression can only be non- γ -rapidly decreasing if this point belongs to $\text{singsupp}_g^\gamma(A)$. Hence we have proved the following

Proposition 21. \square

$$\text{WF}_g^\gamma(U|_{V_2}) \subseteq \{(x, t) \in V_2 \mid x + c_2t \in \text{singsupp}_g^\gamma(A)\} \times \{(r, c_2r) \mid r \neq 0\}.$$

We point out that this is valid for any Colombeau initial value A of γ -type. In particular, any distribution of finite order can be modeled in this way by convolution with a γ -scaled delta net of mollifiers.

It is worth noting that we can even recover the exact shape of $\text{WF}_g^\gamma(U)$ in this region if we model a distributional initial value $a \in \mathcal{D}'(\mathbb{R})$ in an appropriate way. As we saw above within this region we have for ε small

$$u^\varepsilon(x, t) = a^\varepsilon(x + c_2t) = c_2^* a^\varepsilon(x, t)$$

with the slight abuse of the notation c_2^* for the pull-back by the map $c_2(x, t) = x + c_2t$. Assume that A is modeling a distribution a via $a(\phi) = a * \chi_2^\gamma(\phi)$ as in Thm. 10 which we will denote after ε -insertion by $a^\varepsilon = a * \chi_2^\varepsilon$. Then we may rewrite

$$c_2^* a(\phi, x, t) = a * \chi_2^\gamma(\phi)(x + c_2t) = \langle a(y), \chi_2^\gamma(\phi, x + c_2t - y) \rangle.$$

We wish to consider this as a γ -modeling of the distribution $c_2^* a \in \mathcal{D}'(\mathbb{R}^2)$ in order to apply Thm. 10. This can be achieved by the following construction. Choose $\beta_0 \in \mathcal{A}_0(\mathbb{R})$ arbitrary and define

$$\chi_2 = \beta_0 * \frac{1}{c_2} \beta_0\left(\frac{\cdot}{c_2}\right). \tag{45}$$

χ_2 is smooth with compact support and by the simple property (8), (1.3.4), it follows that $\int \chi_2 = 1$. Finally, we set

$$\beta_2(x, t) = \beta_0(x) \frac{1}{c_2} \beta_0\left(\frac{t}{c_2}\right)$$

which defines a test function in \mathbb{R}^2 with integral 1. In the following we use the notation $\beta_0^\gamma(\phi)$, $\beta_2^\gamma(\phi)$, and $\chi_2^\gamma(\phi)$ as in the modeling map of Thm. 10.

Consider the two dimensional convolution

$$c_2^* a * \beta_2^\gamma(\phi)(x, t) = \langle c_2^* a(y, s), \beta_2^\gamma(\phi, x - y, t - s) \rangle.$$

By formula (8), (6.1.1), for the pull-back (e.g., with the function $h(z, r) = (z - r, r/c_2)$ in the notation of the cited equation), then this can be rewritten as

$$\langle (a \otimes 1)(y, s), \beta_2^\gamma(\phi, x - y + s, t - s/c_2)/c_2 \rangle = \langle a(y), \int \beta_2^\gamma(\phi, x - y + s, t - s/c_2) ds/c_2 \rangle.$$

Substituting $t - s/c_2 = r/c_2$ in the integral and using the definition of β_2 via β_0 we finally arrive at

$$c_2^* a * \beta_2^\gamma(\phi)(x, t) = \langle a(y), \left(\beta_0^\gamma(\phi, \cdot) * \frac{1}{c_2} \beta_0^\gamma(\phi, \frac{\cdot}{c_2}) \right)(x + c_2t - y) \rangle$$

which matches exactly the above expression for $c_2^*a(\phi, x, t)$ if χ_2 is given by (45) (observe that a simple computation shows that indeed $\chi_2^\gamma(\phi) = \beta_0^\gamma(\phi) * \beta_0^\gamma(\phi, \cdot/c_2)/c_2$).

To summarize this construction we may state the following.

Proposition 22. [If the initial value A models a distribution a over \mathbb{R}_+ by $A = \iota_{\chi_2}^\gamma(a)$ (in the notation of Thm. 10) with χ_2^γ given by (45) then within the region V_2 we have

$$U = \iota_{\beta_2}^\gamma(c_2^*a) \quad \text{and} \quad \text{WF}_g^\gamma(U) = \text{WF}(c_2^*a) .$$

Note that adapting the arguments of (8), p. 270, it is easy to compute $\text{WF}(c_2^*a)$ explicitly in terms of $\text{WF}(a)$ which recovers the classically expected result within V_2

$$\text{WF}_g^\gamma(U) = \text{WF}(c_2^*a) = \{(x, t; \eta, c_2\eta) \mid (x + c_2t, \eta) \in \text{WF}(a)\} .$$

Note that the situation for $(x_0, t_0) \in V_1$ would be completely. In stating the corresponding results one only has to replace c_2 , resp. V_2 , by c_1 , resp. V_1 , in all the propositions.

$(x_0, t_0) \in W_1$ Note that this case only occurs if $c_1 > 0$ (otherwise $W_1 = \emptyset$). We now use formula (44) where we assume that $\varphi(x, t) = \varphi_1(x)\varphi_2(t)$. Furthermore, let K be some fixed compact set containing $\{y \mid \exists t \in \text{supp}(\varphi_2) : \varphi_1(\sigma^\varepsilon(y, 0; t)) \neq 0\}$ (this is possible by the properties of the characteristic flow established in 3.3). Then we may insert an additional factor $\psi(y)$ in the integrand of (44) where ψ is a test function with $\psi = 1$ on K without changing the value of the integral. We interpret $\varphi_1(\sigma^\varepsilon(y, 0; t))$ as the inverse Fourier transform of $\widehat{\varphi}_1$ evaluated at $\sigma^\varepsilon(y, 0; t)$ giving

$$\widehat{(\varphi u^\varepsilon)}(\xi, \tau) = \frac{1}{2\pi} \iiint e^{-i((\xi-\eta)\sigma^\varepsilon(y, 0; t) + \tau t)} \psi(y) a^\varepsilon(y) \widehat{\varphi}_1(\eta) \varphi_2(t) d\eta dy dt .$$

Since we may assume that ε is small and φ has support concentrated in a small neighborhood of (x_0, t_0) we know that y will vary near $y_0 = \sigma^\varepsilon(x_0, t_0; 0) > 0$. Hence y will stay strictly positive in the support of the integrand. Furthermore, $y_0 = \sigma^\varepsilon(x_0, t_0; 0) \leq x_0 + c_2t_0 < c_2t_0$ since $x_0 < 0$. Therefore we can choose the support of φ so small that $0 < y < c_2t$ whenever $\varphi(\sigma^\varepsilon(y, 0; t), t) \neq 0$. Using the notation from the proof of Prop. 19, case $x > \eta_\varepsilon$, we thus get

$$\sigma^\varepsilon(y, 0; t) = c_1 s_\varepsilon(y) - \eta_\varepsilon - c_1 t$$

on the support of the (original) integrand. We insert this into the above integral formula and interchange the order of integration to obtain

$$\frac{e^{i\xi\eta_\varepsilon}}{2\pi} \int e^{-i\eta_\varepsilon\eta} \widehat{\varphi}_1(\eta) \widehat{\varphi}_2(c_1\eta + \tau - c_1\xi) \underbrace{\int e^{-ic_1(\xi-\eta)s_\varepsilon(y)} \psi(y) a^\varepsilon(y) dy}_{f_\varepsilon(\xi, \eta)} d\eta$$

where the smooth function $f_\varepsilon(\xi, \eta)$ has the property

$$|\partial_\eta^k f_\varepsilon(\xi, \eta)| \leq c_1^k \int |s_\varepsilon(y)|^k |\psi(y) a^\varepsilon(y)| dy \leq C_k \gamma(\varepsilon)^N$$

for C_k independent of ε and some N independent of k because A is assumed to be of γ -type and ψ has compact support. In the integral above we now use $\widehat{\varphi}_2(c_1\eta + \tau - c_1\xi) = e^{i\eta(\tau - c_1\xi)} \widehat{\varphi}_2(c_1\eta)$ and obtain

$$\widehat{(\varphi u^\varepsilon)}(\xi, \tau) = \frac{e^{i\xi\eta_\varepsilon}}{2\pi} \int e^{-i\eta(\eta_\varepsilon - (\tau - c_1\xi))} \widehat{\varphi}_1(\eta) \widehat{\varphi}_2(c_1\eta) f_\varepsilon(\xi, \eta) d\eta .$$

Here, the integrands normalized by $\gamma(\varepsilon)^{-N}$ constitute a bounded family of functions, say $(g^\varepsilon)_\varepsilon$, in $\mathcal{S}(\mathbb{R})$ (w.r.t. the variable η) — recall that this means that every seminorm defining the topology of \mathcal{S} (cf. (8), Def. 7.1.2) is bounded independent of ε . This is guaranteed by the property of f_ε together with the fact that $\varphi_j \in \mathcal{S}$. Therefore if we interpret the above integral as

$$\gamma(\varepsilon)^{-N} \widehat{(\varphi u^\varepsilon)}(\xi, \tau) = \frac{e^{i\xi\eta_\varepsilon}}{2\pi} \widehat{g^\varepsilon}(\eta_\varepsilon - (\tau - c_1\xi))$$

then by continuity of the Fourier transform on \mathcal{S} the family $(\widehat{g^\varepsilon})_\varepsilon$ is bounded in the same sense and we may estimate for arbitrary $k \in \mathbb{N}_0$

$$\gamma(\varepsilon)^{-N} |\widehat{(\varphi u^\varepsilon)}(\xi, \tau)| \leq C_k (1 + |\eta_\varepsilon - (\tau - c_1 \xi)|)^{-k}$$

with a constant C_k independent of ε . If (ξ, τ) vary in closed cones disjoint to $\tau = c_1 \xi$ then since $\eta_\varepsilon \rightarrow 0$ ($\varepsilon > 0$) this estimate proves γ -rapid decrease for $\widehat{(\varphi u^\varepsilon)}(\xi, \tau)$. Thus no cotangent directions different from $\tau = c_1 \xi$ can occur in the γ -wave front set within the region W_1 .

As for the singular support we can use the viewpoint of considering the Colombeau Cauchy problem in W_1 with “initial” value $U|_{x=0} = B$ where a representative $(b(\phi))_\phi$ of B is given by (33). Then we know that by the convergence properties of σ^ε within W_1 we have eventually as ε is getting small enough (let $(x + c_1(t - s), s)$ flow back to the right until its first argument becomes 0)

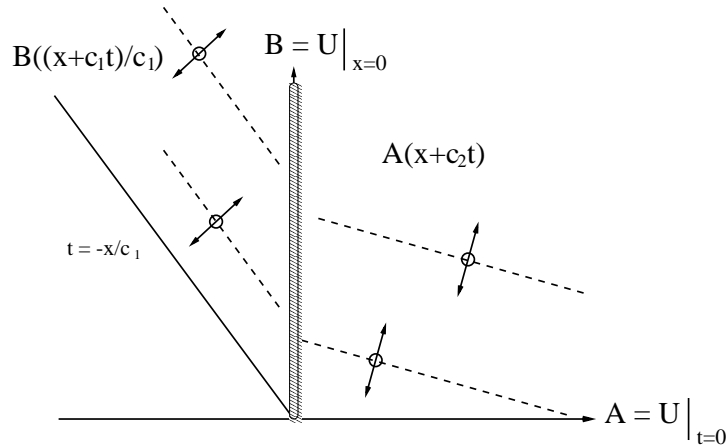
$$u^\varepsilon(x, t) = b^\varepsilon\left(t + \frac{x}{c_1}\right).$$

This shows that any γ -singular behavior of B around a point $r > 0$ can only be transported parallel to the line $x + c_1 t = c_1 r$ into the region W_1 . To summarize we have proved the following.

Proposition 23. \square

$$\text{WF}_g^\gamma(U|_{W_1}) \subseteq \left\{ (x, t) \in W_1 \mid \frac{x + c_1 t}{c_1} \in \text{singsupp}_g^\gamma(B) \right\} \times \{ (r, c_1 r) \mid r \neq 0 \}.$$

We illustrate the results about the regions W_1 and V_2 in a simple qualitative figure (dashed lines denote propagating singularities and solid arrows indicate cotangent directions of the wave front sets).



$x_0 = 0, t_0 > 0$ The precise estimates of cotangent directions for γ -singular behavior will follow from a generalization of Hörmander’s stationary phase approach (8), Thm. 7.7.1, and will be published elsewhere. Here, we conjecture that these directions are bounded by the characteristic cones.

5 DISTRIBUTIONAL SHADOWS

We investigate in short the situation where the initial value A models a given distribution a , i.e. that $A \approx a$. First we state the most general results we can obtain in the regions of undisturbed essentially linear characteristic flow from the horizontal axis without crossing the medium discontinuity.

Proposition 24. \square If $\mathcal{G}(\mathbb{R}) \ni A \approx a \in \mathcal{D}'(\mathbb{R})$ then we have

$$U|_{V_1} \approx c_1^* a \quad U|_{V_2} \approx c_2^* a$$

where c_j^* denotes the distributional pullback via $(x, t) \mapsto x + c_j t$.

Proof. This actually follows from the general consistency result presented in 3.1 when considering the Cauchy problems in the regions V_1 and V_2 separately. However it is easy to prove it directly. If ψ is a test function with $\text{supp}(\psi) \subset V_j$ then we have for the representative $(u(\phi))_\phi$ given in (26) and ε small enough

$$\langle u^\varepsilon, \psi \rangle = \iint a^\varepsilon(x + c_j t) \psi(x, t) dx dt = \langle c_j^* a^\varepsilon, \psi \rangle \rightarrow \langle c_j^* a, \psi \rangle \quad (\varepsilon \rightarrow 0).$$

□

Finally we assume that $a \in L^1_{loc}(\mathbb{R})$. In this case we can take full advantage of Prop. 19 combined with formula (32). The case $c_1 = 0$ is completely covered in (14) and (16), Ex. 17.6, and we presented the result already in subsection 2.1. Therefore we assume $c_1 > 0$. Since $a^\varepsilon \rightarrow a$ in the sense of $L^1_{loc}(\mathbb{R})$ and by the uniform boundedness of $\psi(\sigma^\varepsilon(y, 0; t), t)$ together with Prop. 19 we conclude that as $\varepsilon \rightarrow 0$

$$\langle u^\varepsilon, \psi \rangle \rightarrow \iint a(y) \psi(\sigma(y, 0; t), t) dy dt.$$

Here we can split the integration according to the different regions defined in (41) yielding

$$\begin{aligned} & \int_{-\infty}^0 \int_{-\infty}^{c_1 t} a(y) \psi(y - c_1 t, t) dy dt + \int_0^\infty \int_{-\infty}^0 a(y) \psi(y - c_1 t, t) dy dt \\ & \quad + \int_{-\infty}^0 \int_0^\infty a(y) \psi(y - c_2 t, t) dy dt + \int_0^\infty \int_{c_2 t}^\infty a(y) \psi(y - c_2 t, t) dy dt \\ & \quad + \int_{-\infty}^0 \int_{c_1 t}^0 a(y) \psi\left(\frac{c_2}{c_1}(y - c_1 t), t\right) dy dt + \int_0^\infty \int_{c_2 t}^\infty a(y) \psi\left(\frac{c_1}{c_2}(y - c_2 t), t\right) dy dt \end{aligned}$$

(where the the first two pairs of integrals correspond to the first two lines in (41) respectively). Upon substitution in each integral to obtain a second factor of the form $\psi(x, t)$ and careful inspection of the integral limits one checks that this can be rewritten as (combining then the first two pairs of integrals above into one)

$$\begin{aligned} & \iint H(-x)H(-x - c_1 t)a(x + c_1 t)\psi(x, t) dx dt \\ & \quad + \iint H(x)H(x + c_2 t)a(x + c_2 t)\psi(x, t) dx dt \\ & \quad + \iint H(x)H(-x - c_2 t)a\left(\frac{c_1}{c_2}(x + c_2 t)\right)\psi(x, t) dx dt \\ & \quad + \iint H(-x)H(x + c_1 t)a\left(\frac{c_2}{c_1}(x + c_1 t)\right)\psi(x, t) dx dt. \end{aligned}$$

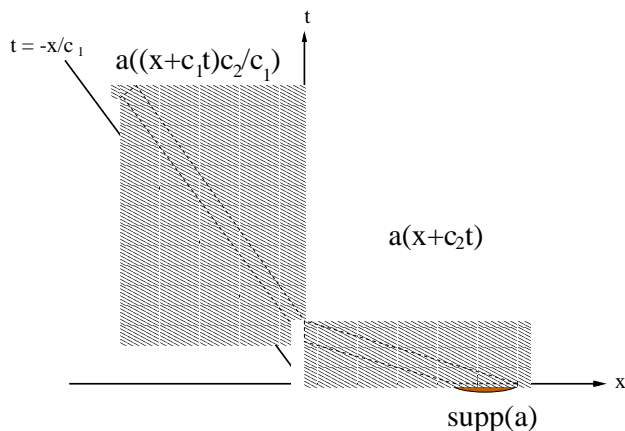
Thus we have proved the following result

Proposition 25. [If $\mathcal{G}(\mathbb{R}) \ni A \approx a \in L^1_{loc}(\mathbb{R})$ then the unique Colombeau solution U to problem (8)-(9) admits a distributional shadow $w \in L^1_{loc}(\mathbb{R}^2)$ which is given by

$$\begin{aligned} w(x, t) = & H(-x)H(-x - c_1 t)a(x + c_1 t) + H(x)H(x + c_2 t)a(x + c_2 t) \\ & + H(x)H(-x - c_2 t)a\left(\frac{c_1}{c_2}(x + c_2 t)\right) + H(-x)H(x + c_1 t)a\left(\frac{c_2}{c_1}(x + c_1 t)\right). \end{aligned}$$

(Here all products are to be understood as products of measurable functions.)

Note that this perfectly resembles the “guess” we made for a distributional solution of problem (1)-(2) in the course of the nonexistence proof in Thm. 2. Assume in particular that a is an approximation to a delta-like source, e.g. a function with small support concentrated around a point $x_0 > 0$, to the right of the medium. Then we can observe that the distributional shadow of the corresponding Colombeau solution looks like a refraction of an incoming signal at the medium discontinuity. Due to the scaling factor c_2/c_1 inside the function argument the support of the signal will be compressed while the amplitude remains unchanged.



Remark 26. [Assume that $\text{supp}(a) \subseteq (0, \infty)$. Then we observe that the wave front set of the limit distribution w , the distributional shadow of U , is given by

$$\begin{aligned} \text{WF}(w|_{V_2}) &= \text{WF}(c_2^*a|_{V_2}) = \{(x, t; \eta, c_2\eta) \in V_2 \times \mathbb{R}^2 \mid (x + c_2t, \eta) \in \text{WF}(a)\} \\ \text{WF}(w|_{W_1}) &= \text{WF}(c_1^*(a(\frac{c_2}{c_1} \cdot))|_{W_1}) \\ &= \{(x, t; \eta, c_1\eta) \in W_1 \times \mathbb{R}^2 \mid (\frac{c_2(x + c_1t)}{c_1}, \eta) \in \text{WF}(a)\} \\ \text{WF}(w)|_{x=0} &= \{(0, c_2t; \xi, \tau) \mid \tau = c_1\eta \text{ or } \tau = c_2\eta, \text{ and } (t, \xi) \in \text{WF}(a)\}. \end{aligned}$$

(The last line denotes the restriction to the submanifold $x = 0$ in the base space only.) The first two lines are immediate. For the third line we note that by direct computation similarly to the first part of the proof of Thm. 2 one obtains for a tensor product of test functions $\varphi_1(x)\varphi_2(t)$ the expression

$$\begin{aligned} \widehat{(\varphi w)}(\xi, \tau) &= \int_{-\infty}^0 e^{-ix(c_2\xi - \tau)/c_2} \varphi_1(x) \langle a, \varphi_2(\cdot - x)/c_2 \rangle e^{-i\tau/c_2} \frac{dx}{c_2} \\ &\quad + \int_0^\infty e^{-ix(c_1\xi - \tau)/c_1} \varphi_1(x) \langle a, \varphi_2(\cdot/c_2 - x/c_1) \rangle e^{-i\tau/c_2} \frac{dx}{c_2}. \end{aligned}$$

This is rapidly decreasing in cones avoiding $\tau = c_1\xi$ and $\tau = c_2\xi$.

We once more emphasize that this completely classical picture was obtained in a situation where no distributional solution to the model equation can exist. The methods of Colombeau theory provided us with a unique solution in the form of a generalized function which can be analyzed in various ways.

ACKNOWLEDGEMENT We would like to thank Michael Oberguggenberger for his valuable comments on section 2 and very helpful hints for the proof of Theorem 2.

REFERENCES

- J.F. Colombeau, *Elementary Introduction to New Generalized Functions*, North-Holland Mathematics Studies 113, Elsevier Science Publishers 1985.
- J.F. Colombeau, Une multiplication générale des distributions. C.R. Acad. Sci. Paris, Sér.I **296**, 357-360 (1983).
- N. Dapic, S. Pilipovic, D. Scarpalezos, Microlocal Analysis of Colombeau's Generalized Functions — Propagation of Singularities, Jour. d'Analyse Math. **75**, 51-66 (1998).
- P. Dierolf, J. Voigt, Convolution and S' -convolution of distributions, Collect. Math. **29**, 185-196 (1978).
- M. Grosser, On the foundations of nonlinear generalized functions II, preprint, University of Vienna 1999.
- M. Grosser, G. Hörmann, M. Kunzinger, M. Oberguggenberger, *Nonlinear theories of generalized functions*, Proceedings of the workshop at the Erwin Schrödinger Institute for Mathematical Physics, Vienna 1997, Chapman & Hall/CRC, Boca Raton, 1999.
- L. Hörmander, Fourier integral operators I, Acta Math. **127**, 79-183 (1971).
- L. Hörmander, *The Analysis of Linear Partial Differential Operators*, Volume I, Springer 1983, 2nd Ed. 1990.
- G. Hörmann, Integration and microlocal analysis in Colombeau algebras, J. Math. Anal. Appl. **239**, 332-348 (1999).
- A. E. Hurd, D. H. Sattinger, Questions of existence and uniqueness for hyperbolic equations with discontinuous coefficients, Trans. Amer. Math. Soc. **132**, 159-174 (1968).
- F. Lafon, M. Oberguggenberger, Generalized solutions to symmetric hyperbolic systems with discontinuous coefficients: the multidimensional case, J. Math. Anal. Appl. **160**, 93-106 (1991).
- J. Mikusiński, Sur la méthode de généralisation de M. Laurent Schwartz sur la convergence faible. Fund. Math. **35**, 235-239 (1948)
- M. Nedeljkov, S. Pilipović, D. Scarpalezos, *The Linear Theory of Colombeau Generalized Functions*, Pitman Research Notes in Mathematics 385, Longman Scientific & Technical 1998.
- M. Oberguggenberger, Hyperbolic systems with discontinuous coefficients: examples. In: B. Stanković, E. Pap, S. Pilipović, V.S. Vladimirov (Eds.), *Generalized Functions, Convergence Structures, and Their Applications*, 257-266, Plenum Press, New York 1988.
- M. Oberguggenberger, Hyperbolic systems with discontinuous coefficients: generalized solutions and a transmission problem in acoustics, J. Math. Anal. Appl. **142**, 452-467 (1989).
- M. Oberguggenberger, *Multiplication of distributions and applications to partial differential equations*, Pitman Research Notes in Mathematics 259, Longman Scientific & Technical 1992.
- M. Oberguggenberger, Nonlinear theories of generalized functions. In: S. Albeverio, W. A. J. Luxemburg, M. P. H. Wolff (Eds.), *Advances in Analysis, Probability, and Mathematical Physics - Contributions from Nonstandard Analysis*. Kluwer, 56 - 74, Dordrecht 1994.
- M. Oberguggenberger, M. Kunzinger, Characterization of Colombeau generalized functions by their point values. Math. Nachr. **203**, 147-157 (1999).
- L.Schwartz, Sur l'impossibilité de la multiplication des distributions. C. R. Acad. Sci. Paris **239**, 847-848 (1954).
- F. Trèves, *Introduction to Pseudodifferential and Fourier Integral Operators*, Volume 1, Plenum Press, New York and London 1980.

